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Invariant Measure of the Infinite Dimensional Rotation Group

By

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§1. Preliminary Discussions

Let H be a real Hilbert space, and $O(H)$ be its rotation group, namely the group of all orthogonal operators of H. We intend to construct an invariant probability measure on $O(H)$. However, such measure does not exist as proved below.

Let $\{e_n\}$ be a CONS (=complete orthonormal system) of H, and α_{nm} be the mean of $(Ue_n, e_m)^2$ with respect to a measure μ on $O(H)$, namely

$$\alpha_{nm}=\int(Ue_n, e_m)^2\, d\mu(U).$$

If μ is left invariant, for any orthogonal operator U_0 we have

$$\alpha_{nm}=\int(U_0 Ue_n, e_m)^2\, d\mu(U)=\int(Ue_n, U_0^* e_m)^2\, d\mu(U),$$

hence especially we have $\alpha_{nm}=\alpha_{n1}$, therefore α_{nm} does not depend on m. Thus $\sum_{m=1}^{\infty}\alpha_{nm}$ must be 0 or ∞, according to $\alpha_{n1}=0$ or >0.

On the other hand, for any $U\in O(H)$ we have $\sum_{m=1}^{\infty}(Ue_n, e_m)^2=||Ue_n||^2=1$, so that integrating both hand sides with respect to μ we get $\sum_{m=1}^{\infty}\alpha_{nm}=1$, which is impossible. Therefore, left invariant measure can not exist on $O(H)$.

In a similar but rather complicated way, we can prove that $O(H)$-invariant measure does not exist on $\mathcal{L}(H)$, the set of all linear continuous

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operators of H, except the Dirac measure concentrated on zero operator. The latter is trivially $O(H)$-invariant.

Hence, in order to construct an $O(H)$-invariant measure, we must extend the group $O(H)$ to some space which is larger enough than $\mathcal{L}(H)$.

§2. Gaussian Measure

If we fix a CONS $\{e_n\}$ of H, H can be identified with (l^2), the space of all square summable sequences. The space (l^2) containes R_0^∞, and is contained in R^∞, where $R_0^\infty=\{(x_1, x_2, \cdots) | {}^\exists n, x_{n+1}=x_{n+2}=\cdots=0\}$ and R^∞ is the space of all sequences.

Any linear operator A from R_0^∞ to R^∞ is determined uniquely from the double sequence (a_{nm}) by the relation:

$$e_n=(0, 0, \cdots, 1, 0, \cdots) \xrightarrow{A} (a_{n1}, a_{n2}, \cdots, a_{nm}, \cdots)$$

Thus the set $\mathcal{L}(R_0^\infty, R^\infty)$ of all linear operators from R_0^∞ to R^∞ can be identified with $R^{\infty\infty}$, the space of all double sequences. However, since $R^{\infty\infty}$ is isomorphic with R^∞, we can consider Gaussian measure with variance 1 on the space $R^{\infty\infty}$. This is a measure on $R^{\infty\infty}$ such that for any finite number of $a_{n_1m_1}, \cdots, a_{n_km_k}$, their joint distribution becomes k-dimensional Gaussian measure with unit variance.

Let $O_0(H)$ be the group of such orthogonal operators of H that keep invariant R_0^∞. If $U\in O_0(H)$, U can be considered as an operator on R_0^∞, and its adjoint operator U^* can be considered as an operator on R^∞. In this sense, the above constructed Gaussian measure is $O_0(H)$-invariant as proved below.

If $A\in\mathcal{L}(R_0^\infty, R^\infty)$ corresponds to (a_{nm}), for any $U\in O_0(H)$ AU corresponds to $(\sum_{k=1}^{\infty} u_{nk}a_{mk})$ and U^*A corresponds to $(\sum_{k=1}^{\infty} a_{nk}u_{mk})$ where $Ue_n=\sum_{k=1}^{\infty} u_{nk}e_k$ which is actually a finite sum. Thus, multiplication of U from right or of U^* from left induces a rotation of row or column vectors of (a_{nm}). Since Gaussian measure is rotationally invariant, such transformation keeps it invariant.

Hereafter, Gaussian measure with variance 1 on $R^{\infty\infty}$ will be denoted with g.

§ 3. Other Invariant Measures

Consider a triplet $E_1 \subset H \subset E_2$ of separable Hilbert spaces where E_1 is continuously and densely imbedded in H while H is continuously and densely imbedded in E_2. Let $\mathcal{L}(E_1, E_2)$ be the set of all linear continuous operators from E_1 to E_2. It contains $\mathcal{L}(H) = \mathcal{L}(H, H)$.

Let $O_1(H)$ be the group of such orthogonal operators of H that keep E_1 invariant and act homeomorphically on E_1, and $O_2(H)$ be the group of such orthogonal operators of H that can be extended to a homeomorphic operator on E_2. We intend to construct a measure on $\mathcal{L}(E_1, E_2)$ which is right invariant with respect to $O_1(H)$ and left invariant with respect to $O_2(H)$.

Let $\{e_n\}$ be a CONS of H such that $e_n \in E_1$ for any n. Identifying H with H^* (=the dual space of H), the space E_2^* can be continuously and densely imbedded in H. Let $\{e_n'\}$ be another CONS of H such that $e_n' \in E_2^*$ for any n.

Now, for any $A \in \mathcal{L}(E_1, E_2)$, Ae_n belongs to E_2. So that the mapping $A \to Ae_n$ is a mapping from $\mathcal{L}(E_1, E_2)$ to E_2. Therefore any given measure μ on $\mathcal{L}(E_1, E_2)$ induces a measure μ_n on E_2. Namely, for any measurable set B (=the set which belongs to the smallest σ-field which contains all Borel cylinders) of E_2,

$$\mu_n(B) = \mu(\{A;\ Ae_n \in B\}).$$

(We suppose that μ is defined on the smallest σ-field of $\mathcal{L}(E_1, E_2)$ which makes (Ax, ξ) measurable for any $x \in E_1$, $\xi \in E_2^*$.)

For any $U \in O_2(H)$, $UAe_n \in B$ is equivalent with $Ae_n \in U^{-1}B$. Therefore if μ is left $O_2(H)$-invariant, μ_n is also an $O_2(H)$-invariant measure on E_2. Hence, μ_n must be a superposition of Gaussian measures with different variances and E_2 must be a nuclear extension of H.[1][2] (Except the trivial case of Dirac measure.)

On the other hand, for any $A\in\mathscr{L}(E_1, E_2)$, $A^*e'_n$ belongs to E_1^*. Hence, from the measure μ the mapping $A\to A^*e'_n$ induces a measure μ'_n on E_1^*. For any $U\in O_1(H)$, $(AU)^*e'_n\in B$ is equivalent with $A^*e'_n \in U^{*-1}B$. Therefore if μ is right $O_1(H)$-invariant, μ'_n is also an $O_1(H)$-invariant measure on E_1^*. Hence, μ'_n must be a superposition of Gaussian measures with different variances and E_1 must be nuclearly imbedded in H.

Theorem. *If any infinite dimensional rotationally invariant measure exists on $\mathscr{L}(E_1, E_2)$ (except Dirac measure), E_2 must be a nuclear extension of H, and E_1 must be nuclearly imbedded in H.*

The converse is also true.

We shall prove the converse. Let $E_1\subset H\subset E_2$ be a nuclearly imbedded triplet. We choose CONS $\{e_n\}$ of H in E_1 and $\{e'_n\}$ in E_2^*.

Any $A\in\mathscr{L}(E_1, E_2)$ is uniquely determined from the double sequence (a_{nm}) by the relation $a_{nm}=(Ae_n, e'_m)$. Hence $\mathscr{L}(E_1, E_2)$ can be identified with a subspace of $R^{\infty\infty}$. Consider the Gaussian measure g on $R^{\infty\infty}$. If $g(\mathscr{L}(E_1, E_2))=1$, g can be identified with a measure on $\mathscr{L}(E_1, E_2)$. Right $O_1(H)$-invariance and left $O_2(H)$-invariance of this measure are easily checked.

Let $\mathscr{H}(E_1, E_2)$ be the set of all Hilbert-Schmidt operators from E_1 to E_2, namely

$$A\in\mathscr{H}(E_1, E_2) \Leftrightarrow \sum_{n=1}^{\infty}\|Af_n\|_{E_2}^2<\infty \qquad \text{for CONS } \{f_n\} \text{ of } E_1.$$

Since $\mathscr{H}(E_1, E_2)\subset\mathscr{L}(E_1, E_2)$, it is sufficient to prove $g(\mathscr{H}(E_1, E_2))=1$.

Without loss of generality, we can suppose that $\{e_n\}$ is a common orthogonal system of H and E_1 with $\|e_n\|_{E_1}=\alpha_n$, and $\{e'_n\}$ is a common orthogonal system of H and E_2^* with $\|e'_n\|_{E_2^*}=\beta_n$. From the assumption of nuclear imbedding, we get $\sum_{n=1}^{\infty}\frac{1}{\alpha_n^2}<\infty$ and $\sum_{n=1}^{\infty}\frac{1}{\beta_n^2}<\infty$. Now, $A\in\mathscr{H}(E_1, E_2)$ is equivalent with $\sum_{n=1}^{\infty}\frac{1}{\alpha_n^2}\|Ae_n\|_{E_2}^2<\infty$, hence with $\sum_{n,m=1}^{\infty}\frac{1}{\alpha_n^2}\left(Ae_n, \frac{e'_m}{\beta_m}\right)^2=\sum_{n,m=1}^{\infty}\frac{1}{\alpha_n^2}\frac{1}{\beta_m^2}a_{nm}^2<\infty$. Since $\sum_{n,m=1}^{\infty}\frac{1}{\alpha_n^2}\frac{1}{\beta_m^2}<\infty$, $\mathscr{H}(E_1, E_2)$ is identical with a nuclear extension of $(l^2)_2$, the space of square summable double sequences. So that the Gaussian measure g lies on $\mathcal{H}(E_1, E_2)$, namely $g(\mathcal{H}(E_1, E_2))=1$.

§4. Properties of the Gaussian Measure

Gaussian measure g on $\mathcal{L}(E_1, E_2)$ is right $O_1(H)$-invariant and left $O_2(H)$-invariant. Moreover;

1°) g is right $O_1(H)$-ergodic and left $O_2(H)$-ergodic.

For the proof of left $O_2(H)$-ergodicity, we shall show that for any bounded measurable function $f(A)$, the relation $f(UA)=f(A)$ for any $U\in O_2(H)$ implies $f(A)=$constant modulo g-null set.

Since we suppose that g is defined on the smallest σ-field that makes (Ax, ξ) measurable for $x\in E_1$ and $\xi\in E_2^*$, any bounded measurable function $f(A)$ can be approximated with a tame function. Namely, for given $\varepsilon>0$ there exist finite number of $x_1, x_2, \cdots, x_n\in E_1$ and $\xi_1, \xi_2, \cdots, \xi_n\in E_2^*$ and a function φ of n real variables such that

$$\int|f(A)-\varphi((Ax_1, \xi_1), \cdots, (Ax_n, \xi_n))|^2 dg(A)<\varepsilon.$$

Since g is left $O_2(H)$-invariant, if $f(UA)=f(A)$ we have

$$\int|f(A)-\varphi((Ax_1, U^*\xi_1), \cdots, (Ax_n, U^*\xi_n))|^2 dg(A)<\varepsilon.$$

Therefore we have

$$\int|\varphi((Ax_1, \xi_1), \cdots, (Ax_n, \xi_n)) \\ -\varphi((Ax_1, U^*\xi_1), \cdots, (Ax_n, U^*\xi_n))|^2 dg(A)<4\varepsilon.$$

If $U^*\xi_1, \cdots, U^*\xi_n$ are orthogonal with $\xi_1, \xi_2, \cdots, \xi_n$, the left hand side becomes

$$2\int|\varphi((Ax_1, \xi_1), \cdots, (Ax_n, \xi_n))-m|^2 dg(A),$$

where

$$m=\int\varphi((Ax_1, \xi_1), \cdots, (Ax_n, \xi_n)) dg(A).$$

Hence, $\varphi((Ax_1, \xi_1), \cdots, (Ax_n, \xi_n))$ is approximated with a constant function, so that $f(A)$ is approximated with a constant function. Therefore, letting $\varepsilon \to 0$, $f(A)$ must be a constant function.

Right $O_1(H)$-ergodicity of g is proved in a similar way.

2°) Since the variance of any matrix element of A is 1, we have

$$\int (Ax, \xi)^2 dg(A) = 1 \qquad \text{for } x \in E_1, \|x\| = 1 \quad \text{and} \quad \xi \in E_2^*, \|\xi\| = 1,$$

3°) If $x_1, x_2, \cdots, x_n \in E_1$ are mutually orthogonal in H, the distributions of $Ax_1, Ax_2, \cdots, Ax_n$ are mutually independent. Namely,

$$g(\{A;\ Ax_1 \in B_1, \cdots, Ax_n \in B_n\}) = \prod_{k=1}^{n} g(\{A;\ Ax_k \in B_k\}).$$

This result comes from the fact that using the matrix representation of A for suitable CONS, Ax_k corresponds to k-th row vector.

4°) Though the measure g can not be constructed on $O(H)$, the orthogonality of A is assured in the following sense. For any $x, y \in E_1$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (Ax, e_k')\,(Ay, e_k') = (x, y) \qquad \text{for } g\text{-almost all } A.$$

This can be regarded as the orthogonality in mean. The proof is obtained from the strong law of large numbers.

§5. Uniqueness

Let $E_1 \subset H \subset E_2$ be a nuclearly imbedded triplet of separable Hilbert spaces, and μ be a right $O_1(H)$-invariant and left $O_2(H)$-invariant measure on $\mathcal{L}(E_1, E_2)$. Let $\{e_n\}$ and $\{e_n'\}$ be CONS of H contained in E_1 and E_2^* respectively.

As mentioned in §3, the mapping $A \to Ae_n$ induces a measure μ_n on E_2. Since μ_n is $O_2(H)$-invariant, it is a superposition of Gaussian measures with different variances. If we assume that μ is left $O_2(H)$-ergodic, then μ_n is also $O_2(H)$-ergodic so that μ_n must be a single Gaussian measure with some variance. Since μ is right $O_1(H)$-invariant, measures μ_n do not depend on n. Therefore they are the same Gaussian measure.

The variance is determined as 1, if we assume that for any $x \in E_1$, $\|x\|=1$ and $\xi \in E_2^*$, $\|\xi\|=1$

$$\int (Ax, \xi)^2 \, d\mu(A)=1.$$

Moreover, assume that if $x_1, x_2, \cdots, x_n \in E_1$ are mutually orthogonal in H, the distributions of $Ax_1, Ax_2, \cdots, Ax_n$ are mutually independent. Then, the joint distribution of $Ae_1, Ae_2, \cdots, Ae_n$ is the direct product of the distributions of Ae_k, which are the Gaussian measure with unit variance. Thus, μ must be identical with the measure g on $\mathfrak{L}(E_1, E_2)$.

Theorem. *A measure μ on $\mathfrak{L}(E_1, E_2)$ is determined uniquely as g under the following three conditions.*

(1) *μ is right $O_1(H)$-invariant and left $O_2(H)$-invariant.*

(2) *μ is left $O_2(H)$-ergodic.*

or (2)′ *μ is right $O_1(H)$-ergodic.*

(3) *For any $x \in E_1$, $\|x\|=1$ and $\xi \in E_2^*$, $\|\xi\|=1$, we have*

$$\int (Ax, \xi)^2 \, d\mu(A)=1.$$

Remark that the mutually independence of $Ae_1, Ae_2, \cdots, Ae_n$ is not necessary for the uniqueness of the invariant measure. From the next section on, we shall prove the theorem.

§6. Characteristic Function

Consider the following characteristic function of μ.

$$\chi(T)=\int \exp[i \sum_{n,m} t_{nm}(Ae_n, e'_m)] \, d\mu(A),$$

where $T=(t_{nm})$ is a double sequence which vanishes except finite number of (n, m).

If μ is right $O_1(H)$-invarinat and left $O_2(H)$-invariant, $\chi(T)$ is invariant under any transformation of the form $T \to UTV$, where U and

V are orthogonal matrices which correspond to elements of $O_1(H)$ and $O_2(H)$ respectively.

The relation: $T \equiv T' \Leftrightarrow {}^{\exists}U, V;\ T' = UTV$ is evidently an equivalence relation, and $\chi(T)$ is constant on any equivalence class. We shall remark that T is equivalent with T' if and only if the eigenvalues of T^*T are identical with those of T'^*T'. Therefore any T is equivalent with a diagonal matrix Δ. Thus, we have

$$\chi(T) = \chi(\Delta) = \int \exp[i \sum_n \lambda_n(Ae_n, e_n')] d\mu(A),$$

where λ_n^2 are the eigenvalues of T^*T.

Put $\chi(T) = \varphi(\lambda_1^2, \lambda_2^2, \cdots, \lambda_n^2, \cdots)$. If we prove that $\varphi(\alpha_1, \alpha_2, \cdots) = \exp\left[-\frac{1}{2}\sum_n \alpha_n\right]$, then for any T we have $\chi(T) = \exp\left[-\frac{1}{2}\sum_{n,m} t_{nm}^2\right]$ because $\sum_{n,m} t_{nm}^2 = Tr(T^*T)$, thus μ must be the Gaussian measure with unit variance, namely $\mu = g$.

In order to determine the function $\varphi(\alpha)$, we need a lemma.

Let $[0, \infty)^\infty$ be the set of all non-negative sequences, and $[0, \infty)_0^\infty$ be a subspace of $[0, \infty)^\infty$ such that

$$\alpha = (\alpha_n) \in [0, \infty)_0^\infty \Leftrightarrow {}^{\exists}n;\ \alpha_{n+1} = \alpha_{n+2} = \cdots = 0.$$

Definition. *A real function $\varphi(\alpha)$ defined on $[0, \infty)_0^\infty$ is called completely monotonic if it satisfies;*

$$(-1)^m \Delta_{\alpha_1} \Delta_{\alpha_2} \cdots \Delta_{\alpha_m} \varphi(\alpha) \geqq 0$$

for any $m = 0, 1, 2, \cdots$, ${}^{\forall}\alpha \in [0, \infty)_0^\infty$, ${}^{\forall}\alpha_k \in [0, \infty)_0^\infty$, where $\Delta_{\alpha_1}\varphi(\alpha) = \varphi(\alpha + \alpha_1) - \varphi(\alpha)$.

Lemma (*Infinite dimensional Bernstein's theorem*). *Let $\varphi(\alpha)$ be completely monotonic on $[0, \infty)_0^\infty$ and right continuous at $\alpha = 0$. Then, there exists uniquely a finite measure $m(\beta)$ on $[0, \infty)^\infty$ such that*

$$\varphi(\alpha) = \int \exp[-\sum_{n=1}^\infty \alpha_n \beta_n] dm(\beta) \qquad \text{for } {}^{\forall}\alpha \in [0, \infty)_0^\infty.$$

The proof, omitted here, is obtained from the corresponding theorem for finite dimensional case.

For a while, we assume the complete monotonicity of $\varphi(\alpha)$, which we shall prove later. Then from the lemma, there exists a corresponding measure $m(\beta)$ on $[0, \infty)^{\infty}$.

As mentioned in §5, under the conditions (1), (2) and (3) of §5, the measure μ_1 of Ae_1 is a Gaussian measure with variance 1. Therefore we have

$$\int \exp[i\lambda_1(Ae_1, e_1')]d\mu(A)=\int \exp[i\lambda_1(x, e_1')]d\mu_1(x)=e^{-\frac{1}{2}\lambda_1^2},$$

thus $\varphi(\alpha_1, 0, 0, \cdots, 0, \cdots)=e^{-\frac{1}{2}\alpha_1}$.

Comparing with the result of the lemma, this means $\beta_1=\dfrac{1}{2}$ for m-almost all β.

In the same way, we have $\beta_2=\beta_3=\cdots=\beta_n=\cdots=\dfrac{1}{2}$ for m-almost all β. Therefore we get

$$\varphi(\alpha)=\exp\left[-\frac{1}{2}\sum_n \alpha_n\right],$$

which was our final purpose.

§7. Complete Monotonicity

Let $R_0^{\infty\infty}$ be the set of infinite dimensional matrices which vanish except finite number of elements, and $\chi(T)=\varphi(\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots)$ be a function on $R_0^{\infty\infty}$ where α_k are eigenvalues of T^*T.

A finite subset $\{T_1, T_2, \cdots, T_n\}$ of $R_0^{\infty\infty}$ is called admissible if $(T_j-T_k)^*(T_j-T_k)$ is diagonal for any $1\leqq j, k\leqq n$.

Proposition. *If $\varphi(\alpha)$ is positive definite for any admissible subset, then*

a) $\varphi(\alpha)\geqq 0$ *for* $\forall\alpha\in[0, \infty)_0^{\infty}$,

b) $\Delta_{\alpha_1}\varphi(\alpha)$ *is negative definite for any admissible subset.*

The complete monotonicity of $\varphi(\alpha)$ is easily derived from this proposition.

Proof of a). For $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_m, 0, \cdots)$, put $T_k=(t_{pq}^{(k)})$ where $t_{pq}^{(k)}=\sqrt{\alpha_q}\,\delta_{p-km,q}$. Then $(T_j-T_k)^*(T_j-T_k)$ is a diagonal matrix whose (p, p) element is $2\alpha_p$. From the positive definiteness, we have

$$0\leqq \sum_{j,k}\chi(T_j-T_k)=n\varphi(0)+n(n-1)\varphi(2\alpha),$$

hence $\varphi(2\alpha)\geqq-\dfrac{1}{n-1}\varphi(0)$. Letting $n\to\infty$, we have $\varphi(2\alpha)\geqq 0$.

Proof of b). We shall prove that for any admissible set $\{T_k\}$ and any complex numbers $\{\lambda_k\}$, we have $\sum\lambda_j\bar{\lambda}_k\Delta_{\alpha_1}\chi(T_j-T_k)\leqq 0$.

We can find such T_0 that satisfies $T_0{}^*T_k=T_k{}^*T_0=0$ for any $1\leqq k\leqq n$, and $T_0{}^*T_0$ is a diagonal matrix whose diagonal elements are $\alpha_1=(\alpha_1^{(1)}, \alpha_2^{(1)}, \cdots)$.

Put

$$S_k=T_k, \quad \lambda_k'=\lambda_k \qquad \text{for } 1\leqq k\leqq n$$

and

$$S_k=T_{k-n}+T_0, \quad \lambda_k'=-\lambda_{k-n} \qquad \text{for } n+1\leqq k\leqq 2n.$$

Then, $\{S_k\}$ is an admissible set. From the positive definiteness of χ for $\{S_k\}$ we have

$$2\sum_{j,k=1}^{n}\lambda_j\bar{\lambda}_k\chi(T_j-T_k)-2\sum_{j,k=1}^{n}\lambda_j\bar{\lambda}_k\chi(T_j-T_k-T_0)\geqq 0.$$

However, the diagonal elements of $(T_j-T_k-T_0)^*(T_j-T_k-T_0)$ are evidently the corresponding diagonal elements of $(T_j-T_k)^*(T_j-T_k)$ plus α_1. Thus, we have proved b).

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