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Representing Handlebodies by Plumbing and Surgery*

By

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Introduction

Let $W=D^m\bigcup_{\{f_i\}}\{\bigcup_{i=1}^{r}D_i^n\times D_i^{m-n}\}$ be a handlebody. Although W has various such representations, we fix the representation. If $2n<m$, $m\geq 6$ and $n\geq 2$, W is diffeomorphic to the sum of $(m-n)$-disk bundles over n-spheres. This is obtained by constructing disjoint spheres S_i^n each of which has $D_i'^n=D_i^n\times o$ as the one hemisphere and a disk D_i^n in D^m as the other hemisphere and by tying with bands the tubular neighbourhoods of the n-sheres $S_i^n=D_i^n\cup D_i'^n$. But, if $2n\geq m$, there arise non-empty intersections of the n-disks D_i^n. In this paper, we consider how handlebodies can be constructed from $(m-n)$-disk bundles over n-spheres in the case when $2n\geq m$. This problem was motivated to analyze the structure of such m-manifolds with non-trivial homology groups only at dimensions 0, k, $m-k$, and m, $(m\neq 2k)$. The results are given in the section 6 as an application.

In [13], J. Milnor constructed the manifolds which are parallelizable and have exotic spheres as their boundaries. There, some $2k$-disk bundles over $2k$-spheres are plumbed correspondingly with a suitable matrix, and in order to kill the fundamental group of the plumbing manifold a proper disk is attached to the boundary. In [11], M. Kerevaire also constructed a manifold like that by plumbing 5-disk bundles over 5-spheres.

In this paper, we generalize those constructions of J. Milnor and M. Kervaire. The generalization is done by plumbing $(m-n)$-disk bundles over n-spheres along $(2n-m)$-spheres or tori which are imbedded in the n-spheres,

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the base spaces of the disk bundles, and have normal $(m-n)$-framings, and then, by attaching proper disks to the boundary of the plumbing manifold. Those disk bundles are plumbed correspondingly to a matrix of which components are the elements of $\Pi_{2n-m}=\pi_{n-1}(S^{m-n-1})$ $(2m\geq 3n+3)$.

Thus we can show that handlebodies W can be obtained by these generalized constructions under the condition that $2m\geq 3n+3$, $n\geq 3$. The results are given in the section 5.

To study the intersections of n-disks in handlebodies, we use Wells' proposition [19], p. 393. Using it, we can modify the intersections to be simpler. The plumbing along spheres (possibly exotic) and the plumbing along tori, in place of spheres, are considered in the sections 3 and 4.

Our results are shown in the differentiable case. For PL case, we must obtain the matter corresponding to Wells' proposition.

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1. Intersection of n-disks

Let $W=D^m\bigcup_{\{f_1,f_2\}}\{D_1^n\times D_1^{m-n}\cup D_2^n\times D_2^{m-n}\}$ be a handlebody with two handles, where $f_i\colon \partial D_i^n\times D_i^{m-n}\to\partial D^m$, $i=1, 2$, are attaching maps. We fix the representation. Consider the link $f_1(S_1^{n-1}\times o)\cup f_2(S_2^{n-1}\times o)$ in S^{m-1}, where $S_i^{n-1}=\partial D_i^n$, $i=1, 2$, $S^{m-1}=\partial D^m$. If $m-n\geq 3$, $f_2(S_1^{n-1}\times o)$ in $S^{m-1}-f_1(S_2^{n-1}\times o)$ induces the linking element λ_{12} which is the element of $\pi_{n-1}(S^{m-n-1})$. Similarly λ_{21} is defined and we know that $\lambda_{21}=(-1)^n\lambda_{12}$.

Let $2m\geq 3n+3$. Then, in D^m, each $f_i(S_i^{n-1}\times o)$ bounds an n-disk D_i^n. If D_1^n and D_2^n can be chosen disjointly, W is diffeomorphic to the sum of two $(m-n)$-disk bundles over n-spheres. If D_1^n and D_2^n intersect by any means, we may assume that they intersect transversely and $K^{2n-m}=D_1^n\cap D_2^n$ is a submanifold of D_1^n and D_2^n. We wish to take D_1^n and D_2^n so that each components of K^{2n-m} are simpler and the number of the components is less. In the following arguments, we show that the linking element λ_{12} is the key to the problem.

Let M^m be a m-dimensional differentiable manifold with boundary. Let D^p and D^q be imbedded disks in M^m of dimension p and q such that ∂D^p,

$\partial D^q \subset \partial M^m$ and $\partial D^p \cap D^q = D^p \cap \partial D^q = \phi$. We assume that D^p and D^q intersect transversely. Take a field $\mathscr{F}$ of normal $(m-p)$-frames to D^p. Then the restriction of $\mathscr{F}$ to $K^{p+q-m} = D^p \cap D^q$ defines a field of normal $(m-p)$-frames to K^{p+q-m} in D^q. We denote it by $\mathscr{F}/K$. $\mathscr{F}/K$ does not depend on the choice of $\mathscr{F}$ up to homotopy under a fixed orientation. Applying the Pontrjagin-Thom construction, we obtain an element of $\pi_q(S^{m-p})$, which we denote by $\lambda(D^p, D^q)$. Similarly $\lambda(D^q, D^p)$ is defined. They are equal up to sign under the stable suspension.

Haefliger's Theorem. *If M is $(p+q-m+1)$-connected and $2m \geq \max(p, q)+p+q+3$, there is an isotopy h_t of $D^p \subset M$, fixed on ∂D^p, such that $h_t(D^p) \cap \partial D^q = \phi$ and $h_1(D^p) \cap D^q = \phi$ if and only if $\lambda(D^p, D^q) = 0$.* ([5], p. 245–07)

Applying this to our case, we have

Theorem 1.1. *If $2m \geq 3n+3$, there is an isotopy h_t of $D_1^n \subset D^m$, fixed on ∂D_1^n, such that $h_t(D_1^n) \cap \partial D_2^n = \phi$ and $h_1(D_1^n) \cap D_2^n = \phi$ if and only if $\lambda(D_1^n, D_2^n) = 0$.*

On the other hand, Wall defined in [18] a bilinear map $\lambda: \pi_n(W) \times \pi_n(W) \rightarrow \pi_n(S^{m-n})$ which is $(-1)^n$-symmetric. Let e_i, $i=1, 2$, be the basis of $\pi_n(W)$ defined by $S_i^n = D_i^n \cup D_i^n \times o$, $i=1, 2$. Then, from the definition of the bilinear map, $\lambda(e_1, e_2)$ coincides with $\lambda(D_1^n, D_2^n)$, and in [18], p. 260, it is known that $\lambda(e_1, e_2) = S\lambda_{12}$, where $S: \pi_{n-1}(S^{m-n-1}) \rightarrow \pi_n(S^{m-n})$ is the suspension homomorphism. We note that $\pi_{n-1}(S^{m-n-1})$ is stable when $2m \geq 3n+3$. Thus we know

Lemma 1.2. *If $2m \geq 3n+3$, $\lambda(D_1^n, D_2^n)$ is equal to λ_{12} as the elements of the stable homotopy group $\prod_{2n-m} = \pi_{n-1}(S^{m-n-1})$.*

Thus, if λ_{12} is not zero, by any means, D_1^n and D_2^n intersect each other. To modify the intersection $K^{2n-m} = D_1^n \cap D_2^n$, we use the following theorem which we call Wells' proposition.

Let M^m be a m-dimensional differentiable manifold with bonudary. Let

D^p and D^q be imbedded disks[1] in M^m of dimension p and q such that ∂D^p, $\partial D^q \subset \partial M^m$ and $\partial D^p \cap D^q = D^p \cap \partial D^q = \phi$. Let $q \geq 2(p+q-m)+3$ and U be an arbitrary open set in M^m such that $U \cap D^q$ is diffeomorphic to q-dimensional Euclidean space R^q. We may assume that D^p and D^q intersect transversely in U. Let $K = D^p \cap D^q$ and let $(L; K, K')$ be an elementary cobordism, that is, L is a cobordism of K and K' and has a Morse function which is equal to o on K and 1 on K' and has exactly one critical point. When we are given an isotopy $h_t\colon D^p \to M^m$ of the inclusion $D^p \subset M^m$, fixed on ∂D^p, define the imbedding $H\colon D^p \times I \to M^m \times I$ as $H(x, t) = (h_t(x), t)$, where I is the unit interval. Then the quadruple $(h_t; D^p, D^q, M^m)$ is called to *realize* $(L; K, K')$ if and only if the following conditions are satisfied;

1) H is transverse regular along $D^q \times I$

2) $L_1 = H(D^p \times I) \cap (D^q \times I)$ is diffeomorphic to L.

We may assume that $L_1 \subset (U \cap D^q) \times I$. A normal $(m-p)$-framing $\mathscr{F}$ of $D^p \times o$ in $M^m \times o$ can be canonically extended to $\mathscr{F}_1$ that of $D^p \times I$ in $M^m \times I$. Define $\sum_1$ by $\mathscr{F}_1/L_1$ and σ_1 by $\mathscr{F}/K$. Under a diffeomorphism $U \cap D^q \to R^q$, the framing σ_1 corresponds to the framing σ in R^q. Let $L \subset R^q \times I$ and let $\sum$ be a normal $(m-p)$-framing which extends σ in $R^q \times o$. Then, $(h_t; D^p, D^q, M)$, a realization of $(L; K, K')$, is called to *realize the framed elementary cobordism* $(L, \sum; K, K')$ if and only if the following condition is satisfied;

2′) Under some diffeomorphism $(U \cap D^q) \times I \to R^q \times I$, L_1 is diffeomorphic to L and the framing $\sum_1$ corresponds to the framing $\sum$.

Wells' Proposition. *If M^m is $(p+q-m+1)$-connected and p, $q \geq 2(p+q-m)+3$, then any framed elementary cobordism $(L, \sum; K, K')$ which extends the canonical framing of K can be realized by $(h_t; D^p, D^q, M)$ where h_t is an isotopy of $D^p \subset M$ fixed on ∂D^p.* [19], p. 393.

Thus any framed cobordism which has (K, σ) as the one boundary can be realized by the composition of finitely many isotopies of $D^p \subset M$ each fixed on ∂D^p, because any framed cobordism can be expressed as a com-

1) Although the submanifolds intersecting in M^m are more general in the original proposition of Wells, we restrict them to the case when they are disks for simplicity.

position of finitely many framed elementary cobordisms. The nullity of $\lambda(D^p, D^q)$ is a necessary and sufficient condition to separate D^p from D^q by an isotopy fixed on ∂D^p. So that, applying the proposition to our case, by Lemma 1.2 we have

Theorem 1.3. *Let f_i: $S_i^{n-1} \to S^{m-1} = \partial D^m$, $i=1, 2$, be two disjoint imbeddings and let $\lambda_{12} \in \pi_{n-1}(S^{m-n-1})$ be the linking element. We assume that $2m \geq 3n+3$. Let D_i^n, $i=1, 2$, be imbedded n-disks in D^m bounded by $f_i(S_i^{n-1})$, $i=1, 2$, such that $D_1^n \cap \partial D_2^n = \partial D_1^n \cap D_2^n = \phi$. Assume that D_i^n, $i=1, 2$, intersect transversely at $K^{2n-m} = D_1^n \cap D_2^n$. Let K'^{2n-m} be any framed manifold in the framed cobordism class which corresponds to $\lambda_{12} \in \Pi_{2n-m} \cong \Omega_{2n-m}^{\text{framed}}$. Then there exists an isotopy h_t of $D_1^n \subset D^m$, fixed on ∂D_1^n, such that $h_t(D_1^n) \cap \partial D_2^n = \phi$ and $h_1(D_1^n) \cap D_2^n = K'^{2n-m}$. Especially, D_1^n can be separated from D_2^n by such an isotopy if and only if $\lambda_{12} = 0$.*

2. Special Cases

Let f_i: $S_i^{n-1} \to S^{m-1} = \partial D^m$, $i=1, 2$, be two disjoint imbeddings and let $\lambda_{12} \in \pi_{n-1}(S^{m-n-1})$ be the linking element. We suppose that $2m \geq 3n+3$. Let D_i^n, $i=1, 2$, be imbedded n-disks in D^m bounded by $f_i(S_i^{n-1})$, $i=1, 2$, such that $D_1^n \cap \partial D_2^n = \partial D_1^n \cap D_2^n = \phi$. Assume that D_i^n, $i=1, 2$, intersect transversely and let $K^{2n-m} = D_1^n \cap D_2^n$. In this section, we discuss the special cases of m up to $2n-m \leq 7$, and we study what K^{2n-m} can be when we move D_1^n by isotopies.

We have the following table, where J: $\pi_{2n-m}(SO_{m-n-1}) \to \pi_{n-1}(S^{m-n-1})$ is the J-homomorphism.

Table 2.1

Case	$m=\dim W$	$2m \geq 3n+3$	K; $\lambda_{12} \neq 0$, $\lambda_{12}=0$	Π_{2n-m}	$\Pi_{2n-m}/\text{Im. } J$
0	$2n$	$n \geq 3$	$\lvert\lambda_{12}\rvert$ Points, ϕ	Z	Z
1	$2n-1$	$n \geq 5$	S^1, ϕ	Z_2	0
2	$2n-2$	$n \geq 7$	$S^1 \times S^1$, ϕ	Z_2	Z_2
3	$2n-3$	$n \geq 9$	S^3, ϕ	Z_{24}	0
4	$2n-4$	$n \geq 11$	ϕ	0	0
5	$2n-5$	$n \geq 13$	ϕ	0	0
6	$2n-6$	$n \geq 15$	$S^3 \times S^3$, ϕ	Z_2	Z_2
7	$2n-7$	$n \geq 17$	S^7, ϕ	Z_{240}	0

In the cases 4) and 5), clearly we can choose D_1^n and D_2^n so that $K=\phi$ by Theorem 1.1. In the case 0), $\lambda(D_1^n, D_2^n)$, therefore λ_{12} by Lemma 1.2, represents the intersection number of D_1^n and D_2^n. So, the situation is classic and by only Whitney's method we can choose D_1^n and D_2^n so that K consists of $|\lambda_{12}|$ points.

About the cases 1), 3) and 7), we have the following theorem.

Theorem 2.2. *Let $2m\geq 3n+3$ and J: $\pi_{2n-m}(SO)\rightarrow\prod_{2n-m}$ be the J-homomorphism. If $\lambda_{12}\neq 0$ and belongs to the J-image, then we can choose D_1^n and D_2^n so that K is the $(2n-m)$-dimensional ordinal sphere S^{2n-m}.*

Proof. Let $P(S^{2n-m})$ be the subset of $\prod_{2n-m}$ obtained by applying the Pontrjagin-Thom construction to all the normal $(m-n)$-framings over S^{2n-m}. It is known that $P(S^{2n-m})$ coincides with the J-image (Kervaire [10]). So that λ_{12} belongs to $P(S^{2n-m})$. Therefore, by Theorem 1.3, there exists an isotopy h_t of $D_1^n\subset D^m$ such that $h_1(D_1^n)\cap D_2^n=S^{2n-m}$.

Corollary 2.3. *In the cases 1), 3) and 7), if $\lambda_{12}\neq 0$, K can be ordinal spheres S^1, S^3 and S^7 respectively.*

For the cases 2) and 6), we must study the inverse images of Hopf maps. Let η: $S^3\rightarrow S^2$ be the Hopf map and $\eta_k=E^{k-3}\eta$: $S^k\rightarrow S^{k-1}$ be the $(k-3)$-fold suspension of η. Then, $\eta_{n-1}\circ\eta_n$: $S^n\rightarrow S^{n-2}$ generates $\pi_n(S^{n-2})\cong Z_2$ for $n\geq 6$. Since η is the projection of the Hopf bundle, $\eta^{-1}(e)$ is the standard circle S^1, where $e=[1, 0]\in CP^1=S^2$. So that, $(\eta_{n-1}\circ\eta_n)^{-1}(e)=\eta_4^{-1}(\eta_3^{-1}(e))=\eta_4^{-1}(S^1)=\eta_3^{-1}(S^1)=\eta^{-1}(S^1)$. Since the Hopf bundle is a product over S^1, we know that $\eta_3^{-1}(e)=S^1\times S^1$. Thus, by Theorem 1.3, we can choose D_1^n so that $K=D_1^n\cap D_2^n=S^1\times S^1$, when $\lambda_{12}\neq 0$.

For the case 6), let ν: $S^7\rightarrow S^4$ be the Hopf map, and let $\nu_k=E^{k-4}\nu$. Then $\nu_{n-3}\circ\nu_n$ generates $\pi_n(S^{n-6})\cong Z_2$ for $n\geq 14$. And analogously we know that the inverse image of a point by $\nu_{n-3}\circ\nu_n$ is $S^3\times S^3$. So that, we have the similar result.

Generally, assuming that Arf $M^m=0$ if $m=2^l-2$ (Browder [1]), any connected closed m-dimensional π-manifold M^m is framed cobordant to a homotopy sphere, if $m\neq 2, 6, 14$. Hence, by Theorem 1.3, we can choose D_1^n and D_2^n so that $K^{2n-m}=D_1^n\cap D_2^n$ consists of some homotopy $(2n-m)$-spheres if $\lambda_{12}\neq 0$. Since these homotopy spheres can be modified to a homotopy $(2n-m)$-sphere by framed surgeries of 0-dimensional spheres, K^{2n-m} is framed cobordant to a homotopy $(2n-m)$-sphere. Thus, again by Theorem 1.3, we can choose D_1^n and D_2^n so that $D_1^n\cap D_2^n$ is only a homotopy $(2n-m)$-sphere. Conversely, we note the following.

Theorem 2.4. *For any given homotopy sphere $\tilde{S}^{2n-m}$, $2m\geq 3n+3$, there exist n-disks D_1^n, D_2^n imbedded in D^m such that $D_1^n\cap D_2^n=\tilde{S}^{2n-m}$.*

Proof. Imbed $\tilde{S}^{2n-m}$ in Euclidean space R^n, and take a normal $(m-n)$-framing φ to $\tilde{S}^{2n-m}$. Let λ be the element of Π_{2n-m} corresponding to the framed cobordism class of $\{\tilde{S}^{2n-m}, \varphi\}$ in $\Omega_{2n-m}^{\text{framed}}$. Let f: $S^{n-1}\rightarrow S^{m-1}$ be the standard imbedding and denote $f(S^{n-1})$ by S_1^{n-1}. Since $S^{m-1}-S_1^{n-1}$ has the homotopy type of S^{m-n-1}, there exists an imbedding g: $S^{n-1}\rightarrow S^{m-1}-S_1^{n-1}$ which corresponds to a representative of λ of $\Pi_{2n-m}=\pi_{n-1}(S^{m-n-1})$. Denote $g(S^{n-1})$ by S_2^{n-1}. Let D_1^n, D_2^n be imbedded n-disks in D^m bounded by S_1^{n-1}, S_2^{n-1} respectively. We may assume that D_1^n, D_2^n intersect transversely at $K^{2n-m}=D_1^n\cap D_2^n$. Then, $\lambda(D_1^n, D_2^n)=\lambda$ by Lemma 1.2, and Pontryagin-Thom map $P(K^{2n-m}, \mathscr{F}/K)$ represents $\lambda(D_1^n, D_2^n)$, where $\mathscr{F}$ is a normal $(m-n)$-framing to D_1^n and $\mathscr{F}/K$ is the restriction to K^{2n-m}. So that, $\{K^{2n-m}, \mathscr{F}/K\}$ is framed cobordant to $\{\tilde{S}^{2n-m}, \varphi\}$, and therefore by Theorem 1.3, there exists an isotopy h_t of $D_1^n\subset D^m$ such that $h_1(D_1^n)\cap D_2^n=\tilde{S}^{2n-m}$. This completes the proof.

Remark. If $m=2n-14$, $n\geq 31$, and $\lambda_{12}=a\sigma^2+b\kappa$ ($b\neq 0$, mod 2) $\in\Pi_{14}\cong Z_2+Z_2$ ([17]), then it is not obvious what K^{2n-m} can be. But, if $\lambda_{12}=\sigma^2$, K^{2n-m} can be taken as $S^7\times S^7$.

3. Plumbing along Spheres

Ordinarily, the plumbing manifold is constructed by 'plumbing' the disk bundles over spheres at some given points of the base spaces. Let (B_i, p_i, S_i^n, D_i^n), $i=1, 2, \cdots, r$, be n-disk bundles over n-spheres, and let $T=(t_{ij})$ be an $r\times r$ matrix with integer entries which is $(-1)^n$-symmetric and has no diagonal elements. Then, if each S_i^n and the disk bundles are suitably oriented, (B_i, p_i, S_i^n, D^n), $i=1, 2, \cdots, r$, can be plumbed so that S_i^n and S_j^n intersect at $|t_{ij}|$ points with intersection number $S_i^n\cdot S_j^n=t_{ij}$ in the plumbing manifold $B_T=\bigvee_T\{B_i;\ i=1, 2, \cdots, r\}$. The details are shown in [2], V-2.1.

We may assume that they are plumbed at the points of D_i^n, where $S_i^n=D_i^n\cup D_i'^n$. In D_i^n, there are the sets of $|t_{ij}|$ points for $j=1, 2, \cdots, r$. In each set, join the $|t_{ij}|$ points by arcs of D_i^n in some order, and then join those sets by arcs of D_i^n in order of j. Then these arcs for $i=1, 2, \cdots, r$ form a 1-dimensional cell complex $\mathfrak{F}$ which is a deformation retract of $B_T-\bigcup_{i=1}^{r} D_i'^n\times D^n$. $\mathfrak{F}$ is determined only by T. Thus the following is easily seen.

Lemma 3.1. *B_T has the homotopy type of $\mathfrak{F}\cup(e_1^n\cup e_2^n\cup\cdots\cup e_r^n)$, where e_i^n denotes an n-cell.*

Now, we consider to plumb disk bundles over spheres along some given spheres. Let $(B_i, p_i, S_i^n, D^{m-n})$, $i=1, 2$, be $(m-n)$-disk bundles over n-spheres. We assume that $S_i^n=D_i^n\cup D_i'^n$ and $B_i=D_i^n\times D^{m-n}\cup D_i'^n\times D^{m-n}$, where we identify D_i^n, $D_i'^n$, and S_i^n with $D_i^n\times o$, $D_i'^n\times 0$, and $D_i^n\times o\cup D_i'^n\times o$, respectively. Let S^{2n-m} be the $(2n-m)$-dimensional sphere $(2n-m>0)$ and φ_i, $i=1, 2$, be imbeddings of $S^{2n-m}\times D^{m-n}$ to the interior of $D_i^n\times o\subset B_i$. Each φ_i can be canonically extended to an imbedding $\bar{\varphi}_i\colon S^{2n-m}\times D^{m-n}\times D^{m-n}\to D_i^n\times D^{m-n}\subset B_i$ such that $\bar{\varphi}_i(p, x, o)=\varphi_i(p, x)$. Put $S_i^{2n-m}=\varphi_i(S^{2n-m}\times o)$. Now, construct the quotient space of the disjoint sum $B_1\cup B_2$ by identifying the subset $\bar{\varphi}_1(S^{2n-m}\times D^{m-n}\times D^{m-n})$ of B_1 with the sebset $\bar{\varphi}_2(S^{2n-m}\times D^{m-n}\times D^{m-n})$ of B_2 in such a way that $\bar{\varphi}_1(p, x, y)\sim\bar{\varphi}_2(p, y, x)$ for all p, x and y. The resulting space is a manifold except along the corner $S^{2n-m}\times S^{n-m-1}\times S^{n-m-1}=\bar{\varphi}_1(S^{2n-m}\times S^{m-n-1}\times S^{m-n-1})=\bar{\varphi}_2(S^{2n-m}\times S^{m-n-1}\times S^{m-n-1})$. But we can correct it as in [12], p. 519 since the corner has a neighborhood $S^{2n-m}\times S^{m-n-1}\times S^{m-n-1}\times Q$, where $Q\subset R^2$ denotes the three-quarter disk consisting of all $(r\cos\theta, r\sin\theta)$ with $0\leq r\leq 1$, $0\leq\theta\leq 3\pi/2$. We denote the resulting manifold by $B_1\underset{S^{2n-m}}{\bigvee}B_2$, calling as 'the manifold plumbed along the $(2n-m)$-sphere'.

Generally, let $(B_i, p_i, S_i^n, D^{m-n})$, $i=1, 2, \cdots, r$, be $(m-n)$-disk bundles over n-spheres, $S_i^n=D_i^n\cup D_i'^n$, and $B_i=D_i^n\times D^{m-n}\cup D_i'^n\times D^{m-n}$. Let $\Phi=(\varphi_{ij})$ be a matrix without diagonal elements consisting of the disjoint imbeddings $\varphi_{ij}\colon S^{2n-m}\times D^{m-n}\to D_j^n\times o\subset B_j$, $i, j=1, 2, \cdots, r$, $i\neq j$. We denote as $\varphi_{ij}=0$ if φ_{ij} is not defined and we understand that $\varphi_{ji}=0$ if $\varphi_{ij}=0$. Then, we can plumb these disk bundles along the given spheres $S_{ij}^{2n-m}=\varphi_{ij}(S^{2n-m}\times o)$ correspondingly to the matrix Φ.

Let $\bar{\varphi}_{ij}\colon S^{2n-m}\times D^{m-n}\times D^{m-n}\to D_j^n\times D^{m-n}\subset B_j$ be the canonical extension of $\varphi_{ij}\neq 0$. Construct the quotient space of the disjoint union $B_1\cup B_2\cup\cdots\cup B_r$ by identifying the subset $\bar{\varphi}_{ij}(S^{2n-m}\times D^{m-n}\times D^{m-n})$ of B_j with the subset $\bar{\varphi}_{ji}(S^{2n-m}\times D^{m-n}\times D^{m-n})$ of B_i in such a way that $\bar{\varphi}_{ij}(p, x, y)\sim\bar{\varphi}_{ji}(p, y, x)$ for all (p, x, y) and for all $\varphi_{ij}\neq 0$, $\varphi_{ji}\neq 0$. Similarly, we correct the corners. The resulting manifold is denoted by $B_\Phi=\underset{\Phi}{\bigvee}\{B_k;\, k=1, 2, \cdots, r\}$ and is called as 'the manifold plumbed along the $(2n-m)$-spheres'. We note that by Wall's pairing $\lambda\colon i_n(B_\Phi)\times\pi_n(B_\Phi)\to\pi_n(S^{m-n})$ ([18], p. 255), which is a homotopical extension of the intersection number pairing, Φ corresponds to a matrix $\Lambda=(\lambda(e_i, e_j))$, where e_i, $i=1, 2, \cdots, r$, are defined by S_i^n.

About the homotopy properties of $B_1\underset{S^{2n-m}}{\bigvee}B_2$, B_Φ, and those boundaries, we have the following lemmas.

Lemma 3.2. *$B_1\underset{S^{2n-m}}{\bigvee}B_2$ has the homotopy type of $S^{2n-m+1}\cup e_1^n\cup e_2^n$ if $m-n\geq 3$.*

Proof. Since $m-n\geq 3$, S_i^{2n-m} is combinatorially unknotted in $D_i^n=D_i^n\times o$ and so bounds a $(2n-m+1)$-dimensional cell E_i in D_i^n, for $i=1, 2$. So that, since $S_1^{2n-m}=S_2^{2n-m}$, $S^{2n-m+1}=E_1\cup E_2$ is a topological sphere and a strong deformation retract of $D_1^n \bigcup_{S_1^{2n-m}=S_2^{2n-m}} D_2^n$, the quotient space of the disjoint union $D_1^n \cup D_2^n$ identified at $S_1^{2n-m}=S_2^{2n-m}$. Hence, $B_1 \bigvee_{S^{2n-m}} B_2 \simeq S_1^n \bigcup_{S_1^{2n-m}=S_2^{2n-m}} S_2^n = (D_1^n \bigcup_{S_1^{2n-m}=S_2^{2n-m}} D_2^n) \cup D_1'^n \cup D_2'^n \simeq S^{2n-m+1} \cup e_1^n \cup e_2^n$.

The homotopy type of B_Φ is some more complicated. In D_j^n, there are disjoint imbedded spheres $S_{ij}^{2n-m}=\varphi_{ij}(S^{2n-m}\times o)$, $i=1, 2, \cdots, r$, $i\neq j$. We understand that $S_{ij}^{2n-m}=S_{ji}^{2n-m}=\phi$ (empty set) if $\varphi_{ij}=\varphi_{ji}=0$. Assume that $2m\geq 3n+2$. Then, since S_{1j}^{2n-m} bounds an imbedded $(2n-m+1)$-disk E_{1j} in D_j^n which does not intersect with S_{ij}^{2n-m}, $i>1$, there is an n-cell C_1^n which includes E_{1j} in its interior and includes S_{ij}^{2n-m}, $i>1$, in its complement. Similarly, considering S_{2j}^{2n-m} in $D_j^n-\operatorname{Int} C_1^n$, S_{2j}^{2n-m} bounds an imbedded $(2n-m+1)$-disk E_{2j} in $D_j^n-\operatorname{Int} C_1^n$ which does not intersect with S_{ij}^{2n-m}, $i>2$, and there exists an n-cell C_2^n which includes E_{2j} in its interior and S_{ij}^{2n-m}, $i>2$, in its complement. Repeating this, we obtain disjoint $(2n-m+1)$-disks E_{ij}, $i=1, 2, \cdots, r$, $i\neq j$, in D_j^n, each bounded by S_{ij}^{2n-m}. We define $E_{ij}=\phi$ if $S_{ij}^{2n-m}=\phi$.

Let $\varphi_{i_1 j}, \varphi_{i_2 j}, \cdots, \varphi_{i_s j}$, $1\leq i_1<i_2<\cdots<i_s\leq r$, $s=s(j)$, be all the non-zero components of Φ in the i-th row. Then $E_{i_1 j}, E_{i_2 j}, \cdots, E_{i_s j}$ are all the non-empty disks in D_j^n. Tie $E_{i_k j}$ and $E_{i_{k+1} j}$ by an imbedded arc $w^j(i_k, i_{k+1})$ in D_j^n, $k=1, 2, \cdots, s-1$, with the end points in the boundaries of the disks. We can take the arcs to be disjoint except their end points. Then, $E_{i_1 j}\cup w^j(i_1, i_2)\cup E_{i_2 j}\cup\cdots\cup E_{i_{s-1} j}\cup w(i_{s-1}, i_s)\cup E_{i_s j}$ is a strong deformation retract of D_j^n, $j=1, 2, \cdots, r$. Since we may assume that final or initial points in S_{ij}^{2n-m} are all common, these arcs form a 1-dimensional cell complex $\mathfrak{K}$. Thus we have

Lemma 3.3. *If $2m\geq 3n+2$, B_Φ has the homotopy type of $(\bigcup_{i>j} S_{ij}^{2n-m+1}) \cup \mathfrak{K} \cup (e_1^n \cup e_2^n \cup \cdots \cup e_r^n)$, where $S_{ij}^{2n-m+1}=E_{ij}\cup E_{ji}$, $i, j=1, 2, \cdots, r$, $i\neq j$, are $(2n-m+1)$-spheres if $\varphi_{ij}\neq 0$ and empty if $\varphi_{ij}=0$.*

Proof. Let $S_\Phi^{2n-m}=\{S_{ij}^{2n-m}; i, j=1, 2, \cdots, r, i\neq j\}$, $\bigcup_{i=1}^r S_i^n=$ the union of S_i^n, $i=1, 2, \cdots, r$, identified at S_Φ^{2n-m}, and $\bigcup_{i=1}^r D_i^n=$ the union of D_i^n, $i=1, 2, \cdots, r$, identified at S_Φ^{2n-m}. Then, $B_\Phi \simeq \bigcup_{i=1}^r S_i^n = (\bigcup_{i=1}^r D_i^n) \cup (\bigcup_{i=1}^r D_i'^n)$, and $\bigvee_{i=1}^{r} D_i^n \simeq (\bigvee_{i>j} S_{ij}^{2n-m+1}) \cup \mathfrak{R}$. This completes the proof.

We note that $\mathfrak{R}$ is determined only by Φ: Take the independent points a_{ij} $(i>j)$ in sufficiently high dimensional Euclidean space correspondingly to the non-zero elements φ_{ij} of Φ, and let $a_{ij}=a_{ji}$ if $i<j$. If $a_{i_1 j}, a_{i_2 j}, \cdots, a_{i_s j}$, $1 \leq i_1 < i_2 < \cdots < i_s \leq r$, $s=s(j)$, are all the points when j is fixed, let $\overline{a_{i_1 j} a_{i_2 j}}, \overline{a_{i_2 j} a_{i_3 j}}, \cdots, \overline{a_{i_{s-1} j} a_{i_s j}}$ be the 1-simplexes for $j=1, 2, \cdots, r$. Then we have a 1-dimensional cell complex which is isomorphic to $\mathfrak{R}$.

The above lemma shows that B_Φ may have non-trivial fundamental group.

Lemma 3.4. *The homomorphism $i_*: \pi_k(\partial B_\Phi) \to \pi_k(B_\Phi)$, where $i: \partial B_\Phi \to B_\Phi$ is the inclusion map, is surjective if $k \leq m-n-1$ and injective if $k \leq m-n-2$. Especially, if $2m \geq 3n+2$, $n \geq 3$, and $\mathfrak{R}$ is connected, then $\pi_1(\partial B_\Phi)$, $\pi_1(B_\Phi)$, and $\pi_1(\mathfrak{R})$ are isomorphic.*

Proof. If $k \leq m-n-1$, a representative map of an element of $\pi_k(B_\Phi)$ can be taken so that the image is detached from $\bigvee_{i=1}^{r} S_i^n$. Since ∂B_Φ is a strong deformation retract of $B_\Phi - \bigvee_{i=1}^{r} S_i^n$, this implies that i_* is surjective. Similarly, we know that i_* is injective if $k \leq m-n-2$. The latter of the lemma is obtained by Lemma 3.3 and using Van Kampen theorem.

Lemma 3.5. *If $m-n \geq 3$ and $n \geq 2$, the boundary of $B_1 \mathop{\mathbb{V}}_{S^{2n-m}} B_2$ is simply connected.*

Proof. This is obtained from Lemma 3.2 and Lemma 3.4.

4. Plumbing along Tori

Analogously to the section 3, we can plumb the disk bundles over spheres along tori.

Let $(B_i, p_i, S_i^n, D^{m-n})$, $i=1, 2, \cdots, r$, be the $(m-n)$-disk bundles over n-spheres, and let $S_i^n = D_i^n \cup D_i'^n$, $B_i = D_i^n \times D^{m-n} \cup D_i'^n \times D^{m-n}$. Let $\Psi = (\psi_{ij})$

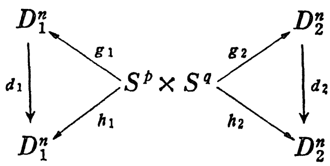
be a matrix without diagonal consisting of the disjoint imbeddings ψ_{ij}: $S^p\times S^q\times D^{m-n}\to \mathrm{Int}\ D_j^n\subset S_j^n$, $p+q=2n-m$, $i, j=1, 2, \cdots, r$, $i\neq j$. We denote as $\psi_{ij}=0$ if ψ_{ij} is not defined and understand that $\psi_{ji}=0$ if $\psi_{ij}=0$. $\psi_{ij}(S^p\times S^q\times o)$ is denoted by $(S^p\times S^q)_{ij}$. Let $\bar{\psi}_{ij}$: $S^p\times S^q\times D^{m-n}\times D^{m-n}\to D_j^n\times D^{m-n}\subset B_j$ be the canonical extension of $\psi_{ij}\neq 0$. Then 'the manifold plumbed along the tori' denoted by $B_\Psi=\bigvee_{\Psi}\{B_k;\, k=1, 2, \cdots, r\}$ is defined as the quotient space of the disjoint union $B_1\cup B_2\cup\cdots\cup B_r$ by identifying the subset $\bar{\psi}_{ij}(S^p\times S^q\times D^{m-n}\times D^{m-n})$ of B_j with the subset $\bar{\psi}_{ji}(S^p\times S^q\times D^{m-n}\times D^{m-n})$ of B_i in such a way that $\bar{\psi}_{ij}(u, v, x, y)\sim\bar{\psi}_{ji}(u, v, y, x)$ for all (u, v, x, y) and for all $\psi_{ij}, \psi_{ji}\neq 0$, where the corners are corrected similarly as in the section 3. If $r=2$, we denote B_Ψ by $B_1\bigvee_{S^p\times S^q}B_2$ and call as 'the manifold plumbed along $S^p\times S^q$.

Also we note that by Wall's pairing λ: $i_n(B_\Psi)\times\pi_n(B_\Psi)\to\pi_n(S^{m-n})$ ([18], p. 255) Ψ corresponds to a matrix $\Lambda=(\lambda(e_i, e_j))$, where e_i, $i=1, 2, \cdots, r$, are defined by S_i^n.

Homotopy types of these manifolds plumbed along the tori are different from those of the manifolds plumbed along the spheres. About the homotopy properties of $B_1\bigvee_{S^p\times S^q}B_2$, B_Ψ, and those boundaries we have the following results.

Lemma 4.1. *Let* $0<p\leq q$ *and* $n>p+2q+1$. *Then* $B_1\bigvee_{S^p\times S^q}B_2$ *has the homotopy type of* $(S^{p+q+1}\vee S^{p+1}\vee S^{q+1})\cup e_1^n\cup e_2^n$.

Proof. Take a small standard $D_i^{p+1}\times D_i^{q+1}$, the product of the $(p+1)$- and $(q+1)$-disks, in D_i^n for $i=1, 2$, and let g_i: $D^{p+1}\times D^{q+1}\to D_i^{p+1}\times D_i^{q+1}\subset D_i^n$, $i=1, 2$, be the imbeddings shrinking $D^{p+1}\times D^{q+1}$ similarly to $D_i^{p+1}\times D_i^{q+1}$. Define the imbeddings h_i: $S^p\times S^q\to D_i^n$, $i=1, 2$, as $h_i(u, v)=\psi_i(u, v, o)$, where $\psi_1=\psi_{21}$ and $\psi_2=\psi_{12}$. Then, from the assumption of p, q, g_i: $S^p\times S^q\to D_i^n$ is isotopic to h_i, using Haefliger [4] or Zeeman [20]. And by the isotopy extension theorem ([15], [7]), there exists a diffeomorphism or a PL-homeomorphism d_i of D_i^n onto itself such that $h_i=d_i\circ g_i$. Those relations are shown in the following commutative diagram.



Thus $d_1 \cup d_2$ maps $D_1^n \underset{S_1^p \times S_1^q = S_2^p \times S_2^q}{\bigcup} D_2^n$, the quotient space of the disjoint union $D_1^n \cup D_2^n$ identified as $g_1(u, v) = g_2(u, v)$, homeomorphically onto $D_1^n \underset{(S^p \times S^q)_1 = (S^p \times S^q)_2}{\bigcup} D_2^n$, the quotient space of the disjoint union $D_1^n \cup D_2^n$ identified as $h_1(u, v) = h_2(u, v)$, where $(S^p \times S^q)_i$ denotes $h_i(S^p \times S^q)$ for $i = 1, 2$. $(S^p \times S^q)_1 = h_1(S^p \times S^q) = \partial d_1(S_1^p \times D_1^{q+1})$ and $(S^p \times S^q)_2 = h_2(S^p \times S^q)$ $= \partial d_2(D_2^{p+1} \times S_2^q)$. The subspace $d_1(S_1^p \times D_1^{q+1}) \cup d_2(D_2^{p+1} \times S_2^q)$ of $D_1^n \underset{(S^p \times S^q)_1 = (S^p \times S^q)_2}{\bigcup} D_2^n$ is homeomorphic to the subspace $S^{p+q+1} = S_1^p \times D_1^{q+1}$ $\cup D_2^{p+1} \times S_2^q$ of $D_1^n \underset{S_1^p \times S_1^q = S_2^p \times S_2^q}{\bigcup} D_2^n$.

$d_1(S_1^p \times D_1^{q+1}) \cup d_1(D_1^{p+1} \times y)$, $y \in S_1^q$ and $d_2(D_2^{p+1} \times S_2^q) \cup d_2(x \times D_2^{q+1})$, $x \in S_2^p$ are the strong deformation retracts of D_1^n and D_2^n respectively. So that $d_1(S_1^p \times D_1^{q+1}) \cup d_2(D_2^{p+1} \times S_2^q) \cup d_1(D_1^{p+1} \times y) \cup d_2(x \times D_2^{q+1})$ is a strong deformation retract of $D_1^n \underset{(S^p \times S^q)_1 = (S^p \times S^q)_2}{\bigcup} D_2^n$, and the former has the homotopy type of $S^{p+q+1} \vee S^{p+1} \vee S^{q+1}$.

Hence $B_1 \underset{S^p \times S^q}{\mathbb{V}} B_2 \simeq S_1^n \underset{(S^p \times S^q)_1 = (S^p \times S^q)_2}{\bigcup} S_2^n = (D_1^n \underset{(S^p \times S^q)_1 = (S^p \times S^q)_2}{\bigcup} D_2^n) \cup D'^n_1 \cup D'^n_2$ $\simeq (S^{p+q+1} \vee S^{p+1} \vee S^{q+1}) \cup e_1^n \cup e_2^n$. This completes the proof.

Now, let us consider the general case, the homotopy type of B_{Ψ}. In D_j^n, there are the imbedded disjoint tori $(S^p \times S^q)_{ij} = \psi_{ij}(S^p \times S^q \times o)$, $i = 1, 2, \cdots, r$, $i \neq j$, where $p + q = 2n - m$ and $(S^p \times S^q)_{ij} = \phi$ if $\psi_{ij} = 0$. Assume that $0 < p \leq q$ and $n > p + 2q + 1$. Then, as in Lemma 4.1, each $(S^p \times S^q)_{ij}$ bounds $d_{ij}(D_j^{p+1} \times S_j^q)$ if $i < j$ and $d_{ij}(S_j^p \times D_j^{q+1})$ if $i > j$ in D_j^n, where each d_{ij} is a diffeomorphism of D_i^n onto itself. We denote $d_{ij}(D_j^{p+1} \times S_j^q)$ $\cup d_{ij}(x_j \times D_j^{q+1})$, $x_j \in S_j^p$, $i < j$ or $d_{ij}(S_j^p \times D_j^{q+1}) \cup d_{ij}(D_j^{p+1} \times y_j)$, $y_j \in S_j^q$, $i > j$, simply by F_{ij}.

Let $\psi_{i_1 j}, \psi_{i_2 j}, \cdots, \psi_{i_s j}$, $1 \leq i_1 < i_2 < \cdots < i_s \leq r$, $s = s(j)$, be all the non-zero components of Ψ in the j-th row. If $n > 2(p + q + 1)$, then we can take $F_{i_1 j}, F_{i_2 j}, \cdots, F_{i_s j}$ to be disjoint. But in order that they are disjoint it is sufficient that $n > 2(p + q) + 1$. Because, considering the tubular neighborhood of $F_{i_1 j}$, there exists an n-cell C_1^n in D_j^n which includes $(S^p\times S^q)_{ij}$ $i>i_1$ in its complement. So that, considering $(S^p\times S^q)_{i_2 j}$ in $D_j^n-\mathrm{Int}\,C_1^n$, $F_{i_2 j}$ can be taken in $D_j^n-\mathrm{Int}\,C_1^n$. Repeating this as in Lemma 3.3, we obtain the disjoint sets $F_{i_1 j}, F_{i_2 j}, \cdots, F_{i_s j}$.

In D_j^n, tie $F_{i_k j}$ and $F_{i_{k+1} j}$ by an imbedded arc $w^j(i_k, i_{k+1})$ with the initial point in $(S^p\times S^q)_{i_k j}$ and the final point in $(S^p\times S^q)_{i_{k+1} j}$. We may assume that each arc $w^j(i_k, i_{k+1})$ does not intersect with other arcs and each F_{ij} except at the end points. Then, $F_{i_1 j}\cup w^j(i_1, i_2)\cup F_{i_2 j}\cup\cdots\cup F_{i_{s-1} j}\cup w^j(i_{s-1}, i_s)\cup F_{i_s j}$ is a strong deformation retract of D_j^n for each j. Since we may assume that final or initial points in $(S^p\times S^q)_{ij}$ are all common, these arcs form a 1-dimensional cell complex $\mathfrak{L}$, which is determined only by Ψ as in the section 3.

Lemma 4.2. *Let $0<p\leq q$ and $n>2p+2q+1$. Then, B_Ψ has the homotopy type of $\{\bigcup_{i>j}(S_{ij}^{p+q+1}\vee S_{ij}^{p+1}\vee S_{ij}^{q+1})\}\cup\mathfrak{L}\cup(e_1^n\cup e_2^n\cup\cdots\cup e_r^n)$, where S_{ij}^{p+1}, S_{ij}^{q+1}, and S_{ij}^{p+q+1} are respectively the copies of $(p+1)$-, $(q+1)$-, and $(p+q+1)$-dimensional spheres, and we understand that they are empty if $\psi_{ij}=0$.*

Proof. Let $(S^p\times S^q)_\Psi=\{(S^p\times S^q)_{ij};\ i,\ j=1, 2, \cdots, r,\ i\neq j\}$, $\bigvee_{i=1}^{r} S_i^n=$ the union of S_i^n, $i=1, 2, \cdots, r$, identified at $(S^p\times S^q)_\Psi$, and $\bigvee_{r=1}^{i} D_i^n=$ the union of D_i^n, $i=1, 2, \cdots, r$, identified at $(S^p\times S^q)_\Psi$.

Then, $B_\Psi\simeq\bigvee_{i=1}^{r} S_i^n=(\bigvee_{i=1}^{r} D_i^n)\cup(\bigvee_{i=1}^{r} D'^n_i)$, and $\bigvee_{i=1}^{r} D_i^n\simeq\{\bigvee_{i>j}(S_{ij}^{p+q+1}\vee S_{ij}^{p+1}\vee S_{ij}^{q+1})\}\cup\mathfrak{L}$. This completes the proof.

Lemma 4.3. *The homomorphism $i_*\colon\pi_k(\partial B_\Psi)\to\pi_k(B_\Psi)$, where $i\colon\partial B_\Psi\to B_\Psi$ is the inclusion map, is surjective if $k\leq m-n-1$ and injective if $k\leq m-n-2$. Especially, if $0<p\leq q$, $n>2p+2q+1$, and $\mathfrak{L}$ is connected, then $\pi_1(\partial B_\Psi)$, $\pi_1(B_\Psi)$, and $\pi_1(\mathfrak{L})$ are isomorphic.*

Proof. ∂B_Ψ is also a strong deformation retract of $B_\Psi-\bigvee_{i=1}^{r} S_i^n$. So, the proof is quite similar to that of Lemma 3.4.

Lemma 4.4. *Let $0<p\leq q$ and $n>p+2q+1$. Then the boundary* *of* $B_1 \bigvee_{S^p \times S^q} B_2$ *is simply connected if* $m-n \geq 3$.

Proof. This is obtained from Lemma 4.1 and Lemma 4.3.

5. Representing Handlebodies by Plumbing and Surgery

Let $W = D^m \bigcup_{\{f_i\}} \{\bigcup_{i=1}^{r} D_i^n \times D_i^{m-n}\}$ $(2n \geq m)$ be a handlebody and fix the representation. In this section we consider to represent W by plumbing D^{m-n}-bundles over n-spheres along $(2n-m)$-spheres or tori and by surgeries at the boundary of the plumbing manifold. We consider it by every case given at the section 2.

$f_1(S_1^{n-1} \times o) \cup f_2(S_2^{n-1} \times o) \cup \cdots \cup f_r(S^{n-1} \times o)$ is an r-link in S^{m-1}. Let $1 < n < m-2$. Then $f_i(S^{n-1} \times o) \subset X_i = S^{m-1} - \bigcup_{j \neq i} f_j(S_j^{n-1} \times o)$ defines an element of $\pi_{n-1}(X_i) \cong \pi_{n-1}(S^{m-n-1}) + \cdots + \pi_{n-1}(S^{m-n-1})$ (the direct sum of $(r-1)$ copies of $\pi_{n-1}(S^{m-n-1})$). The element $\lambda_i = \lambda_{1i} + \cdots + \lambda_{ji} + \cdots + \lambda_{ri}$ $(j \neq i)$ is called 'the linking element of $f_i(S_i^{n-1} \times o)$ to the others'. λ_{ji} is the linking element of $f_i(S_i^{n-1} \times o)$ to $f_j(S_j^{n-1} \times o)$ in the 2-link $f_i(S_i^{n-1} \times o) \cup f_j(S_j^{n-1} \times o)$, and $\lambda_{ij} = (-1)^n \lambda_{ji}$. If $2m \geq 3n+3$, the equivalence class of the r-link $f_1(S_1^{n-1} \times o) \cup \cdots \cup f_r(S^{n-1} \times o)$ is determined by the linking elements λ_i, $i = 1, 2, \cdots, r$ (Haefliger [6]).

Thus, if $2m \geq 3n+3$, a $(-1)^n$-symmetric $r \times r$ matrix $\Lambda = (\lambda_{ij})$ with the components in $\prod_{2n-m} = \pi_{n-1}(S^{m-n-1})$ and without diagonal elements corresponds to the handlebody W. If $\Lambda = \mathrm{diag}(\Lambda_1, \Lambda_2, \cdots, \Lambda_k)$ for some $k > 1$, under some exchanges of suffixes of handles, where Λ_i is an $r_i \times r_i$ matrix, we say as Λ is *decomposable*. If Λ is not decomposable, we say as Λ is *undecomposable*.

Throughout the following arguments, we assume that $2m \geq 3n+3$.

i) Let us consider the case 0), that is, $m = 2n$, $n \geq 3$. We can stretch an n-disk D_i^n to $f_i(S_i^{n-1} \times o)$, $i = 1, 2, \cdots, r$, in D^m. Going inductively, we can stretch them so that each intersection $D_i^n \cap D_j^n = |\lambda_{ij}|$ points $(i \neq j)$, especially $D_i^n \cap D_j^n = \phi$ if $\lambda_{ij} = 0$, $i \neq j$. Let $S_i^n = D_i^n \cup D_i'^n$, $D_i'^n = D_i^n \times o \subset D_i^n \times D_i^{m-n}$, and take the tubular neighborhoods B_i of S_i^n, $i = 1, 2, \cdots, r$. Then the ordinary plumbing manifold $B_\Lambda = \bigvee_{\Lambda} \{B_i;\ i = 1, 2,$

$\cdots, r\}$ is imbedded in W^{2n}. If $\Lambda=\text{diag}(\Lambda_1, \Lambda_2, \cdots, \Lambda_k)$, $k>1$, and each Λ_i is an undecomposable $r_i\times r_i$ matrix, B_Λ is the disjoint union of $B_{\Lambda_1}=\bigvee_{\Lambda_1}\{B_i;\ i=1, 2, \cdots, r_1\}, \cdots, B_{\Lambda_k}=\bigvee_{\Lambda_k}\{B_i;\ i=r-r_k+1, \cdots, r\}$, which are the connected components of B_Λ. Since $\pi_1(\partial B_{\Lambda_i})\cong\pi_1(B_{\Lambda_i})\cong\pi_1(\mathfrak{J}_i)$ and $n\geq 3$, where $\mathfrak{J}_i$ is the 1-dimensional complex associated to Λ_i, we can represent the generators of $\pi_1(\mathfrak{J}_i)$ by the imbedded 1-spheres on the boundary of B_{Λ_i} and in the interior of D^m, $i=1, 2, \cdots, r$. In $(W-\text{Int}\, B_\Lambda)\cap\text{Int}\, D^m$, we can attach the disjointly imbedded 2-disks to those 1-spheres transversely to ∂B_Λ. Then, by thickening the 2-disks to 2-handles in the interior of D^m, we have m-dimensional submanifolds W_i of W, $i=1, 2, \cdots, r$, with simply connected boundary. Join $W_1, W_2, \cdots, W_k$ with bands in W, and denote it by W'. Then clearly $(W-\text{Int}\, W';\ \partial W, \partial W')$ is an h-cobordism, and we have

Theorem 5.1. *Let* $W=D^{2n}\bigcup_{\{f_i\}}\{\bigcup_{i=1}^{r} D_i^n\times D_i^n\}$ $(n\geq 3)$ *be a handlebody with the matrix* Λ *of linking elements,* $\Lambda=\text{diag}(\Lambda_1, \Lambda_2, \cdots, \Lambda_k)$ $k\geq 1$, *where* Λ_i *is an undecomposable* $r_i\times r_i$ *matrix. Then,* W *is diffeomorphic to* $W_1\natural W_2\natural\cdots\natural W_k$, *the boundary connected sum of handlebodies* W_i *with* r_i *handles,* $i=1, 2, \cdots, k$, *which are obtained from the ordinary plumbing manifolds* $B_{\Lambda_i}=\bigvee_{\Lambda_i}\{B_j;\ j=r_1+\cdots+r_{i-1}+1, \cdots, r_1+r_2+\cdots+r_i\}$ *by attaching 2-handles to* ∂B_{Λ_i} *(surgery at* ∂B_{Λ_i}*) so as to kill* $\pi_1(\partial B_{\Lambda_i})\cong\pi_1(\mathfrak{J}_i)$, *where* B_j *are the tubular neighborhoods of* S_j^n, *with, as hemispheres,* D_j^n *in* D^{2n} *and* $D_j'^n=D_j^n\times o$ *in the* j*-th handle.*

Corollary 5.2. *Under the above assumptions,* ∂W *is diffeomorphic to* $\partial W_1\#\partial W_2\#\cdots\#\partial W_k$, *the connected sum of* $\partial W_1, \partial W_2, \cdots, \partial W_k$, *and each* ∂W_i *is obtained from* ∂B_{Λ_i} *by the surgery of* $\pi_1(\partial B_{\Lambda_i})\cong\pi_1(\mathfrak{J}_i)$.

We note that the above is also an epitome of the following arguments.

Remark. Let Λ be the 8×8 matrix given by J. Milnor [13], p. 11. Define $f_i\colon \partial D_i^{2k}\times D_i^{2k}\to\partial D^{4k}$, $i=1, 2, \cdots, 8$, so that the link $f_1(S_1^{2k-1}\times o)\cup\cdots\cup f_8(S_8^{2k-1}\times o)$ corresponds to the matrix Λ and $B_1, B_2, \cdots, B_8$, the tubular neighbourhoods of $S_i^{2k}=D_i^{2k}\cup D_i^{2k}\times o$, $i=1, 2, \cdots, 8$, have $(T)\in\pi_{2k-1}(SO_{2k})$ as those characteristic classes, where (T) is the generator of $\mathrm{Ker}(\pi_{2k-1}SO_{2k}\to\pi_{2k-1}SO)$. Then the handlebody $W=D^{4k}\bigcup_{\{f_i\}}\{\bigcup_{i=1}^{8}D_i^{2k}\times D_i^{2k}\}$ coincides with Milnor's manifold W_0 [13], p. 13.

ii) Let us consider the case that λ_{ij} are all in the J-image. It includes the cases 1), 3), 4), 5), and 7) in the section 2. Firstly, assume that $r=2$, that is, $W=D^m\bigcup_{\{f_1,f_2\}}\{D_1^n\times D_1^{m-n}\cup D_2^n\times D_2^{m-n}\}$. Let D_i^n, $i=1, 2$, be the imbedded n-disks in D^m which are bounded by $f_i(S_i^{n-1}\times o)$, $i=1, 2$. By Theorem 2.2 or Theorem 1.1, we may assume that they intersect transversely at the $(2n-m)$-sphere S^{2n-m} if $\lambda_{12}\neq 0$, or that they are disjoint in D^m if $\lambda_{12}=0$. Let $\lambda_{12}\neq 0$ and B_i, $i=1, 2$, be the tubular neighborhoods of $S_i^n=D_i^n\cup D_i'^n$, $D_i'^n=D_i^n\times o\subset D_i^n\times D_i^{m-n}$, $i=1, 2$. Take a normal $(m-n)$-framing $\mathscr{F}_i$ on D_i^n, and let σ_i be the restriction of $\mathscr{F}_i$ to S^{2n-m}, $i=1, 2$. σ_1, σ_2 are normal $(m-n)$-framings of S^{2n-m} in D_2^n, D_1^n respectively. Then $B_i=D_i^n\times D_i^{m-n}\cup D_i'^n\times D_i^{m-n}$, $i=1, 2$, and by Lemma 1.2, (S^{2n-m}, σ_1) in D_2^n and (S^{2n-m}, σ_2) in D_1^n correspond respectively to the elements of Π_{2n-m}, $\lambda(D_1^n\, D_2^n)=\lambda_{12}$ and $\lambda(D_2^n D_1^n)=\lambda_{21}=(-1)^n\lambda_{12}$. So that there exist the imbeddings φ_1: $S^{2n-m}\times D^{m-n}\to D_1^n$ and φ_2: $S^{2n-m}\times D^{m-n}\to D_2^n$ corresponding respectively to λ_{21} and λ_{12}. Thus the manifold $B_1\underset{S^{2n-m}}{\bigwedge}B_2$ plumbed along S^{2n-m} is imbedded in the interior of W. The resulting situation is illustrated schematically in Figure[2] 1.

Let $n\geq 2$. From Lemma 3.2, $B_1\underset{S^{2n-m}}{\bigvee}B_2$ has the homotopy type of $S^{2n-m+1}\cup e_1^n\cup e_2^n$, where $S^{2n-m+1}=E_1\cup E_2$, E_i, $i=1, 2$, are $(2n-m+1)$-disks, and $E_1\cap E_2=S^{2n-m}$. So, by Lemma 3.4, there is an imbedded $(2n-m+1)$-sphere T^{2n-m+1} on $\partial(B_1\underset{S^{2n-m}}{\bigvee}B_2)\cap\mathrm{Int}\,D^m$ which is homotopic to S^{2n-m+1}. In $\{W-\mathrm{Int}(B_1\underset{S^{2n-m}}{\bigvee}B_2)\}\cap\mathrm{Int}\,D^m$, we can attach a $(2n-m+2)$-dimensional disk D^{2n-m+2} to T^{2n-m+1} transversely to the boundary of $B_1\underset{S^{2n-m}}{\bigvee}B_2$. Then, thickening the disk D^{2n-m+2} to a handle $D^{2n-m+2}\times D^{2m-2n-2}$ in the interior of D^m, we obtain an m-dimensional submanifold $W'=(B_1\underset{S^{2n-m}}{\bigvee}B_2)\cup D^{2n-m+2}\times D^{2m-2n-2}$ of W. Since $\partial W'$, ∂W, and

2) The dotted curves mean that $D_1^n\,(D_2^n)$ can not be separated from $D_2^n\,(D_1^n)$.

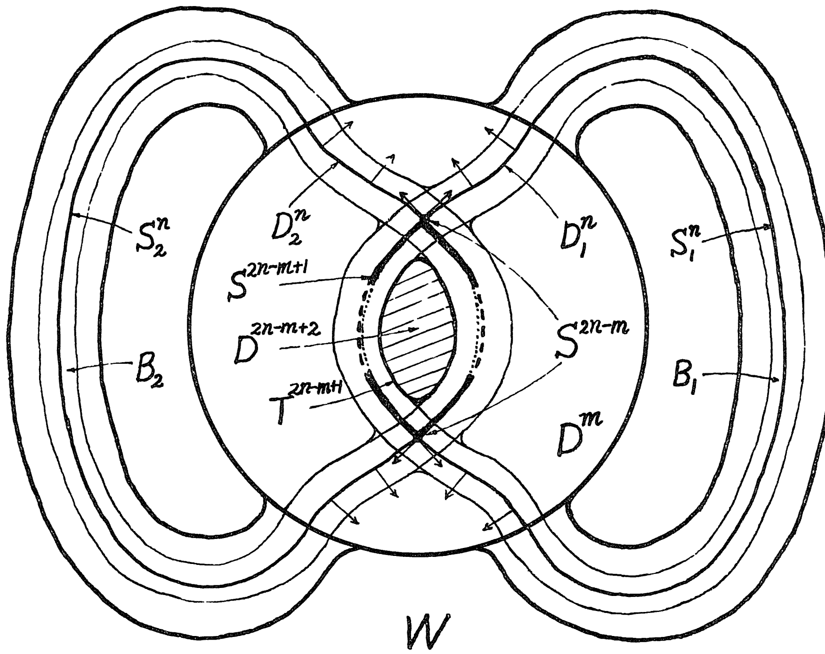


Figure 1

$W-\text{Int}\,W'$ are simply connected and $H_*(W-\text{Int}\,W', \partial W')\cong H_*(W, W')\cong 0$, $(W-\text{Int}\,W; \partial W', \partial W)$ is an h-cobordism. Thus we have

Theorem 5.3. *Let $W=D^m \bigcup_{\{f_1, f_2\}} \{D_1^n\times D_1^{m-n}\cup D_2^n\times D_2^{m-n}\}$ be a handlebody with two handles and let $2m\geq 3n+3$, $m\geq 6$, and $n\geq 2$. If the linking element $\lambda_{12}\in\prod_{2n-m}$ is a non-zero element of the J-image, then W is diffeomorphic to the manifold W' which is obtained from the manifold $B_1 \bigvee_{S^{2n-m}} B_2$ plumbed along S^{2n-m} by attaching a $(2n-m+2)$-handle to $\partial(B_1 \bigvee_{S^{2n-m}} B_2)$ (surgery at the boundary) so as to kill $\pi_{2n-m+1}(B_1 \bigvee_{S^{2n-m}} B_2)\cong Z$, where B_i, $i=1, 2$, are the tubular neighborhoods of S_i^n with, as hemispheres, D_i^n in D^m and $D_i'^n=D_i^n\times o$ in the i-th handle.*

Corollary 5.4. *Under the above assumptions, ∂W is diffeomorphic to $\partial W'$ obtained from $\partial(B_1 \bigvee_{S^{2n-m}} B_2)$ by the surgery to kill $\pi_{2n-m+1}(\partial(B_1 \bigvee_{S^{2n-m}} B_2))\cong Z$.*

iii) Now let us consider the case when $r\geq 3$ and λ_{ij} are all in the J-image. We denote $D_2^n\cap D_1^n$ by S_{21}^{2n-m}, which is a $(2n-m)$-sphere if $\lambda_{21}\neq 0$ and ϕ if $\lambda_{21}=0$. Let D_3^n be an imbedded n-disk in D^m bounded by $f_3(S^{n-1}\times o)$. Since $(2n-m)+n<m$, we can take D_3^n so that it does not meet with S_{21}^{2n-m}, and by moving D_3^n isotopically, fixing ∂D_3^n, in its tubular neighborhood in $D^m-S_{21}^{2n-m}$, we may assume that D_3^n is transverse to D_1^n and D_2^n. Let $K_{3j}^{2n-m}=D_3^n\cap D_j^n$, $j=1, 2$, and let U_j, $j=1, 2$, be sufficiently small open neighborhoods of K_{3j} in $D^m-S_{21}^{2n-m}$ such that $U_1\cap U_2=\phi$ and $U_j\cap D_j^n$, $j=1, 2$, are diffeomorphic to R^n, respectively.

Let $(L, \sum; K_{31}, K_{31}')$ be a framed elementary cobordism which extends the canonical framing σ of K_{31}. Using Wells' method [19], p. 401, there are the isotopies ζ_t of $D_1^n\subset D^m$ and η_t of $D_3^n\subset D^m$ such that $(\zeta_t, D_1^n, \eta_1(D_3^n), D^m)$ realizes $(L, \sum; K_{31}, K_{31}')$. ζ_t is an isotopy extending the isotopy sliding D_1^n along the $(n+1)$-th coordinate in terms of the coordinates in U_1. There is a Morse function defined on L with exactly one critical point of index $p+1$, $-1\leq p\leq 2n-m$. K_{31}' is obtained from K_{31} by a surgery of a p-dimensional sphere $f(S^p)$ of K_{31}, where $f\colon S^p\to K_{31}$ is the attaching map of the $(p+1)$-disk D_L^{p+1} such that $K\bigcup_f D_L^{p+1}$ is a deformation retract of L. Take the imbeddings $f_1\colon D^{p+1}\to D_1^n-S_{21}^{2n-m}$ and $f_2\colon D^{p+1}\to D_3^n$ extending f respectively. Then, an imbedding $g\colon \mathscr{D}^{p+2}\to D^m-S_{21}^{2n-m}$ such that $g(\mathscr{D}^{p+2})\cap(D_1^n\cup D_3^n)=f_1(D^{p+1})\cup f_2(D^{p+1})$ can be defined, where $\mathscr{D}^{p+2}$ is a homeomorph of D^{p+2}. And a $(n-p-1)$-dimensional vector bundle ξ is defined on $g(\mathscr{D}^{p+2})$. ξ is trivial and is a sub-bundle of the normal bundle of $g(\mathscr{D}^{p+2})$ in D^m. Then, choosing some field of $(n-p-1)$-frames in ξ, $\mathscr{D}^{p+2}\times D^{n-p-1}$ is imbedded in D^m and the isotopy η_t is constructed. Similarly for a given framed elementary cobordism $(L', \sum'; K_{32}, K_{32}')$ which extends the canonical framing σ' of K_{32}. There exists isotopies ζ_t' of $D_2^n\subset D^m$ and η_t' of $D_3^n\subset D^m$ such that $(\zeta_t', D_2^n, \eta_1'(D_3^n), D^m)$ realizes the given framed elementary cobordism of K_{32}.

We note that since $2m\geq 3n+3$ and p, $p'\leq 2n-m$, $g(\mathscr{D}^{p+2})$ and $g'(\mathscr{D}^{p'+2})$ can be separated from the other in D^m, where $g'\colon \mathscr{D}^{p'+2}\to D^m-S_{21}^{2n-m}$ is the imbedding corresponding to $g\colon \mathscr{D}^{p+2}\to D^m-S_{21}^{2n-m}$.

Hence, the pairs of the isotopies (ζ_t, η_t), (ζ'_t, η'_t) can be taken so that they do not affect each other and S_{21}^{2n-m}. Thus D_3^n can be taken so that $D_3^n \cap D_j^n = S_{3j}^{2n-m}$, $j=1, 2$, which are $(2n-m)$-spheres if $\lambda_{3j} \neq 0$ and ϕ if $\lambda_{3j}=0$. Repeating this, we can take D_i^n, $i=1, 2, \cdots, r$, in D^m, bounded respectively by $f_i(S_i^n \times o)$, $i=1, 2, \cdots, r$, so that for each (i, j), $i \neq j$, D_i^n and D_j^n intersect transversely at

$$D_i^n \cap D_j^n = S_{ij}^{2n-m} = \begin{cases} \text{a } (2n-m)\text{-sphere} & \text{if } \lambda_{ij} \neq 0 \\ \phi \quad \text{if } \lambda_{ij}=0, \end{cases}$$

which are disjoint in D^m. The intersection is denoted as S_{ij}^{2n-m} when it is considered in D_j^n and as S_{ji}^{2n-m} when considered in D_i^n.

Let $S_i^n = D_i^n \cup D_i'^n$, $D_i'^n = D_i^n \times o \subset D_i^n \times D_i^{m-n}$, and B_i be the tubular neighborhood of S_i^n in W, $i=1, 2, \cdots, r$. Take a normal $(m-n)$-framing $\mathscr{F}_i$ on each D_i^n, and let σ_{ij} be the restriction of $\mathscr{F}_i$ to S_{ij}^{2n-m} $(i \neq j)$ in D_j^n. Then $B_i = D_i^n \times D^{m-n} \cup D_i'^n \times D^{m-n}$, $i=1, 2, \cdots, r$, and by Lemma 1.2 each $(S_{ij}^{2n-m}, \sigma_{ij})$ in D_j^n corresponds to $\lambda(D_i^n, D_j^n) = \lambda_{ij} \in \Pi_{2n-m}$. So that, there exist the imbeddings $\varphi_{ij}: S^{2n-m} \times D^{m-n} \rightarrow D_j^n$, $i, j=1, 2, \cdots, r$, $i \neq j$, corresponding respectively to λ_{ij}. Let $\Phi = (\varphi_{ij})$, the matrix corresponding to Λ. Then $B_\Phi = \bigvee_\Phi \{B_i;\ i=1, 2, \cdots, r\}$, the manifold plumbed along $(2n-m)$-spheres, is imbedded in the interior of W. Firstly, we assume that Λ is undecomposable, that is, B_Φ is connected. A typical situation is schematically illustrated in the Figure[3] 2. There, $k=1$, $r=3$, and all λ_{ij} are assumed to be non-zero.

By Lemma 3.3, B_Φ has the homotopy type of $(\bigcup_{i>j} S_{ij}^{2n-m+1}) \cup \mathfrak{K} \cup (e_1^n \cup e_2^n \cup \cdots \cup e_r^n)$, where $S_{ij}^{2n-m+1} = \phi$ if $\varphi_{ij}=0$, and from the assumption, $\mathfrak{K}$ is connected. Although B_Φ and ∂B_Φ are not necessarily simply connected, we can take the imbedded $(2n-m+1)$-spheres T_{ij}^{2n-m+1} $(i>j)$ which are respectively homotopic to S_{ij}^{2n-m+1}, disjointly on ∂B_Φ, as follows; let $B_o = B_\Phi - \bigcup_{k \neq i,j} B_k = B_i \bigvee_{S^{2n-m}} B_j - \{\bigcup_{k \neq j} \bar{\varphi}_{ki}(S^{2n-m} \times D^{m-n} \times D^{m-n}) + \bigcup_{k \neq i} \bar{\varphi}_{kj}(S^{2n-m} \times D^{m-n} \times D^{m-n})\}$, and let $S_o = S_i^n \cup S_j^n - \{\bigcup_{k \neq j} \bar{\varphi}_{ki}(S^{2n-m} \times D^{m-n} \times o) + \bigcup_{k \neq i} \bar{\varphi}_{kj}(S^{2n-m} \times D^{m-n} \times o)\}$. Then S_o and ∂B_o is strong deformation

3) The dotted curves also mean that each D_i^n can not be separated from the other.

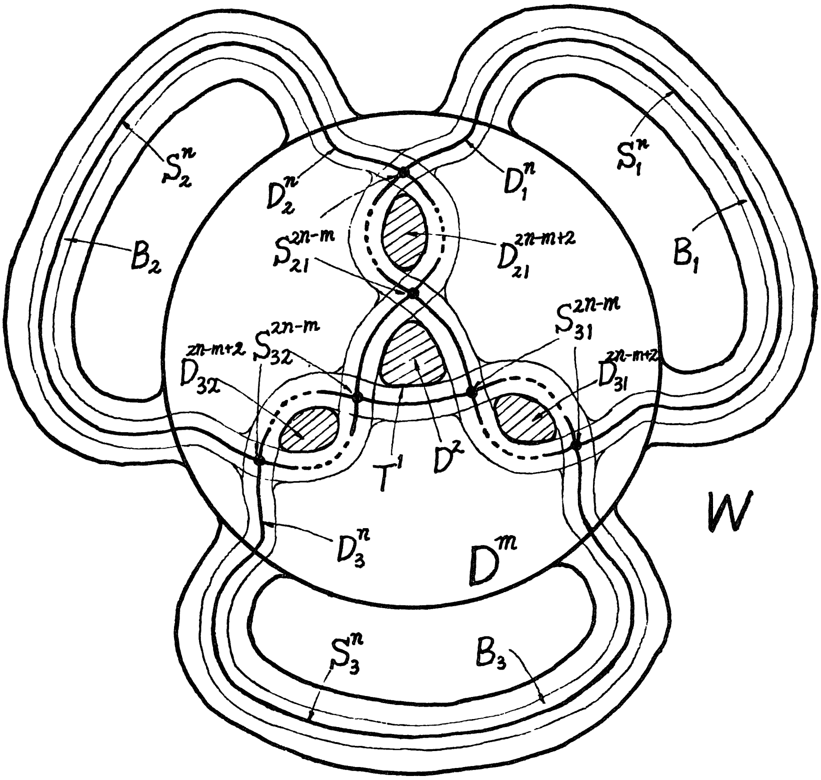


Figure 2

retracts of B_o and $B_o - S_o$, respectively. So that, as in Lemma 3.4, and by Lemma 3.2, we know that $\pi_k(\partial B_o) \cong \pi_k(B_o) \cong \pi_k(S_o) \cong \pi_k(S_i^n \cup S_j^n)$ $\cong \pi_k(S^{2n-m+1} \cup e_1^n \cup e_2^n)$, if $k \leq 2n-m+1$. Thus ∂B_o is $(2n-m)$-connected if $n \geq 2$. Since it may be assumed that D_{ij}^{2n-m+1}, D_{ji}^{2n-m+1}, and hence S_{ij}^{2n-m+1} are contained in S_o, we can take the imbedded $(2n-m+1)$-sphere T_{ij}^{2n-m+1} homotopic to S_{ij}^{2n-m+1} on ∂B_o, hence on ∂B_ϕ. (Haefliger [4]). From the assumption that $2m \geq 3n+3$, they can be disjoint.

On the other hand, $\pi_1(\partial B_\phi) \cong \pi_1(B_\phi) \cong \pi_1(\mathfrak{K})$ $(n \geq 3)$, by Lemma 3.4. Therefore, we can again take the imbedded 1-spheres T_l^1, disjointly on ∂B_ϕ, which represent respectively the generators of $\pi_1(\mathfrak{K})$. Thus, on ∂B_ϕ, there are the imbedded spheres T_{ij}^{2n-m+1} and T_l^1, which are disjoint and can be assumed to be in $\partial B_\phi \cap \operatorname{Int} D^m$. In $(W - \operatorname{Int} B_\phi) \cap \operatorname{Int} D^m$, we can attach $(2n-m+2)$-dimensional disks D_{ij}^{2n-m+2} and 2-disks D_l^2 respectively to T_{ij}^{2n-m+1} and T_l^1, transversely to ∂B_Φ. These disks can be taken disjointly for all i, j and l. Then, thickening these disks to the handles $D_{ij}^{2n-m+2} \times D_{ij}^{2m-2n-2}$ and $D_l^2 \times D_l^{m-2}$ in the interior of D^m, we have an m-dimensional submanifold W' imbedded in the interior of W.

Generally, let $\Lambda = \mathrm{diag}(\Lambda_1, \Lambda_2, \cdots, \Lambda_k)$ and $\Phi = \mathrm{diag}(\Phi_1, \Phi_2, \cdots, \Phi_k)$, $k \geq 1$, where each Λ_i is an undecomposable $r_i \times r_i$ matrix and Φ_i corresponds to Λ_i. Then B_Φ is the disjoint union of $B_{\Phi_1} = \bigvee_{\Phi_1} \{B_i;\ i=1, 2, \cdots, r_1\}$, $B_{\Phi_2} = \bigvee_{\Phi_k} \{B_i;\ i=r_1+1, \cdots, r_1+r_2\}, \cdots, B_{\Phi_k} = \bigvee_{\Phi_2} \{B_i;\ i=r-r_k+1, \cdots, r\}$, which are the connected components of B_Φ. So, constructing the manifold W_t' as above for each B_{Φ_t}, $t=1, 2, \cdots, k$, we have the m-dimensional submanifolds $W_1', W_2', \cdots, W_k'$, disjointly imbedded in the interior of W. Join these manifolds with thin bands, resulting an m-manifold W' imbedded in the interior of W, which is diffeomorphic to $W_1' \natural W_2' \natural \cdots \natural W_k'$, the boundary connected sum of W_t', $t=1, 2, \cdots, k$.

Clearly, $\pi_1(\partial W') \cong 0$, $\pi_1(\partial W) \cong 0$, and since $\pi_1(W') \cong 0$, $\pi_1(W - \mathrm{Int}\, W') \cong 0$, using Van Kampen theorem. And, the following diagram shows that $H_*(W - \mathrm{Int}\, W', \partial W') \cong H_*(W, W') \cong 0$, where V denotes $W' - \bigcup_{i=1}^{r} (\mathrm{Int}\, D_n'^{n}) \times D_i^{m-n}$ which is contractible.

$$\begin{array}{ccc} H_i(W') & \longrightarrow & H_i(W) \\ \downarrow \cong & (i>0) & \downarrow \cong \\ H_i(W', V) & \xrightarrow{\cong} & H_i(W, D^m) \end{array}$$

Therefore we know that $(W - \mathrm{Int}\, W'; \partial W', \partial W)$ is an h-cobordism. Thus we obtain the following.

Theorem 5.5. *Let* $W = D^m \bigcup_{\{f_i\}} \{\bigcup_{i=1}^{r} D_i^n \times D_i^{m-n}\}$ *be a handlebody and let* $2m \geq 3n+3$, *and* $n \geq 3$. *Assume that the linking elements* $\lambda_{ij} \in \Pi_{2n-m}$ *are all in the J-image. Let* $\Lambda = (\lambda_{ij}) = \mathrm{diag}(\Lambda_1, \Lambda_2, \cdots, \Lambda_k)$ *and* $\Phi = \mathrm{diag}(\Phi_1, \Phi_2, \cdots, \Phi_k)$, $k \geq 1$, *where each* Λ_i *is an undecomposable* $r_i \times r_i$ *matrix and* Φ_i *corresponds to* Λ_i. *Then* W *is diffeomorphic to the manifold* $W' = W_1' \natural W_2' \natural \cdots \natural W_k'$, *the boundary connected sum of handlebodies* W_i' *with* r_i *handles,* $i=1, 2, \cdots, k$, *which are obtained from the manifolds* B_{Φ_i}

$= \bigvee_{\Phi_i} \{B_j;\, j=r_1+\cdots+r_{i-1}+1, \cdots, r_1+\cdots+r_i\}$ *by attaching* $(2n-m+2)$-*handles and* 2-*handles to* ∂B_{Φ_i} (*surgeries at* ∂B_{Φ_i}) *so as to kill, on* ∂B_{Φ_i}, *the* $(2n-m+1)$-*spheres homotopic to* S_{ij}^{2n-m+1} $(i>j)$ *and the* 1-*spheres representing the generators of* $\pi_1(\mathfrak{K}_i)$, *where* S_{ij}^{2n-m+1} *and* $\mathfrak{K}_i$ *are those for* Φ_i *denoted in Lemma* 3.3, *and* B_j *are the tubular neighborhoods of* S_j^n *with, as hemispheres,* D_j^n *in* D^m *and* $D_j'^n = D_j^n \times o$ *in the j-th handle.*

Corollary 5.6. *Under the above assumptions* ∂W *is diffeomorphic to* $\partial W_1' \# \partial W_2' \# \cdots \# \partial W_k'$, *and each* W_i' *is obtained from* ∂B_{Φ_i} *by the surgeries of the* $(2n-m+1)$-*spheres and* 1-*spheres corresponding respectively to* S_{ij}^{2n-m+1} $(i>j)$ *and the generators of* $\pi_1(\mathfrak{K}_i)$, *where* S_{ij}^{2n-m+1} *and* $\mathfrak{K}_i$ *are those for* Φ_i *denoted in Lemma* 3.3 *and Lemma* 3.4.

We note that the above argument includes the cases 4), 5) in the section 2.

Corollary 5.7. *Let* $W = D^m \bigcup_{\{f_i\}} \{\bigcup_{i=1}^r D_i^n \times D_i^{m-n}\}$ *be a handlebody and let* $2m \geq 3n+3$ *and* $n \geq 3$. *If the linking elements* λ_{ij} *are all zero, then* W *is diffeomorphic to the boundary connected sum* $B_1 \natural B_2 \natural \cdots \natural B_r$, *where* B_i *are the tubular neighborhoods of* S_i^n *with, as hemispheres,* D_i^n *in* D^m *and* $D_i'^n = D_i^n \times o$ *in the i-th handle.*

iv) Let us consider the case 2) and 6) in the section 2. The argument is similar to the above. But in these cases S_{ij}^{2n-m} must be replaced by $(S^p \times S^q)_{ij}$, where $p=q=1$ or $p=q=3$, and B_i, $i=1, 2, \cdots, r$, must be plumbed along the tori $(S^p \times S^q)_{ij}$, $i, j=1, 2, \cdots, r$, $i \neq j$. By the Lemmas 4.1 and 4.2, the resulting manifold B_Ψ has the homotopy type of $\{\bigcup_{i>j}(S_{ij}^{p+q+1} \vee S_{ij}^{p+1} \vee S_{ij}^{q+1})\} \cup \mathfrak{L} \cup (e_1^n \cup \cdots \cup e_r^n)$, especially if $r=2$, the homotopy type of $(S^{p+q+1} \vee S^{p+1} \vee S^{q+1}) \cup (e_1^n \cup e_2^n)$. So that, the surgeries at ∂B_Ψ or $\partial(B_1 \bigvee_{S^p \times S^q} B_2)$ must be done so as to kill those spheres and the generators of $\pi_1(\mathfrak{L})$.

Theorem 5.8. *Let* $W = D^m \bigcup_{\{f_i\}} \{\bigcup_{i=1}^r D_i^n \times D_i^{m-n}\}$ *be a handlebody* $(2m \geq 3n+3)$. *Assume that the linking elements* $\lambda_{ij} \in \Pi_{2n-m}$ *are all represented* *respectively by the framed manifolds* $(S^p\times S^q)_{ij}\in\Omega_{2n-m}^{\text{framed}}$, *where* $p+q=2n-m$, $0<p\leq q$, *and* $2p+2q+1<n$. *Let* $\Lambda=(\lambda_{ij})=\text{diag}(\Lambda_1, \Lambda_2, \cdots, \Lambda_k)$ *and* $\Psi=\text{diag}(\Psi_1, \Psi_2, \cdots, \Psi_k)$, $k\geq 1$, *where each* Λ_i *is an undecomposable* $r_i\times r_i$ *matrix and* Ψ_i *corresponds to* Λ_i. *Then* W *is diffeomorphic to the manifold* $W'=W'_1\natural W'_2\natural\cdots\natural W'_k$, *the boundary connected sum of handlebodies* W'_i *with* r_i *handles*, $i=1, 2, \cdots, k$, *which are obtained from the manifolds* $B_{\Psi_i}=\bigvee_{\Psi_i}\{B_j;\ j=r_1+\cdots+r_{i-1}+1, \cdots, r_1+\cdots r_i\}$ *by attaching the* $(p+q+2)$-, $(p+2)$-, $(q+2)$-, *and* 2-*handles to* ∂B_{Ψ_i} (*surgeries at* ∂B_{Ψ_i}) *so as to kill, on* ∂B_{Ψ_i}, *the* $(p+q+1)$-, $(p+1)$-, *and* $(q+1)$-*spheres respectively homotopic to* S_{ij}^{p+q+1}, S_{ij}^{p+1}, *and* S_{ij}^{q+1} $(i>j)$ *and* 1-*spheres representing the generators of* $\pi_1(\mathfrak{L}_i)$, *where* S_{ij}^{p+q+1}, S_{ij}^{p+1}, S_{ij}^{q+1}, *and* $\mathfrak{L}_i$ *are those for* Ψ_i *denoted in Lemma* 4.2, *and* B_j *are the tubular neighborhoods of* S_j^n *with, as hemispheres*, D_j^n *in* D^m *and* $D_j'^n=D_j^n\times o$ *in the* j-*th handle.*

Corollary 5.9. *Under the above assumptions* ∂W *is diffeomorphic to* $\partial W'_1\#\partial W'_2\#\cdots\#\partial W'_k$, *and each* W'_i *is obtained from* ∂B_{Ψ_i} *by the surgeries of the* $(p+q+1)$-, $(p+1)$-, *and* $(q+1)$-*spheres corresponding respectively to* S_{ij}^{p+q+1}, S_{ij}^{p+1}, *and* S_{ij}^{q+1} $(i>j)$ *and* 1-*spheres representing the generators of* $\pi_1(\mathfrak{L}_i)$, *where* S_{ij}^{p+q+1}, S_{ij}^{p+1}, S_{ij}^{q+1}, *and* $\mathfrak{L}_i$ *are those for* Ψ_i *denoted in Lemma* 4.2 *and Lemma* 4.3.

We note that in Theorem 5.8 and Corollary 5.9, if $r=2$ and $\lambda_{12}\neq 0$, the condition $2p+2q+1<n$ may be replaced by the condition $p+2q+1<n$ by Lemma 4.1 and we need not kill the elements of $\pi_1(\mathfrak{L})$ since $\mathfrak{L}=$ a point. So, we have the results corresponding to Theorem 5.3 and Corollary 5.4.

v) Generally, including the case when $2n-m\geq 8$, if $2n-m\neq 2, 6$, and 14, we can take each intersection $D_i^n\cap D_j^n$ to be a homotopy $(2n-m)$-sphere or empty according as $\lambda_{ij}\neq 0$ or $\lambda_{ij}=0$, as mentioned at the section 2. Of course, we must assume that $\text{Arf}\,K^{2n-m}=0$, if $2n-m=2^l-2$, for any connected closed $(2n-m)$-dimensional π-manifold K^{2n-m}. Therefore, a homotopy $(2n-m)$-sphere corresponds to each $\lambda_{ij}\neq 0$ and we can consider the manifold $B_{\tilde{\Phi}}$ plumbed along the homotopy $(2n-m)$-spheres. Then, treating those homotopy spheres combinatorially, we can decide the homotopy type of $B_{\tilde{\Phi}}$ analogously to the section 3, and consequently have the results quite similar to those of the section 3. Note that the complex $\mathfrak{K}$ remain unchanged. So, we have,

Theorem 5.10. *Let* $W=D^m\bigcup_{\{f_i\}}\{\bigcup_{i=1}^{r}D_i^n\times D_i^{m-n}\}$ *be a handlebody and let* $2m\geq 3n+3$, $n\geq 3$, *and* $2n-m\neq 2, 6, 14$. *Assume that* $\operatorname{Arf}K^{2n-m}=0$ *for any connected closed* π*-manifold* K^{2n-m} *if* $2n-m=2^l-2$. *Let* $\Lambda=(\lambda_{ij})=\operatorname{diag}(\Lambda_1, \Lambda_2, \cdots, \Lambda_k)$ *and* $\tilde{\Phi}=\operatorname{diag}(\tilde{\Phi}_1, \tilde{\Phi}_2, \cdots, \tilde{\Phi}_k)$, $k\geq 1$, *where each* Λ_i *is an undecomposable* $r_i\times r_i$ *matrix and* $\tilde{\Phi}_i$ *corresponds to* Λ_i. *Then* W *is diffeomorphic to the manifold* $W'=W'_1\natural W'_2\natural\cdots\natural W'_k$, *the boundary connected sum of handlebodies* W'_i *with* r_i *handles*, $i=1, 2, \cdots, k$, *which are obtained from the manifolds* $B_{\tilde{\Phi}_i}=\bigvee_{\tilde{\Phi}_i}\{B_j; j=r_1+\cdots+r_{i-1}+1, \cdots, r_1+\cdots+r_i\}$ *by attaching* $(2n-m+2)$*-handles and* 2*-handles to* $\partial B_{\tilde{\Phi}_i}$ *(surgeries at* $\partial B_{\tilde{\Phi}_i}$*) so as to kill, on* $\partial B_{\tilde{\Phi}_i}$, *the* $(2n-m+1)$*-spheres homotopic to* S_{ij}^{2n-m+1} $(i>j)$ *and the* 1*-spheres representing the generators of* $\pi_1(\mathfrak{K}_i)$, *where* S_{ij}^{2n-m+1} *and* $\mathfrak{K}_i$ *are those for* $\tilde{\Phi}_i$ *denoted in Lemma* 3.2 *and Lemma* 3.3, *and* B_j *are the tubular neighborhoods of* S_j^n *with, as hemispheres,* D_j^n *in* D^m *and* $D_j'^n=D_j^n\times o$ *in the* j*-th handle.*

6. Applications

In this section, we study the structure of some manifolds as an application which was the motivation of this paper. Firstly, we give some lemmas on surgeries.

Let M^m be a connected m-dimensional manifold, $\varphi: S^k\times D^{m-k}\to M^m$ an imbedding, and $\lambda\in H_k(M)$ the homology class of $\varphi(S^k\times o)$. Let $M'^m=\chi(M^m, \varphi)$ be the modified manifold, $\varphi': D^{k+1}\times S^{m-k-1}\to M'^m$ the dual of φ, $\lambda'\in H_{m-k-1}(M')$ the homology class of $\varphi'(o\times S^{m-k-1})$. Let $M_o=M-\operatorname{Int}\varphi(S^k\times D^{m-k})=M'-\operatorname{Int}\varphi'(D^{k+1}\times S^{m-k-1})$.

Lemma 6.1. *If* $1\leq k\leq\left[\frac{m}{2}\right]-2$, *the inclusion maps* $i: M_o\subset M$, $i': M_o\subset M'$ *induce the isomorphisms*

$$H_pM\xleftarrow{i^*}H_pM_o\xrightarrow{i'_*}H_pM'$$

for $k+2\leq p\leq m-k-2$. *If the order of* λ *is infinite and* $m=2n$, $1\leq k\leq n-2$ *or* $m=2n+1$, $1\leq k\leq n-1$, *then the homomorphisms*

$$H_{k+1}M \xleftarrow{i_*} H_{k+1}M_o \xrightarrow{i_*} H_{k+1}M'$$

are isomorphisms and λ' *is zero or a torsion element.*

Proof. The first half of the lemma is easily shown using the homology exact sequences of the pairs (M, M_o), (M', M_o), where we note that $H_*(M, M_o)\cong H_*(S^k\times D^{m-k}, S^k\times S^{m-k-1})$ and $H_*(M', M_o)\cong H_*(D^{k+1}\times S^{m-k-1}, S^k\times S^{m-k-1})$. The latter half of the lemma is obtained from the following diagram, a portion of the homology exact sequences of the pairs (M, M_o), (M', M_o),

$$\begin{array}{ccccccccc}
0 & & H_{k+1}M & \longrightarrow & 0\cong H_{k+1}(M, M_o) & & & H_kM' & \longrightarrow 0 \\
& \searrow & \nearrow{\scriptstyle i_*\,\cong} & & & \searrow & \nearrow{\scriptstyle i'_*} & & \\
& H_{k+1}M_o & & & & & H_kM_o & & \\
& \nearrow & \searrow{\scriptstyle i'_*} & & & \nearrow & \searrow{\scriptstyle i_*\,\cong} & & \\
0 & & H_{k+1}M' & \xrightarrow{\cdot\lambda'} & Z\cong H_{k+1}(M', M_o) & & & H_kM & \longrightarrow 0
\end{array}$$

where $1\in Z$ corresponds to $i_*^{-1}(\lambda)$ and $\cdot\lambda'$ denotes the homomorphism which carries each element of $H_{k+1}M'$ into the intersection number with λ'.

From this diagram, we also have

Lemma 6.2. *If the order of* λ *is finite and* $m=2n$, $1\leq k\leq n-2$ *or* $m=2n+1$, $1\leq k\leq n-1$, *then*

$$\operatorname{rank} H_{k+1}M' = \operatorname{rank} H_{k+1}M+1$$

and the order of λ' *is infinite.*

Remark. For the lemmas when $m=2n$, $k=n-1$, see [9].

Let $M^m (m\geq 5)$ be a 1-connected closed m-dimensional C^∞-manifold satisfying the following conditions;

$$\text{i)}\qquad H_pM\cong\begin{cases} Z & \text{if } p=0, m,\\ \overbrace{Z+\cdots+Z}^{r} & \text{if } p=k, m-k\ (k<m-k),\\ 0 & \text{otherwise}\end{cases}$$

ii) the normal bundles of the imbedded k-spheres which represent the generators of $H_k M$ are trivial.

Then, we can kill $H_k M$ so that the surgeries do not affect $H_p M$, $k+1 \leq p \leq m-k-1$, by Lemma 6.1 if $m=2n$, $1 \leq k \leq n-2$ or $m=2n+1$, $1 \leq k \leq n-1$, and by Proposition 2.3 of [9] if $m=2n$, $k=n-1$. So that, by killing $H_k M$, M^m can be modified to a homotopy m-sphere $\sum^m$, therefore $(-\sum^m)\# M^m$ to the standard m-sphere S^m. Reversing this construction, we see that $(-\sum^m)\# M^m$ can be obtained from S^m by surgery on a disjoint set of imbeddings f_i: $S^{m-k-1}\times D^{k+1} \to S^m$, $i=1, 2, \cdots, r$. That is, $(-\sum^m)\# M^m = \partial W^{m+1}$, where W^{m+1} is a handlebody $D^{m+1} \bigcup_{\{f_i\}} \{\bigcup_{i=1}^{r} D_i^{m-k}\times D_i^{k+1}\}$. Thus, we have

Theorem 6.3. *Let $M^m (m \geq 5)$ be a 1-connected closed m-dimensional C^∞-manifold satisfying the above conditions* i), ii). *Then, $M^m = \sum^m \# \partial W^{m+1}$, where $\sum^m$ is a homotopy m-sphere and W^{m+1} is a handlebody $D^{m+1} \bigcup_{\{f_i\}} \{\bigcup_{i=1}^{r} D_i^{m-k}\times D_i^{k+1}\}$. Therefore, if $\frac{m}{3} < k < \frac{m}{2}$, such manifolds are obtained, modulo homotopy m-spheres, as the boundaries of manifolds constructed from r $(k+1)$-disk bundles over $(m-k)$-spheres by the plumbing constructions and surgeries mentioned at the previous sections.* (Compare [9], [16].)

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