Publ. RIMS, Kyoto Univ.
7 (1971/72), 511-539

Some Aspects of Ornstein's Theory of Isomorphism Problems in Ergodic Theory

By

Gishiro MARUYAMA*

§0. Introduction

In a series of recent papers [4]-[7], D. Ornstein and Friedman carried out fundamentally important investigations about metrical isomorphism in ergodic theory. Motivated by the papers, the authors of [1] and the present paper were engaged in allied isomorphism problems, especially those when generators are not finite and entropies are infinite. Using Ornstein's ideas, the authors of the two papers independently have arrived at similar results. However, the approaches and techniques in the both papers are different.

In the present paper, basically we only assume several fundamental results from [7]. In several respects we can simplify and refine the techniques of [7], thus not using the method in [5], we prove in §4 a generalized version of the main theorem of [5].

The basic probability measure space $(\Omega, m, \mathcal{B})$ is isomorphic with [0, 1] endowed with ordinary Lebesgue measure, the Lebesgue unit interval. A partition of Ω is a system of disjoint subsets of Ω whose union is Ω. From now on partitions are measurable. Script capitals $\mathcal{P}, \mathcal{Q}, \mathcal{A}, \cdots$ will stand for measurable partitions; by writing $\mathcal{P}\subset\mathcal{Q}$ it is meant that $\mathcal{Q}$ refines $\mathcal{P}$, and by $p\in\mathcal{P}$ that p is a generic element of $\mathcal{P}$ referred to as a cell of $\mathcal{P}$. A $\mathcal{P}$-set is that which is represented as a

Received June 29, 1971.
Communicated by K. Yosida.
* Department of Applied Mathematical Science, Tokyo University of Education, Otsuka, Tokyo, 112, Japan.

union of $\mathcal{P}$-cells. To avoid notational complexity, $\mathcal{P}$ will sometimes stand for the σ-algebra consisting of all measurable $\mathcal{P}$-sets. For further developments we are requested to single out three classes of partitions, i.e., finite, countable, and general measurable ones, respectively denoted by $\mathcal{Z}_0, \mathcal{Z}_1, \mathcal{Z}_2$, so that $\mathcal{Z}_0 \subset \mathcal{Z}_1 \subset \mathcal{Z}_2$. $\mathcal{P} \in \mathcal{Z}_1$ is written in the form $\mathcal{P} = (p_1, p_2, \cdots)$. As in [7] every $\mathcal{P} \in \mathcal{Z}_1$ is ordered, and $|\mathcal{P}|$ will stand for the number of $\mathcal{P}$-cells p with $m(p) > 0$.

For $\mathcal{P} \in \mathcal{Z}_2$ and an automorphism T, we frequently use the abbreviations

$$(\mathcal{P}^T)_a^b = \mathcal{P}_a^b = \bigvee_{i=a}^{b} T^i \mathcal{P}$$

so that $\mathcal{P}_a^b$ is different from $\mathcal{P}_b^a$ as ordered partitions; a generator $\mathcal{P} \in \mathcal{Z}_2$ for T is such that $\mathcal{P}_{-\infty}^{\infty} = \mathcal{B}$. To generalize the notion of weak Bernoulli automorphism [7] we prepare several steps of definitions.

Definition. (i) Let T be an automorphism and $\mathcal{P} \in \mathcal{Z}_0$, then $(T, \mathcal{P})$ is called a weak Bernoulli pair (WBP) if

$$\lim_{k\to\infty} \sum_{\substack{p \in P_{-n}^{0} \\ q \in P_k^{k+n}}} |m(p \cap q) - m(p)m(q)| = 0 \tag{0.1}$$

uniformly in n.

(ii) T is called a weak Bernoulli automorphism (WBA) if there exists a generator $\mathcal{R} \in \mathcal{Z}_2$ for T such that we can find an increasing sequence of partitions $\mathcal{R}_n \in \mathcal{Z}_0$, $n \leqq 1$, for which $(T, \mathcal{R}_n)$ are WBP's and $\bigvee_{n\geq 1} \mathcal{R}_n = \mathcal{R}$. Such a generator of T is called a weak Bernoulli generator (WBG).

Especially, if $T^i\mathcal{R}$, $-\infty < i < \infty$, are independent, T is a Bernoulli automorphism (BA), with Bernoulli generator (BG) $\mathcal{R}$.

Let $\mathcal{P} \in \mathcal{Z}_0$, $\mathcal{Q} \in \mathcal{Z}_2$, then $\mathcal{P}$ is ε-approximable by $\mathcal{Q}$, in symbols

$$\delta(\mathcal{P}, \mathcal{Q}) < \varepsilon$$

if for any $p \in \mathcal{P}$, there exists a measurable $\mathcal{Q}$-set A such that

$$m(p \Delta A) < \varepsilon.$$

§1. Classifications and Reductions

First of all, using the notations introduced in §0, we distinguish three classes of automorphisms (α), (β), (γ) according as their WBG's satisfy the conditions: $(\alpha)\mathcal{R}\in\mathcal{Z}_0$ $(\beta)\mathcal{R}\in\mathcal{Z}_1$ $(\gamma)\mathcal{R}\in\mathcal{Z}_2$. When $T\in(\alpha)$, $\mathcal{R}_n=\mathcal{R}$ for a sufficiently large $n, h(T)<\infty$, and in this case, the structure problem of T from the isomorphism view point is settled in [7]. As was mentioned earlier this paper is a basis for our further developments.

To make reductions desired in the following arguments it is convenient to consider the class (δ) of WBA's: $T\in(\delta)$ if $\mathcal{R}\in\mathcal{Z}_1$ is a WBG for T and for any finite subpartition $\mathcal{R}_0$ of $\mathcal{R}$, $(T,\mathcal{R}_0)$ is a WBP. Clearly (δ) is a subclass of (β).

In each case of classiffications

$$h(T)=\lim_{n\to\infty}h(T,\mathcal{R}_n).$$

1° If $h(T,\mathcal{R}_1)=h(T,\mathcal{R}_2)=\cdots$, $T\in(\alpha)$.

Proof. Given positive numbers $\varepsilon_1,\varepsilon_2,\cdots$, with $\sum_{i=1}^{\infty}\varepsilon_i<\infty$, then by a repeated use of Lemma 14 [7], we can successively assign partitions $Q_1, Q_2, \cdots$ and natural numbers $k_1, k_2, \cdots$ satisfying the following conditions:

$$Q_1\subset(\mathcal{R}_2)_{-\infty}^{\infty},\ (T,\mathcal{R}_1)\sim(T,Q_1),^{(*)}$$

$$D(\mathcal{R}_1,Q_1)<\varepsilon_1,\ \delta(\bigvee_{-k_1}^{k_1}T^iQ_2,\mathcal{R}_2)<2^{-1};$$

$$Q_2\subset(\mathcal{R}_3)_{-\infty}^{\infty},\ (T,\mathcal{R}_1)\sim(T,Q_2),$$

$$D(Q_1,Q_2)<\varepsilon_2,\ \delta(\bigvee_{-k_1}^{k_1}T^iQ_2,\mathcal{R}_2)<2^{-1},$$

$$\delta(\bigvee_{-k_2}^{k_2}T^iQ_2,\mathcal{R}_3)<2^{-2};$$

in general

(*) For the meaning of this expression cf. p. 367, [7].

$$Q_n \subset (\mathcal{R}_{n+1})_{-\infty}^{\infty}, \ (T, R_1) \sim (T, Q_n),$$

$$D(Q_{n-1}, Q_n) < \varepsilon_n,$$

$$\delta(\bigvee_{-k_1}^{k_1} T^i Q_n, \mathcal{R}_2) < 2^{-1}, \ \delta(\bigvee_{-k_2}^{k_2} T^i Q_n, \mathcal{R}_3) < 2^{-2},$$

$$\cdots, \ \delta(\bigvee_{-k_n}^{k_n} T^i Q_n, \mathcal{R}_{n+1}) < 2^{-n}.$$

Let $Q_\infty = \lim_{n\to\infty} Q_n$, then

$$(T, \mathcal{R}_1) \sim (T, Q_\infty), \ \delta(\bigvee_{-k_{p-1}}^{k_{p-1}} T^i Q_\infty, \mathcal{R}_p) < 2^{-p+1},$$

and

$$\delta(\bigvee_{-\infty}^{\infty} T^i Q_\infty, \mathcal{R}_m) \leq 2^{-p+1} \qquad \text{for } m \leq p,$$

whence

$$(Q_\infty)_{-\infty}^{\infty} \supset \mathcal{R}_m \qquad \text{for } m = 1, 2, \cdots,$$

that is Q_∞ is a generator for T. This implies that $T \in (\alpha)$.

The remaining essential cases will be settled in the proposition after Theorem 1 and Theorem 2.

So that from now on without loss of generality we may assume that

(1.1) $\quad h(T, \mathcal{R}_1) > 0$ and $h(T, \mathcal{R}_n)$ is strictly increasing with n.

It is easy to show that a direct product of a finite or countable number of WBA's is a WBA. By using the fact that for WBA's in (α) the entropy is a complete invariant it is straightforward to show:

2° Automorphisms S, T represented as direct products of the same finite or countable number of WBA's from (α) are isomorphic if $h(S) = h(T)$.

The second step of reduction is to show the following proposition.

3° To every $T \in (\gamma)$, there is an $S \in (\delta)$ which is isomorphic with T.

Proof. By the above observations we may assume that (1. 1) holds true.

Divide the Lebesgue unit interval [0, 1] into disjoint intervals $I_{11}, \cdots, I_{1s_1+1}$ so that we have a partition $\mathcal{A}_1$ with $H(\mathcal{A}_1)=\sum_{j=1}^{s_1+1}-p_j \log p_j=h(T, \mathcal{R}_1)$, $p_j=\text{meas.}(I_{1j})$, $1\leqq j\leqq s_1+1$. Subdivide I_{1s_1+1} into disjoint subintervals $I_{21}, \cdots, I_{2s_2+1}$ so that they together with $I_{11}, \cdots, I_{1s_1}$ form a partition $\mathcal{A}_2$ with $H(\mathcal{A}_2)=h(T, \mathcal{R}_2)$. Continuing this procedure we have a sequence of partitions $\mathcal{A}_n$, $1\leqq n<\infty$, in which $\mathcal{A}_{n+1}$ is obtained by subdividing the last cell of $\mathcal{A}_n$, and $H(\mathcal{A}_n)=h(T, \mathcal{R}_n)$. Write

$$\mathcal{A}=\bigvee_{n\geq 1}\mathcal{A}_n,$$

then $\mathcal{A}$ is a countable partition with

$$H(\mathcal{A})=\lim_{n\to\infty}H(\mathcal{A}_n)=\lim_{n\to\infty}h(T, \mathcal{R}_n)=h(T).$$

On the basis of this fact and making use of a shift transformation on a direct product measure space one can define a BA S with BG $Q=(q_1, q_2, \cdots)$ such that there exists a sequence of natural numbers n_k, and $Q_k=(q_1, \cdots, q_{n_k}, q_{n_k+1}\cup q_{n_k+2}\cup\cdots)$ satisfies

$$h(S, Q_k)=h(T, \mathcal{R}_k), \qquad k=1, 2, \cdots.$$

By means of the *proof* of Theorem 2 in §4 T is isomorphic with S. Then obviously S is the desired automorphism of type (δ).

§2. Terminologies and Preliminary Lemmas

I owe several concepts, terminologies and notations to Friedman-Ornstein [7]. For details I refer the reader to this paper. All partitions from $\mathcal{Z}_0$, $\mathcal{Z}_1$ are ordered. Given a partition $\mathcal{P}$ and a set E, there is its induced partition $\mathcal{P}|E$ on E. $\mathcal{Z}_1$ becomes a complete metric space by the distance

$$D(\mathcal{P}, Q)=\sum_{k=1}^{\infty}m(p_k \varDelta q_k)$$

$$\mathcal{P}=(p_1, p_2, \cdots),\ Q=(q_1, q_2, \cdots)\in\mathcal{Z}_1,$$

where $\varDelta$ is the symmetric difference. Writing $(p_1, \cdots, p_n, \phi, \phi, \cdots)$ for the partition $\mathcal{P}_0=(p_1, \cdots, p_n)\in\mathcal{Z}_0$, $\mathcal{Z}_0$ is imbedded in $\mathcal{Z}_1$. The size of

$\mathcal{P}\in\mathcal{Z}_1$ is defined to be the number of $\mathcal{P}$-cells p with $m(p)>0$ and denoted by $|\mathcal{P}|$. We write $d(\mathcal{P})$, $d(\mathcal{P}, \mathcal{Q})$ for the distribution of $\mathcal{P}\in\mathcal{Z}_1$, the distance between the distributions of $\mathcal{P}, \mathcal{Q}\in\mathcal{Z}_1$ respectively. The distance $d(a, b)$ between probability vectors $a=(a_1, a_2, \cdots)$, $b=(b_1, b_2, \cdots)$ is likewise defined by

$$d(a, b)=\sum_{i=1}^{\infty}|a_i-b_i|.$$

$\bigvee_{n=1}^{k}\mathcal{P}_n$ can be considered as a member of $\mathcal{Z}_1$ according to the lexicographical ordering. The equality $\bigvee_{n=1}^{k}\mathcal{P}_n=\bigvee_{n=1}^{k}\mathcal{Q}_n$ means that the both sides, when ordered in this way, are equal in $\mathcal{Z}_1$. On the other hand the equality $\mathcal{P}\vee\mathcal{Q}=\mathcal{R}$ means that the both members are equal in reference to suitable orderings of the cells on the both sides.

Let $\mathcal{P}\subset\mathcal{Q}$ be partitions from $\mathcal{Z}_1$. $q\in\mathcal{Q}$, $m(q)>0$, is a proper $\mathcal{Q}$-cell if there exists $p\in\mathcal{P}$, with $p\supset q$, $m(p)>m(q)$, while $p\in\mathcal{P}$, $m(p)>0$, is a proper $\mathcal{P}$-cell, if p contains no proper $\mathcal{Q}$-cell. Improper $\mathcal{P}$-cells and the union of the proper $\mathcal{P}$-cells, when ordered together, form a partition $\mathcal{B}_0$, such that $\mathcal{P}\vee\mathcal{B}_0=\mathcal{P}$; proper $\mathcal{Q}$-cells and the union of proper $\mathcal{P}$-cells, when ordered together, form a partition $\mathcal{B}_1$ such that $\mathcal{P}\vee\mathcal{B}_1=\mathcal{Q}$. A proper $\mathcal{P}$-cell is an improper $\mathcal{Q}$-cell and vice versa. In this case, $\mathcal{P}, \mathcal{Q}$ are said to be properly ordered, if they are ordered in such a way that a proper $\mathcal{P}$-cell corresponds to itself as an improper $\mathcal{Q}$-cell, every improper $p\in\mathcal{P}$ to one of $\mathcal{Q}$-cells contained in p, while $\phi\in\mathcal{P}$ to every $\mathcal{Q}$-cell remained in p. $D(\mathcal{P}, \mathcal{Q})$ can be minimized when they are properly ordered. Obviously $\mathcal{B}_0\subset\mathcal{B}_1$, and if $\mathcal{P}, \mathcal{Q}, \mathcal{B}_0, \mathcal{B}_1$ are properly ordered, it holds that

$$D(\mathcal{P}, \mathcal{Q})=D(\mathcal{B}_0, \mathcal{B}_1)=D(\mathcal{P}\vee\mathcal{B}_0, \mathcal{P}\vee\mathcal{B}_1).$$

For instance, consider $\mathcal{R}=(r_1, r_2, \cdots)$ and its subpartitions $\mathcal{P}=(p_1, p_2, \cdots)$, $\mathcal{Q}=(q_1, q_2, \cdots)$, with $p_i=r_i$ $(1\leqq i\leqq n_1)$, $p_{n_1+1}=\bigcup_{k=n_1+1}^{\infty}r_k$, $p_i=\phi\,(n_1+1<i<\infty)$, $q_i=r_i$ $(1\leqq i\leqq n_2)$, $q_{n_2+1}=\bigcup_{k=n_2+1}^{\infty}r_k$, $q_i=\phi\,(n_2+1<i<\infty)$, and $n_1\leqq n_2$. Then $\mathcal{P}\subset\mathcal{Q}$, and if we put

$$\mathcal{B}_0=(\bigcup_{i=1}^{n_1}r_i,\ \bigcup_{i=n_1+1}^{\infty}r_i,\ \phi,\ \cdots), \tag{2.1}$$

$$\mathcal{B}_1=(\bigcup_{i=1}^{n_1} r_i,\ r_{n_1+1},\ \cdots,\ r_{n_2},\ \bigcup_{i=n_2+1}^{\infty} r_i,\ \phi,\ \cdots)$$

one obtains

$$D(\mathcal{B}_0,\ \mathcal{B}_1)=2\sum_{i\geq n_1+2} m(r_i)=D(\mathcal{P},\ \mathcal{Q}).$$

The following elementary inequality is useful for evaluating $|h(T,\ \mathcal{A})-h(T,\ \mathcal{C})|$.

0° Let $\mathcal{A}=(a_1,\ \cdots,\ a_l)$, $\mathcal{C}=(c_1,\ \cdots,\ c_k)\in\mathcal{Z}_0$, and T be an arbitrary automorphism, then

$$|h(T,\ \mathcal{A})-h(T,\ \mathcal{C})|<E(\mathcal{A},\ \mathcal{C}), \tag{2.2}$$

where

$$E(\mathcal{A},\ \mathcal{C})=-D(\mathcal{A},\ \mathcal{C})\log D(\mathcal{A},\ \mathcal{C})+D(\mathcal{A},\ \mathcal{C})(\log|\mathcal{A}|+\log|\mathcal{C}|+2)$$

Proof. If $\varepsilon_i>0$, $1\leq i\leq n$, a well-known inequality for entropy gives us

$$\sum_1^n -\varepsilon_i\log\varepsilon_i\leq -S\log S+S\log n, \tag{2.3}$$

where

$$S=\sum_{i=1}^n \varepsilon_i.$$

Now by the definition of conditional entropy

$$H(\mathcal{A}\,|\,\mathcal{C})=\sum_{i=1}^k m(c_i)H(\mathcal{A}\,|\,c_i),$$

$$\begin{aligned} H(\mathcal{A}\,|\,c_i)&=\sum_{j=1}^l -m(a_j\,|\,c_i)\log m(a_j\,|\,c_i)\\ &=\sum_{j\neq i} -m(a_j\,|\,c_i)\log m(a_j\,|\,c_i)\\ &\quad -m(a_i\,|\,c_i)\log m(a_i\,|\,c_i). \end{aligned} \tag{2.4}$$

Since $(1-x)+x\log x\geqq 0$ for $0\leqq x\leqq 1$,

$$(2.5) \qquad -m(a_i\,|\,c_i)\log m(a_i\,|\,c_i)\leqq m(a_i^c\,|\,c_i) \quad (a_i^c=\text{complement of } a_i).$$

Applying (2.3), (2.5) to the right-hand member of (2.4), it is easy to show that

$$H(\mathcal{A}\,|\,c_i)\leqq -m(a_i^c\,|\,c_i)\log m(a_i^c\,|\,c_i)+m(a_i^c\,|\,c_i)\log l+m(a_i^c\,|\,c_i),$$

$$H(\mathcal{A}\,|\,\mathcal{C})\leqq \sum_{i=1}^{k}-m(a_i^c\,c)\log m(a_i^c\,c_i)+(\sum_{i=1}^{k}m(a_i^c\,c_i))(\log l+1)$$

$$\leqq -(\sum_{i=1}^{k}m(a_i^c\,c_i))\log(\sum_{i=1}^{k}m(a_i^c\,c_i))$$

$$(2.6) \qquad +(\sum_{i=1}^{k}m(a_i^c\,c_i))(\log k+\log l+1),$$

and similarly with $H(\mathcal{C}\,|\,\mathcal{A})$.

A fundamental inequality estimating discrepancy between entropies is given by

$$|h(T,\mathcal{A})-h(T,\mathcal{C})|<\rho(\mathcal{A},\mathcal{C}),$$

where

$$(2.7) \qquad \rho(\mathcal{A},\mathcal{C})=H(\mathcal{A}\,|\,\mathcal{C})+H(\mathcal{C}\,|\,\mathcal{A}).$$

Inserting (2.6) and a corresponding inequality for $H(\mathcal{C}\,|\,\mathcal{A})$ into (2.7) and using (2.3) again

$$\rho(\mathcal{A},\mathcal{C})\leqq -\sum_{i=1}^{k}m(a_i^c\,c_i)+\sum_{j=1}^{l}m(c_j^c\,a_j))$$

$$\times\log(\sum_{i=1}^{k}m(a_i^c\,c_i)+\sum_{j=1}^{l}m(c_j^c\,a_j))$$

$$+(\sum_{i=1}^{k}m(a_i^c\,c_i)+\sum_{j=1}^{l}m(c_j^c\,a_j))(\log 2+\log k+\log l+1)$$

$$\leqq D(\mathcal{A},\mathcal{C})\{-\log D(\mathcal{A},\mathcal{C})+\log k+\log l+2\}$$

Remarks. Since $-(x+y)\log(x+y)\leqq -x\log x-y\log y$ for $0\leqq x$, $y<\infty$, it is easy to show that the function $E(\mathcal{A},\mathcal{C})$ is dominated by a metric on $\mathcal{Z}_0$.

We now turn to summarize several lemmas from [7], rephrasing and sometimes refining them.

Suppose that we have a certain naming system which let to every $\omega \in \Omega$ correspond a finite or infinite sequence of non-negative integers, the name of ω. Such a system is often connected with a partition.

Suppose we are given a superposition of partitions $\mathcal{P}=\mathcal{P}^0 \vee \mathcal{P}^1 \vee \cdots$, $\mathcal{P}^i=(p_1^i, p_2^i, \cdots)$. The $\mathcal{P}$-name of $\omega \in \Omega$ is the vector $(k_0, k_1, \cdots)$ so determined that

$$\omega \in \bigcap_{i=0}^{\infty} p_{k_i}^i$$

k_i is a ω-function, written as $k_i(\omega)$. Then we can speak of the $\mathcal{P}$-name $(k_0(p), k_1(p), \cdots)$ of a $\mathcal{P}$-cell p, what is the same thing as the $\mathcal{P}$-name of an ω contained in p.

1° Suppose we are given a superposition of partitions Q besides $\mathcal{P}$ in the above, $Q=Q^0 \vee Q^1 \vee \cdots$, and let the Q-name of ω be $(l_0, l_1, \cdots)$, then

$$\text{(2.8)} \qquad m(\omega: \ \mathcal{P}, Q\text{-names of } \omega \text{ differ at more than } e \text{ places})$$
$$\leqq e^{-1} \sum_{i=0}^{\infty} D(\mathcal{P}^i, Q^i).$$

Define indeed, a function δ on non-negative integers, so that $\delta(0)=0$, $\delta(i)=1$ for $i \neq 0$, then the left-hand side of (2.8) being equal to $m(\omega: \sum_0^{\infty} \delta(|k_i(\omega)-l_j(\omega)|) \geqq e)$, (2.8) is a version of Tchebychev's inequality.

2° Let $I_k^m=\begin{pmatrix} 1 \\ 2 \\ \vdots \\ k \end{pmatrix}^m$ be an m-fold product space, $q=(q_0, \cdots, q_u) \in I_k^{u+1}$ be a $u+1$-dimensional vector with integer entries, $\pi=(\pi_q, q \in I_k^{u+1})$, a probability vector. Let S be an automorphism, $B \in \mathcal{B}$ be such that $B, SB, \cdots, S^{n-1}B$ are disjoint, and put $\Omega_1=\bigcup_0^{n-1} S^i B$. There is given a finite partition $\mathcal{W}$ on B and to each $w \in \mathcal{W}$ there corresponds its name $l(w)=(l_0(w), \cdots, l_{n-1}(w)) \in I_k^n$. Define a partition $\mathcal{P}$ so that $\mathcal{P}|\Omega_1=(p_1, \cdots, p_k)$ and $\mathcal{P}|\Omega_1$-cells are determined by

$$p_a=\bigcup_{w \in \mathcal{W}} \Big(\bigcup_{l_i(w)=a} S^i w \Big), \qquad 1 \leqq a \leqq k,$$

and $\mathcal{P}$ is arbitrary on Ω_1^c. Define next the frequency of q in $l(w)$ to be

$$f(q\,|\,l(w))=\frac{\#\{0\leqq i\leqq n-u-1: q=(l_i(w), \cdots, l_{i+u}(w))\}}{n-u}.$$

Then if $\max_{w\in\mathcal{W}} d(f(q\,|\,l(w)), \pi)\leqq\varepsilon_1$, $m(\Omega_1^c)\leqq\varepsilon_2$, we have

$$d(\pi, d(\bigvee_{i=0}^{u} S^{-i}\mathcal{P}))\leqq 2\left(\varepsilon_1+\varepsilon_2+\frac{u}{n}\right).$$

3° Let S, B and $\mathcal{W}$ be as in 2°. Suppose that we have two naming systems, (1) $w\in\mathcal{W}\to k(w)=(k_0, \cdots, k_{n-1})\in I_k^n$, (2) $w\in\mathcal{W}\to l(w)=(l_0, \cdots, l_{n-1})\in I_k^n$. Consider two partitions $\mathcal{P}_1$, $\mathcal{P}_2$ constructed as in 2° by the systems (1), (2) respectively. If

$$\max_{w\in\mathcal{W}}\sum_{i=0}^{n-1}|k_i(w)-l_i(w)|\leqq n\varepsilon,$$

then

$$D(\mathcal{P}_1, \mathcal{P}_2)\leqq m(\Omega_1^c)+\varepsilon.$$

In the following arguments, if $\mathcal{P}$ is a partition, its subfamily of $\mathcal{P}$-cells is denoted by a script capital like $\mathcal{C}\subset\mathcal{P}$, whereas the corresponding $\mathcal{B}$-measurable set $\bigcup_{p\in\mathcal{C}} p$ is denoted by C.

4° Let S, A play the same role as S, B in 2°, $\mathcal{W}\in\mathcal{Z}_0$, and $\mathcal{W}_n=\bigvee_0^{n-1} S^{-i}\mathcal{W}$. Suppose that to every $w\in\mathcal{W}_n|A$ there corresponds a name $(l_0(w), \cdots, l_{n-1}(w))$ in a 1-1 way.

Define a partition $\mathcal{P}$ on Ω_1 as in 2°, whereas in this case Ω_1^c is classified into a cell of $\mathcal{P}$. Then

$$|h(S, \mathcal{P})-h(S, \mathcal{W})|\leqq K_0(\mathcal{W})\{L(m(A))+L(m(\Omega_1^c))\}, \tag{2.9}$$

where $L(x)=x-x\log x$ and $K_i(\mathcal{W})$ $(0\leqq i\leqq 1)$ are constants depending only on $|\mathcal{W}|$.

Proof. The uniqueness of the name of w implies that w is a cell of the partition $\{A, A^c\}\vee(\bigvee_0^{n-1} S^{-i}\mathcal{P})$. Therefore

$$\mathcal{Q}=\{\Omega_1^c \text{ and } S^i w: w\in\mathcal{W}_n|A, \quad 0\leqq i\leqq n-1\}$$

$$\subset \bigvee_{-n}^{n} S^i(\{A, A^c\} \vee \mathcal{P}),$$

$$Q \supset \mathcal{W}' = (\mathcal{W} \mid \Omega_1, \Omega_1^c).$$

Then by 0°

$$|h(S, \mathcal{W}) - h(S, \mathcal{W}')| \leqq K_1(\mathcal{W}) L(m(\Omega_1^c)),$$

$$|h(S, \mathcal{P}) - h(S, \mathcal{P} \vee \{A, A^c\})| \leqq L(m(A)).$$

Thus

$$h(S, Q) \geqq h(S, \mathcal{W}) - K_1(\mathcal{W}) L(m(\Omega_1^c)),$$

$$\begin{aligned} h(S, \mathcal{P}) &\geqq h(S, \mathcal{P} \vee \{A, A^c\}) - L(m(A)) \\ &\geqq h(S, Q) - L(m(A)) \\ &\geqq h(S, \mathcal{W}) - \{L(m(A)) + K_1 L(m(\Omega_1^c))\}. \end{aligned}$$

On the other hand, since

$$\mathcal{P} \subset \bigvee_{i=0}^{n-1} S^i \mathcal{W}_n \vee \{A, A^c\}) \subset \bigvee_{-n}^{n} S^i (\mathcal{W} \vee \{A, A^c\}),$$

$$h(S, \mathcal{P}) \leqq h(S, \mathcal{W} \vee \{A, A^c\}) \leqq h(S, \mathcal{W}) + 2L(m(A)),$$

(2.9) holds with $K_0(\mathcal{W}) = K_1(\mathcal{W}) + 1$.

In the following arguments by γ we denote a non-negative constant which can be made arbitrarily small, dependently on adjustable parameters.

5° Let $\mathcal{P} \in \mathcal{Z}_0$ with $|\mathcal{P}| = k$, S be an ergodic automorphism, I_k^n be as in 2°, $\mathcal{P}_n = \bigvee_{0}^{n-1} S^{-i} \mathcal{P}$ and $\xi(p) = (\xi_0, \cdots, \xi_{n-1})$ be the $\mathcal{P}_n$-name of $p \in \mathcal{P}_n$. Let $\eta = (\eta_0, \cdots, \eta_u) \in I_k^{u+1}$ and define

$$p_\eta = \bigcap_{i=0}^{u} S^{-i} p_{\eta_i}.$$

Then for an arbitrary $\gamma > 0$, if n is large we have

$$m(p: \sum_\eta |f(\eta \mid \xi(p)) - m(p_\eta)| < \gamma) > 1 - \gamma.$$

§3. Sinai's Theorem

Theorem 1 (*Sinai*). *Let $S\in(\gamma)$ and T be an ergodic automorphism with $h(S)\leqq h(T)\leqq\infty$. Then S is a factor of T.*

Proof. By 3° of §1, without loss of generality, we will assume that $S\in(\delta)$. Let $\mathcal{R}=(r_1, r_2, \cdots)$ be its WBG, and increasing natural numbers $s_1, s_2, \cdots$ be taken so sparsely that

$$\mathcal{R}_m=(r_1, \cdots, r_{s_m}, r_{s_m+1}\cup r_{s_m+2}\cup\cdots, \phi, \cdots)$$

satisfies

$$\sum_{m=1}^{\infty}\delta_m<\infty,$$

where

$$\delta_m^2=D(\mathcal{R}_m, \mathcal{R}_{m+1})=2\sum_{i\geq s_m+2}m(r_i).$$

Let $\varepsilon_1, \varepsilon_2, \cdots$ be a sequence of positive numbers with

$$\sum_{m=1}^{\infty}\varepsilon_m<\infty,$$

and let us put $n_k=n_1(\varepsilon_k^2, \mathcal{R}_k, S)$, $n_k'=n_2(\varepsilon_k^2, \mathcal{R}_k, S)$, $\eta_k=\eta(\varepsilon_k^2, \mathcal{R}_k, S)$, where n_1, n_2, η are the functions of ε, $\mathcal{R}$, S, $n_1=n_1(\varepsilon^2, \mathcal{R}, S)$, etc. in Lemma 12, [7].

We will first choose $\mathcal{P}_1\in\mathcal{Z}_0$ such that

$$\text{(3.1)}\qquad \begin{gathered} d(\bigvee_0^{n_1}S^i\mathcal{R}_1, \bigvee_0^{n_1}T^i\mathcal{P}_1)<\eta_1 \\ 0<h(S, \mathcal{R}_1)-h(T, \mathcal{P}_1)<\eta_1. \end{gathered}$$

For this purpose let $k=|\mathcal{R}_1|$, and choose $\mathcal{W}_0\in\mathcal{Z}_0$ satisfying

$$\text{(3.2)}\qquad 0<h(S, \mathcal{R}_1)-h(T, \mathcal{W}_0)<\eta_1$$

and write $\mathcal{R}_n^1=\bigvee_0^{n-1}S^{-i}\mathcal{R}_1$, $\mathcal{W}_n^0=\bigvee_0^{n-1}T^{-i}\mathcal{W}_0$.

As in 5°, denote by $\xi(r)$ the $\mathcal{R}_n^1$-name of $r\in\mathcal{R}_n^1$, define r_η for

$\eta \in I_k^{n_1+1}$, $k \equiv s_1+1$, and put

$$E=\{r \mid \sum_{\eta} |f(\eta \mid \xi(r))-m(r_\eta)| < \gamma\},$$

$$F=\{\omega \in E \mid e_n(a+\varepsilon) < m(w(\omega)) < e_n(a-\varepsilon), \tag{3.3}$$
$$e_n(b+\varepsilon) < m(r(\omega)) < e_n(b-\varepsilon)\},$$

where $w(\omega)$, $r(\omega)$, are the $\mathcal{W}_n^0$-, $\mathcal{R}_n^1$-cells containing ω, $\gamma>0$ is arbitrarily fixed and

$$e_n(x)=e^{-nx}, \quad a=h(T, \mathcal{W}_0), \quad b=h(S, \mathcal{R}_1),$$

$$0<\varepsilon<\min\left(a, \frac{b-a}{2}\right),$$

then by 5° and Shannon-Mcmillan-Breiman's theorem, if n is large

$$m(F)>1-\gamma. \tag{3.4}$$

Define

$$\mathcal{C}=\{w: m(w \cap F)>0\}, \ \mathcal{C}'=\{r: m(r \cap F)>0\}, \tag{3.5}$$

then (3.4) implies

$$m(C), \ m(C')>1-\gamma.$$

Thus

$$\#\{w: w \in \mathcal{C}\} \ll \#\{r: r \in \mathcal{C}'\},$$

to every $w \in \mathcal{C}$ we can let correspond a $\xi(r)$, $r \in \mathcal{C}'$, in a 1–1 way.

Take an I such that $(*)$ $T^i I$, $0 \leqq i \leqq n-1$, are disjoint, $m(G)>1-\gamma$, where $G=\bigcup_0^{n-1} T^i I$. There exists i_0 satisfying $m(T^{i_0} I \cap C^c)<\gamma/n$, and hence $m(T^{i_0} I \cap C)>(1-2\gamma)/n$. Set $A=T^{i_0} I \cap C$, $\Omega_1=\bigcup_0^{n-1} T^i A$, then $m(\Omega_1)>1-2\gamma$, and $T^i A$, $0 \leqq i \leqq n-1$, are disjoint; to every $w \in \mathcal{W}_n^0 | A$ there is assigned a $\xi(r)$ in a 1–1 way. Define $\mathcal{P}_1$ as in 2°, then if n is large

$$d(\bigvee_0^{n_1} S^{-i} \mathcal{R}_1, \ \bigvee_0^{n_1} T^{-i} \mathcal{P}_1) \leqq 2\left(3\gamma+\frac{n_1}{n}\right)<\eta_1. \tag{3.6}$$

Also by 4°, if γ, $1/n$ are small

$$(3.7) \qquad |h(T, \mathcal{W}_0)-h(T, \mathcal{P}_1)| \leqq K_0(\mathcal{W}_0)(L(1/n)+L(2\gamma)),$$

and the right-hand member can be made arbitrarily small. Collecting (3.2), (3.6), (3.7) we get (3.1).

Next choose $\mathcal{W}_1 \supset \mathcal{P}_1$ such that

$$0<h(S, \mathcal{R}_2)-h(T, \mathcal{W}_1)<\eta_2.$$

By Lemma 9, [7], (3.1) implies that there exist partitions Q_i, $0 \leqq i \leqq n-1$, such that

$$(3.8) \qquad \begin{gathered} d(\bigvee_0^{n-1} Q_i)=d(\bigvee_0^{n-1} S^{-i}\mathcal{R}_1), \\ \sum_{i=0}^{n-1} D(Q_i, T^{-i}\mathcal{P}_1) \leqq n\varepsilon_1^2 \quad (n \geqq n_1'). \text{ (*)} \end{gathered}$$

Now engrave each $\bigvee_0^{n-1} Q_i$-cell to have $\tilde{Q}_i$, $0 \leqq i \leqq n-1$ so that $\tilde{Q}_i \supset Q_i$,

$$d(\bigvee_0^{n-1} \tilde{Q}_i)=d(\bigvee_0^{n-1} S^{-i}\mathcal{R}_2).$$

Then

$$D(Q_i, \tilde{Q}_i)=D(S^{-i}\mathcal{R}_1, S^{-i}\mathcal{R}_2)=D(\mathcal{R}_1, \mathcal{R}_2)=\delta_1^2, \sum_0^{n-1} D(\tilde{Q}_i, T^{-i}\mathcal{P}_1)$$

$$(3.9) \qquad \leqq \sum_0^{n-1} D(\tilde{Q}_i, Q_i)+\sum_0^{n-1} D(Q_i, T^{-i}\mathcal{P}_1)$$

$$\leqq n(\varepsilon_1^2+\delta_1^2)<n(\varepsilon_1+\delta_1)^2 \qquad (n \geqq n_1').$$

Let $|\mathcal{R}_2|=l\equiv s_2+1$; as in the first step, write $\xi(\tilde{q})$ for the $\tilde{Q}_n$-name of $\tilde{q} \in \tilde{Q}_n$, $\tilde{Q}_n=\bigvee_0^{n-1} Q_i$; define $\tilde{q}_\eta$ for $\eta \in I_l^{n_2+1}$, and put

$$\tilde{E}=\{\tilde{q}: \sum_\eta |f(\eta|\xi(\tilde{q}))-m(\tilde{q}_\eta)|<\gamma\}.$$

Write $\mathcal{W}_n^1=\bigvee_0^{n-1} T^{-i}\mathcal{W}_1$, $\mathcal{P}_n^1=\bigvee_0^{n-1} T^{-i}\mathcal{P}_1$, and define $\tilde{F}$ after (3.3) with

(*) In [7], use was made of conclusions somewhat different from (3.8). For the present use we prefer (3.8) to those; (3.8) itself easily derives from (9.3)-(9.5), [7] after a simple application of an isomorphism.

$\tilde{E}$, $\mathcal{W}_n^1$, $\tilde{q}$, $h(T, \mathcal{W}_1)$, $h(S, \mathcal{R}_2)$ in place of E, $\mathcal{W}_n^0$, r, a, b, then

$$m(\tilde{F})>1-\gamma.$$

Setting $H=\{\omega: \tilde{Q}_n\text{-name of } \omega \text{ differs from } \mathcal{P}_n^1\text{-name at less than } n(\varepsilon_1+\delta_1) \text{ places}\}$, by (3.9) and 1°

$$m(H^c)<\varepsilon_1+\delta_1,\ m([H\cap\tilde{F}]^c)<\varepsilon_1+\delta_1+\gamma;$$

setting

$$\mathcal{C}=\{w\in\mathcal{W}_n^1: m(w\cap\tilde{F})>0\},$$

$$\mathcal{D}=\{w\in\mathcal{W}_n^1: m(w\cap H\cap\tilde{F})>\frac{1}{2}\, m(w)\},$$

$$\mathcal{D}'=\{\tilde{q}: m(\tilde{q}\cap H\cap\tilde{F})>0\},$$

one has

$$m(C)>1-\gamma,\ m(D)>1-2(\varepsilon_1+\delta_1+\gamma),$$
$$m(D')>1-(\varepsilon_1+\delta_1+\gamma).$$

Take I as in $(*)$ in the above, then since $m(G\cap C^c)<\gamma$, $m(G\cap D^c)<2(\varepsilon_1+\delta_1+\gamma)$, there is an i_0 such that $I_0=T^{i_0}I$ satisfies

$$m(I_0\cap C^c)<\frac{3\gamma}{n},\ m(I_0\subset D^c)<\frac{4(\varepsilon_1+\delta_1+\gamma)}{n}$$

so that

$$m(I_0\cap C)>\frac{1-4\gamma}{n},\ m(I_0\cap D)>\frac{1-4(\varepsilon_1+\delta_1)-5\gamma}{n}.$$

As in pp. 384–385, [7], to a $w\in\mathcal{D}$, we can let correspond $\xi(\tilde{q})$ of a $\tilde{q}\in\mathcal{D}'$, with $m(w\cap\tilde{q})>0$, and to a $w\in\mathcal{C}\backslash\mathcal{D}$ that of a remaining $\tilde{q}\in\mathcal{D}'$ in a 1–1 way; w has the same $\mathcal{P}_n^1$-name as $p\in\mathcal{P}_n^1$ with $p\supset w$; we apply 3° and 4° to $B=I_0\cap D$, $A=I_0\cap C$, to have a partition $\mathcal{P}_2$ satisfying

$$(3.10)\qquad |h(T, \mathcal{P}_2)-h(T, \mathcal{W}_1)|\leqq k_0(\mathcal{W}_1)\{L(1/n)+L(\gamma)\}$$

$$(3.11)\qquad D(\mathcal{P}_1, \mathcal{P}_2)<4(\varepsilon_1+\delta_1)+5\gamma+(\varepsilon_1+\delta_1)<6(\varepsilon_1+\delta_1).$$

Apply 2° to A, then

$$d(\overset{n_2}{\underset{0}{\vee}} S^i \mathcal{R}_2, \overset{n_2}{\underset{0}{\vee}} T^i \mathcal{P}_2) \leqq 2\left(5\gamma + \frac{n_2}{n}\right). \tag{3.12}$$

The right-hand members of (3.10), (3.12) being able to be made arbitrarily small, we have shown that there exists a $\mathcal{P}_2$ such that

$$0 < h(S, \mathcal{R}_2) - h(T, \mathcal{P}_2) < \eta_2, \tag{3.13}$$

$$\overset{n_2}{\underset{0}{\vee}} S^i \mathcal{R}_2, \overset{n_2}{\underset{0}{\vee}} T^i \mathcal{P}_2 < \eta_2, \tag{3.14}$$

$$D(\mathcal{P}_1, \mathcal{P}_2) < 6(\varepsilon_1 + \delta_1). \tag{3.15}$$

Continuing in this way, we obtain partitions $\mathcal{P}_1, \mathcal{P}_2, \cdots$ such that

$$0 < h(S, \mathcal{R}_m) - h(T, \mathcal{P}_m) < \eta_m, \tag{3.16}$$

$$d(\overset{n_m}{\underset{0}{\vee}} S^i \mathcal{R}_m, \overset{n_m}{\underset{0}{\vee}} T^i \mathcal{P}_m) < \eta_m, \tag{3.17}$$

$$D(\mathcal{P}_{m-1}, \mathcal{P}_m) < 6(\varepsilon_{m-1} + \delta_{m-1}), \tag{3.18}$$

$$2 \leqq m < \infty.$$

Then $\mathcal{P}_m \to \mathcal{P} \in \mathcal{Z}_1$, $m \to \infty$, and for an arbitrary a and m

$$d(\overset{a}{\underset{0}{\vee}} S^i \mathcal{R}_m, \overset{a}{\underset{0}{\vee}} T^i \mathcal{P}_m) < \eta_m.$$

On making $m \to \infty$

$$d(\overset{a}{\underset{0}{\vee}} S^i \mathcal{R}, \overset{a}{\underset{0}{\vee}} T^i \mathcal{P}) = 0,$$

i.e.

$$(S, \mathcal{R}) \sim (T, \mathcal{P}),$$

which proves the theorem.

Remarks. Obviously we can apply the same method to the case when $\mathcal{R}_1 = \mathcal{R}_2 = \cdots$, thus having the proof of Lemma 13, [7]. Doing so, the presence of the first inequality sign in (3.16) which rests on the estimates (2.2), (2.9) simplify the final steps in the proof of Lemma 12, [7].

Using the proof of Theorem 1 we can prove the following proposition.

Proposition. *For* WBA'*s with finite entropies entropy is a complete invariant.*

Proof. Let T_1, T_2 be WBA's with $h(T_1)=h(T_2)<\infty$. To show $T_1\sim T_2$, the case when $T_1, T_2\in(\alpha)$ is settled in [7], whereas the case when $T_i\in(\delta)$ and the defining generators $\mathcal{R}_i$ of T_i $(i=1, 2)$ satisfy $\mathcal{R}_1\in\mathcal{Z}_1$, $|\mathcal{R}_1|=|\mathcal{R}_2|=\infty$ is included in Theorem 2. So that, in view of reductions in §1, we are sufficed to deal with the case where $T_1\in(\alpha)$, $T_2\in(\delta)$ with $|\mathcal{R}_2|=\infty$.

A key to the proof of this case is the following lemma.

Lemma. Let $(T_1, \mathcal{R})$ be a WBP. Let T be a BA with BG $\mathcal{P}=(p_1, p_2, \cdots)$, $|\mathcal{P}|=\infty$ and $h(T_1, \mathcal{R})=h(T)$. Let Q be a partition such that

(3.19) $$(T_1, \mathcal{R})\sim(T, Q).$$

Given $\varepsilon>0$, there exist Q_1, $\bar{\mathcal{P}}$, k such that

(3.20) $$(T_1, \mathcal{R})\sim(T, Q)$$

(3.21) $$D(\mathcal{P}, \bar{\mathcal{P}})<\varepsilon,\ \bar{\mathcal{P}}\subset\bigvee_{-k}^{k}T^iQ_1,$$

(3.22) $$|\bar{\mathcal{P}}|=|\mathcal{P}|,\quad D(Q_1, Q)<\varepsilon.$$

Lemma 14, [7] is modified to the above, adapted to the present situation, with the notations preserved.

Let us write T_Q for the automorphism T reduced to $Q_{-\infty}^{\infty}$.

To prove our lemma, we need a slight change in Ornstein's proof as will be mentioned below.

Let $\mathcal{P}=(p_1, p_2, \cdots)$ and define a partition $\mathcal{P}'\subset Q_{-\infty}^{\infty}$ as in p. 388, [7]. Define

$$\mathcal{R}_k=(p_1, \cdots, p_{n_k}, p_{n_k+1}\cup p_{n_k+2}\cup\cdots, \phi, \cdots)$$

and similarly $\mathcal{R}_k'$ from $\mathcal{P}'=(p_1', p_2', \cdots)$, $k=1, 2, \cdots$, so that

$$\sum_1^\infty \delta_k<\infty,\ \delta_k^2=D(\mathcal{R}_k, \mathcal{R}_{k+1}).$$

n_1 is taken so large that

$$m(p_i')=m(p_i) \qquad \text{for} \quad i \geqq n_1+1.$$

Obviously $\mathcal{R}_i' \subset Q_{-\infty}^{\infty}$ and by the above $h(T, \mathcal{R}_1') \leqq H(\mathcal{R}_1') < H(\mathcal{R}_1) = h(T, \mathcal{R}_1)$. Moreover $\mathcal{P}'$ can be defined so close to $\mathcal{P}$ that

$$d(\bigvee_0^{n_1} T^i \mathcal{R}_1, \bigvee_0^{n_1} T_Q^i \mathcal{R}_1') < \eta_1$$

$$0 < h(T, \mathcal{R}_1) - h(T_Q, \mathcal{R}_1') \leqq H(\mathcal{R}_1 | \mathcal{R}_1')$$
$$+ H(\mathcal{R}_1' | \mathcal{R}_1) < \eta_1,$$

where $n_1 = n_1(\varepsilon_1^2, \mathcal{R}_1, T)$, $\eta_1 = \eta(\varepsilon_1^2, \mathcal{R}_1, T)$.

Apply the method of the proof of Theorem 1 with $(T, \mathcal{R}_k, \delta_k)$ in place of $(S, \mathcal{R}_k, \delta_k)$, and with T_Q, $\mathcal{R}_1'$ in place of T, $\mathcal{P}_1$, then one obtains $\mathcal{P}_1 \subset Q_{-\infty}^{\infty}$ such that

$$(T, \mathcal{P}) \sim (T_Q, \mathcal{P}_1) = (T, \mathcal{P}_1)$$

$$D(\mathcal{R}_1', \mathcal{P}_1) \leqq 6 \sum_{i=1}^{\infty} (\delta_i + \varepsilon_i),$$

where ε_i, $1 \leqq i \leqq \infty$, are positive numbers which can be chosen as in the proof of Theorem 1 so small that

$$\sum_{i=1}^{\infty} \varepsilon_i \leqq \sum_{i=1}^{\infty} \delta_i.$$

Now

$$D(\mathcal{P}', \mathcal{P}_1) \leqq D(\mathcal{P}', \mathcal{R}_1') + D(\mathcal{R}_1', \mathcal{P}_1)$$
$$\leqq \delta_{n_1}^2 + 7 \sum_{n_1}^{\infty} \delta_i,$$

which can be made arbitrarily small.

The remaining arguments in the proof of Lemma 14, [7] are available without essential change. Then along the same line as in the proof of Lemma 15, [7], we are led to the conclusion of the proposition.

§4. Isomorphism Theorem

Finally we will prove the following theorem of Ornstein in a different way from his own proof, but using results in [7].

Theorem 2 (*Ornstein*). *Entropy is a complete characteristic for* WBA's.

Let S, T be such two automorphisms. In view of 3° of §1, the only case we have to deal with is that $S\in(\delta)$, $h(S)=h(T)\leqq\infty$, $h(S,\mathcal{R}_n)=h(T,\mathcal{U}_n)$, $1\leqq n<\infty$, where $\mathcal{R}_n$, $\mathcal{U}_n\in\mathcal{Z}_0$, $\mathcal{R}$, $\mathcal{U}$ are WBG's for S, T, $(S,\mathcal{R}_n)$, $(T,\mathcal{U}_n)$ are WBP's, $\mathcal{R}_n\uparrow\mathcal{R}$, $\mathcal{U}_n\uparrow\mathcal{U}$. With this remark in mind, we will begin by proving

Lemma. *Let $\mathcal{P}_1\subset\mathcal{P}$, $\mathcal{R}_1\subset\mathcal{R}_2$ be finite partitions and $\mathcal{R}_1$, $\mathcal{R}_2$ be properly ordered (like $\mathcal{P}$, Q preceding (2.1)), $(S,\mathcal{R}_i)$ $(1\leqq i\leqq 2)$, $(T,\mathcal{P}_1)$, $(T,\mathcal{P})$ be* WBP'*s with $0<h(S,\mathcal{R}_1)<h(S,\mathcal{R}_2)=h(T,\mathcal{P})$, and let $(S,\mathcal{R}_1)\sim(T,\mathcal{P}_1)$.*

Then for any $\eta>0$, natural number u, we can find a $\mathcal{D}\in\mathcal{Z}_0$, $\mathcal{D}\subset(\mathcal{P}^T)_{-\infty}^{\infty}$ subject to

$$d\Big(\bigvee_0^u S^i\mathcal{R}_2,\ \bigvee_0^u T^i(\mathcal{P}_1\vee\mathcal{D})\Big)<\eta \tag{4.1}$$

$$0<h(S,\mathcal{R}_2)-h(T,\mathcal{P}_1\vee\mathcal{D})<\eta \tag{4.2}$$

$$D(\mathcal{P}_1,\mathcal{P}_1\vee\mathcal{D})<6\delta_1, \tag{4.3}$$

where $\delta_1^2=D(\mathcal{R}_1,\mathcal{R}_2)$.

Proof. After p. 516, choose $\mathcal{B}_0\subset\mathcal{B}_1$ with $\mathcal{R}_1=\mathcal{R}_1\vee\mathcal{B}_0$, $\mathcal{R}_2=\mathcal{R}_1\vee\mathcal{B}_1$ and define $\mathcal{A}_i^0$, $\mathcal{A}_i^1\in\mathcal{Z}_0$ such that

$$d\Big(\bigvee_0^{n-1}S^{-i}(\mathcal{R}_1\vee\mathcal{B}_0)\Big)=d\Big(\bigvee_0^{n-1}(Q_i\vee\mathcal{A}_i^0)\Big),$$

$$d\Big(\bigvee_0^{n-1}S^{-i}(\mathcal{R}_1\vee\mathcal{B}_1)\Big)=d\Big(\bigvee_0^{n-1}(Q_i\vee\mathcal{A}_i^1)\Big),$$

where $Q_i=T^{-i}\mathcal{P}_1$.

Let $|\mathcal{Q}_i|=|\mathcal{P}_1|=k$, $|\mathcal{A}_i^1|=|\mathcal{B}_1|=p$, $\tilde{\mathcal{Q}}_n=\bigvee_0^{n-1}(\mathcal{Q}_i\vee\mathcal{A}_i^1)$, $\mathcal{Q}_n=\bigvee_0^{n-1}(\mathcal{Q}_i\vee\mathcal{A}_i^0)$, $\mathcal{Q}_i=(q_{i1},\cdots,q_{ik})$, $\mathcal{P}_1=(p_{11},\cdots,p_{1k})$, $\mathcal{A}_i^1=(a_{i1}^1,\cdots,a_{ip}^1)$; take $\mathcal{W}\in\mathcal{Z}_0$ subject to the conditions

$$\mathcal{P}_{-\infty}^{\infty}\supset\mathcal{W}\supset\mathcal{P}_1,\quad 0<h(S,\mathcal{R}_2)-h(T,\mathcal{W})<\eta.$$

From the representations of $q\in\mathcal{Q}_n$, $\tilde{q}\in\tilde{\mathcal{Q}}_n$

$$q=\bigcap_0^{n-1}q_{i\xi_i}=\bigcap_0^{n-1}T^{-i}p_{1\xi_i},\ \tilde{q}=\bigcap_0^{n-1}(q_{i\xi_i}\cap a_{i\xi_i}^1),$$

write $\xi(q)=(\xi_0(q),\cdots,\xi_{n-1}(q))$, $(\xi(\tilde{q}),\zeta(\tilde{q}))=(\xi_0(\tilde{q}),\cdots,\xi_{n-1}(\tilde{q});\zeta_0(\tilde{q}),\cdots,\zeta_{n-1}(\tilde{q}))$ for the names of q, $\tilde{q}$; $\xi_i(q)=\xi_i(\tilde{q})$ for $q\supset\tilde{q}$. For $(\alpha,\beta)=(\alpha_0,\cdots,\alpha_u;\beta_0,\cdots,\beta_u)\in I_k^{u+1}\times I_k^{u+1}$, let

$$\tilde{q}_{\alpha\beta}=\bigcap_{i=0}^{u}(q_{i\xi_i}\cap a_{i\zeta_i}^1)$$

and put

$$E=(\tilde{q}:\sum_{\alpha\beta}|f((\alpha,\beta)|(\xi(\tilde{q}),\zeta(\tilde{q}))-m(\tilde{q}_{\alpha\beta})|<\gamma);$$

write $a=h(T,\mathcal{P}_1)$, $b=h(T,\mathcal{W})$, $c=h(S,\mathcal{R}_2)$, $\mathcal{W}_n=\bigvee_0^{n-1}T^{-i}\mathcal{W}$, take $0<\varepsilon<\min\left(a,\dfrac{b-a}{2},\dfrac{c-b}{2}\right)$, and define

$$\begin{aligned}F=\{\omega\in E;\ &e_n(a+\varepsilon)<m(q(\omega))<e_n(a-\varepsilon),\\ &e_n(b+\varepsilon)<m(w(\omega))<e_n(b-\varepsilon),\\ &e_n(c+\varepsilon)<m(q(\omega))<e_n(c-\varepsilon)\},\ w(\omega)\in\mathcal{W}_n;\end{aligned}$$

put

$$\mathcal{C}_1=\{q: m(q\cap F)>0\},\quad \mathcal{C}_2=\{w: m(w\cap F)>0\}$$
$$\mathcal{C}_3=\{\tilde{q}: m(\tilde{q}\cap F)>0\},\qquad w\in\mathcal{W}_n.$$

Then if n is large

(4.4) $$m(E),\ m(F),\ m(C_i)>1-\gamma/4\qquad (i=1,2,3).$$

If we drop a subset of measure zero from F, and pick an ω from the rest, we have always $m(\tilde{q}(\omega)\cap w(\omega)\cap F)>0$.

Define

$$\mathscr{C}_{11}=\{q: \#(\tilde{q}\in\mathscr{C}_3: \tilde{q}\subset q)<\#(w\in\mathscr{C}_2: w\subset q)\},$$

and write $\mathscr{C}_{10}=\mathscr{C}_1\backslash\mathscr{C}_{11}$, then if n is large

$$\begin{aligned} m(C_{11}) &\leqq m(C_3^c)+\sum_{q\in\mathscr{C}_{11}} m(q\cap C_3)<\gamma/4 \\ &+\sum_{q\in\mathscr{C}_{11}}\sum_{\substack{\tilde{q}\subset q\\ \tilde{q}\in\mathscr{C}_3}} m(\tilde{q})<\frac{\gamma}{4}+\sum_{q\in\mathscr{C}_{11}} e_n(c-\varepsilon) \\ &\times\#(w: w\in\mathscr{C}_2,\ w\subset q) \\ &\leqq \gamma/4+e_n(c-b-2\varepsilon)<\gamma/2. \end{aligned} \tag{4.5}$$

In accordance with the representation of $q\in Q_n$ as

$$q=\bigcap_0^{n-1}(T^{-i}p_{1\xi_i}\cap a^0_{i\eta_i}),$$

where $a^0_{i\eta_i}$ are $\mathscr{A}^0_i$-cells, we will write $(\xi_0(q),\cdots,\xi_{n-1}(q);\eta_0(q),\cdots,\eta_{n-1}(q))$ for the name of q. Define

$H=\{\omega: \tilde{Q}_n$-name of ω differs from its Q_n-name at less than $n\delta_1$ places$\}$,

$\overline{\mathscr{C}}_2=\{w\in\mathscr{W}_n: m(w\cap F\cap H)>0\}$,

$\overline{\mathscr{C}}_3=\{\tilde{q}: m(\tilde{q}\cap F\cap H)>0\}$, $\overline{\mathscr{C}}_{11}=\{q: \#(\tilde{q}\in\overline{\mathscr{C}}_3: \tilde{q}\subset q)$
$<\#(w\in\mathscr{C}_2: w\subset q)\}$;

put $\bar{C}=C_2\cap\bar{C}_{10}$, $C=C_2\cap C_{10}$, where $\overline{\mathscr{C}}_{10}=\mathscr{C}_1\backslash\overline{\mathscr{C}}_{11}$. Since obviously $\mathscr{C}_{11}\subset\overline{\mathscr{C}}_{11}$, $\overline{\mathscr{C}}_{10}\subset\mathscr{C}_{10}$, we have $\bar{C}\subset C$.

Recall

$$\begin{aligned} D(Q_i\vee\mathscr{A}^1_i,\ Q_i\vee\mathscr{A}^0_i)&=D(\mathscr{A}^1_i,\ \mathscr{A}^0_i) \\ &=D(\mathscr{B}_0,\ \mathscr{B}_1)=D(\mathscr{R}_1,\ \mathscr{R}_2)=\delta_1^2 \end{aligned}$$

and use 1° of §2 to get

$$m(H^c)<\delta_1,\ m([F\cap F]^c)<\delta_1+\gamma/2,$$

whence

$$m(\bar{C}_2),\ m(\bar{C}_3)>1-\delta_1-\gamma/2;$$

notice that

$$m(C)>1-\gamma$$

from (4.4), (4.5). From this, correspondingly to (4.5) we have

$$m(\bar{C}_{11})\leqq m(\bar{C}_3^c)+\sum_{q\in\mathscr{Q}_{11}}\sum_{\bar{q}\subset q,\, \bar{q}\in\bar{\mathscr{Q}}_3} m(q)<\delta_1+\gamma/2.$$

Therefore

$$m(\bar{C}_{10})\geqq m(C_1)-m(\bar{C}_{11})>1-3\gamma/4-\delta_1,$$

$$m(\bar{C})>1-\delta_1-\gamma.$$

From these observations, if $q\in\bar{\mathscr{Q}}_{10}$, to every $w\in\mathscr{Q}_2$, $w\subset q$, we can assign the name $(\xi(\tilde{q}), \zeta(\tilde{q}))$ of a $\tilde{q}\in\bar{\mathscr{Q}}_3$, whereas if $q\in\bar{\mathscr{Q}}_{10}\backslash\mathscr{Q}_{10}$, to every $w\in\mathscr{Q}_2$, $w\subset q$, that of a $\tilde{q}\in\mathscr{Q}_3$, in a 1-1 way.

Take I as in $(*)$ of §3, then likewise we have $A=T^{i_0}I\cap C$, $B=T^{i_0}I\cap\bar{C}$ subject to the conditions that $m(A)\geqq(1-4\gamma)/n$, $m(B)\geqq(1-4\delta_1-4\gamma)/n$, T^iA, $0\leqq i\leqq n-1$, are disjoint, and the same with B.

Now turn to define a partition $\mathscr{D}=(d_0, \cdots, d_p)$ as follows: let

$$T^iw\subset d_{\zeta_i}\ (0\leqq i\leqq n-1), \qquad 1\leqq\zeta_i\leqq p,$$

if the assigned name for $w\in\mathscr{Q}_2|A$ is $(\xi_0, \cdots, \xi_{n-1}; \zeta_0, \cdots, \zeta_{n-1})$, and let

$$d_0=\Omega\backslash\Omega_1.$$

Since $(\xi_0, \cdots, \xi_{n-1})$ is nothing but the name of $q\supset w$, one obtains

$$w\subset A\cap d_{\zeta_0}\cap p_{1\xi_0}$$

$$Tw\subset d_{\zeta_1}\cap p_{1\xi_1}$$

$$\cdots$$

$$T^{n-1}w\subset d_{\zeta_{n-1}}\cap p_{1\xi_{n-1}}$$

or

$$w\subset A\bigcap_{i=0}^{n-1}T^{-i}(d_{\zeta_i}\cap p_{1\xi_i}).$$

Since the naming for w is a 1-1 map, as in 4° we can easily check that the last inclusion is actually an equality, then $T^i w$ $(0 \leqq i \leqq n-1$, $w \in \mathcal{O}_2 | A)$ are cells of $T^i(\{A, A^c\} \bigvee_0^{n-1} T^{-i}(\mathcal{P}_1 \vee \mathcal{D}))$. In this case the partition $\{\Omega_1^c, T^i w : w \in \mathcal{O}_2 | A, 0 \leqq i \leqq n-1\}$ playing the same role as Q in the proof of 4°, by the same argument as there, we obtain

$$|h(T, \mathcal{W}) - h(T, \mathcal{P}_1 \vee \mathcal{D})| \leqq K_0(\mathcal{W})\{L(m(A)) + L(m(\Omega_1^c))\} \leqq K_0(\mathcal{W})(L(1/n) + L(4\gamma)),$$

and hence if n is large

$$0 < h(S, \mathcal{R}_2) - h(T, \mathcal{P}_1 \vee \mathcal{D}) < \eta. \tag{4.6}$$

Applying 2° to A and 3° to B we obtain

$$d(\bigvee_0^u S^i(\mathcal{R}_1 \vee \mathcal{B}_1), \bigvee_0^u T^i(\mathcal{P}_1 \vee \mathcal{D})) < 2\left(4\gamma + \gamma + \frac{u}{n}\right), \tag{4.7}$$

$$D(\mathcal{P}_1 \vee \mathcal{A}_0, \mathcal{P}_1 \vee \mathcal{D}) \leqq (\delta_1 + 4\delta_1 + 4\gamma) < 6\delta_1, \tag{4.8}$$

where $\mathcal{A}_0 = \mathcal{A}_0^0$. (4.7), (4.8) imply (4.1), (4.3).

Proof of the theorem. Take positive numbers $\varepsilon_1, \varepsilon_2, \cdots$ such that

$$\sum_{k=1}^{\infty} \varepsilon_k < \infty.$$

Let T be a given WBA with WBG $\mathcal{U} \in \mathcal{Z}_2$, $\mathcal{U}_n \in \mathcal{Z}_0$ such that $\mathcal{U}_n \uparrow \mathcal{U}$ and $(T, \mathcal{U}_n)$, $0 \leqq n < \infty$, are WBP's. Then according to 3°, §1, there is a BA S with BG $\mathcal{R} \in \mathcal{Z}_1$, $\mathcal{R}_n \in \mathcal{Z}_0$ such that $\mathcal{R}_n \uparrow \mathcal{R}$ and $h(S, \mathcal{R}_n) = h(T, \mathcal{U}_n)$. Without loss of generality we may assume that

$$\sum_{n=1}^{\infty} \delta_n < \infty, \qquad \delta_n = D(\mathcal{R}_n, \mathcal{R}_{n+1}).$$

Obviously $S \in (\delta)$. For the proof it is sufficient to show that S is isomorphic with T.

By the isomorphism theorem in [7], we can find $\mathcal{P}_1 \in \mathcal{Z}_0$ such that

$$(S, \mathcal{R}_1) \sim (T, \mathcal{P}_1), \ (\mathcal{P}_1)_{-\infty}^{\infty} \equiv (\mathcal{P}_1^T)_{-\infty}^{\infty} = (\mathcal{U}_1)_{-\infty}^{\infty}, \tag{4.9}$$

hence an integer k_1 with

$$\delta(\mathcal{U}_1, (\mathcal{P}_1)_{-k_1}^{k_1}) < 2^{-1}.$$

At this stage apply the above lemma to have (4.1)–(4.3) with $u=n_1(\varepsilon_1^2, \mathcal{R}_2, S)$, $\eta=\eta(\varepsilon_1^2, \mathcal{R}_2, S)$, where n_1, η are the functions introduced in the proof of Theorem 1. Next apply Lemma 12, [7] to (4.1), (4.2) and find $\mathcal{D}_1\equiv\mathcal{D}_{\varepsilon_1}$, $\mathcal{P}_2\in\mathcal{Z}_2$ such that

$$(S, \mathcal{R}_2)\sim(T, \mathcal{P}_2),\ (\mathcal{P}_2)_{-\infty}^{\infty}=(\mathcal{U}_2)_{-\infty}^{\infty} \tag{4.10}$$

$$D(\mathcal{P}_2, \mathcal{P}_1\vee\mathcal{D}_1)<6\varepsilon_1,\ D(\mathcal{P}_1, \mathcal{P}_2)<6(\varepsilon_1+\delta_1), \tag{4.11}$$

whence k_2 with

$$\delta(\mathcal{U}_2, (\mathcal{P}_2)_{-k_2}^{k_2})<2^{-2}. \tag{4.12}$$

ε_1 being made arbitrarily small we may assume that

$$\delta(\mathcal{U}_1, (\mathcal{P}_2)_{-k_1}^{k_1})<2^{-1}+2^{-2}. \tag{4.13}$$

With (4.10) as a starting condition in place of (4.9), apply the same procedure as above and find out $\mathcal{D}_2=\mathcal{D}_{\varepsilon_2}$,

$$(S, \mathcal{R}_3)\sim(T, \mathcal{P}_3),\quad (\mathcal{P}_3)_{-\infty}^{\infty}=(\mathcal{U}_3)_{-\infty}^{\infty}, \tag{4.14}$$

$$D(\mathcal{P}_3, \mathcal{P}_2\vee\mathcal{D}_2)<6\varepsilon_2,\ D(\mathcal{P}_2, \mathcal{P}_3)<6(\varepsilon_2+\delta_2), \tag{4.15}$$

$$\delta(\mathcal{U}_3, (\mathcal{P}_3)_{-k_3}^{k_3})<2^{-3} \tag{4.16}$$

$$\delta(\mathcal{U}_2, (\mathcal{P}_3)_{-k_2}^{k_2})<2^{-2}+2^{-3}, \tag{4.17}$$

$$\delta(\mathcal{U}_1, (\mathcal{P}_3)_{-k_1}^{k_1})<2^{-1}+2^{-2}+2^{-3}. \tag{4.18}$$

Proceeding in this way, we conclude that there exist $\mathcal{P}_1, \mathcal{P}_2, \cdots$, $k_1, k_2, \cdots$ satisfying

$$(S, \mathcal{R}_n)\sim(T, \mathcal{P}_n) \tag{4.19}$$

$$\delta(\mathcal{U}_p, (\mathcal{P}_n)_{-k_p}^{k_p})<2\times 2^{-p}$$

for any $p<n$. On making $n\to\infty$, and writing $\mathcal{P}=\lim_{n\to\infty}\mathcal{P}_n$

$$\delta(\mathcal{U}_p, \mathcal{P}_{-\infty}^{\infty})\leqq\delta(\mathcal{U}_p, \mathcal{P}_{-k_p}^{k_p})\leqq 2\times 2^{-p}$$

for any p. This implies that

$$\mathcal{U}\subset\mathcal{P}_{-\infty}^{\infty}. \tag{4.20}$$

On the other hand from (4.19) we have

$$(S, \mathcal{R}) \sim (T, \mathcal{P}). \tag{4.21}$$

Combination of (4.20), (4.21) proves the theorem.

§5. Remarks Related to Stationary Processes

In this concluding section we will discuss about applications of isomorphism theorems to stationary processes. Introduce a measurable space $(E, \mathcal{F})$, with σ-algebra $\mathcal{F}$ on an abstract space E, as the state space of stochastic processes which will appear in the following considerations. Suppose it is separable in the sense that there is a sequence of increasing finite subalgebras $\mathcal{F}_p$, $1 \leqq p < \infty$, and $\mathcal{F} = \bigvee_{p \geq 1} \mathcal{F}_p$. Consider a strictly stationary sequence $X = \{x_n(\omega), -\infty < n < \infty\}$ on a probability space $(\Omega, \mathcal{A}, P)$ with state space $(E, \mathcal{F})$; denote by $\mathcal{A}_a^b$ the subalgebra of $\mathcal{A}$ generated by $x_n(\omega)$, $a \leqq n \leqq b$, and ${}_p\mathcal{A}_a^b$ that generated by the sets like $\{x_n(\omega) \in \Gamma_n, a \leqq n \leqq b\}$, $\Gamma_k \in \mathcal{F}_p$; write $\mathcal{A}_{-\infty}^b = \lim_{a \to -\infty} \mathcal{A}_a^b$, $\mathcal{A}_{-\infty}^{\infty} = \lim_{a \to -\infty} \lim_{b \to \infty} \mathcal{A}_a^b$. We are concerned with the measure preserving transformation T acting on $\mathcal{A}_{-\infty}^{\infty}$ produced by X; $\mathcal{A}_0^0$ is a generator of T.

In several papers has been dealt with the representation problem: Under what conditions is it possible that $x_n(\omega)$ is representable as a function of the shift of a sequence of independent identically distributed random variables ξ_n, $-\infty < n < \infty$, (Bernoulli sequence), thus $x_n = f(\cdots, \theta^n \xi_{-1}, \theta^n \xi_0, \theta^n \xi_1, \cdots)$, $\theta^n \xi_k = \xi_{n+k}$? Except for Gaussian processes, no effective answer has been brought out. X is regular if

$$\lim_{n \to -\infty} \mathcal{A}_{-\infty}^n \text{ is trivial.} \tag{5.1}$$

If this is the case, T is a Kolmogorov automorphism. According to [6] and Ornstein's recent result that there exists a Kolmogorov automorphism which is not a BA, a necessary and sufficient condition that x_n should admit the specified representation is that T is a WBA, whereas the condition (5.1) is neither necessary nor sufficient. Strengthening (5.1) Rosenblatt proposed the condition

(5.2) $$\lim_{k\to\infty} \sup_{\substack{B\in\mathcal{A}_k^\infty \\ A\in\mathcal{A}_{-\infty}^0}} |P(A\cap B)-P(A)P(B)|=0$$

Up to now, notwithstanding its intimate connection with central limit theorem [2], no relation between this and the representation problem has been revealed. Ibragimov strengthened Rosenblatt's condition to the following:

(5.3) $$\varphi(k)=\sup_{B\in\mathcal{A}_k^\infty} \text{ess.}\sup|P(B|\mathcal{A}_{-\infty}^0)-P(B)|\to 0. \qquad \text{as } k\to\infty.$$

Take $\boldsymbol{\Gamma}_k\in\mathfrak{F}_p$ and consider

(5.4) $$\boldsymbol{\varDelta}_n=\sup_{\Gamma\text{'s}\in\mathfrak{F}_p}\sum|P(x_{-n}\in\boldsymbol{\Gamma}_{-n},\cdots,x_0\in\boldsymbol{\Gamma}_0,x_k\in\boldsymbol{\Gamma}_k,\cdots,x_{k+n}\in\boldsymbol{\Gamma}_{k+n})$$
$$-P(x_{-n}\in\boldsymbol{\Gamma}_{-n},\cdots,x_0\in\boldsymbol{\Gamma}_0)P(x_k\in\boldsymbol{\Gamma}_k,\cdots,x_{k+n}\in\boldsymbol{\Gamma}_{k+n})|,$$

where $\boldsymbol{\Gamma}_k$ runs over all $\mathfrak{F}_p$-atoms. After an elementary reflection $\boldsymbol{\varDelta}_n$ is shown to be equal to

$$\sup_{\Gamma\text{'s}}\sum\int|P(x_k\in\boldsymbol{\Gamma}_k,\cdots,x_{k+n}\in\boldsymbol{\Gamma}_{k+n}|\tilde{\mathcal{A}}_{-n}^0)$$
$$-P(x_k\in\boldsymbol{\Gamma}_k,\cdots,x_{k+n}\in\boldsymbol{\Gamma}_{k+n})|P(d\omega)$$
$$=2\int\max_{A\in\tilde{\mathcal{A}}_k^{k+n}}|P(A|\tilde{\mathcal{A}}_{-n}^0)-P(A)|P(d\omega)$$
$$=2\int\max_{A\in\tilde{\mathcal{A}}_0^n}|P(A|\tilde{\mathcal{A}}_{-n-k}^{-k})-P(A)|P(d\omega),$$

where $\tilde{\mathcal{A}}_a^b={}_p\mathcal{A}_a^b$.

Now

$$Q_{l,m}=\max_{A\in\tilde{\mathcal{A}}_0^l}|P(A|\tilde{\mathcal{A}}_{-m}^{-k})-P(A)|,\qquad 0\leqq l<\infty,\ n+k\leqq m<\infty,$$

being submartingale in the obvious ordering, there holds

$$\boldsymbol{\varDelta}_{n'}\leqq 2\int Q_{lm}P(d\omega)\leqq\boldsymbol{\varDelta}_n$$
$$n'+k\leqq m\leqq n+k,\ n'\leqq l\leqq n,$$

whence

(5.5) $$\sup_n \varDelta_n$$

$$=\lim_{l\to\infty}\lim_{m\to\infty} 2\int Q_{lm} P(d\omega)=2\int \lim_{l\to\infty}\lim_{m\to\infty} Q_{lm} P(d\omega);$$

$$\lim_{l\to\infty}\lim_{m\to\infty} Q_{lm}=\lim_{l\to\infty}\max_{A\in\tilde{\mathcal{A}}_0^l}|P(A|\tilde{\mathcal{A}}_{-\infty}^{-k})-P(A)|.$$

The probability space $(\Omega, \mathcal{A}_{-\infty}^{\infty}, P)$ being separable, with no loss of generality it may be considered as a Lebesgue space, then Rohlin's Lebesgue space theory [8] applies to the set function $A\to\tilde{Q}(A)=P(A|\tilde{\mathcal{A}}_{-\infty}^{-k})-P(A)$ considered as a "canonical system of signed measures"; thus

$$2\lim_{l\to\infty}\max_{B\in\tilde{\mathcal{A}}_0^l}|\tilde{Q}(A)|\leqq 2\sup_{A\in\tilde{\mathcal{A}}_0^\infty}|\tilde{Q}(B)|=\int_\Omega|\tilde{Q}(d\omega)|,$$

and for $\varepsilon>0$, there exist l, $A\in\tilde{\mathcal{A}}_0^l$, $B\in\tilde{\mathcal{A}}_0^\infty$ such that

$$\int_\Omega|\tilde{Q}(d\omega)|-\varepsilon\leq 2|\tilde{Q}(B)|\leq 2|\tilde{Q}(A)|+\varepsilon$$

$$\leq 2\lim_{l\to\infty}\max_{A\in\tilde{\mathcal{A}}_0^l}|\tilde{Q}(A)|+\varepsilon.$$

Therefore

(5.6) $$2\lim_{l\to\infty}\lim_{m\to\infty} Q_{lm}=2\sup_{B\in\tilde{\ }_0^\infty}|\tilde{Q}(B)|=\int_\Omega|\tilde{Q}(d\omega)|,$$

each member existing with probability 1.

Let us write

(5.7) $${}_p\alpha_{-\infty}^{-k}(\omega)=\sup_{B\in\tilde{\mathcal{A}}_0^\infty}|\tilde{Q}(B)|,$$

and in the same spirit, define

(5.8) $$2\alpha_{-\infty}^{-k}=2\sup_{A\in\tilde{\mathcal{A}}_0^\infty}|Q(A)|=\int_\Omega|Q(d\omega)|,$$

where

$$Q(A)=P(A|\mathcal{A}_{-\infty}^{-k})-P(A),$$

then

$$\int_{\Omega} {}_p\alpha_{-\infty}^{-k} P(d\omega) \leqq \int_{\Omega} \alpha_{-\infty}^{-k} P(d\omega) \leqq \varphi(k).$$

By martingale property, there exist the limits $\lim_{k\to\infty} {}_p\alpha_{-\infty}^{-k}$, $\lim_{k\to\infty} \alpha_{-\infty}^{-k}$ with probability 1.

1° There holds:

$$\lim_{k\to\infty} {}_p\alpha_{-\infty}^{-k} = 0 \quad \text{a.e. iff} \tag{5.9}$$

$$\lim_{k\to\infty} \int {}_p\alpha_{-\infty}^{-k} P(d\omega) = 0, \qquad p=1, 2, \cdots;$$

$$\lim_{k\to\infty} \alpha_{-\infty}^{-k} = 0 \quad \text{a.e. iff} \tag{5.10}$$

$$\lim_{k\to\infty} \int \alpha_{-\infty}^{-k} P(d\omega) = 0 \quad (*);$$

and the implications

$$(5.3) \Rightarrow (5.10) \Rightarrow (5.9). \tag{5.11}$$

Either of the above three conditions is sufficient for X to admit the specified representation by a Bernoulli sequence.

It is possible that (5.3) is not true but (5.9) is true.

Now we turn our attention to stationary Markov sequences. Let $\{x_n(\omega), -\infty < n < \infty\}$ be such a sequence with state space $(E, \mathfrak{F})$, determined by the stationary probability measure $p(dx)$ on E and k-step transition probability $p^{(k)}(x, dy)$. An easy calculation shows that

$$2\alpha_{-\infty}^{-k}(\omega) = \int_E |p^{(k)}(x_{-k}(\omega), dx) - p(dx)|,$$

$$2\int \alpha_{-\infty}^{-k} P(d\omega) = \int_E p(dx) \int_E |p^{(k)}(x, dy) - p(dy)|.$$

It is known [2] that $\varphi(k) \to 0$, $k \to \infty$, iff

(*) This condition is what the authors of [3] call absolute regularity. They derived interesting results for Gaussian stationary processes, which rest on this notion.

$$\sup_x \int_E |p^{(k)}(x, dy) - p(dy)| = O(\rho^k), \tag{5.12}$$

where ρ is a constant with $0<\rho<1$. As is well-known this condition is certainly assured by Doeblin's condition. When X is a mixing stationary sequence constructed by the n-step transition matrix $p_{ij}^{(n)}$ and its stationary probability measure $\{p_j\}$, using the sole fact that $p_{ij}^{(n)} \to p_j$ as $n \to \infty$, it is straightforward to show that

$$2\int \alpha_{-\infty}^{-k} P(d\omega) = \sum_i p_i \sum_j |p_{ij}^{(k)} - p_j| \to 0$$

as $k \to \infty$.

Summarizing the above arguments we have the following criterion.

2° If the transition probability has the stationary probability measure $p(dx)$ and satisfies

$$\lim_{k\to\infty} \int_E p(dx) \int_E |p^{(k)}(x, dy) - p(dy)| = 0,$$

then the generated stationary process X is representable by the shifts of a Bernoulli sequence of random variables. It is the case when X is generated by a transition probability measure satisfying Doeblin's condition or by a mixing transition matrix $p_{ij}^{(n)}$ which has the stationary probability measure.

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