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Absolute Continuity of Hamiltonian Operators with Repulsive Potentials

By

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1. Introduction

The purpose of the present note is to improve the results of R.B. Lavine [3] on the absolute continuity of a Hamiltonian operator $H=-\Delta+V$ in $L_2(R^n)$ with repulsive potential V (where Δ is the Laplacian and V is the operation of multiplication by a real function $V(x)$). If the potential $V(x)$ satisfies

(1) $$\partial V/\partial r \leqq 0$$

where $r=|x|$, then it is said to be repulsive.

Lavine [3] shows that if the potential V satisfies not only the assumption (1) but also

(2) $$\partial V/\partial r \leqq -ar^{-3+\varepsilon} \qquad \text{for large } r$$

for some positive constants a and ε, then $H=-\Delta+V$ is absolutely continuous for $n=1, 3$. Our aim is to extend his results in two directions: One is to remove the restriction on the dimension n of the space, and the other is to remove the assumption (2). This will be accomplished except for the cases $n=1$ and 2, where we must impose an assumption somewhat weaker than (2).

Our method is that of Lavine [3] which is based on an abstract theory of Putnam [4] on commutators of pairs of selfadjoint operators.

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2. Notations and Results

Let T be a selfadjoint operator in a Hilbert space $\mathfrak{H}$ and $E(\lambda)$ be the spectral family associated with T. Denote by $\mathfrak{H}_{ac}(T)$ the set of all vectors ϕ such that $||E(\lambda)\phi||^2$ is absolutely continuous with respect to the Lebesgue measure. Then $\mathfrak{H}_{ac}(T)$ is a closed subspace which reduces T; cf. [2], Chapter X, Theorem 1.5. Denote by T_{ac} the restriction of T in $\mathfrak{H}_{ac}(T)$. The spectrum of T_{ac} is called the absolutely continuous spectrum of T. If $T=T_{ac}$, that is, $\mathfrak{H}_{ac}(T)=\mathfrak{H}$, then we say that T is absolutely continuous.

Let $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_n)$ be a vector with nonnegative integral coordinates and $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n$. We denote by $D^\alpha\phi$ the partial derivative

$$D^\alpha\phi = \frac{\partial^{|\alpha|}\phi}{\partial^{\alpha_1}x_1\partial^{\alpha_2}x_2\cdots\partial^{\alpha_n}x_n}$$

in the distribution sense.

Let X be a set of functions defined in a domain $\Omega\subset R^n$. We denote by $\mathcal{E}^p_X(\Omega)$ the set of all functions ϕ such that all the derivatives $D^\alpha\phi$ in the distribution sence for $0\leqq|\alpha|\leqq p$ belong to the set X.[1)] In case $\Omega=R^n$, we sometimes write $\mathcal{E}^p_X$ instead of $\mathcal{E}^p_X(R^n)$.

Let $\mathfrak{H}=L_2(R^n)$ be the Hilbert space with the ordinary inner product

$$(\phi, \psi)=\int\phi(x)\psi(x)^*\ dx,$$

where the asterisk means the complex conjugate. Let H_0 be the selfadjoint operator $H_0=-\Delta$ with domain $D(H_0)=\mathcal{E}^2_{L_2}$.

Let $Q_\alpha(\alpha>0)$ be the set of real functions $V(x)$ satisfying the assumption

$$\int_{|x-y|\leqq 1}|V(y)|^2\,dy\leqq M \qquad (n=1,2,3)$$

$$\int_{|x-y|\leqq 1}|V(y)|^2|x-y|^{4-n-\alpha}\,dy\leqq M \qquad (n\geqq 4)$$

1) In the sequel, we shall use this notation in the case $X=L_2$, L_∞ and Q_α.

for some positive constant M dependent on V. Let $V\in Q_\alpha$. Then it is known (cf. [6], Satz 4.2) that for any given $\varepsilon>0$, there exists a constant C_ε such that

$$||V\phi||\leqq\varepsilon||H_0\phi||+C_\varepsilon||\phi|| \qquad \text{for any} \quad \phi\in C_0^\infty, \tag{3}$$

where the notation $\phi\in C_0^\infty$ means that ϕ is infinitely differentiable and has a compact support. By virtue of this inequality, the operator $-\Delta+V$ defined on C_0^∞ is essentially selfadjoint, that is, its closure, which will be denoted by H, is selfadjoint. Moreover its domain $D(H)$ coincides with $D(H_0)$, (3) holds for $\phi\in D(H)$, and the graph norms of H_0 and H are equivalent; cf. [2], Chap. V, Theorem 4.5.

We shall prove the following

Theorem 1. *Let V be a real function of class $\mathcal{E}^1_{L_\infty}$ and repulsive. In case $n=1$ and 2, we assume in addition that there exist constants a, b and m such that $0<a<b$,*

$$m>a^{-1}(b-a)^{-2}, \tag{4}$$

and

$$\begin{cases} \partial V/\partial r\leqq -m & \text{in} \quad a\leqq r\leqq b \\ \partial V/\partial r<0 & \text{in} \quad b\leqq r \end{cases} \qquad (n=1) \tag{5}$$

$$\begin{cases} \partial V/\partial r+\frac{1}{2}r^{-3}\leqq -\frac{1}{2}m & \text{in} \quad a\leqq r\leqq b \\ \partial V/\partial r+\frac{1}{2}r^{-3}<0 & \text{in} \quad b\leqq r. \end{cases} \qquad (n=2) \tag{6}$$

Then H is absolutely continuous.

Corollary 1. *Let V satisfy the assumptions of Theorem 1 and*

$$\lim_{|x|\to\infty}\int_{|x-y|\leqq 1}|V(y)|^2\,dy=0. \tag{7}$$

Then the spectrum of the absolutely continuous operator H is the interval

$[0, \infty)$.

Corollary 2. *Assume that* $V=V_1+V_2$ *satisfies the following conditions;*

i) *Each* V_i *is of class* $\mathcal{E}_{Q_\alpha}^{2(k-1)}$, *where* k *is an integer strictly larger than* $n/4$.

ii) *For large* r, *say for* $r \geqq R$, V_1 *is of class* $\mathcal{E}_{L_\infty}^1$ *and* $\partial V_1/\partial r \leqq 0$ $(n \neq 2)$, $\partial V_1/\partial r + \frac{1}{2}r^{-3} \leqq 0 \quad (n=2)$.

iii) V_1 *satisfies* (7) *with* V *replaced by* V_1.

iv) $V_2 \in L_1$.

Then the absolutely continuous spectrum of $H=-\Delta+V$ *is* $[0, \infty)$.

3. Preliminaries

In this section we assume that $V \in \mathcal{E}_{L_\infty}^1$ so that $V \in Q_\alpha$.

Let P_j's $(j=1, 2, \cdots, n)$ be the differential operators given by

$$P_j\phi = -i\partial\phi/\partial x_j$$

with domain $D(P_j)=\mathcal{E}_{L_2}^1$. Then P_j maps its domain into L_2 and $H_0=\sum_{i=1}^n P_j^2$.

Let $[A, B]$ be the commutator $AB-BA$ in the strict operator theoretical sense. If $f \in \mathcal{E}_{L_\infty}^1$, then we have

(8) $$i[P_j, f]\phi = (\partial f/\partial x_j)\phi \qquad \text{for } \phi \in \mathcal{E}_{L_2}^1.$$

Let g_j's $(j=1, 2, \cdots, n)$ be real valued functions of class $\mathcal{E}_{L_\infty}^1(R^n)$ and put

(9) $$A=(H-i)^{-1}\left(\sum_{j=1}^n (g_jP_j+P_jg_j)\right)(H+i)^{-1}.$$

Since $(g_jP_j+P_jg_j)(H+i)^{-1}$ is bounded by the closed graph theorem, A and HA are bounded so that $AH \subset (HA)^*$ is also bounded on $D(H)$. Thus the operator $i[H, A]$ is defined on $D(H)$ and bounded. Put $C=i(HA-(HA)^*)$. Then it is bounded and selfadjoint and the closure of

$i[H, A]$. Since $[H, A]=-iC$ on $D(H)$ and A is bounded, the following lemma is a special case of a theorem of Putnam [4, Theorem 2.13.2].

Lemma 1. *If there exists an operator A such that the operator C is nonnegative and* 0 *is not an eigenvalue of C, then H is absolutely continuous.*

In the next section we shall construct such g_j's that the operator A defined by (9) satisfies the assumptions of Lemma 1.

Lemma 2. *Let $g(r)$ be a real function defined on the half line $r\geqq 0$ of class $\mathcal{E}^3_{L_\infty}(0, \infty)$ such that $g(r)=\mathrm{const}\, r$ for small r. Put*

$$g_j(x)=g(|x|)x_j/|x|. \tag{10}$$

Then we have for $\phi\in L_2$,

$$\begin{aligned} C\phi=4\sum_{j,k=1}^{n}(H-i)^{-1}\, P_j x_j x_k g'\, r^{-2}P_k(H+i)^{-1}\phi+ \\ +4\sum_{j,k=1}^{n}(H-i)^{-1}\, P_j(\delta_{jk}-x_jx_kr^{-2})gr^{-1}\, P_k(H+i)^{-1}\phi- \\ -(H-i)^{-1}G(x)\,(H+i)^{-1}\phi,\end{aligned} \tag{11}$$

where $r=|x|$,

$$G(x)=g'''+2(n-1)r^{-1}g''+(n-1)\,(n-3)r^{-3}(rg'-g)+2g\partial V/\partial r, \tag{12}$$

$g=g(|x|)$ *and* $'=d/dr$.

Proof. We note thet $g_j\in\mathcal{E}^3_{L_\infty}$. Let $\phi\in C_0^\infty$. Then $(g_jP_j+P_jg_j)\phi$ are of class $\mathcal{E}^2_{L_\infty}$ and have compact supports so that they belong to $D(H_0)$, and we have

$$i[H_0, \sum_{j=1}^{n}(g_jP_j+P_jg_j)]\,\phi=\sum_{j,k=1}^{n}i[P_k^2,\ g_jP_j+P_jg_j]\,\phi$$

$$=\sum i\,\{P_k[P_k,g_jP_j+P_jg_j]+[P_k,\ g_jP_j+P_jg_j]P_k\}\,\phi$$

$$=\sum i\,\{P_k([P_k,\ g_j]P_j+g_j[P_k,\ P_j]+[P_k,\ P_j]\,g_j+P_j[P_k,\ g_j])+$$

$$+([P_k,\ g_j]P_j+g_j[P_k,\ P_j]+[P_k,\ P_j]g_j+P_j[P_k,\ g_j])P_k\}\ \phi$$

$$=\sum i\ \{P_k[P_k,\ g_j]P_j+P_kP_j[P_k,\ g_j]+[P_k,\ g_j]P_jP_k+P_j[P_k,\ g_j]P_k\}\phi,$$

where we used the identities $[P_j,\ P_k]=0$. Using (8) with f replaced by g_j, we have

$$i\,[H_0,\ \sum (g_jP_j+P_jg_j)]\,\phi$$

$$=\sum \{P_k\partial g_j/\partial x_kP_j+P_j\partial g_j/\partial x_kP_k+P_jP_k\partial g_j/\partial x_k+\partial g_j/\partial x_kP_kP_j\}\ \phi$$

$$=\sum \{2(P_k\partial g_j/\partial x_kP_j+P_j\partial g_j/\partial x_kP_k)+P_j[P_k,\ \partial g_j/\partial x_k]-$$

$$-[P_k,\ \partial g_j/\partial x_k]P_j\}\ \phi$$

$$=\sum \{2(P_k\partial g_j/\partial x_kP_j+P_j\partial g_j/\partial x_kP_k)+[P_j,\ [P_k,\ \partial g_j/\partial x_k]]\}\ \phi$$

$$=2\sum \{P_k\partial g_j/\partial x_kP_j+P_j\partial g_j/\partial x_kP_k\}\ \phi-\{\Delta(\sum_j \partial g_j/\partial x_j)\}\ \phi.$$

On the other hand

$$i\,[V,\ \sum (g_jP_j+P_jg_j)]\,\phi=-2(\sum g_j\partial V/\partial x_j)\,\phi.$$

Thus we have

$$i\,[H,\ \sum_j (g_jP_j+P_jg_j)\phi= \tag{13}$$
$$=2\sum_{j,k}\{P_k\partial g_j/\partial x_kP_j+P_j\partial g_j/\partial x_kP_k\}\ \phi-$$
$$-\{\ \Delta\ (\ \sum_j (\partial g_j/\partial x_j)+2\ \sum_j g_j\partial V/\partial x_j\}\ \phi$$

for $\phi\in C_0^\infty$.

Let ϕ and ψ be such that $(H+i)^{-1}\phi$, $(H+i)^{-1}\psi\in C_0^\infty$. Then ϕ and ψ run over a dense set since H restricted on C_0^∞ is essentially selfadjoint. Noting that $\sum (g_jP_j+P_jg_j)\,(H+i)^{-1}\phi\in D(H)$ and using formula (13), we have

$$(C\phi,\ \psi)=i\ (((HA)-(HA)^*)\phi,\ \psi)$$

$$=i\ ((H-i)^{-1}\ H(\ \sum\ (g_jP_j+P_jg_j))\ (H+i)^{-1}\phi,\ \psi)$$

$$-i\,(\phi, (H-i)^{-1} H(\sum (g_jP_j+P_jg_j))\,(H+i)^{-1}\,\psi)$$
$$=i\,([H, \sum (g_jP_j+P_jg_j)]\,(H+i)^{-1}\phi, \ (H+i)^{-1}\,\psi)$$
$$=((H-i)^{-1}\,2\sum (P_k\partial g_j/\partial x_kP_j+P_j\partial g_j/\partial x_kP_k)\,(H+i)^{-1}\,\phi, \psi)$$
$$-((H-i)^{-1}\{\Delta(\sum \partial g_j/\partial x_j)+2\sum g_j\partial V/\partial x_j\}\,(H+i)^{-1}\,\phi, \psi).$$

The operator C is bounded as was noted above. Since $\partial g_j/\partial x_k\in\mathcal{E}^2_{L_\infty}$ and $\Delta(\sum \partial g_j/\partial x_j)+2\sum g_j\partial V/\partial x_j\in L_\infty$, the two operators in the last member of this formula:

$$C_1=2(H-i)^{-1}\{\sum (P_k\partial g_j/\partial x+P_j\partial g_j/\partial x_kP_k)\}(H+i)^{-1}$$
$$C_2=(H-i)^{-1}\{\Delta(\sum \partial g_j/\partial x_j)+2\sum g_j\partial V/\partial x_j\}(H+i)^{-1}$$

are also bounded. Thus since ϕ and ψ run over a dense set, we have

$$C\phi=C_1\phi-C_2\phi \qquad \text{for} \quad \phi\in L_2. \tag{14}$$

Now since $g_j(x)=g(r)x_j/r$, we have

$$\partial g_j/\partial x_k=x_jx_kg'r^{-2}+(\delta_{jk}-x_jx_kr^{-2})\,gr^{-1}$$

and

$$\Delta(\sum \partial g_j/\partial x_j)=(d^2/dr^2+(n-1)r^{-1}\,d/dr)\,(g'+(n-1)r^{-1}g)=$$
$$=g'''+2(n-1)r^{-1}g''+(n-1)(n-3)(r^{-2}g'-r^{-3}g).$$

Thus we have

$$\begin{aligned} C_1&=4\sum (H-i)^{-1}\{P_jx_jx_kg'r^{-2}P_k+P_j(\delta_{jk}-x_jx_kr^{-2})gr^{-1}P_k\}\,(H+i)^{-1},\\ C_2&=(H-i)^{-1}\{g'''+2(n-1)r^{-1}g''+(n-1)\,(n-3)r^{-3}(rg'-g)+\\ &\quad +2g\partial V/\partial r\}(H+i)^{-1}\\ &=(H-i)^{-1}G(x)\,(H+i)^{-1}, \end{aligned} \tag{15}$$

which with (14) proves the lemma.

Lemma 3. *Let $g(r)$ satisfy the assumptions of Lemma 2. Assume also that*

$$g \geqq 0,\ g' \geqq 0,\ G \leqq 0 \qquad (r \neq 0), \tag{16}$$

$$g' > 0 \quad (r \leqq b), \qquad G < 0 \quad (r \geqq a), \tag{17}$$

for some constants a and b $(0<a<b)$. Then the operator C is nonnegative and zero is not an eigenvalue of C.

Proof. First we show that $C \geqq 0$. We note the formula (15). We have $C_2 \leqq 0$ since $G \leqq 0$ and $C_1 \geqq 0$ since g, $g' \geqq 0$ and the matrices $(x_j x_k)$ and $(\delta_{jk} - x_j x_k r^{-2})$ are nonnegative. Thus we have $C = C_1 - C_2 \geqq 0$.

Next we show that zero is not an eigenvalue of C. If $C\phi = 0$, then since the second term of (11) is nonnegative as was shown above, we have

$$\begin{aligned} 0 = (C\phi, \phi) &\geqq 4 \sum ((H-i)^{-1} P_j x_j x_k g' r^{-2} P_k (H+i)^{-1} \phi, \phi) \\ &\quad - ((H-i)^{-1} G (H+i)^{-1} \phi, \phi) \\ &= 4\int |\sum x_j P_j (H+i)^{-1} \phi|^2 g' r^{-2}\, dx + \int G |(H+i)^{-1}\phi|^2\, dx \geqq 0, \end{aligned}$$

so that by virtue of (17), $u(x) \equiv ((H+i)^{-1}\phi)(x)$ satisfies

$$\partial u/\partial r(x) = i(\sum x_j r^{-1} P_j (H+i)^{-1}\phi)(x) = 0 \quad \text{for almost all } |x| \leqq b,$$

and

$$u(x) = 0 \qquad \text{for almost all } |x| \geqq a.$$

Thus $u(x) = 0$ for almost all x since $u(x) = -\int_1^{a/|x|} \frac{du(tx)}{dt} dt = 0$ for $0 < |x| \leqq a$. Thus we have $\phi = 0$, which shows that zero is not an eigenvalue of the operator C.

4. Proof of Theorem 1

Now let us construct the function g satisfying the assumptions of

Lemma 3. Then by virtue of Lemma 1, the proof of Theorem 1 will be completed.

First we treat the case $n \geqq 3$. Let a and b be some constants such that $0<a<b$. Let k be a number such that $2<k<2(n-1)$. Put

$$g''(r)=\begin{cases} 0 & (0 \leqq r \leqq a) \\ -\dfrac{b^{-k}}{b-a}(r-a) & (a \leqq r \leqq b) \\ -r^{-k} & (b \leqq r), \end{cases}$$

$$g'(r)=-\int_r^{\infty} g''(r)\,dr$$

and

$$g(r)=\int_0^r g'(r)\,dr.$$

Then

$$g'''(r)=\begin{cases} 0 & (0 \leqq r \leqq a) \\ -\dfrac{b^{-k}}{b-a} & (a \leqq r \leqq b) \\ kr^{-k-1} & (b \leqq r) \end{cases}$$

is bounded. g'' is bounded and nonpositive. $g'(r)$ is bounded and positive since $g'' \in L_1(0, \infty)$ and $g'' \leqq 0$. $g(r)$ is positive since $g'>0$, and bounded since

$$g(r)=\int_0^b g'dr-(k-1)^{-1}(k-2)^{-1}(r^{-k+2}-b^{-k+2}) \qquad \text{in } b \leqq r,$$

and $k>2$.

Thus $g \in \mathcal{E}_{L_\infty}^3$ and g satisfies (16) and (17) except the assertions on G. Now let us show that G satisfies the assertions (16) and (17). We note that the third term $(n-1)(n-3)r^{-3}(rg'-g)$ of G in (12) is nonpositive since $(rg'-g)'=rg'' \leqq 0$ and $(rg'-g)(0)=0$. In $0 \leqq r \leqq a$, the first three terms in (12) are zero so that $G=2g\partial V/\partial r \leqq 0$ by the assumption (1). In $a \leqq r \leqq b$, the first term g''' is negative and the other terms are nonpositive so that $G<0$. In $b \leqq r$,

$$G \leqq g''' + 2(n-1)r^{-1}g'' = kr^{-k-1} - 2(n-1)r^{-k-1}$$
$$= (k-2(n-1))r^{-k-1} < 0$$

since $k < 2(n-1)$. Thus we have shown that this function g is desired one in case $n \geqq 3$.

Next we treat the case $n=1, 2$. Let a and b be the numbers in the assumption of Theorem 1 and c be such that $a<c<b$. Put

$$g(r) = \begin{cases} \frac{1}{2}(c-a)(b-c)r & (0 \leqq r \leqq a) \\ \frac{1}{2}(c-a)(b-c)r - \frac{b-c}{6(b-a)}(r-a)^3 & (a \leqq r \leqq c) \\ \frac{1}{6}(a+b+c)(c-a)(b-c) + \frac{c-a}{6(b-a)}(r-b)^3 & (c \leqq r \leqq b) \\ \frac{1}{6}(a+b+c)(c-a)(b-c) & (b \leqq r). \end{cases}$$

Then $g \in \mathcal{E}^3_{L_\infty}(0, \infty)$ and $g, g', -g'' \geqq 0$ for $r \geqq 0$ and $g' > 0$ for $r < b$. Let us show that $G \leqq 0$ $(r \geqq 0)$ and $G < 0$ $(r \geqq a)$ for c sufficiently near to a.

First let $n=1$. Then $G = g''' + 2g\partial V/\partial r$ and

$$g''' = \begin{cases} 0 & (0 \leqq r \leqq a) \\ -(b-c)/(b-a) < 0 & (a \leqq r \leqq c) \\ (c-a)/(b-a) > 0 & (c \leqq r \leqq b) \\ 0 & (b \leqq r) \end{cases}$$

so that $G \leqq 0$ in $0 \leqq r \leqq a$ and $G < 0$ in $a \leqq r \leqq c$ and $b \leqq r$ by the second assertion (5). In $c \leqq r \leqq b$, using the estimate $g(r) \geqq g(a) = \frac{1}{2}(c-a)(b-c)a$ and the assumption (5), we have

$$G = (c-a)/(b-a) + 2g\partial V/\partial r \leqq (c-a)/(b-a) - (c-a)(b-c)am$$
$$= -(c-a)(b-c)a\{m - a^{-1}(b-c)^{-1}(b-a)^{-1}\},$$

which is negative for c sufficiently near to a by the assumption (4). Thus (16) and (17) are verified for $n=1$.

Next let $n=2$. Since

$$G=g'''+2r^{-1}g''-r^{-2}g'+r^{-3}g+2g\partial V/\partial r,$$

using the assumptions of Theorem 1, we have

$$G\begin{cases} =2g\partial V/\partial r\leqq 0 \quad (0\leqq r\leqq a), \\ \leqq g(r^{-3}+2\partial V/\partial r)\leqq -mg<0 \quad (a\leqq r\leqq c), \\ \leqq g'''+g(r^{-3}+2\partial V/\partial r)\leqq -(c-a)(b-c)a\{m-a^{-1}(b-c)^{-1}(b-a)^{-1}\} \\ \qquad <0 \quad (c\leqq r\leqq b), \\ =r^{-3}g+2\partial V/\partial r<0 \quad (b\leqq r), \end{cases}$$

for c sufficiently near to a. Thus (16) and (17) are now verified for $n=2$.

Thus we have constructed the function g which have the desired properties, which yields Theorem 1.

5. Proof of Corollaries

Proof of Corollary 1. Since the potential V belongs to Q_α and satisfies the assumption (7), the essential spectrum of $H=H_0+V$ is $[0, \infty)$ (cf. [5]). On the other hand, by virtue of Theorem 1, H is absolutely continuous. Thus the spectrum of H is $[0, \infty)$.

For the proof of Corollary 2, we use the following theorem due to Birman [1]:

Let H_i $(i=1, 2)$ be selfadjoint operators in a Hilbert space $\mathfrak{H}$ with the same domain. If the operator $(H_2+i)^{-k}(H_2-H_1)(H_1+i)^{-k}$ is of trace class for some positive number k, then the complete wave operators $W_\pm(H_2, H_1)$ exist so that the absolutely continuous spectrum of H_1 and H_2 coincide with each other. (For the definition and natures of the wave operators, see e.g. [2], Chapter X.)

Proof of Corollary 2. Let $V=V_1+V_2$ satisfy the assumptions of this corollary. First we note that we may assume without loss of generality that the function V_1 satisfies the assumptions of Corollary 1 with V replaced by V_1. Indeed, let $h_1(r)$ and $h_2(r)$ be sufficiently smooth real functions of $r\geqq 0$ such that $h_1(r)=0$ $(r\leqq R)$; $=1$ $(r\geqq 2R)$, and $h_2(r)=1$ $\left(r\leqq\frac{1}{2}R\right)$, $h_2'(r)<0$ $\left(\frac{1}{2}R<r\leqq 3R\right)$ and $h_2(r)=e^{-r}$ $(r\geqq 3R)$, where R is taken sufficiently large $\left(R>\frac{1}{3}\right)$ so that the assmption (ii) of this corollary is satisfied. Put

$$\bar{V}_1=h_1V_1+c\,h_2, \quad \bar{V}_2=V_2+(1-h_1)V_1-c\,h_2.$$

Then the assmptions of Corollary 2 with V_1 and V_2 replaced by $\bar{V}_1$ and $\bar{V}_2$, respectively, are satisfied and those of Theorem 1 with V replaced by $\bar{V}_1$ are also satisfied for sufficiently large c.

Put $H_1=H_0+V_1$ and $H_2=H_0+V$. Then by virtue of the theorem of Birman stated above, it is sufficient to show that the operator $(H_2+i)^{-k}\,V_2\,(H_1+i)^{-k}$ is of trace class for some k since H_1 is absolutely continuous with spectrum $[0, \infty)$ by Corollary 1.

We denote by $\hat{u}$ the Fourier transform of u. Since

$$((H_0+i)^{-k}u)\,(\xi)=(|\xi|^2+i)^{-k}\hat{u}(\xi),$$

we have

$$((H_0+i)^{-k}\,u)(x)=(2\pi)^{-n/2}\int\exp\,(i\xi x)\hat{u}(\xi)\,(|\xi|^2+i)^{-k}\,d\xi.$$

Let k be the integer in the assumption (i), that is, $k>n/4$. Then $\exp(i\xi x)(|\xi|^2+i)^{-k}\in L_2$ so that we can apply the Parseval formula with the result that

$$((H_0+i)^{-k}u)\,(x)=\int K(x-y)\,u(y)\,dy,$$

where $K(x)=(2\pi)^{-n/2}\int\exp\,(i\xi x)\,(|\xi|^2+i)^{-k}\,d\xi\in L_2$. Thus the operator $V_2^{\frac{1}{2}}(H_0+i)^{-k}$ is of the Hilbert-Schmidt class with the Hilbert-Schmidt norm $||V_2||_{L_1}\,||K||_{L_2}$ since $V_2\in L_1$ by the assumption (iv).

As will be shown later (in Lemma 4), the operators $(H_0+i)^k(H_1+i)^{-k}$ and $(H_0+i)^k(H_2-i)^{-k}$ are bounded. Thus the operators

$$V_2^{\frac{1}{2}}(H_1+i)^{-k}=V_2^{\frac{1}{2}}(H_0+i)^{-k}(H_0+i)^k(H_1+i)^{-k}$$

and

$$(H_2+i)^{-k}V_2^{\frac{1}{2}}\subset(V_2^{\frac{1}{2}}(H_2-i)^{-k})^*=(V_2^{\frac{1}{2}}(H_0+i)^{-k}(H_0+i)^k(H_2-i)^{-k})^*$$

are of the Hilbert-Schmidt class so that $(H_2+i)^{-k}\,V_2(H_1+i)^{-k}$ is of trace class. Thus we can complete the proof of Corollary 2 if we prove the following

Lemma 4. *Let* $V\in\mathcal{E}^{2(k-1)}_{Q_\alpha}$ *and* $H=H_0+V$. *Then the operators* $(H_0+i)^k(H\pm i)^{-k}$ *are bounded.*

Before proving this lemma, we prepare the following

Lemma 5. *Let* $V_i\in\mathcal{E}^{2(i-1)}_{Q_\alpha}$ *and* $\phi\in D(H_0^k)=\mathcal{E}^{2k}_{L^2}$. *Then for any* $\varepsilon>0$ *there exists a constant* C_ε *such that the inequality*

$$\|(\prod_{i=1}^{k} V_i)\phi\|\leqq\varepsilon\|H_0^k\phi\|+C_\varepsilon\|\phi\| \tag{18}$$

holds.

Proof. In case $k=1$, the inequality (18) is obvious by the inequality (3) and the assertion just after it. Let $\phi\in D(H_0^k)$. Since

$$(-\Delta)^{k-1}V_k\phi=\sum_{\substack{|\alpha|+|\beta|\leqq 2(k-1)\\ |\beta|\neq 2(k-1)}}C_{\alpha,\beta}(D^\alpha V_k)\,(D^\beta\phi)+V_kH_0^{k-1}\phi,$$

$$\|(D^\alpha V_k)(D^\beta\phi)\|\leqq \text{const}\,(\|H_0D^\beta\phi\|+\|\phi\|)$$

$$(|\alpha|\leqq 2(k-1),\ |\beta|<2(k-1))$$

and

$$\|V_kH_0^{k-1}\phi\|\leqq\varepsilon\|H_0^k\,\phi\|+C_\varepsilon\|H_0^{k-1}\,\phi\|,$$

by virtue of (18) with $k=1$, we have that $V_k\,\phi\in D(H_0^{k-1})$

and

$$\|H_0^{k-1}V_k\,\phi\| \leqq \varepsilon\|H_0^k\,\phi\| + \sum_{|\beta|\leqq 2k-1} C_\varepsilon\|D^\beta\phi\|. \tag{19}$$

Now we assume that the lemma holds with $k=1, 2, \cdots, k-1$ by the assumption of induction. Since $V_k\,\phi \in D(H_0^{k-1})$, we have

$$\begin{aligned}\|(\prod_{i=1}^{k} V_i)\,\phi\| &= \|(\prod_{i=1}^{k-1} V_i)V_k\,\phi\| \leqq \varepsilon\|H_0^{k-1}V_k\,\phi\| + C_\varepsilon\|V_k\,\phi\| \\ &\leqq \varepsilon\|H_0^{k-1}V_k\,\phi\| + C_\varepsilon(\|H_0\,\phi\| + \|\phi\|),\end{aligned} \tag{20}$$

by (18) with $k=k-1$ and $k=1$. The well known inequality

$$\|D^\beta\phi\| \leqq \varepsilon\|H_0^k\,\phi\| + C_\varepsilon\|\phi\| \quad (|\beta| \leqq 2k-1) \tag{21}$$

and the inequalities (19) and (20) show that (18) holds with $k=k$, which yields the result by the induction method.

Proof of Lemma 4. Let $\phi \in C_0^\infty$. Then we have

$$(-\Delta+V)^k\,\phi = \sum_{\substack{j\leqq k\\ \Sigma|\alpha_i|+|\beta|=2(k-j)}} C_{\alpha,\beta}\,(\prod_{i=1}^{j} D^{\alpha_i}V)\,D^\beta\phi = W\phi + H_0^k\,\phi,$$

where

$$W\phi = \sum_{\substack{0<j\leqq k\\ \Sigma|\alpha_i|+|\beta|=2(k-j)\\ \Sigma|\alpha_i|\neq 0}} C_{\alpha,\beta}(\prod_{i=1}^{j} D^{\alpha_i}V)D^\beta\phi + \sum_{j=1}^{k} V^jH_0^{k-j}\,\phi,$$

and α_j and β are multi-indices. Since

$$D^{\alpha_i}V \in \mathcal{E}_{Q_\alpha}^{2(k-1)-|\alpha_i|} \subset \mathcal{E}_{Q_\alpha}^{2(k-1)-\Sigma|\alpha_i|} = \mathcal{E}_{Q_\alpha}^{2(j-1)+|\beta|} \subset \mathcal{E}_{Q_\alpha}^{2(j-1)},$$

using (18) with $k=j$, we have

$$\|(\prod_{i=1}^{j} D^{\alpha_i}V)\,(D^\beta\phi)\| \leqq \varepsilon\|H_0^jD^\beta\phi\| + C_\alpha\|D^\beta\phi\| \qquad (2j+|\beta| \leqq 2k-1),$$

and

$$\|V^jH_0^{k-j}\,\phi\| \leqq \varepsilon\|H_0^j\,H_0^{k-j}\,\phi\| + C_\varepsilon\|H_0^{k-j}\,\phi\| \quad (1 \leqq j \leqq k).$$

By virtue of the inequality (21), we have

(22) $$\|W\phi\| \leqq \varepsilon\|H_0^k \phi\| + C_\varepsilon\|\phi\| \qquad \text{for } \phi \in C_0^\infty.$$

Since $H^k\phi = H_0^k \phi + W\phi$ for $\phi \in C_0^\infty$ and (22) holds, it holds that $D(H^k) = D(H_0^k)$ and

(23) $$\|H_0^k \phi\| + \|\phi\| \leqq \text{const } (\|H^k\phi\| + \|\phi\|), \quad \phi \in D(H_0^k),$$

by virtue of the assertion just after the inequality (3).

The well known inequalities

$$\|(H_0 + i)^k \phi\| \leqq \text{const } (\|H_0^k \phi\| + \|\phi\|)$$

and

$$\|H^k \phi\| + \|\phi\| \leqq \text{const } \|(H \pm i)^k \phi\|$$

and the inequality (23) show that

$$\|(H_0 + i)^k \phi\| \leqq \text{const } \|(H \pm i)^k \phi\|,$$

which shows that the operators $(H_0 + i)^k (H \pm i)^{-k}$ are bounded.

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