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The Principle of Limiting Absorption for Second-order Differential Equations with Operator-valued Coefficients*

By

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§0. Introduction

Let us consider differential operators of the form

$$L=-\frac{d^2}{dr^2}+B(r)+C(r) \qquad (0<r<\infty), \tag{0.1}$$

where for each $r\in(0, \infty)$ $B(r)$ and $C(r)$ are operators in a Hilbert space X. L acts on X-valued funcitons on $(0, \infty)$.

The purpose of the present paper is to justify *the principle of limiting absorption* for the equation

$$(L-(\lambda+i\mu))u=f. \tag{0.2}$$

The essence of the above principle consists in the following: Let $u_{\lambda+i\mu}$ be the solution of (0.2), where f is a given X-valued function on $(0, \infty)$. Then a solution u_λ of the equation

$$(L-\lambda)u=f \tag{0.3}$$

is given by $u_\lambda=\lim_{\mu\to 0} u_{\lambda+i\mu}$. The meaning of the limit is to be determined suitably. For the literature of the principle of limiting absorption see, for example, Eidus [1].

Jäger [5] considers the differential operator L and gives, among others, the following result: Let $B(r)$ be a non-negative self-adjoint

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operator in X and let $C(r)$ behave like $0(r^{-\frac{3}{2}-\varepsilon})$ $(\varepsilon>0)$ at infinity. Then with some other conditions imposed on $B(r)$ and $C(r)$ the principle of limiting absorption holds for equation (0.2) with boundary condition

$$u(0)=0 \tag{0.4}$$

and the "*radiation condition*"

$$\int_0^\infty |u'(r)-i\sqrt{z}\,u(r)|^2 dr<\infty \quad (z=\lambda+i\mu), \tag{0.5}$$

where $|\quad|$ means the norm of X. He uses the above results to construct an eigenfunction expansion associated with L.

We shall extend Jäger's results to L with $C(r)$ which behaves like $0(r^{-1-\varepsilon})$ $(\varepsilon>0)$ at infinity. In our case the radiation condition (0.5) will be replaced by

$$\int_0^\infty (1+r)^{-1+\varepsilon}|u'(r)-i\sqrt{z}\,u(r)|^2 dr<\infty, \tag{0.6}$$

which is weaker than (0.5).

As an application we shall prove the principle of limiting absorption for the Schrödinger operator $-\Delta+q(y)$ in $\mathbf{R}^n$ $(n\geqq 3)$ with $q(y)=0(|y|^{-1-\varepsilon})$ at infinity. In this case $X=L^2(S^{n-1})$ and

$$\begin{cases} B(r)=\dfrac{1}{r^2}\left\{-\Lambda_n+\dfrac{(n-3)(n-1)}{4}\right\} \\ C(r)=q(r\omega)\times \qquad \left(r=|y|,\ \omega=\dfrac{y}{r}\in S^{n-1}\right), \end{cases} \tag{0.7}$$

where S^{n-1} is $(n-1)$-sphere, and Λ_n is the Laplace-Beltrami operator on S^{n-1}.

In §1 we state conditions imposed on $B(r)$ and $C(r)$ and prove some inequalities which will be used to obtain various a priori estimates for the solution of equation (0.2) in §3. §2 and §3 are devoted to showing the existence and uniqueness of the solution u of the equation

$$(L-k^2)\,u=f \qquad (\mathrm{Im}\,k\geqq 0) \tag{0.8}$$

which satisfies the boundary condition (0. 4) and the radiation condition (0.6). Moreover we show that the solution u continuously depends on k. Thus the principle of limiting absorption is justified. We discuss in §4 the dependency on $C(r)$ of the solution of equation (0.8). In §5 we apply these results to the Schrödinger operator in $\mathbf{R}^n$ $(n \geqq 3)$.

Using the results obtained in this paper we can develop a spectral and scattering theory for the differential operator L with an application to Schrödinger operators $-\Delta+q(y)$ in $\mathbf{R}^n$, where $q(y)=0(|y|^{-1-\varepsilon})$ at infinity. We shall discuss these elsewhere.[1]

Recently we have been informed by Prof. T. Ikebe that the following very extensive results have been obtained by S. Agmon: Let

$$L=\sum_{|\alpha|\leqq m} a_\alpha D^\alpha = L_0+B \tag{0.9}$$

be an elliptic operator in $\mathbf{R}^n$ which has a unique self-adjoint extension in $L^2(\mathbf{R}^n)$, where $L_0=\sum_{|\alpha|\leqq m} a_\alpha^0 D^\alpha$ is an elliptic operator with constant coefficients, and $B=\sum_{|\alpha|\leqq m} b_\alpha D^\alpha$ is a differential operator with $b(x)=0(|x|^{-1-\varepsilon})$ as $|x| \to \infty$. Assume that $\lambda>0$ does not belong to an exceptional set which is discrete in $(-\infty, \infty)$ and contains all the eigenvalues of L. Then the principle of limiting absorption holds good for λ, i.e., we have

$$\begin{cases} v_{\lambda\pm i\mu} \to v_{\lambda\pm i0} \ \text{ as } \ \mu \downarrow 0 \ \text{ in } L_2(\mathbf{R}^n, (1+|x|)^{-1-\varepsilon}dx), \\ \int_{R^n}(1+|x|)^{-1-\varepsilon}|v_{\lambda\pm i\mu}(x)|^2 dx \leqq C\int_{R^n}(1+|x|)^{1+\varepsilon}|f(x)|^2 dx, \end{cases} \tag{0.10}$$

where $v_{\lambda\pm i\mu}=(L-(\lambda\pm i\mu))^{-1}f$. In his method any radiation condition is unnecessary. These results are used to construct an eigenfunction expansion for L.

§1. Assumptions and Preliminary Lemmas

Let X be a Hilbert space with the norm $|\ \ |$ and inner product $(\ ,\ )$. For an open interval J in $\mathbf{R}$[2] and $\beta\in\mathbf{R}$ we denote by $H^\beta(J, X)$ the

1) See Y. Saitō [7].
2) $\mathbf{R}$ is the set of all real numbers.

Hilbert space of all (equivalence classes of) X-valued function on J with the norm and inner product

$$(1.1)\qquad \begin{cases} \|f\|_{\beta,J}=[((f,f))_{\beta,J}]^{\frac{1}{2}}, \\ ((f,g))_{\beta,J}=\int_J (f(r),\, g(r))(1+|r|)^{\beta}dr. \end{cases}$$

Let Y be a linear topological space, let m be a non-negative integer, and let $J=(a_1, a_2)$ be an open interval in $\mathbf{R}$. $C^m(J, Y)$ denotes the set of all Y-valued functions on J having m strong continuous derivatives. We denote by $C^m(\bar{J}, Y)$[3] the set of all Y-valued functions $f(r)$ such that $f\in C^m(J, Y)$ and $\frac{d^jf}{dr^j}$ $(j=0, 1, \cdots, m)$ can be extended to continuous functions on $\bar{J}$. $C^m_{0,a_i}(J, Y)$ $(i=1, 2)$ denotes the set of all $f\in C^m(J, Y)$ satisfying $f(r)=0$ in some neighborhood of a_i. We put $C^m_0(J, Y)$ $=C^m_{0,a_1}(J, Y)\cap C^m_{0,a_2}(J, Y)$. If $Y=\mathbf{C}$,[4] we omit $\mathbf{C}$ as in $C^m(J)=C^m(J, \mathbf{C})$.

Let $I=(0, \infty)$ and let $B(r)$ and $C(r)$ be operator-valued functions on I. For local properties of $B(r)$ and $C(r)$ we make the following

Assumption 1.1. (a) *For each $r\in I$ $B(r)$ is a non-negative, self-adjoint operator in X such that its domain $\mathscr{D}(B(r))=D$[5] does not depend on r, and $B(r)x\in C^0(I, X)$ for any $x\in D$.*

(b) *Let $x, y\in D$. Then $(B(r)x, y)\in C^2(I)$ and for any compact interval $M\subset I$ there exists a constant $c_1(M)>0$ satisfying*

$$(1.2)\qquad \left|\frac{d^j}{ds^j}(B(s)x, y)\right|\leqq c_1(M)(|x|+|B^{\frac{1}{2}}(r)x)(|y|+|B^{\frac{1}{2}}(r)y|),$$

where $r, s\in M$ and $j=1, 2$.

(c) *For each $r\in I$ $C(r)$ is a symmetric operator in X with $\mathscr{D}(C(r))$ $=D$ such that $C(r)x\in C^1(I, X)$ for any $x\in D$.*

(d) *Let M be a compact interval in I. Then there exists a constant $c_2(M)>0$ such that*

3) $\bar{J}$ means the closure of J.
4) $\mathbf{C}$ is all complex numbers.
5) $\mathscr{D}(T)$ means the domain of T.

$$\left|\frac{d}{dr}C(r)x\right| \leqq c_2(M)(|x|+|B^{\frac{1}{2}}(r)x|), \tag{1.3}$$

holds for any $x \in D$ *and any* $r \in M$.

We introduce the norm $\| \ \|_{B,J}$ and inner product $((\ ,\ ))_{B,J}$ by

$$\|f\|_{B,J}=[((f, f))_{B,J}]^{\frac{1}{2}}, \tag{1.4}$$

$$((f, g))_{B,J}=((f', g'))_{0,J}+((Bf, g))_{0,J}+((f, g))_{0,J}.^{6)} \tag{1.5}$$

We denote by $C^{2,B}(J, X)$ ($C^{2,B}_{0,a_i}(J, X)$, $i=1, 2$) the linear space spanned by the set of all $\varphi \in C^2(J, X)$ having the form $\varphi=\psi x$, where $x \in D$, $\psi \in C^2(J)$ ($\psi \in C^2_{0,a_i}(J)$, $i=1, 2$) and $\|\varphi\|_{B,J}<\infty$. We denote $C^{2,B}_{0,a_1}(J, X) \cap C^{2,B}_{0,a_2}(J, X)$ by $C^{2,B}_0(J, X)$. We define Hilbert spaces $H^{1,B}(J, X)$, $H^{1,B}_0(J, X)$ and $H^{1,B}_{0,a_i}(J, X)$ ($i=1, 2$), respectively, by the completion of $C^{2,B}(J, X)$, $C^{2,B}_0(J, X)$ and $C^{2,B}_{0,a_i}(J, X)$ ($i=1, 2$) in the norm $\| \ \|_{B,J}$. Let us denote by $\mathrm{loc}\, H^0(\bar{I}, X)$ the set of all X-valued functions $f(r)$ on I such that $f \in H^0((0, b), X)$ for any $b>0$. In a similar way $\mathrm{loc}\, H^{1,B}(\bar{I}, X)$ and $\mathrm{loc}\, H^{1,B}_0(\bar{I}, X)$ are also defined.

Assumption 1.2.[7] (a) *There exist constants* $\rho_1>0$ *and* $c_1>1$ *such that*

$$-\frac{d}{dr}(B(r)x, x) \geqq \frac{c_1}{r}(B(r)x, x) \tag{1.6}$$

holds for any $x \in D$ *and any* $r \geqq \rho_1$.

(b) *For each finite* $b \in I$ *the natural imbedding*

$$H^{1,B}_0((0, b), X) \to H^0((0, b), X) \tag{1.7}$$

is compact.

(c) *There exists* $c_2>0$ *such that*

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6) Here and in the sequel u' and u'' mean $\frac{du}{dr}$ and $\frac{d^2u}{dr^2}$, respectively.

7) The conditions imposed on $B(r)$ and $C(r)$ are the same as in Jäger [5] except (c) of Assumption 1.2. Jäger [5] assumes that

$$|C(r)x| \leqq c_2(1+r)^{-\frac{3}{2}-\varepsilon}(|x|+|B^{\frac{1}{2}}(r)x|), \quad (r \in I, \ x \in D)$$

instead of (1.8).

(1.8) $$|C(r)x| \leqq c_2(1+r)^{-1-\varepsilon}(|x|+|B^{\frac{1}{2}}(r)x|), \qquad (r\in I,\ x\in D)$$

with some $0<\varepsilon<1$.

For an open interval $J\subset I$ $\mathscr{U}(J)$ denotes the set of all linear, continuous functionals on $H_0^{1,B}(J, X)$. $\mathscr{U}(J)$ is a Banach space with the norm

(1.9) $$|||l|||_J=\sup\{|<l, \varphi>|\ ;\ \varphi\in C_0^{2,B}(J, X),\ \|\varphi\|_B=1\}.$$

For example, for $g\in H^0(J, X)$ we define $l[g]\in\mathscr{U}(J)$ by

(1.10) $$<l[g], \varphi>=((g, \varphi))_{0,J} \qquad (\varphi\in H_0^{1,B}(J, X)).$$

Then we can easily see

(1.11) $$|||l[g]|||_J\leqq \|g\|_{0,J}.$$

Definition 1.3. Let $l\in\mathscr{U}(I)$, $u\in H^{1,B}(I, X)$ and $k\in\mathbf{C}^+$ be given, where

(1.12) $$\mathbf{C}^+=\{k\,|\,k\in\mathbf{C},\ \mathrm{Im}\,k\geqslant 0 \quad \text{and} \quad \mathrm{Re}\,k\neq 0\}.$$ [8]

Then $v\in\mathrm{loc}\,H^{1,B}(\bar{I}, X)$ is called a radiative function for $\{L, k, l, u\}$, if the following three conditions hold:

(a) $v-u\in\ \mathrm{loc}\,H_0^{1,B}(\bar{I}, X)$.

(b) $v'-ikv\in H^{-1+\varepsilon}(I, X)$ (the "radiation condition")[9]

(c) For all $\varphi\in C_0^{2,B}(I, X)$ we have

(1.13) $$((v, (L-\bar{k}^2)\varphi))_{0,I}=<l, \varphi>.$$

We shall give a lemma which will be used to prove the existence theorem of the radiative fucntion.

Lemma 1.4.[10] *Let* $I_0=(b, \infty)$, $b>0$. *Eor each* $r\in I_0$ $B(r)$ *is assumed*

8) $\mathrm{Im}\,k$ and $\mathrm{Re}\,k$ mean the imaginary and real, respectively.

9) In Jäger [5] the radiation condition is defined by $v'-ikv\in H^0(I, X)$.

10) Cf. Jäger [3], Hilfssatz 4 (p. 68).

to be a non-negative, self-adjoint operator in X with $\mathscr{D}(B(r))=D$ constant in r. Suppose that $(B(r)x, x)\in C^1(I_0)$ for any $x\in D$ and that we have

$$(1.14)\qquad -\frac{d}{dr}(B(r)x, x)\geqq\frac{e_0}{r}(B(r)x, x)\qquad (x\in D, r\geqq b_0)$$

with constants $b_0>b$ and $e_0>1$. Let $C(r)$, $r\in I_0$, be a symmetric operator with $\mathscr{D}(C(r))=D$. Let $v(r)$ be an X-valued function on I_0 which satisfies the following (i)～(iii):

(i) $v\in C^2(I_0, D)$,[11] $Bv, Cv\in C^0(I_0, X)$, *and*

$$(1.15)\qquad \left(-\frac{d^2}{dr^2}+B(r)+C(r)-k^2\right)v(r)=g(r)\qquad (r\in I_0)$$

with $k\in\mathbf{C}^+$ and $g\in H^{1+\varepsilon}(I_0, X)$.

(ii) $v'-ikv\in H^{-1+\varepsilon}(I_0, X)$ *and* $v\in H^{-1-\varepsilon}(I_0, X)$.

(iii) *We have*

$$(1.16)\qquad |C(r)v(r)|^2\leqq e_1r^{-2-2\varepsilon}(|v'(r)|^2+|B^{\frac{1}{2}}(r)v(r)|^2)\qquad (r\geqq b)$$

with constants $e_1>0$, $0<\varepsilon<1$.

Then there exist constants $\delta_0>0$ and $r_0\geqq b_0+1$ which do not depend on $v(r)$ and $g(r)$ such that

$$(1.17)\qquad \int_{r_0+1}^{\infty}r^{-1+\varepsilon}\{|v'(r)-ikv(r)|^2+(B(r)v(r), v(r))\}dr$$
$$\leqq\delta_0\int_{r_0-1}^{\infty}(r^{-1-\varepsilon}|v(r)|^2+r^{1+\varepsilon}|g(r)|^2)dr.$$

Moreover δ_0 and r_0, as functions of k, are bounded on any bounded set in $\mathbf{C}^+$.

For the proof of this lemma we need the following lemma due to Jäger [5] (Lemma 4.1).

Lemma 1.5. *Let $-\infty\leqq a_2<a_1<b_1<b_2\leqq\infty$ and put $I_i=(a_i, b_i)$,*

11) $C^2(I_0, D)$ is the set of all $\varphi(r)\in C^2(I_0, X)$ such that $\varphi(r)\in D$ for any $r\in I_0$.

$i=1, 2$. *Let* $B(r)$ *be a non-negative, self-adjoint operator in* X *for each* $r\in I_2$ *with* $\mathscr{D}(B(r))=D$ *constant in* r. *Let* $C(r)$ *be a symmetric operator in* X *with* $\mathscr{D}(C(r))=D$ *for each* $r\in I_2$. *Suppose that* $v\in H^{1,B}(I_2, X)$ *satisfies for any* $\varphi\in C_0^{2,B}(I_2, X)$

$$(1.18)\qquad \left(\left(v, \left(-\frac{d^2}{dr^2}+B(\cdot)+C(\cdot)-k^2\right)\varphi\right)\right)_{0,I_2}=<l, \varphi>,$$

where $k\in\mathbf{C}^+$ *and* $l\in\mathscr{U}(I_2)$. *Suppose, further, that* v *satisfies*

$$(1.19)\qquad |C(r)v(r)|\leqq c_2(|v'(r)|+|B^{\frac{1}{2}}(r)v(r)|+|v(r)|), \qquad (r\in I_2)$$

with a constant $c_2=c_2(I_2)>0$. *Then there exists a constant* $K=K(I_1, I_2, k)>0$ *such that*

$$(1.20)\qquad ||v||_{B,I_1}\leqq K(||v||_{0,I_2}+|||l|||_{I_2})$$

holds. Further if we assume $v\in H^{1,B}_{0,a_2}(I_2, X)$ *and* $a_2>-\infty$ *then the conclusion is valid for* $a_2\leqq a_1$.

Proof of Lemma 1.4. Take $r_0\geqq b_0+1$, where b_0 is given in (1.14). Let $\psi\in C^1(I_0)$ such that $0\leqq\psi\leqq 1$, $\psi'(r)\geqq 0$, and

$$(1.21)\qquad \varphi(r)=\begin{cases}0 & \text{for } r\in(b, r_0],\\ 1 & \text{for } r\in[r_0+1, \infty).\end{cases}$$

Then we have for $r\geqq r_0$

$$\begin{aligned}(1.22)\qquad &\frac{d}{dr}(r^\varepsilon\psi(r)|v'(r)-ikv(r)|^2)\\ &=\varepsilon r^{-1+\varepsilon}\psi(r)|v'(r)-ikv(r)|^2+r^\varepsilon\psi'(r)|v'(r)-ikv(r)|^2\\ &\quad+2r^\varepsilon\psi(r)\operatorname{Re}(v''(r)-ikv'(r), v'(r)-ikv(r))\\ &\geqq\varepsilon r^{-1+\varepsilon}\psi(r)|v'-ikv|^2+2r^\varepsilon\psi(r)\operatorname{Re}(v''-ikv', v'-ikv),\end{aligned}$$

since we have assumed that $\psi'(r)\geqq 0$. Noting that $\operatorname{Im} k\geqq 0$ we have

$$(1.23)\qquad \operatorname{Re}(v''(r)-ikv'(r), v'(r)-ikv(r))$$

$$=\mathrm{Re}\,(v''(r)-B(r)\,v(r)+k^2\,v(r),\ v'(r)-ikv(r))$$

$$+(\mathrm{Im}\,k)\{|v'(r)-ikv(r)|^2+(B(r)\,v(r),\ v(r))\}$$

$$+\mathrm{Re}\,(B(r)\,v(r),\ v'(r))$$

$$\geqq \mathrm{Re}\,(v''(r)-B(r)\,v(r)+k^2\,v(r),\ v'(r)-ikv(r))$$

$$+\mathrm{Re}\,(B(r)\,v(r),\ v'(r)).$$

We estimate $2r^{\varepsilon}\,\mathrm{Re}\,(v''-Bv+k^2 v,\ v'-ikv)$ as follows:

$$(1.24)\qquad 2r^{\varepsilon}\,\mathrm{Re}\,(v''(r)-B(r)\,v(r)+k^2\,v(r),\ v'(r)-ikv(r))$$

$$\geqq -2r^{\varepsilon}\,|v''(r)-B(r)\,v(r)+k^2\,v(r)|\,|v'(r)-ikv(r)|$$

$$\geqq -\frac{r^{2\beta}}{\alpha}|v''(r)-B(r)\,v(r)+k^2\,v(r)|^2-\alpha r^{2\eta}|v'(r)-ikv(r)|^2$$

$$(\alpha>0,\ \beta+\eta=\varepsilon)$$

$$\geqq -\frac{\delta_1}{\alpha}[r^{-2-2\varepsilon+2\beta}\{|v(r)|^2+|v'(r)-ikv(r)|^2+(B(r)\,v(r),\ v(r))\}$$

$$+r^{2\beta}|g(r)|^2]-\alpha r^{2\eta}|v'(r)-ikv(r)|^2,$$

since we have by (1.15) and (1.16)

$$(1.25)\qquad |v''(r)-B(r)\,v(r)+k^2\,v(r)|^2\leqq \delta_1[r^{-2-2\varepsilon}\{|v(r)|^2+|v'(r)-ikv(r)|^2$$

$$+(B(r)\,v(r),\ v(r))\}+|g(r)|^2]$$

with a constant $\delta_1=\delta_1(k)>0$. We obtain from (1.14)

$$(1.26)\qquad 2\mathrm{Re}\,(B(r)\,v(r),\ v'(r))=\frac{d}{dr}(B(r)\,v(r),\ v(r))-\frac{d}{dr}(B(r)x,\ x)\Big|_{x=v(r)}$$

$$\geqq \frac{d}{dr}(B(r)\,v(r),\ v(r))+\frac{e_0}{r}(B(r)\,v(r),\ v(r)).$$

(1.22), (1.23), (1.24) and (1.26) are combined to give

$$(1.27)\qquad \frac{d}{dr}(r^{\varepsilon}\psi(r)\,|v'(r)-ikv(r)|^2)$$

$$\geqq \psi(r)\Big\{\Big(\varepsilon r^{-1+\varepsilon}-\frac{\delta_1}{\alpha}r^{-2-2\varepsilon+2\beta}-\alpha r^{2\eta}\Big)|v'(r)-ikv(r)|^2$$

$$+\left(e_0 r^{-1+\varepsilon}-\frac{\delta_1}{\alpha} r^{-2-2\varepsilon+2\beta}\right)(B(r)\,v(r),\ v(r))$$

$$+r^{\varepsilon}\frac{d}{dr}(B(r)\,v(r),\ v(r))-\frac{\delta_1}{\alpha} r^{-2-2\varepsilon+2\beta}\,|v(r)|^2$$

$$-\frac{\delta_1}{\alpha} r^{2\beta}\,|g(r)|^2\Big\}.$$

Putting $\eta=\frac{1}{2}(-1+\varepsilon)$, $\beta=\frac{1}{2}(1+\varepsilon)$ and $\alpha=\frac{1}{2}\varepsilon$, we integrate (1.27) from r_0 to R $(R\geqq r_0+2)$ to obtain

$$(1.28)\qquad R^{\varepsilon}\,|v'(R)-ikv(R)|^2$$
$$\geqq\int_{r_0}^{R}\psi(r)\Big(\frac{\varepsilon}{2}r^{-1+\varepsilon}-\frac{2\delta_1}{\varepsilon}r^{-1-\varepsilon}\Big)\,|v'(r)-ikv(r)|^2\,dr$$
$$+\int_{r_0}^{R}\psi(r)\Big((e_0-\varepsilon)\,r^{-1+\varepsilon}-\frac{2\delta_1}{\varepsilon}r^{-1-\varepsilon}\Big)(B(r)\,v(r),\ v(r))\,dr$$
$$-\frac{2\delta_1}{\varepsilon}\int_{r_0}^{R}\psi(r)(r^{-1-\varepsilon}\,|v(r)|^2+r^{1+\varepsilon}\,|g(r)|^2)\,dr$$
$$-\int_{r_0}^{r_0+1}r^{\varepsilon}\psi'(r)\,(B(r)\,v(r),\ v(r))\,dr,$$

where we have made use of the estimate

$$(1.29)\qquad \int_{r_0}^{R}r^{\varepsilon}\psi(r)\frac{d}{dr}(B(r)\,v(r),\ v(r))\,dr$$
$$=R^{\varepsilon}(B(R)\,v(R),\ v(R))-\int_{r_0}^{R}\Big[\frac{d}{dr}(r^{\varepsilon}\psi(r))\Big](B(r)\,v(r),\ v(r))\,dr$$
$$\geqq-\int_{r_0}^{R}\varepsilon r^{-1+\varepsilon}\,\psi(r)\,(B(r)\,v(r),\ v(r))\,dr$$
$$-\int_{r_0}^{r_0+1}r^{\varepsilon}\psi'(r)\,(B(r)\,v(r),\ v(r))\,dr.$$

Now we take r_0 $(\geqq b_0+1)$ so large that we have with a constant $\delta_2>0$

$$(1.30)\qquad \begin{cases}\dfrac{\varepsilon}{2}r^{-1+\varepsilon}-\dfrac{2\delta_1}{\varepsilon}r^{-1-\varepsilon}\geqq\delta_2 r^{-1+\varepsilon}\\[2ex] (e_0-\varepsilon)\,r^{-1+\varepsilon}-\dfrac{2\delta_1}{\varepsilon}r^{-1-\varepsilon}\geqq\delta_2 r^{-1+\varepsilon}\end{cases}$$

for all $r \geqq r_0$. On the other hand, using Lemma 1.5 with $I_1=(r_0, r_0+1)$ and $I_2=(r_0-1, r_0+2)$, we obtain the following estimate with constants $K>0$ and $\delta_3>0$:

$$(1.31)\qquad \int_{r_0}^{r_0+1} r^{\varepsilon}\psi'(r)(B(r)v(r), v(r))\,dr$$
$$\leqq (r_0+1)^{\varepsilon}(\max_{r_0\leqq r\leqq r_0+1}\psi'(r))\int_{r_0}^{r_0+1}(B(r)v(r), v(r))\,dr$$
$$\leqq (r_0+1)^{\varepsilon}(\max_{r_0\leqq r\leqq r_0+1}\psi'(r))K\int_{r_0-1}^{r_0+2}(|v(r)|^2+|g(r)|^2)\,dr$$
$$<\delta_3\int_{r_0-1}^{R}(r^{-1-\varepsilon}|v(r)|^2+r^{1+\varepsilon}|g(r)|^2)\,dr,$$

where we used (1.11). It follows from (1.28), (1.30) and (1.31) that

$$(1.32)\qquad \delta_2\int_{r_0+1}^{R} r^{-1+\varepsilon}\{|v'(r)-ikv(r)|^2+(B(r)v(r), v(r))\}\,dr$$
$$\leqq R^{\varepsilon}|v'(R)-ikv(R)|^2$$
$$+\left(\frac{2\delta_1}{\varepsilon}+\delta_3\right)\int_{r_0-1}^{R}(r^{-1-\varepsilon}|v(r)|^2+r^{1+\varepsilon}|g(r)|^2)\,dr.$$

Since $r^{-1+\varepsilon}|v'(r)-ikv(r)|^2$ is integrable on I_0, we have

$$(1.33)\qquad R_j^{\varepsilon}|v'(R_j)-ikv(R_j)|^2\to 0,\qquad j\to\infty$$

for some sequence $R_j\to\infty$. Thus we obtain (1.17) from (1.32). Q.E.D.

Lemma 1.6. *Let us assume Assumption* 1.1 *and* (a) *and* (c) *of Assumption* 1.2. *Let* $k\in\mathbf{C}^+$ *and let* $v(r)$ *be a radiative function for* $\{L, k, l[g], 0\}$ *with* $v\in \mathrm{loc}\, H_0^{1,B}(\bar{I}, X)\cap H^{-1-\varepsilon}(I, X)$ *and* $g\in \mathrm{loc}\, H^{1,B}(\bar{I}, X)\cap H^{1+\varepsilon}(I, X)$. *Then there exists a constant* $\delta>0$ *such that*

$$(1.34)\qquad ||v'-ikv||_{-1+\varepsilon}+||B^{\frac{1}{2}}v||_{-1+\varepsilon}\leqq\delta(||v||_{-1-\varepsilon}+||g||_{1+\varepsilon}),^{12)}$$

where δ *depends only on* k *and is bounded on any bounded set in* $\mathbf{C}^+$.

12) Here and in the sequel we put $\|\ \|_{\beta,I}=\|\ \|_{\beta}$ and $\|\ \|_{B,I}=\|\ \|_{B}$ for the sake of simplicity.

Proof. It follows from Assumption 1.1 that we can apply the regularity theorem of Jäger [5] (Satz 3.1, p. 76) to see that $v \in C^2(I, D)$, Bv, $Cv \in C^0(I, X)$, and v satisfies (1.15) for all $r \in I$. From Lemma 1.4 we obtain

$$(1.35) \qquad \int_{r_0+1}^{\infty} (1+r)^{-1+\varepsilon}\{|v'(r)-ikv(r)|^2+(B(r)v(r), v(r))\}dr$$
$$\leqq K_1\int_{r_0-1}^{\infty} \{(1+r)^{-1-\varepsilon}|v(r)|^2+r^{1+\varepsilon}|g(r)|^2\}dr$$

with constants $r_0 \geqq \rho_1+1$ and $K_1>0$. Since $v \in \mathrm{loc}\, H_0^{1,B}(\bar{I}, X)$, we can use the last statement of Lemma 1.5 with $I_1=(0, r_0+1)$ and $I_2=(0, r_0+2)$ to obtain

$$(1.36) \qquad \int_{0}^{r_0+1} (1+r)^{-1+\varepsilon}\{|v'(r)-ikv(r)|^2+(B(r)v(r), v(r))\}dr$$
$$\leqq K_2\int_{0}^{r_0+2} \{(1+r)^{-1-\varepsilon}|v(r)|^2+(1+r)^{1+\varepsilon}|g(r)|^2\}dr$$

with a constant $K_2>0$. (1.34) follows from (1.35) and (1.36). Q.E.D.

§2. The Uniqueness Theorem

We shall show the uniqueness of the radiative function using arguments due to Jäger [5].

Lemma 2.1. *Let* $B(r)$ *satisfy* (a) *and* (b) *of Assumption* 1.1. *Let* $C(r)$ *be a symmetric operator in* X *for each* $r \in I$ *such that* $C(r)$ *satisfies* (c) *and* (d) *of Assumption* 1.1. *and*

$$(2.1) \qquad |C(r)x| \leqq c(|x|+|B^{\frac{1}{2}}(r)x|), \qquad (x \in D, r \in I)$$

with a constant $c>0$. *Let* $v \in \mathrm{loc}\, H_0^{1,B}(\bar{I}, X)$ *satisfy*

$$(2.2) \qquad ((v, (L-\bar{k}^2)\varphi))_0=((g, \varphi))_0 \qquad (\varphi \in C_0^{2,B}(I, X))$$

with $g \in \mathrm{loc}\, H^{1,B}(\bar{I}, X)$ *and* $k \in \mathbf{C}^+$, *where*

$$(2.3) \qquad L=-\frac{d^2}{dr^2}+B(r)+C(r).$$

Then we have for all $r\in I$,

$$|v'(r)-ikv(r)|^2 = |v'(r)+(\mathrm{Im}\,k)v(r)|^2+(\mathrm{Re}\,k)^2|v(r)|^2 + 4(\mathrm{Re}\,k)^2(\mathrm{Im}\,k)\|v\|^2_{0,(0,r)}+2(\mathrm{Re}\,k)\mathrm{Im}((g,v))_{0,(0,r)}. \tag{2.4}$$

Proof. As we have seen in the proof of Lemma 1.6, it follows from the regularity theorem of Jäger [5] (p. 76) that

$$\begin{cases} v\in C^2(I,D) \quad \text{and} \quad Bv,\ Cv\in C^0(I,X) \\ (L-k^2)v(r)=g(r) \qquad (r\in I). \end{cases} \tag{2.5}$$

On the other hand we obtain from the fact that $v\in \mathrm{loc}\,H_0^{1,B}(I,X)$[13)]

$$\begin{cases} v\in C^0(\bar{I},X) \\ v(0)=0. \end{cases} \tag{2.6}$$

From (2.5) and (2.6) we see that

$$\begin{aligned} \int_0^r (g(t),\varphi(t))\,dt &= \int_0^r ((L-k^2)v(t),\varphi(t))\,dt \\ &= \int_0^r \{(v'(t),\varphi'(t))+((B(t)+C(t)-k^2)v(t),\varphi(t)\}\,dt \\ &\quad -(v'(r),\varphi(r)) \end{aligned} \tag{2.7}$$

holds for any $\varphi\in C_0^{2,B}(I,X)$. Since $v\in\mathrm{loc}\,H_0^{1,B}(\bar{I},X)$, for any $r>0$ there is a sequence $\{\varphi_n\}$ in $C_0^{2,B}(I,X)$ such that

$$\begin{cases} \|\varphi_n-v\|_{B,(0,r+1)}\to 0, \\ \varphi_n(t)\to v(t) \qquad \text{in } X \qquad (t\in[0,r+1]) \end{cases} \tag{2.8}$$

as $n\to\infty$. Replacing φ by φ_n in (2.7) and letting $n\to\infty$, we have

13) Note that $H^{1,B}(I,X)$ is continuously imbedded in $C^0(\bar{I},X)$. See Jäger [5], p. 69.

$$(2.9)\quad \int_0^r (g(t),\ v(t))dt = \int_0^r \{|v'(t)|^2 + ((B(t)+C(t)-k^2)v(t),\ v(t))\}dt - (v'(r),\ v(r)).$$

Hence we obtain

$$(2.10)\qquad \operatorname{Im}(v'(r),\ v(r)) = -\operatorname{Im}((g,\ v))_{0,(0,r)} - 2(\operatorname{Re}k)(\operatorname{Im}k)\|v\|^2_{0,(0,r)}.$$

Using (2.10), we calculate $|v'(r)-ikv(r)|^2$ as follows:

$$\begin{aligned}(2.11)\qquad |v'(r)-ikv(r)|^2 &= |v'(r)+(\operatorname{Im}k)v(r)|^2 + (\operatorname{Re}k)^2|v(r)|^2 - 2(\operatorname{Re}k)\operatorname{Im}(v'(r),\ v(r))\\ &= |v'(r)+(\operatorname{Im}k)v(r)|^2 + (\operatorname{Re}k)^2|v(r)|^2\\ &\quad + 4(\operatorname{Re}k)^2(\operatorname{Im}k)\|v\|^2_{0,(0,r)} + 2(\operatorname{Re}k)\operatorname{Im}((g,\ v))_{0,(0,r)}.\end{aligned}$$

Q.E.D.

Theorem 2.2. *Let us assume Assumption* 1.1 *and* (a) *and* (c) *of Assumption* 1.2. *Let* $l \in \mathscr{U}(I)$, $k \in \mathbf{C}^+$ *and* $u \in H^{1,B}(I, X)$ *be given. Then the radiative function for* $\{L, k, l, u\}$ *is unique.*

Proof. Let v be a radiative function for $\{L, k, 0, 0\}$, where $k \in \mathbf{C}^+$. What we want to show is that v is identically zero.

We start with the relation

$$(2.12)\qquad |v'(r)-ikv(r)|^2 = |v'(r)+(\operatorname{Im}k)v(r)|^2 + (\operatorname{Re}k)^2|v(r)|^2 + 4(\operatorname{Re}k)^2(\operatorname{Im}k)\|v\|^2_{0,(0,r)},$$

which follows from Lemma 2.1.

If $\operatorname{Im}k>0$, then we obtain from (2.12) and the fact that $v'-ikv \in H^{-1+\varepsilon}(I, X)$

$$(2.13)\quad 0 \leqq \|v\|^2_{0,(0,r_j)} \leqq \frac{1}{4(\operatorname{Re}k)^2(\operatorname{Im}k)}|v'(r_j)-ikv(r_j)|^2 \to 0,\ j\to\infty,$$

for some sequence $r_j \to \infty$. Hence we have $\|v\|_0^2 = 0$, i.e., $v \equiv 0$.

Next let us assume that $\mathrm{Im}\, k = 0$. Then we have from (2.12) and the radiation condition $v' - ikv \in H^{-1+\varepsilon}(I, X)$

$$\text{(2.14)} \qquad \varliminf_{r\to\infty} (|v'(r)|^2 + k^2 |v(r)|^2) = \varliminf_{r\to\infty} |v'(r) - ikv(r)|^2 = 0.$$

By the regularity theorem of Jäger [5] (p. 76) and (1.8) we have

$$\text{(2.15)} \qquad \begin{cases} v \in C^2(I, D) \\ |v''(r) - B(r)v(r) + k^2 v(r)|^2 = |C(r)v(r)|^2 \end{cases}$$

$$\leqq 2c_2^2(1+r)^{-2-2\varepsilon}\{|v(r)|^2 + (B(r)v(r), v(r))\} \qquad (r \in I),$$

where $c_2 > 0$ is given in (1.8). (2.14) and (2.15) enable us to apply Hilfssatz 1 of Jäger [3] (p. 66) on the growth property of solutions of the equation $(L-k^2)v=0$ to show that the carrier of v is compact in I. Hence, using Satz 3 of Jäger [4] (p. 32), a unique continuation theorem for solutions of the equation $(L-k^2)v=0$, we see that $v \equiv 0$ on I.

Q.E.D.

§ 3. The Existence Theorems

This section is devoted to showing the existence of the radiative function v for $\{L, k, l, u\}$, where $k \in \mathbf{C}$, $u \in H^{1,B}(I, X)$, and l belongs to a subspace $\mathscr{U}_{1+\varepsilon}(I)$ of $\mathscr{U}(I)$. We shall first prove a priori estimates for radiative functions v for $\{L, k, l, 0\}$, $k \in \mathbf{C}^+$ and $l \in \mathscr{U}_{1+\varepsilon}(I)$ (Lemma 3.1 and Lemma 3.4). This corresponds to Satz 5.3 of Jäger [5]. But it seems that we have to modify its proof in order to obtain the a priori estimates needed in our case. Lemma 3.2 is necessary for this modification. Next we shall prove the existence theorems using our a priori estimates (Theorem 3.7 and Theorem 3.8). At the same time we shall see that the radiative function v for $\{L, k, l, u\}$ depends continuously on k, l and u.

Lemma 3.1. *Let us assume Assumptions* 1.1 *and* 1.2. *Let* K *be a compact set in* $\mathbf{C}^+$. *Let* $k\in K$ *and* $g\in H^{1+\varepsilon}(I, X)\cap \operatorname{loc} H^{1,B}(\bar{I}, X)$. *Let* v *be a radiative function for* $\{L, k, l[g], 0\}$ *such that*

$$v\in \operatorname{loc} H_0^{1,B}(\bar{I}, X)\cap H^{-1-\varepsilon}(I, X). \tag{3.1}$$

Then we have

$$\|v\|_{-1-\varepsilon}+\|v'-ikv\|_{-1+\varepsilon}+\|B^{\frac{1}{2}}v\|_{-1+\varepsilon}\leqq \delta_1\|g\|_{1+\varepsilon}. \tag{3.2}$$

with a constant $\delta_1>0$, *where* δ_1 *depends only on* K *and* L.

To prove this lemma we prepare

Lemma 3.2. *Let* K, g, k *and* v *be as in Lemma* 3.1. *Then there exists a positive number* α_0 *such that*

$$\int_\rho^\infty (1+r)^{-1-\varepsilon}|v(r)|^2 dr\leqq \alpha_0(\|v\|_{-1-\varepsilon}^2+\|g\|_{1+\varepsilon}^2)\rho^{-\varepsilon}, \qquad (\rho\geqq 1), \tag{3.3}$$

where α_0 *depends only on* K *and* L.

Proof. From Lemma 2.1 we obtain

$$(\operatorname{Re} k)^2|v(r)|^2+2(\operatorname{Re} k)\operatorname{Im}((g, v))_{0,(0,r)}\leqq |v'(r)-ikv(r)|^2, \tag{3.4}$$

whence we have

$$\begin{aligned} |v(r)|^2 &\leqq \frac{1}{(\operatorname{Re} k)^2}|v'(r)-ikv(r)|^2+\frac{2}{|\operatorname{Re} k|}\int_0^r |g(t)|\,|v(t)|\,dt \\ &\leqq \frac{1}{(\operatorname{Re} k)^2}|v'(r)-ikv(r)|^2 \\ &\quad +\frac{2}{|\operatorname{Re} k|}\left\{\int_0^r (1+t)^{1+\varepsilon}|g(t)|^2 dt\right\}^{\frac{1}{2}}\left\{\int_0^r (1+t)^{-1-\varepsilon}|v(t)|^2 dt\right\}^{\frac{1}{2}} \\ &\leqq \frac{1}{(\operatorname{Re} k)^2}|v'(r)-ikv(r)|^2+\frac{2}{|\operatorname{Re} k|}\|g\|_{1+\varepsilon}\|v\|_{-1-\varepsilon}. \end{aligned} \tag{3.5}$$

Multiplying both sides of (3.5) by $r^{-1-\varepsilon}$ and integrating from ρ to ∞, we have

$$(3.6)\qquad \int_\rho^\infty r^{-1-\varepsilon}|v(r)|^2dr \leqq \frac{1}{(\mathrm{Re}\,k)^2}\int_\rho^\infty r^{-1-\varepsilon}|v'(r)-ikv(r)|^2dr$$

$$+\frac{2}{\varepsilon|\mathrm{Re}\,k|}\|g\|_{1+\varepsilon}\|v\|_{-1-\varepsilon}\rho^{-\varepsilon}$$

$$\leqq \frac{1}{(\mathrm{Re}\,k)^2}\rho^{-2\varepsilon}\int_\rho^\infty r^{-1+\varepsilon}|v'(r)-ikv(r)|^2\,dr$$

$$+\frac{2}{\varepsilon|\mathrm{Re}\,k|}\|g\|_{1+\varepsilon}\|v\|_{-1-\varepsilon}\rho^{-\varepsilon}.$$

(3.3) follows from (3.6) and Lemma 1.6. Q.E.D.

Proof of Lemma 3.1. It follows from Lemma 1.6 that it is enough to show

$$(3.7)\qquad \|v\|_{-1-\varepsilon} \leqq \alpha\|g\|_{1+\varepsilon}$$

with a constant $\alpha>0$ depending only on K and L. Let us assume that (3.7) is false. Then for each positive integer n we can find $k_n\in K$, $h_n\in \mathrm{loc}\,H^{1,B}(\bar{I}, X)$, and radiative functions u_n for $\{L, k_n, l[h_n], 0\}$ such that

$$(3.8)\qquad \|u_n\|_{-1-\varepsilon} > n\|h_n\|_{1+\varepsilon}.$$

Since we see $\|u_n\|_{-1-\varepsilon}>0$ from (3.8), we obtain radiative functions $v_n = \frac{u_n}{\|u_n\|_{-1-\varepsilon}}$ for $\{L, k_n, l[g_n], 0\}$, $g_n=\frac{h_n}{\|u_n\|_{-1-\varepsilon}}$, with

$$(3.9)\qquad \begin{cases} \|v_n\|_{-1-\varepsilon}=1, \\ \|g_n\|_{1+\varepsilon}<\dfrac{1}{n}. \end{cases}$$

Let $\{k_{n_m}\}$ be a subsequence of $\{k_n\}$ satisfying

$$(3.10)\qquad k_{n_m}\longrightarrow k_0, \qquad m\longrightarrow\infty$$

with $k_0\in K$. Without loss of generality we can assume

$$(3.11)\qquad k_n\longrightarrow k_0, \qquad n\longrightarrow\infty.$$

In view of (3.9) we have for any $R\in I$

$$(3.12)\qquad \begin{cases} \sup_n \|v_n\|_{0,(0,R+1)} < \infty, \\ \sup_n \|g_n\|_{0,(0,R+1)} < \infty. \end{cases}$$

Therefore it follows from Lemma 1.5 that

$$(3.13)\qquad \sup_n \|v_n\|_{B,(0,R)} < \infty$$

for all $R>0$. Since for all $0<R<\infty$ the imbedding $H_0^{1,B}((0, R), X) \to H^0((0, R), X)$ is compact by (b) of Assumption 1.2, we obtain a subsequence of $\{v_n\}$ which is a Cauchy sequence in $H^0((0, R), X)$ for all $R \in I$. Without loss of generality we can assume that $\{v_n\}$ itself is a Cauchy sequence in $H^0((0, R), X)$ for all $R \in I$. The sequence $\{v_n\}$ is a Cauchy sequence in $H_0^{1,B}((0, R), X)$ for all $R \in I$, too. In fact for each pair (n, m) $v_n - v_m$ is the radiative function for $\{L, k, l[g_{nm}], 0\}$, where

$$(3.14)\qquad g_{nm} = g_n - g_m - (k_0^2 - k_n^2)v_n + (k_0^2 - k_m^2)v_m$$

and k_0 is given as in (3.11). From (3.9) and (3.11) we obtain $g_{nm} \to 0$, $n, m \to \infty$ in $H^0((0, R+1), X)$ for any $R>0$. Hence, noting that $\{v_n\}$ is a Cauchy sequence in $H^0((0, R+1), X)$, we can apply Lemma 1.5 to show

$$(3.15)\qquad \|v_n - v_m\|_{B,(0,R)} \leqq \beta(\|v_n - v_m\|_{0,(0,R+1)} + \|g_{n,m}\|_{0,(0,R+1)}) \longrightarrow 0, \qquad n, m \longrightarrow \infty,$$

where $\beta>0$ depends only on R, k and L. Therefore there exists $v \in \mathrm{loc}\, H_0^{1,B}(\bar{I}, X)$ satisfying

$$(3.16)\qquad v_n \longrightarrow v, \qquad n \longrightarrow \infty$$

both in $H_0^{1,B}((0, R), X)$ and in $H^0((0, R), X)$ for any $R \in I$.

Letting $n \to \infty$ in the relation

$$(3.17)\qquad ((v_n, (L - \bar{k}_n^2)\varphi))_0 = ((g_n, \varphi))_0, \qquad (\varphi \in C_0^{2,B}(I, X))$$

we obtain from (3.16), (3.9) and (3.11)

$$(3.18) \qquad ((v, (L-\bar{k}_0^2)\varphi))_0 = 0.$$

Using (3.9), (3.16) and Lemma 1.6 we estimate $\|v'-ik_0 v\|_{-1+\varepsilon,(0,R)}$ as follows:

$$(3.19) \qquad \begin{aligned} \|v'-ik_0 v\|_{-1+\varepsilon,(0,R)} &= \lim_{n\to\infty} \|v_n'-ik_0 v_n\|_{-1+\varepsilon,(0,R)} \\ &\leqq \sup_n \|v_n'-ik_0 v_n\|_{-1+\varepsilon} \\ &\leqq \delta \sup_n \{\|v_n\|_{-1-\varepsilon}+\|g_n\|_{1+\varepsilon}\} \\ &\leqq \delta \sup_n \left(1+\frac{1}{n}\right) \leqq 2\delta, \end{aligned}$$

where $\delta>0$ is as in Lemma 1.6. Since the last member of (3.19) does not depend on n and R, we have $v'-ik_0 v \in H^{-1+\varepsilon}(I, X)$, i.e., v satisfies the radiation condition. Thus v is a radiative function for $\{L, k_0, 0, 0,\}$, and hence $v\equiv 0$ by Theorem 2.2.

From Lemma 3.2 we obtain for $\rho \geqq 1$

$$(3.20) \qquad \begin{aligned} \overline{\lim_{n\to\infty}} \|v_n\|_{-1-\varepsilon}^2 &\leqq \lim_{n\to\infty} \|v_n\|_{-1-\varepsilon,(0,\rho)}^2 + \sup_n \|v_n\|_{-1-\varepsilon,(\rho,\infty)}^2 \\ &\leqq \|v\|_{-1-\varepsilon,(0,\rho)}^2 + \alpha_0 \rho^{-\varepsilon} \sup_n \{\|v_n\|_{-1-\varepsilon}^2 + \|g_n\|_{1+\varepsilon}^2\} \\ &= 0(\rho^{-\varepsilon}), \end{aligned}$$

where we have noted (3.9) and the fact $v\equiv 0$. Since $\rho\geqq 1$ is arbitrary, we obtain $\lim_{n\to\infty} \|v_n\|_{-1-\varepsilon}=0$, which contradicts the assumption that $\|v_n\|_{-1-\varepsilon} =1$. Q.E.D.

Now we introduce a subspace of $\mathscr{U}(I)$.

Definition 3.3. Let $\mathscr{U}_{1+\varepsilon}(I)$ be the set of all $l \in \mathscr{U}(I)$ such that

$$(3.21) \qquad |||l|||_{1+\varepsilon} = \sup \{ |\langle l, (1+r)^{\frac{1+\varepsilon}{2}} \varphi\rangle| \,;\, \varphi \in C_0^{2,B}(I, X),\ \|\varphi\|_B = 1\} < \infty.$$

$\mathscr{U}_{1+\varepsilon}(I)$ is a Banach space with the norm $|||\ |||_{1+\varepsilon}$.

It is easy to see that we have

$$(3.22) \qquad |||l||| \leqq a_0 |||l|||_{1+\varepsilon} \qquad (l\in \mathscr{U}_{1+\varepsilon}(I))$$

with a constant $a_0>0$.

We shall show that the inequality (3.2) also holds for the radiative function for $\{L, k, l, 0\}$, where $l\in\mathscr{U}_{1+\varepsilon}(I)$.

Lemma 3.4. *Let us assume Assumptions* 1.1 *and* 1.2. *Let* K *be as in Lemma* 3.1. *Let* $k\in K$ *and* $l\in\mathscr{U}_{1+\varepsilon}(I)$. *Let* v *be a radiative function for* $\{L, k, l, 0\}$ *such that* $v\in H^{-1-\varepsilon}(I, X)\cap \mathrm{loc}\, H_0^{1,B}(\bar{I}, X)$.

Then there exists a constant $\delta_2>0$ *such that*

$$\|v\|_{-1-\varepsilon}+\|v'-ikv\|_{-1+\varepsilon}+\|B^{\frac{1}{2}}v\|_{-1+\varepsilon}\leqq\delta_2|||l|||_{1+\varepsilon}, \tag{3.23}$$

where δ_2 *depends only on* K *and* L.

To prove this lemma we need

Lemma 3.5.[14] *Let* $B(r)$ *satisfy* (a) *of Assumption* 1.1 *and let* $C(r)$ $(r\in I)$ *be a symmetric operator in* X *with the domain* $\mathscr{D}(C(r))=D$ *such that*

$$|C(r)x|\leqq c(|x|+|B^{\frac{1}{2}}(r)x|) \qquad (r\in I,\ x\in D) \tag{3.24}$$

with a constant $c>0$. *Let* $k_0\in\mathbf{C}^+$ *and* $\mathrm{Im}\,k_0>0$. *Let* $l\in\mathscr{U}(I)$. *Then the equation*

$$((u, (L-\bar{k}_0^2)\varphi))_0=\langle l, \varphi\rangle \qquad (\varphi\in C_0^{2,B}(I, X)) \tag{3.25}$$

has a unique solution u *in* $H_0^{1,B}(I, X)$ *with the estimate*

$$\|u\|_B\leqq\beta_1|||l|||, \tag{3.26}$$

where $\beta_1=\beta_1(k_0)>0$ *is a constant. Further, if* $l\in\mathscr{U}_{1-\varepsilon}(I)$, *then we have* $u\in H^{1+\varepsilon}(I, X)$ *and*

$$\|u\|_{1+\varepsilon}\leqq\beta_2|||l|||_{1+\varepsilon} \tag{3.27}$$

with a constant $\beta_2=\beta_2(k_0)>0$.

Proof. Let us define a bilinear form $\mathscr{B}_{L-k_0^2}[\cdot,\cdot]$ on $H_0^{1,B}(I, X)$

14) Cf. Jäger [5], Lemma 2.3 (p. 75) and the proof of Satz 5.3 (p. 86).

$\times H_0^{1,B}(I, X)$ by

$$(3.28)\qquad \mathscr{B}_{L-k_0^2}[w_1, w_2]=((w_1, w_2))_B+(((C(r)-k_0^2-1)w_1, w_2))_0.$$

Then we shall show

$$(3.29)\qquad d_1\|w\|_B^2 \geqq |\mathscr{B}_{L-k_0^2}[w, w]| \geqq d_2\|w\|_B^2 \qquad (w\in H_0^{1,B}(I, X)),$$

where $d_j=d_j(k_0)>0$ $(j=1, 2)$ are constants. Since $C_0^{2,B}(I, X)$ is dense in $H_0^{1,B}(I, X)$, and we have by integration by parts

$$(3.30)\qquad \mathscr{B}_{L-k_0^2}[\varphi_1, \varphi_2]=((\varphi_1, (L-\bar{k}_0^2)\varphi_2))_0 \qquad (\varphi_1, \varphi_2\in C_0^2(I, X)),$$

it is sufficient to show (3.29) that we show

$$(3.31)\qquad \begin{cases} d_1\|\varphi\|_B^2 \geqq |((\varphi, (L-\bar{k}_0^2)\varphi))_0|, \\ d_2\|\varphi\|_B^2 \leqq |((\varphi, (L-\bar{k}_0^2)\varphi))_0|, \end{cases} \qquad (\varphi\in C_0^{2,B}(I, X)).$$

Let us prove (3.31). From (3.24) we see that

$$(3.32)\qquad \|C\varphi\|_0^2 \leqq \int_I c^2(|\varphi(r)|+|B^{\frac{1}{2}}(r)\varphi(r)|)^2 dr \leqq 2c\|\varphi\|_B^2,$$

whence follows for all $\varphi\in C_0^{2,B}(I, X)$

$$(3.33)\qquad \begin{aligned} |((\varphi, (L-\bar{k}_0^2)\varphi))_0| &\leqq \|\varphi\|_B^2+\sqrt{2c}\,\|\varphi\|_B\|\varphi\|_0+|k_0|^2\|\varphi\|_0^2 \\ &\leqq (1+\sqrt{2c}+|k_0|^2)\|\varphi\|_B^2. \end{aligned}$$

Thus we have shown the first inequality of (3.31) with $d_1=(1+\sqrt{2c}+|k_0|^2)$. On the other hand we have

$$(3.34)\qquad |((\varphi, (L-\bar{k}_0^2)\varphi))_0|^2=\{\|\varphi\|_B^2+(((C-\lambda-1)\varphi, \varphi))_0\}^2+\mu^2\|\varphi\|_0^4,$$

where $\lambda=\mathrm{Re}\,k_0^2$ and $\mu=\mathrm{Im}\,k_0^2\neq 0$. Hence, using (3.32) again, we have

$$(3.35)\qquad \begin{aligned} |((\varphi, (L-\bar{k}_0^2)\varphi))_0|^2 &\geqq \|\varphi\|_B^4-2\|\varphi\|_B^2|(((C-\lambda-1)\varphi, \varphi))_0| \\ &\quad +(((C-\lambda-1)\varphi, \varphi))^2+\mu^2\|\varphi\|_0^4 \\ &\geqq (1-\alpha)\|\varphi\|_B^4-\left(\frac{1}{\alpha}-1\right)(((C-\lambda-1)\varphi, \varphi))_0^2+\mu^2\|\varphi\|_0^4 \end{aligned}$$

with $\alpha>0$. Take $0<\alpha<1$ in (3.35). Then, noting that we obtain from (3.32)

$$(((C-\lambda-1)\varphi,\ \varphi))_0^2 \leqq \|(C-\lambda-1)\varphi\|_0^2\|\varphi\|_0^2 \tag{3.36}$$
$$\leqq 2\{2c^2+(|\lambda|+1)^2\}\|\varphi\|_B^2\|\varphi\|_0^2$$
$$\leqq \{2c^2+(|\lambda|+1)^2\}\left(\beta\|\varphi\|_B^4+\frac{1}{\beta}\|\varphi\|_0^4\right) \qquad (\beta>0),$$

we arrive at

$$|((\varphi,\ (L-\bar{k}^2)\varphi))_0|^2 \geqq \left\{1-\alpha-\frac{1-\alpha}{\alpha}\beta c_1\right\}\|\varphi\|_B^4 \tag{3.37}$$
$$+\left\{\mu^2-\frac{1-\alpha}{\alpha\beta}c_1\right\}\|\varphi\|_0^4 \qquad (0<\alpha<1,\ \beta>0),$$

where we put $c_1=2c^2+(|\lambda|+1)^2$. Putting $\beta=\dfrac{\alpha}{2c_1}$ and taking $1-\alpha>0$ small enough, we obtain from (3.37)

$$|((\varphi,\ (L-\bar{k}^2)\varphi))_0|^2 \geqq \frac{1}{2}(1-\alpha)\|\varphi\|_B^4, \tag{3.38}$$

whence follows the second inequality of (3.31) with $d_2=\sqrt{\dfrac{(1-\alpha)}{2}}$.

Since (3.29) has been justified, we can make use of the Lax-Milgram theorem[15] to show that there exists a unique solution of u in $H_0^{1,B}(I, X)$ of the equation

$$\mathscr{B}_{L-k_0^2}[u,\ w]=<l,\ w> \qquad (w\in H_0^{1,B}(I,\ X)) \tag{3.39}$$

for $l\in\mathscr{U}(I)$. Since $\mathscr{B}_{L-k_0^2}[u,\ \varphi]=((u,\ (L-\bar{k}_0^2)\varphi))_0$ for $\varphi\in C_0^{2,B}(I,\ X)$, it follows from (3.38) that u is a unique solution of (3.25). (3.29) and (3.39) are combined to give

$$\|u\|_B^2 \leqq \frac{1}{d_2}|\mathscr{B}_{L-k_0^2}[u,\ u]| = \frac{1}{d_2}|<l,\ u>| \leqq \frac{1}{d_2}|||l|||\ \|u\|_B, \tag{3.40}$$

which implies (3.26) with $\beta_1=\dfrac{1}{\sqrt{d_2}}$.

15) See, for example, Yosida [6], p. 92.

Next let us show (3.27). Let $\psi \in C^1(\mathbf{R})$ such that $0 \leqq \psi(r) \leqq 1$, $0 \leqq |\psi'(r)| \leqq 1$ and

$$\psi(r) = \begin{cases} 0 & \text{for } r \geqq 2, \\ 1 & \text{for } r \leqq \frac{1}{2}. \end{cases} \tag{3.41}$$

For each $m = 1, 2, \cdots$ we define

$$\begin{cases} \psi_m(r) = (1+r)^{\frac{1+\varepsilon}{2}} \psi\left(\frac{r}{m}\right), \\ u_m = \psi_m u, \end{cases} \tag{3.42}$$

where u is the solution of the equation (3.38). Then we obtain from (3.39) and (3.28)

$$\begin{aligned} \mathscr{B}_{L-k_0^2}[u_m, u_m] &= \|u_m\|_B^2 + (((C(r) - k_0^2 - 1) u_m, u_m))_0 \\ &= (((\psi_m u)', u_m'))_0 + ((B^{\frac{1}{2}} u, B^{\frac{1}{2}} \psi_m u_m))_0 \\ &\quad + (((C(r) - k_0^2) u, \psi_m u_m))_0 \\ &= \mathscr{B}_{L-k_0^2}[u, \psi_m u_m] + ((\psi_m' u, u_m'))_0 - ((\psi_m' u', u_m))_0 \\ &= \langle l, \psi_m u_m \rangle + ((\psi_m' u, u_m'))_0 - ((\psi_m' u', u_m))_0. \end{aligned} \tag{3.43}$$

It follows from (3.43) and (3.29)

$$\begin{aligned} \|u_m\|_B^2 &\leqq \frac{1}{d_2} |\mathscr{B}_{L-k_0^2}[u_m, u_m]| \\ &\leqq \frac{1}{d_2} \left\{ \left| \langle l, (1+r)^{\frac{1+\varepsilon}{2}} \psi\left(\frac{r}{m}\right) u_m \rangle \right| \right. \\ &\quad \left. + \|\psi_m' u\|_0 \|u_m'\|_0 + \|\psi_m' u'\|_0 \|u_m\|_0 \right\} \\ &\leqq \frac{1}{d_2} \left\{ |||l|||_{1+\varepsilon} \left\| \psi\left(\frac{r}{m}\right) u_m \right\|_B + (\|\psi_m' u\|_0 + \|\psi_m' u'\|_0) \|u_m\|_B \right\} \\ &\leqq \frac{2}{d_2} \left\{ |||l|||_{1+\varepsilon} + \left(\frac{1+\varepsilon}{2} + 3\right) \|u\|_B \right\} \|u_m\|_B, \end{aligned} \tag{3.44}$$

where we note $\left|\psi\left(\frac{r}{m}\right)\right| \leqq 1$, $\left|\frac{1}{m} \psi'\left(\frac{r}{m}\right)\right| \leqq 1$ and

(3.45) $$|\psi_m'(r)|\leqq\frac{1+\varepsilon}{2}+3$$

for any $r\in I$ and for any $m=1, 2, \cdots$. Taking account of (3.26) and (3.22), we obtain from (3.44)

(3.46) $$\begin{aligned}||u_m||_B &\leqq\frac{2}{d_2}\left\{|||l|||_{1+\varepsilon}+\left(\frac{1+\varepsilon}{2}+3\right)\beta_1\ |||l|||\right\}\\ &\leqq\frac{2}{d_2}\left\{1+\left(\frac{1+\varepsilon}{2}+3\right)\beta_1 a_0\right\}\ |||l|||_{1+\varepsilon},\end{aligned}$$

which implies with $\frac{2}{d_2}\left(1+\left(\frac{1+\varepsilon}{2}+3\right)\beta_1 a_0\right)=\beta_2$

(3.47) $$\left\|\psi\left(\frac{r}{m}\right)u\right\|_{1+\varepsilon}=||\psi_m u||_0=||u_m||_0\leqq||u_m||_B\leqq\beta_2\ |||l|||_{1+\varepsilon}.$$

Thus, letting $m\to\infty$ in (3.47), we obtain (3.27). Q.E.D.

Proof of Lemma 3.4. Since $l\in\mathscr{U}_{1+\varepsilon}(I)$, it follows from Lemma 3.5 that the equation

(3.48) $$((g, (L+i)\varphi))_0=<l, \varphi> \qquad (\varphi\in C_0^{2,B}(I, X))$$

has a solution $g\in H_0^{1,B}(I, X)\cap H^{1+\varepsilon}(I, X)$. Put $w=g-v$, where v is a radiative function for $\{L, k, l, 0\}$, i.e., v satisfies

(3.49) $$((v, (L-\bar{k}^2)\varphi))_0=<l, \varphi> \qquad (\varphi\in C_0^{2,B}(I, X))$$

and

(3.50) $$||v'-ikv||_{-1+\varepsilon}<\infty.$$

From (3.48) and (3.49) we see that

(3.51) $$((w, (L-\bar{k}^2)\varphi))_0=(k^2-i)((g, \varphi))_0.$$

Noting $g\in H_0^{1,B}(I, X)$ and (3.50), we have

(3.52) $$\begin{aligned}||w'-ikw||_{-1+\varepsilon} &\leqq||v'-ikv||_{-1+\varepsilon}+||g'-ikg||_{-1+\varepsilon}\\ &\leqq||v'-ikv||_{-1+\varepsilon}+||g'||_0+|k|\,||g||_0<\infty.\end{aligned}$$

Hence w is a radiative function for $\{L, k, (k^2-i)l[g]\}$. We make use of Lemma 3.1 to obtain

$$\|w\|_{-1-\varepsilon}+\|w'-ikw\|_{-1+\varepsilon}+\|B^{\frac{1}{2}}w\|_{-1+\varepsilon}\leqq\delta_1(1+|k|^2)\|g\|_{1+\varepsilon}, \tag{3.53}$$

where $\delta_1=\delta_1(K)$ is given in (3.2). It is implied by (3.26) and (3.22) that

$$\begin{aligned}\|g\|_{-1-\varepsilon}+\|g'-ikg\|_{-1+\varepsilon}+\|B^{\frac{1}{2}}g\|_{-1+\varepsilon} &\leqq(1+|k|)\|g\|_0+\|g'\|_0+\|B^{\frac{1}{2}}g\|_0\\ &\leqq(3+|k|)\|g\|_B\leqq(3+|k|)\,|||l|||\leqq(3+|k|)\,a_0\,|||l|||_{1+\varepsilon}.\end{aligned} \tag{3.54}$$

Since $v=g+w$, (3.23) follows from (3.53), (3.54) and (3.27). Q.E.D.

Lemma 3.6. *Let us assume Assumptions* 1.1 *and* 1.2. *Let* $k_m\in\mathbf{C}^+$, $l_m\in\mathscr{U}_{1+\varepsilon}(I)$ *for each* $m=1, 2, \cdots$. *Let* v_m, $m=1, 2, \cdots$ *be radiative functions for* $\{L, k_m, l_m, 0\}$ *such that*

$$v_m\in H^{-1-\varepsilon}(I, X) \qquad (m=1, 2, \cdots). \tag{3.55}$$

Let us assume

$$\begin{cases}\lim\limits_{m\to\infty}k_m=k,\\ \lim\limits_{m\to\infty}|||l-l_m|||_{1+\varepsilon}=0\end{cases} \tag{3.56}$$

with $k\in\mathbf{C}^+$ *and* $l\in\mathscr{U}_{1+\varepsilon}(I)$. *Then there exists the radiative function* v *for* $\{L, k, l, 0\}$ *satisfying*

$$v_m\to v \text{ both in } H^{-1-\varepsilon}(I, X) \text{ and in } \mathrm{loc}\,H_0^{1,B}(\bar{I}, X) \text{ as } m\to\infty. \tag{3.57}$$

Proof. As in the proof of Lemma 3.4 we put $v_m=g_m+w_m$, where $g_m\in H_0^{1,B}(I, X)\cap H^{1+\varepsilon}(I, X)$ is the solution of the equation

$$((g_m, (L+i)\varphi))_0=\langle l_m, \varphi\rangle \qquad (\varphi\in C_0^{2,B}(I, X)), \tag{3.58}$$

and w_m is the radiative function for $\{L, k_m, (k_m^2-i)l[g_m], 0\}$ for each $m=1, 2, \cdots$. For each pair (m, n) we have

$$((g_m - g_n, (L+i)\varphi))_0 = \langle l_m - l_n, \varphi \rangle, \tag{3.59}$$

and hence we obtain, using Lemma 3.5,

$$\begin{cases} \|g_m - g_n\|_B \leqq \beta_1(\sqrt{i})\,|||l_m - l_n||| \to 0 \\ \|g_m - g_n\|_{1+\varepsilon} \leqq \beta_2(\sqrt{i})\,|||l_m - l_n|||_{1+\varepsilon} \to 0 \end{cases} \tag{3.60}$$

as $m, n \to \infty$. We put $g = \lim_{m\to\infty} g_m$. Then $g \in H_0^{1,B}(I, X) \cap H^{1+\varepsilon}(I, X)$, and g is the solution of equation (3.48).

Now we turn to the sequence $\{w_m\}$. Since the sequence $\{l_m\}$ is uniformly bounded in $\mathscr{U}_{1+\varepsilon}(I)$, it follows from Lemma 3.4 and Lemma 3.5 that the sequence $\{\|v_m\|_{-1-\varepsilon} + \|v_m' - ikv_m\|_{-1+\varepsilon} + \|B^{\frac{1}{2}} v_m\|_{-1+\varepsilon}\}$, $\{\|g_m\|_B\}$ and $\{\|g_m\|_{1+\varepsilon}\}$ are also uniformly bounded. Therefore, noting that $w_m = v_m - g_m$, we obtain the uniform estimate

$$\|w_m\|_{-1-\varepsilon} + \|w_m' - ik_m w_m\|_{-1+\varepsilon} + \|B^{\frac{1}{2}} w_m\|_{-1+\varepsilon} \leqq \alpha \qquad (m=1, 2, \dots) \tag{3.61}$$

with a constant $\alpha > 0$. From (3.61) we have

$$\sup_{m=1,2,\dots} \|w_m\|_{B,(0,R)} < \infty \tag{3.62}$$

for any $R \in I$. Hence, proceeding as in the proof of Lemma 3.1, we obtain a subsequence $\{w_{m_j}\}$ of $\{w_m\}$ which converges to w in $\mathrm{loc}\, H_0^{1,B}(\bar{I}, X)$. On the other hand, using Lemma 3.2 and the uniform boundedness of $\{\|g\|_{1+\varepsilon}\}$, we have uniformly with respect to m

$$\begin{aligned} \int_\rho^\infty (1+r)^{-1-\varepsilon} |w_m(r)|^2 dr &\leqq \alpha_0(\|w_m\|_{-1-\varepsilon}^2 + |k_m^2 - i|^2 \|g_m\|_{1+\varepsilon})\rho^{-\varepsilon} \\ &= 0(\rho^{-\varepsilon}) \qquad (\rho \to \infty), \end{aligned} \tag{3.63}$$

where we have noted that $\{k_m\}$ is uniformly bounded and w_m is a radiative function for $\{L, k, (k_m^2 - i)l[g_m], 0\}$. It is implied by (3.63) and the convergence of $\{w_{m_j}\}$ in $\mathrm{loc}\, H_0^{1,B}(\bar{I}, X)$ that w_{m_j} converges to w in $H^{-1-\varepsilon}(I, X)$. Therefore, taking note of (3.61) and $k_m \to k$, $m \to \infty$, we see that w is a radiative function for $\{L, k, (k^2 - i)l[g], 0\}$ and we have

$$w_{m_j} \to w \qquad (j \to \infty) \tag{3.64}$$

both in $\operatorname{loc} H_0^{1,B}(\bar{I}, X)$ and $H^{-1-\varepsilon}(I, X)$.

Finally put $v_{m_j} = g_{m_j} + w_{m_j}$. Then we obtain from (3.60) and (3.64)

(3.65) $$v_{m_j} \to v \qquad (j \to \infty)$$

both in $\operatorname{loc} H_0^{1,B}(\bar{I}, X)$ and $H^{-1-\varepsilon}(I, X)$, where $v = g + w$ is a radiative function for $\{L, k, l, 0\}$. Since v is unique by the uniqueness of the radiative function (Theorem 2.2), it follows from (3.65) that the original sequence $\{v_m\}$ itself converges to v both in $H^{-1-\varepsilon}(I, X)$ and $\operatorname{loc} H_0^{1,B}(\bar{I}, X)$. Q.E.D.

We can now prove the existence theorem of the radiative function for $\{L, k, l, 0\}$, where $k \in \mathbf{C}^+$, $l \in \mathscr{U}_{1+\varepsilon}(I)$.

Theorem 3.7. *Let us assume Assumptions* 1.1 *and* 1.2. *Let* $k \in \mathbf{C}^+$ *and* $l \in \mathscr{U}_{1+\varepsilon}(I)$. *Then there exists a unique radiative function* $v = v(\cdot, k, l)$ *for* $\{L, k, 0\}$ *in* $H^{-1-\varepsilon}(I, X)$. *If* k *belongs to a compact set* K *in* $\mathbf{C}^+$ *then we have*

(3.66) $$\|v\|_{-1-\varepsilon} + \|v' - ikv\|_{-1+\varepsilon} + \|B^{\frac{1}{2}} v\|_{-1+\varepsilon} \leqq \delta_2 |||l|||_{1+\varepsilon}$$

with a constant $\delta_2 > 0$, *depending only on* K. *Denote by* Σ_0 *the mapping*

(3.67) $$\Sigma_0 : \mathbf{C}^+ \times \mathscr{U}_{1+\varepsilon}(I) \ni (k, l) \to v(\cdot, k, l) \in H^{-1-\varepsilon}(I, X) \cap \operatorname{loc} H_0^{1,B}(\bar{I}, X).$$

Then Σ_0 *is continuous as a mapping from* $\mathbf{C}^+ \times \mathscr{U}_{1+\varepsilon}(I)$ *into* $H^{-1-\varepsilon}(I, X)$ *and is also continuous as a mapping from* $\mathbf{C}^+ \times \mathscr{U}_{1+\varepsilon}(I)$ *into* $\operatorname{loc} H_0^{1,B}(\bar{I}, X)$.

Proof. First assume that $\operatorname{Im} k > 0$. Then from Lemma 3.5 we obtain a unique radiative function $v(\cdot, k, l)$ for $\{L, k, l, 0\}$ such that $v \in H^{1+\varepsilon}(I, X) \cap H_0^{1,B}(I, X)$. Next assume that $\operatorname{Im} k = 0$. Then, putting for $m = 1, 2, \cdots$

(3.68) $$\begin{cases} k_m = k + \dfrac{i}{m} \\ v_m = v_m(\cdot, k_m, l), \end{cases}$$

we see from Lemma 3.6 that the radiative function $v=v(\cdot, k, l)$ for $\{L, k, l, 0\}$ is obtained as $v=\lim_{m\to\infty} v_m$. The other statements follow from Lemma 3.4 and Lemma 3.6. Q.E.D.

Finally we prove the existence of the radiative function for $\{L, k, l, u\}$.

Let $v=v(\cdot, k, l, u)$ be a radiative function for $\{L, k, l, u\}$, where $k\in\mathbf{C}^+$, $l\in\mathscr{U}_{1+\varepsilon}(I)$ and $u\in H^{1,B}(I, X)$. We define $l_1\in\mathscr{U}_{1+\varepsilon}(I)$ by

$$<l_1, \varphi>=<l, \varphi>-((\psi u, (L-\bar{k}^2)\varphi))_0 \qquad (\varphi\in C_0^{2,B}(I, X)), \tag{3.69}$$

where $\psi\in C^1(I)$, $0\leqq\psi\leqq 1$ and

$$\psi(r)=\begin{cases}1 & (0<r\leqq 1),\\ 0 & (r\geqq 2).\end{cases} \tag{3.70}$$

Then it is easy to see that $v_0=v-\psi u$ is a radiative function for $\{L, k, l_1, 0\}$. Thus we can reduce the equation with the boundary value $v(0)=u(0)$ to the equation with the boundary value $v_0(0)=0$. Therefore, noting that $l_1=l_1(u)$ is a $\mathscr{U}_{1+\varepsilon}(I)$-valued continuous function on $H^{1,B}(I, X)$, we obtain from Theorem 3.7 the following

Theorem 3.8. *Let us assume Assumptions* 1.1 *and* 1.2. *Let* $k\in\mathbf{C}^+$, $l\in\mathscr{U}_{1+\varepsilon}(I)$ *and* $u\in H^{1,B}(I, X)$. *Then there exists a unique radiative function* $v=v(\cdot, k, l, u)$ *for* $\{L, k, l, u\}$ *in* $H^{-1-\varepsilon}(I, X)$. *Denote by* Σ *the mapping*

$$\Sigma: \mathbf{C}^+\times\mathscr{U}_{1+\varepsilon}(I)\times H^{1,B}(I, X)\ni(k, l, u) \to v(\cdot, k, l, u)\in H^{-1-\varepsilon}(I, X)\cap \mathrm{loc}\, H^{1,B}(\bar{I}, X). \tag{3.71}$$

Then Σ *is continuous as a mapping from* $\mathbf{C}^+\times\mathscr{U}_{1+\varepsilon}(I)\times H^{1,B}(I, X)$ *into* $H^{-1-\varepsilon}(I, X)$ *and is also continuous as a mapping from* $\mathbf{C}^+\times\mathscr{U}_{1+\varepsilon}(I)\times H^{1,B}(I, X)$ *into* $\mathrm{loc}\, H^{1,B}(\bar{I}, X)$.

§4. The Dependency of Radiative Functions on $C(r)$

Let $C_m(r)$, $m=1, 2,\cdots$, be a sequence of operator-valued functions on

I. Let $C(r)$ be as above. In this section we study the relations between radiative functions for L and radiative functions for $L_m=-\frac{d^2}{dr^2}+B(r)+C_m(r)$ when $C_m(r)\to C(r)$ as $m\to\infty$.

Assumption 4.1. (a) *For each* $r\in I$ $C_m(r)$ *is a symmetric operator in* X *with* $\mathscr{D}(C_m(r))=D$ *such that* $C_m(r)x\in C^1(I, X)$ *for any* $x\in D$. *Moreover for any compact interval* M *in* I *there exists a constant* $c^{(m)}(M)>0$ *such that*

$$\left|\frac{d}{dr}C_m(r)x\right|\leqq c^{(m)}(M)(|x|+|B^{\frac{1}{2}}(r)x|) \tag{4.1}$$

holds for any $x\in D$ *and any* $r\in M$.

(b) *There exists a constant* $c_0>0$ *such that*

$$|C_m(r)x|\leqq c_0(1+r)^{-1-\varepsilon}(|x|+|B^{\frac{1}{2}}(r)x|) \qquad (x\in D,\ r\in I) \tag{4.2}$$

for any $m=1, 2, \cdots$, *where* c_0 *does not depend on* m, *and* $0<\varepsilon<1$ *is as given in* (1.8).

(c) *We have*

$$\lim_{m\to\infty}|C(r)x-C_m(r)x|=0 \tag{4.3}$$

for any $x\in D$ *and any* $r\in I$.

Since C_m is assumed to satisfy (a) and (b) of Assumption 4.1 for each $m=1, 2, \cdots, C_m(r)$ is so smooth and tends to zero at $r=\infty$ so rapidly that the results of §2 and §3 can be applied to the operator

$$L_m=-\frac{d^2}{dr^2}+B(r)+C_m(r), \tag{4.4}$$

i.e., there exists a unique radiative function $v_m(r, k, l, u)$ for $\{L_m, k, l, u\}$, where $(k, l, u)\in \mathbf{C}^+\times\mathscr{U}_{1+\varepsilon}(I)\times H^{1,B}(I, X)$.

Theorem 4.2. *Let* $B(r)$ *and* $C(r)$ *satisfy Assumptions* 1.1 *and* 1.2. *Let* $C_m(r)$, $m=1, 2,\cdots$, *satisfy Assumption* 4.1. *Let* K *be a compact set such that* $K\subset\mathbf{C}^+$ *and let* $v_m=v_m(r, k_m, l_m)$, $m=1, 2, \cdots$, *be the radiative* *function for* $\{L_m, k_m, l_m, 0\}$, *where* $k_m \in K$ *and* $l_m \in \mathcal{U}_{1+\varepsilon}(I)$. *Then there exists a constant* $\delta_0 > 0$ *such that*

$$(4.5) \qquad \|v_m\|_{-1-\varepsilon} + \|v_m' - ik_m v_m\|_{-1+\varepsilon} + \|B^{\frac{1}{2}} v_m\|_{-1+\varepsilon} \leqq \delta_0 |||l_m|||_{1+\varepsilon}.$$

δ_0 *depends only on* K.

Proof. Denote by g_m the radiative function for $\{L_m, \sqrt{i}, l_m, 0\}$. We see from Lemma 3.5 that $g_m \in H^{1+\varepsilon}(I, X)$ for each $m=1, 2, \cdots$. We denote by w_m the radiative function for $\{L_m, k_m, (k_m^2 - i) l[g_m], 0\}$. Obviously we have $v_m = g_m + w_m$. Proceeding as in the proof of Lemma 3.5, from (4.2) we obtain uniformly for $m=1, 2, \cdots$,

$$(4.6) \qquad \alpha \|\varphi\|_B^2 \geqq |((\varphi, (L_m + i)\varphi))_0| \geqq \beta \|\varphi\|_B^2 \qquad (\varphi \in C_0^{2,B}(I, X)),$$

with constants $\alpha, \beta > 0$, whence follows that we obtain uniformly for $m = 1, 2, \cdots$

$$(4.7) \qquad \begin{cases} \|g_m\|_B \leqq \eta_0 |||l_m|||, \\ \|g_m\|_{1+\varepsilon} \leqq \eta_0 |||l_m|||_{1+\varepsilon} \end{cases}$$

with a constant $\eta_0 > 0$. Re-examining the proof of Lemma 1.6, we can see from (4.2) that we obtain uniformly for $m=1, 2, \cdots$

$$(4.8) \qquad \|w_m' - ikw_m\|_{-1+\varepsilon} + \|B^{\frac{1}{2}} w_m\|_{-1+\varepsilon} \leqq \eta_1(\|w_m\|_{-1-\varepsilon} + \|g_m\|_{1+\varepsilon}),$$

with a constant $\eta_1 = \eta_1(K) > 0$. Finally, proceeding as in the proof of Lemma 3.1, we can show by reduction to absurdity that we have uniformly for $m=1, 2, \cdots$

$$(4.9) \qquad \|w_m\|_{-1-\varepsilon} \leqq \eta_2 \|g_m\|_{1+\varepsilon}$$

with a positive constant $\eta_2 = \eta_2(K)$. Thus we have (4.5) from (4.7), (4.8), (4.9) and (3.22) as follows:

$$(4.10) \qquad \|v_m\|_{-1-\varepsilon} + \|v_m' - ik_m v_m\|_{-1+\varepsilon} + \|B^{\frac{1}{2}} v_m\|_{-1+\varepsilon}$$
$$\leqq \|w_m\|_{-1-\varepsilon} + \|w_m' - ik_m w_m\|_{-1+\varepsilon} + \|B^{\frac{1}{2}} w_m\|_{-1+\varepsilon}$$
$$+ \|g_m\|_0 + \|g_m'\|_0 + |k_m| \|g_m\|_0 + \|B^{\frac{1}{2}} g_m\|_0$$

$$\leqq \|w_m\|_{-1-\varepsilon} + \eta_1(\|w_m\|_{-1-\varepsilon} + \|g_m\|_{1+\varepsilon}) + (3 + |k_m|)\|g_m\|_B$$

$$\leqq \{\eta_2 + \eta_1(1+\eta_2)\}\|g_m\|_{1+\varepsilon} + (3 + |k_m|)\|g_m\|_B$$

$$\leqq [\{\eta_2 + \eta_1(1+\eta_2)\} + (3+T)a_0] \; |||l_m|||_{1+\varepsilon} \qquad (m=1, 2, \cdots),$$

where we put $T = \sup_{m=1,2,\ldots} |k_m|$, and a_0 is given as in (3.22). Q.E.D.

Theorem 4.3. *Let $B(r)$ and $C(r)$ satisfy Assumptions* 1.1 *and* 1.2. *Let $C_m(r)$, $m=1, 2, \cdots$, satisfy Assumption* 4.1.

(i) *Let $k_m \in \mathbf{C}^+$ and $l_m \in \mathcal{U}_{1+\varepsilon}(I)$ such that*

$$(4.11) \qquad \begin{cases} \lim_{m\to\infty} k_m = k \\ \lim_{m\to\infty} |||l - l_m|||_{1+\varepsilon} = 0 \end{cases}$$

with $k \in \mathbf{C}^+$ and $l \in \mathcal{U}_{1+\varepsilon}(I)$. Denote by $v_m(\cdot, k_m, l_m)$ the radiative function for $\{L_m, k_m, l_m, 0\}$ for each $m=1, 2, \cdots$. Then we have

$$(4.12) \qquad v_m(\cdot, k_m, l_m) \to v(\cdot, k, l)$$

both in $H^{-1-\varepsilon}(I, X)$ and in $\mathrm{loc}\, H_0^{1,B}(\bar{I}, X)$, where $v(\cdot, k, l)$ is the radiative function for $\{L, k, l, 0\}$.

(ii) *Let K be a compact set in $\mathbf{C}^+$ and let M be a compact metric space. For each $m=1, 2, \cdots$, $l_m(k, s)$ is assumed to be a $\mathcal{U}_{1+\varepsilon}(I)$-valued, continuous function on $K\times M$ such that*

$$(4.13) \qquad \lim_{m\to\infty} |||l(k, s) - l_m(k, s)|||_{1+\varepsilon} = 0$$

uniformly on $K\times M$ with a $\mathcal{U}_{1+\varepsilon}(I)$-valued, continuous function $l(k, s)$ on $K\times M$. Denote by $v_m(\cdot, k, s)$ the radiative function for $\{L_m, k, l_m(k, s), 0\}$. Then we have

$$(4.14) \qquad \lim_{m\to\infty} v_m(\cdot, k, s) = v(\cdot, k, s)$$

both in $H^{-1-\varepsilon}(I, X)$ and in $\mathrm{loc}\, H_0^{1,B}(\bar{I}, X)$ uniformly on $K\times M$, where $v(\cdot, k, s)$ is the radiative function for $\{L, k, l(k, s), 0\}$.

Proof. First let us prove (i). Let g_m be the radiative function for

$\{L, \sqrt{i}, l_m, 0\}$ and let w_m be the radiative function for $\{L, k, (k_m^2-i)l[g_m],$ $0\}$. Then we have $v_m=g_m+w_m$. Similarly we have $v=g+w$, where g is the radiative function for $\{L, \sqrt{i}, l, 0\}$ and w is the radiative function for $\{L, k, (k^2-i)l[g], 0\}$. It follows from Lemma 3.5 and the regularity theorem of Jäger [5] that $g, g_m \in H^{1+\varepsilon}(I, X) \cap H_0^{1,B}(I, X) \cap C^2(I, D)$. We can show that

$$\lim_{m\to\infty}\|(C-C_m)g\|_{1+\varepsilon}=0. \tag{4.15}$$

In fact we obtain from (4.3) and the fact that $g(r)\in D$

$$\lim_{m\to\infty}|(C(r)-C_m(r))\,g(r)|=0 \qquad (r\in I), \tag{4.16}$$

and also obtain from (1.8) and (4.2)

$$\begin{aligned} &|(C(r)-C_m(r))g(r)|^2 \\ &\leqq[(c_2+c_0)(1+r)^{-1-\varepsilon}(|g(r)|+|B^{\frac{1}{2}}(r)g(r)|)]^2 \\ &\leqq 2(c_2+c_0)^2(1+r)^{-2-2\varepsilon}(|g(r)|^2+|B^{\frac{1}{2}}(r)g(r)|^2) \\ &\in L^1(I, (1+r)^{1+\varepsilon}dr). \end{aligned} \tag{4.17}$$

(4.15) directly follows from (4.16) and (4.17). Noting that $g-g_m$ satisfies the equation

$$((g-g_m, (L+i)\varphi))_0=<l-l_m, \varphi>+(((C-C_m)g, \varphi))_0 \qquad (\varphi\in C_0^{2,B}(I, X)), \tag{4.18}$$

We see from (4.7) and (4.15) that

$$\begin{cases} \|g-g_m\|_B\leqq\eta_0\{|||l-l_m|||+\|(C-C_m)g\|_0\}\to 0, \\ \|g-g_m\|_{1+\varepsilon}\leqq\eta_0\{|||l-l_m|||_{1+\varepsilon}+\|(C-C_m)g\|_{1+\varepsilon}\}\to 0 \end{cases} \tag{4.19}$$

as $m\to\infty$. Using (4.19) and Theorem 4.2, we can proceed as in the proof of Lemma 3.6 to show that the sequence w_m converges to w both in $H^{-1-\varepsilon}(I, X)$ and in $\operatorname{loc} H_0^{1,B}(\bar{I}, X)$. Thus we have shown that $v_m=g_m+w_m$ converges to $v=g+w$ both in $H^{-1-\varepsilon}(I, X)$ and in $\operatorname{loc} H_0^{1,B}(\bar{I}, X)$ which completes the proof of (i).

Next let us prove (ii). It follows from (i) that for each pair $(k, s) \in K\times M$ $v_m(\cdot, k, s)$ converges to $v(\cdot, k, s)$ both in $H^{-1-\varepsilon}(I, X)$ and in $\mathrm{loc}\, H_0^{1,B}(\bar{I}, X)$. Assume that the convergence of v_m in $H^{-1-\varepsilon}(I, X)$ is not uniform on $K\times M$. Then there exists $\varepsilon_0>0$ and the set of positive integers $\{m_j\}_{j=1}^{\infty}$ and $(k_j, s_j)\in K\times M$ such that $m_j\to\infty$ as $j\to\infty$ and

$$\|v(\cdot, k_j, s_j)-v_{m_j}(\cdot, k_j, s_j)\|_{-1-\varepsilon}\geqq\varepsilon_0. \tag{4.20}$$

Since the set $\{(k_j, s_j)\mid j=1, 2, \cdots\}$ has at least an accumulating point $(k_0, s_0)\in K\times M$, we can assume $k_j\to k_0$ and $s_j\to s_0$ without loss of generality. Then, using the continuity of $l(k, s)$ and the uniform convergence of $l_m(k, s)$, we obtain

$$\begin{aligned} &|||l(k_0, s_0)-l_{m_j}(k_j, s_j)|||_{1+\varepsilon} \\ &\leqq |||l(k_0, s_0)-l(k_j, s_j)|||_{1+\varepsilon}+|||l(k_j, s_j)-l_{m_j}(k_j, s_j)|||_{1+\varepsilon}\to 0, \\ &\qquad j\to\infty. \end{aligned} \tag{4.21}$$

Therefore it follows from (i) that

$$\|v(\cdot, k_0, s_0)-v_{m_j}(\cdot, k_j, s_j)\|_{-1-\varepsilon}\to 0, \qquad j\to\infty. \tag{4.22}$$

On the other hand we obtain from Lemma 3.6

$$\|v(\cdot, k_0, s_0)-v(\cdot, k_j, s_j)\|_{-1-\varepsilon}\to 0, \qquad j\to\infty. \tag{4.23}$$

(4.22) and (4.23) are combined to give $\|v(\cdot, k_j, s_j)-v_{m_j}(\cdot, k_j, s_j)\|_{-1-\varepsilon}\to 0$, $j\to\infty$, which contradicts (4.20). Hence $v_m(\cdot, k, s)$ converges to $v(\cdot, k, s)$ in $H^{-1-\varepsilon}(I, X)$ uniformly for $(k, s)\in K\times M$. Similarly we can show that $v_m(\cdot, k, s)$ converges to $v(\cdot, k, s)$ in $\mathrm{loc}\, H_0^{1,B}(\bar{I}, X)$ uniformly for $(k, s) \in K\times M$. Q.E.D.

By an argument similar to the one used in obtaining Theorem 3.8 from Theorem 3.7, we can show the following

Theorem 4.4. *Let* $B(r)$, $C(r)$ *and* $C_m(r)$, $m=1, 2, \cdots$, *be as in Theorem* 4.3. *Let* $u\in H^{1,B}(I, X)$.

(i) *Let* $k_m\in \mathbf{C}^+$ *and* $l_m\in \mathscr{U}_{1+\varepsilon}(I)$ *satisfy* (4.11). *Denote by* $v_m(\cdot, k_m, l_m, u)$ *the radiative function for* $\{L_m, k_m, l_m, u\}$ *for each* $m=1, 2, \cdots$.

Then we have $v_m(\cdot, k_m, l_m, u) \to v(\cdot, k, l, u)$, $m\to\infty$, *both in* $H^{-1-\varepsilon}(I, X)$ *and in* $\mathrm{loc}\, H^{1,B}(\bar{I}, X)$, *where* $v(\cdot, k, l, u)$ *is the radiative function for* $\{L, k, l, u\}$.

(ii) *Let* k, M, $l_m(k, s)$ *and* $l(k, s)$ *be as in* (ii) *of Theorem* 4.3. *Let* (4.13) *be satisfied. Then we have*

$$\lim_{m\to\infty} v_m(\cdot, k_m, s_m, u) = v(\cdot, k, s, u) \tag{4.24}$$

both in $H^{-1-\varepsilon}(I, X)$ *and in* $\mathrm{loc}\, H^{1,B}(\bar{I}, X)$ *uniformly on* $K\times M$, *where* $v_m(\cdot, k_m, s_m, u)$ *and* $v(\cdot, k, s, u)$ *are the radiative functions for* $\{L_m, k_m, l_m(k, s), u\}$ *and* $\{L, k, l(k, s), u\}$, *respectively.*

§5. The Schrödinger Operator in $\mathbf{R}^n$ $(n\geqq 3)$

In this section we apply the results obtained in the preceding sections to the Schrödinger operator in $\mathbf{R}^n$ $(n\geqq 3)$.

Let $X=L^2(S^{n-1})$, S^{n-1} being $(n-1)$-sphere. We define a unitary operator U from $L^2(\mathbf{R}^n)$ onto $H^0(I, X)$ by

$$(UF)(r) = r^{\frac{n-1}{2}} F(r\omega) \qquad (F(y)\in L^2(\mathbf{R}^n)), \tag{5.1}$$

where $r=|y|$ and $\omega=\dfrac{y}{r}\in S^{n-1}$.

Let us consider the Laplacian on $\mathbf{R}^n$

$$-\Delta F(y) = -\sum_{j=1}^{n} \frac{\partial^2 F}{\partial y_j^2}. \tag{5.2}$$

We denote by H_0 the restriction of $-\Delta$ to $C_0^\infty(\mathbf{R}^n)$, i.e.,

$$\begin{cases} \mathscr{D}(H_0) = C_0^\infty(\mathbf{R}^n),^{16)} \\ H_0\Phi = -\Delta\Phi. \end{cases} \tag{5.3}$$

As is well known, we have for $\Phi\in C_0^\infty(\mathbf{R}^n)$

$$UH_0\Phi = L_0 U\Phi, \tag{5.4}$$

16) $C_0^\infty(\mathbf{R}^n)$ is the set of all infinitely continuously differentiable functions on $\mathbf{R}^n$ with compact carrier.

where

$$(5.5)\qquad \begin{cases} L_0 = -\dfrac{d^2}{dr^2} + B(r) \\ \mathscr{D}(B(r)) = D = \mathscr{D}(\varLambda_n), \\ B(r) = \dfrac{1}{r^2}\Big(-\varLambda_n + \dfrac{(n-1)(n-3)}{4}\Big), \end{cases}$$

and $\varLambda_n$ is the Laplace-Beltrami operator on S^{n-1}. As is well-known $-\varLambda_n$ is a non-negative, self-adjoint operator in $L^2(S^{n-1})$, and hence we can easily see that $B(r)$ satisfies (a) and (b) of Assumptions 1.1 and 1.2. We obtain from (5.4)

$$(5.6)\qquad \begin{cases} U\mathscr{D}^1_{L^2}(\mathbf{R}^n) = H_0^{1,B}(I, X)^{17)} \\ \|F\|_{(1)} = \|UF\|_B \qquad (F \in \mathscr{D}^1_{L^2}(\mathbf{R}^n)). \end{cases}$$

Let $\mathscr{V}(R^n)$ be the set of all linear continuous functionals α on $\mathscr{D}^1_{L^2}(R^n)$. $\mathscr{V}(\mathbf{R}^n)$ is a Banach space with the norm

$$(5.7)\qquad |\alpha| = \sup\{|<\alpha, F>|\,;\, F \in \mathscr{D}^1_{L^2}(\mathbf{R}),\ \|F\|_{(1)} = 1\}.$$

Then a linear mapping $\tilde{U}$ from $\mathscr{V}(\mathbf{R}^n)$ into $\mathscr{U}(I)$ is defined by

$$(5.8)\qquad <\tilde{U}\alpha, \varphi> = <\alpha, U^{-1}\varphi> \qquad (\varphi \in H_0^{1,B}(I, X)).$$

We have

$$(5.9)\qquad \begin{cases} \tilde{U}\mathscr{V}(\mathbf{R}^n) = \mathscr{U}(I), \\ |\alpha| = |||\tilde{U}\alpha|||. \end{cases}$$

Denote by $q(y)$ a real-valued function on $\mathbf{R}^n$. $q(y)$ is assumed to satisfy the following conditions:

(Q) *$q(y)$ is continuously differentiable on $\mathbf{R}^n$ and behaves like $O(|y|^{-1-\varepsilon})$ ($\varepsilon>0$) at infinity, i.e., there exist constants $c>0$, $\rho>0$ such that*

17) The Hilbert space $\mathscr{D}^1_{L^2}(\mathbf{R}^n)$ is defined as the completion of $C_0^\infty(\mathbf{R}^n)$ in the norm

$$\|F\|^2_{(1)} = \int_{\mathbf{R}^n}\Big\{\sum_{j=1}^n \Big|\frac{\partial F}{\partial y_j}\Big|^2 + |F(y)|^2\Big\}dx$$

(5.10) $$|q(y)| \leqq c|y|^{-1-\varepsilon} \qquad (|y| \geqq \rho)$$

with $0<\varepsilon<1$.

Let us define $C(r)$ by

(5.11) $$\begin{cases} C(r)=q(r\omega)\times \\ \mathscr{D}(C(r))=D. \end{cases}$$

It is easy to see that $C(r)$ satisfies Assumptions 1.1 and 1.2.

Define a differential operator H by

(5.12) $$\begin{cases} \mathscr{D}(H)=C_0^\infty(\mathbf{R}^n) \\ H\Phi=-\Delta\Phi+q(y)\Phi. \end{cases}$$

Then we have

(5.13) $$UH\Phi=LU\Phi \qquad (\Phi\in C_0^\infty(\mathbf{R}^n)),$$

where

(5.14) $$L=-\frac{d^2}{dr^2}+B(r)+C(r).$$

Denote by $\mathscr{V}_{1+\varepsilon}(\mathbf{R}^n)$ the set of all $\alpha\in\mathscr{V}(\mathbf{R}^n)$ such that

(5.15) $$|\alpha|_{1+\varepsilon}=\sup\{|<\alpha, (1+r)^{\frac{1+\varepsilon}{2}}F>|\,;\, F\in\mathscr{D}^1_{L^2}(\mathbf{R}^n),\ \|F\|_{(1)}=1\}<\infty.$$

We have $\tilde{U}\mathscr{V}_{1+\varepsilon}(\mathbf{R}^n)=\mathscr{U}_{1+\varepsilon}(I)$ and $|\alpha|_{1+\varepsilon}=|||\tilde{U}\alpha|||_{1+\varepsilon}$ for $\alpha\in U_{1+\varepsilon}(\mathbf{R}^n)$.

We now give the definition of the radiative function for H as follows:

Let $k\in\mathbf{C}^+$ *and* $\alpha\in\mathscr{V}(\mathbf{R}^n)$. *Then* $F\in\mathrm{loc}\,\mathscr{D}^1_{L^2}(\mathbf{R}^n)$[18] *is called the radiative function for* $\{H, k, \alpha\}$, *if* F *satisfies the following conditions*;

(1) *For any* $\Phi\in C_0^\infty(\mathbf{R}^n)$ *we have*

(5.16) $$(F, (H-\bar{k}^2)\Phi)_{L^2(\mathbf{R}^n)}=<\alpha, \Phi>.$$

18) $\mathrm{loc}\,\mathscr{D}^1_{L^2}(\mathbf{R}^n)$ is the set of all $F(y)$ on R^n such that $\psi_n F\in\mathscr{D}^1_{L^2}(\mathbf{R}^n)$ for any $n=1$, 2, ⋯, where $\psi_n\in C_0^\infty(\mathbf{R}^n)$, $0\leqq\psi_n\leqq 1$ and

$$\psi_n(x)=\begin{cases} 1 & \text{for } |x|\leqq n, \\ 0 & \text{for } |x|\geqq n+1. \end{cases}$$

(2) *The "radiation condition"*

$$\int_{|y|\geqq 1}(1+|y|)^{-1+\varepsilon}\left|\frac{\partial F}{\partial|y|}-ikF(y)\right|^2 dy<\infty \tag{5.17}$$

holds.

Let F be the radiative function for $\{H, k, 0\}$, $k\in\mathbf{C}^+$. Then, putting $v=UF\in \mathrm{loc}\, H_0^{1,B}(\bar{I}, X)$, we have

$$((v, (L-\bar{k}^2)\varphi))_0=0 \qquad (\varphi\in C_0^{2,B}(I, X)), \tag{5.18}$$

and

$$\left\|v'-\frac{n-1}{2r}v-ikv\right\|_{-1+\varepsilon,(1,\infty)}<\infty. \tag{5.19}$$

Modifying slightly the proof of Lemma 2.1, we obtain

$$\begin{aligned}\left|v'(r)-\frac{n-1}{2r}v(r)-ikv(r)\right|^2 &=\left|v'+\left(\mathrm{Im}\,k-\frac{n-1}{2r}\right)v(r)\right|^2+(\mathrm{Re}\,k)^2|v(r)|^2\\ &\quad -2(\mathrm{Re}\,k)\,\mathrm{Im}\,(v'(r), v(r))\\ &=\left|v'(r)+\left(\mathrm{Im}\,k-\frac{n-1}{2r}\right)v(r)\right|^2+(\mathrm{Re}\,k)^2|v(r)|^2\\ &\quad +4(\mathrm{Re}\,k)^2(\mathrm{Im}\,k)\|v\|_{0,(0,r)}^2.\end{aligned} \tag{5.20}$$

If $\mathrm{Im}\,k\neq 0$, then we see from (5.20)

$$\begin{aligned}&\lim_{r_j\to\infty}\|v\|_{0,(0,r_j)}^2\\ &\leqq\frac{1}{4(\mathrm{Re}\,k)^2(\mathrm{Im}\,k)}\lim_{r_j\to\infty}\left|v'(r_j)-\frac{n-1}{2r}v(r_j)-ikv(r_j)\right|^2=0\end{aligned} \tag{5.21}$$

along some sequence $\{r_j\}_{j=1}^\infty$, and hence $\|v\|_0=0$, i.e., $v\equiv 0$. If $\mathrm{Im}\,k=0$, then we obtain from (5.20)

$$\begin{aligned}&\left|v'(r)-\frac{n-1}{2r}v(r)-ikv(r)\right|^2\\ &\geqq\left|v'(r)-\frac{n-1}{2r}v(r)\right|^2+(\mathrm{Re}\,k)^2|v(r)|^2\end{aligned} \tag{5.22}$$

$$\geqq \frac{1}{2}|v'(r)|^2+\left(k^2-\frac{(n-1)^2}{4r^2}\right)|v(r)|^2$$

$$\geqq \frac{1}{2}\left\{|v'(r)|^2+k^2|v(r)^2|\right\}, \qquad \left(r\geqq \frac{n-1}{\sqrt{2k}}\right),$$

whence follows $\varliminf_{r\to\infty}(|v'(r)|^2+k^2|v(r)|^2)=0$. Therefore, proceeding as in the proof of Theorem 2.2, we have $v\equiv 0$. Thus the uniqueness of the radiative function for H has been proved.

Next let $\alpha\in\mathscr{V}_{1+\varepsilon}(\mathbf{R}^n)$ and $k\in\mathbf{C}^+$. Since we have $\tilde{U}\alpha\in\mathscr{U}_{1+\varepsilon}(I)$, it follows from Theorem 3.7 that there exists the radiative function $v=v(\cdot, k, \tilde{U}\alpha)$ for $\{L, k, \tilde{U}\alpha, 0\}$. Put

$$F=U^{-1}v(\cdot, k, \tilde{U}\alpha). \tag{5.23}$$

Then $F\in \mathrm{loc}\,\mathscr{D}^1_{L^2}(\mathbf{R}^n)$ and it follows from (5.23) that

$$(F, (H-\bar{k}^2)\Phi)_{L^2(\mathbf{R}^n)}=((v, (L-\bar{k}^2)U\Phi))_0=<\tilde{U}\alpha, U\Phi>=<\alpha, \Phi> \tag{5.24}$$

holds for any $\Phi\in C_0^\infty(\mathbf{R}^n)$. Since $v\in H^{-1-\varepsilon}(I, X)$ and $0<\varepsilon<1$, we have $\frac{n-1}{2r}v\in H^{-1+\varepsilon}((1, \infty), X)$. This together with $v'-ikv\in H^{-1+\varepsilon}(I, X)$ implies that $v'-ikv-\frac{n-1}{2r}v\in H^{-1+\varepsilon}((1, \infty), X)$. Hence we obtain

$$\int_{|y|\geqq 1}(1+|y|)^{-1+\varepsilon}\left|\frac{\partial F}{\partial|y|}-ikF(y)\right|^2 dy \tag{5.25}$$

$$=\left\|v'-ikv-\frac{n-1}{2r}v\right\|^2_{-1+\varepsilon,(1,\infty)}<\infty.$$

Therefore it has been shown that $F=U^{-1}v$ is the radiative function for $\{H, k, \alpha\}$. It follows from $v\in H^{-1-\varepsilon}(I, X)$ that $F\in L^2(\mathbf{R}^n, (1+|y|)^{-1-\varepsilon}dy)$. Thus we obtain

Theorem 5.1. *Let n be an integer such that $n\geqq 3$. Let $q(y)$ satisfy the condition* (Q). *Then for given $k\in\mathbf{C}^+$ and $\alpha\in\mathscr{V}(\mathbf{R}^n)$ the radiative function $F(\cdot, k, \alpha)$ for $\{H, k, \alpha\}$ is unique. For given $k\in\mathbf{C}^+$ and $\alpha\in\mathscr{V}_{1+\varepsilon}(\mathbf{R}^n)$ there exists the radiative function $F(\cdot, k, \alpha)$ for $\{H, k, \alpha\}$ such that $F(\cdot, k, \alpha)\in L^2(\mathbf{R}^n, (1+|y|)^{-1-\varepsilon}dy)$. Denote by σ the mapping*

(5.26) $\quad \sigma : \mathbf{C}^+ \times v_{1+\varepsilon}(\mathbf{R}^n) \ni (k, \alpha)$

$$\rightarrow F(\cdot, k, \alpha) \in L^2(R^n, (1+|y|)^{-1-\varepsilon} dy) \cap \mathrm{loc}\, \mathscr{D}^1_{L^2}(\mathbf{R}^n).$$

Then σ *is continuous as a mapping from* $\mathbf{C}^+ \times \mathscr{V}_{1+\varepsilon}(\mathbf{R}^n)$ *into* $L^2(\mathbf{R}^n, (1+|y|)^{-1-\varepsilon} dy)$ *and is also continuous as a mapping from* $\mathbf{C}^+ \times \mathscr{V}_{1+\varepsilon}(\mathbf{R}^n)$ *into* $\mathrm{loc}\, \mathscr{D}^1_{L^2}(R^n)$.

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