Publ. RIMS, Kyoto Univ.
8 (1972/73), 111-116

On the Limit for the Representation by the Sum of Two Abundant Numbers

By

Sin HITOTUMATU

1. Introduction

For a positive integer n, let us denote by $S(n)$ the sum of all divisors of n including 1 and n itself. If $S(n)>2n$, $=2n$, or $<2n$, n is called the *abundant*, *perfect*, or *deficient* number, respectively. We shall denote by $\mathbf{S}$ the set of all abundant numbers. It has long been known that the natural density $\delta(\mathbf{S})$ of the set $\mathbf{S}$ exists and satisfies

$$0.241<\delta(\mathbf{S})<0.314,$$

and $\mathbf{S}$ constitutes an asymptotic basis of finite order, namely, every sufficiently large integer is a sum of a bounded number of abundant numbers.[1] Here, the abundant numbers are so defined as to include the perfect numbers, without affecting, however, the density of the set of abundant numbers.

Now, L. Moser has proved[2] that every sufficiently large integer $n>n_0$ can be represented as the sum of *two* abundant numbers, viz.

$$n=x+y, \quad x, y\in\mathbf{S}.$$

The limit may be taken, for example, $n_0=83160$. For simplicity, we put

$$\mathbf{M}=\{n \mid n=x+y, \quad x, y\in\mathbf{S}\}.$$

Received October 22, 1971.

1) cf. [3], Satz 9, p. 20. The author wishes to express his gratitude for the referee who has indicated this fact.

2) The author knows the result in the book by Ogilvy [1].

According to Ogilvy's book [1], the precise limit n_0

$$\min[n_0 \mid \{n > n_0\} \subset \mathbf{M}]$$

is not known. In the present paper, the author would like to show that the precise limit n_0 is 20161.

Though some part of the proof is due to a systematic search by computer, careful conversational considerations were necessary during the search,[3] so that, I hope, it will be worthwhile to report the present result.

2. Elementary Properties of Abundant Number

It is well known that if n is decomposed into prime factors

(1) $$n = p_1^{e_1} \cdots p_m^{e_m} \quad (p_i\text{'s are all distinct})$$

then we have

(2) $$\frac{S(n)}{n} = \prod_{i=1}^{m}\left(1 + \frac{1}{p_i} + \cdots + \frac{1}{p_i^{e_i}}\right).$$

Hence, if $n \in \mathbf{S}$, all multiple of n is again in $\mathbf{S}$. If n is a perfect number, then all its multiple other than n itself is in $\mathbf{S}$. Further, if p is a prime and not a Mersenne number, $2^{m-1}p \in \mathbf{S}$, provided that $2^m > p$. For example, every multiple of 6 except 6, every multiple of 28 except 28, and every multiple of 20, 88, $\cdots$ are all abundant numbers.

For even integers, the limit in the Moser's result is very simple. Every even integer $n \geqq 48$ can be represented by one of the forms

$$12 + 6m, \quad 20 + 6m, \quad \text{or} \quad 40 + 6m \qquad (m \geqq 2)$$

and hence it is in $\mathbf{M}$. On the other hand, 46 is not in $\mathbf{M}$.

The least odd abundant number is $945 = 3^3 \cdot 5 \cdot 7$, and the next smallest one is $1575 = 3^2 \cdot 5^2 \cdot 7$. Combining the multiples of 6, 20 or 28 (except 6

---

3) The main part of the present paper has been published in [2], in Japanese.

and 28 itself) with them, it is easy to see that every integer n divisible by 3, 5, or 7 is in **M** if $n>951$, 1555, or 1603 respectively, and that these are the precise limit for the multiples of 3, 5, or 7 respectively.

3. A Reduction of the Limit

Moser's limit 83160 is equal to 945×88; here 945 is the least odd abundant number, and 88 is the least abundant number coprime to it. Hence every integer $n>83160$ can be written in the form

$$n=945l+88m \qquad (l,\ m>0) \tag{3}$$

and both terms in (3) are in **S**.

Now, let us remark that though $315=3^2\cdot 5\cdot 7$ itself is a deficient number, it is quite close to **S**, say

$$\frac{S(315)}{315}=2\left(1-\frac{1}{105}\right).$$

It is easy to see that $315\,l\in S$ provided that l contains at least one prime factor less than 105, and especially for $l=2,\cdots,89$. Therefore, every integer n written in the form

$$n=315\,l+88\,m \qquad (l=2,\ \cdots,\ 89;\ m>0) \tag{4}$$

is in **M**, and this is true for n greater than

$$315\times 89+88=28123. \tag{5}$$

Thus we could reduce the limit quite a lot.

4. Determination of the Precise Limit

After the limit is reduced to (5), our main interest is restricted to odd integers n less than (5) which is not a multiple of 3, 5, or 7 and cannot be represented as (4). They are the numbers of the form

$315l-88m$ ($l=89, 87, \cdots$ (odd); $m>0$). We fix m and look for the smallest odd l_0 such that

$$t_0 = 315(l_0-3)-88m \in \mathbf{S}. \tag{6}$$

If we can find such $l_0<89$, then every $n=315l-88m$ ($l_0 \leqq l \leqq 89$, l odd) may be represented as

$$n = t_0 + 315(l-l_0+3) \in \mathbf{M}. \tag{7}$$

The results by computer[4] for the search of l_0 is given in Table 1. Here we have omitted the case when m is divisible by 3, 5, or 7, since the precise limit may be estimated about 20000, and we have already seen that the multiples of 3, 5, or 7 in such region are all in **M**. In the Table, u is the limit $315\times 89-88m$, and — means that there is no l_0 with (6) in the region $l_0 \leqq 89$.

From Table 1, the only exceptional cases are $m=16$ and 68. For such m, $n=315l-88m$ cannot be represented in the form $315\times s+x$ (s odd, $x\in \mathbf{S}$). It is not difficult to see that all odd abundant numbers s less than $26627=315\times 89-88\times 16$ and not a multiple of 315 are given in Table 2.

We check for each $n=315l-88m$ ($l=89, 87, \cdots$ (odd); $m=16, 68$) whether it is represented as $n=s+t$, where s is in Table 2, and t is an even abundant number. The results for a possible decomposition are given in Table 3; — means that no such representation is possible. Through the considerations up to here, we can conclude that 20161 is the greatest number not in **M**. Further, we have verified that only three numbers, say

20161, 19067, 18437

are not in **M** in the region $n \geqq 18000$.

By similar check, I saw that the next largest integers not in **M** are

17891, 17261

4) In my experience, in the computation of $S(n)$, it was faster to divide n by 2, 3, 4, $\cdots$ up to $\sqrt{n}$ and add the divisor and the quotient when it divides n, than to decompose n into prime factors and use the formula (2).

both correspond to $m=58$ in Table 1, and these are only integers not in $\mathbf{M}$ in the region $n \geqq 17000$.

Table 1

u = upper bound $315 \times 89 - 88\,m$

$t_0 = 315(l_0 - 3) - 88m \in \mathbf{S}$

m	u	l_0	t_0	m	u	l_0	t_0
1	27947	11	2432	61	22667	35	4712
2	27859	19	4864	62	22579	35	4624
4	27683	27	7208	64	22403	43	6968
8	27331	47	13156	67	22139	27	1664
11	27067	27	6592	68	22051	—	—
13	26191	11	1376	71	21787	27	1312
16	26627	—	—	73	21611	47	7436
17	26539	51	13624	74	21523	59	11128
19	26363	27	5888	76	21347	35	3392
22	26099	27	5624	79	21083	27	608
23	26011	19	3016	82	20819	51	7904
26	25747	19	2752	83	20731	59	10336
29	25483	59	15088	86	20467	43	5032
31	25307	47	11132	88	20291	67	12416
32	26219	59	14824	89	20203	75	14848
34	25043	47	10868	92	19939	35	1984
37	24779	43	9344	94	19763	51	6848
38	24691	19	1696	97	19499	75	14144
41	24427	27	3952	101	19147	43	3712
43	24251	27	3776	103	18971	43	3536
44	24163	35	6208	104	18883	35	928
46	23987	19	992	106	18707	83	15872
47	23899	43	8464	107	18619	51	5704
52	23459	19	464	109	18443	43	3008
53	23371	43	7936	113	18091	59	7696
58	22931	75	17576	116	17727	43	2392
59	22843	27	2368				

Table 2

s	prime factors	s	prime factors
5775	$3\cdot5^2\cdot7\cdot11$	15015	$3\cdot5\cdot7\cdot11\cdot13$
6435	$3^2\cdot5\cdot11\cdot13$	19305	$3^3\cdot5\cdot11\cdot13$
6825	$3\cdot5^2\cdot7\cdot13$	19635	$3\cdot5\cdot7\cdot11\cdot17$
7425	$3^3\cdot5^2\cdot11$	21945	$3\cdot5\cdot7\cdot11\cdot19$
8085	$3\cdot5\cdot7^2\cdot11$	22275	$3^4\cdot5^2\cdot11$
8415	$3^2\cdot5\cdot11\cdot17$	23205	$3\cdot5\cdot7\cdot13\cdot17$
8925	$3\cdot5^2\cdot7\cdot17$	25245	$3^3\cdot5\cdot11\cdot17$
9555	$3\cdot5\cdot7^2\cdot13$	25935	$3\cdot5\cdot7\cdot13\cdot19$
12705	$3\cdot5\cdot7\cdot11^2$	26565	$3\cdot5\cdot7\cdot11\cdot23$

Table 3

	$m=16$		$m=68$	
l	n	decomposition	n	decomposition
89	26627	9555+17072	22051	6435+15616
87	25997	8925+17072	21421	8925+12496
85	25367	8415+16952	20791	8415+12376
83	24737	12705+12032	20161	—
81	24107	19635+ 4472	19531	8415+11116
79	23477	8085+15392	18901	8085+10816
77	22847	5775+17072	18271	5775+12496
75	22217	6825+15392	17641	6825+10816
73	21587	9555+12032	17011	8415+ 8596
71	20957	8925+12032	16381	8085+ 8296
69	20327	5775+14552	....	
67	19697	7425+12272		
65	19067	—		
63	18437	—		
61	17807	5775+12032		
59	17177	12705+ 4472		
	....			

References

[1] Ogilvy, C. S., Tomorrow's Math.: Unsolved problems for the amateur, Oxford Univ. Press, 1962.

[2] Hitotumatu, S., On the Moser's problem to represent an integer by the sum of two abundant numbers (in Japanese), *Sugaku* **24**, No. 2, 1972.

[3] Ostmann, H.-H., Additive Zahlentheorie, II, *Erg. d. Math. u. ihrer Grenzgeb. N.F.* **11**, Springer-Verlag, 1956.