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Projective Limit of Haar Measures on $O(n)$

By

Yasuo YAMASAKI

Introduction

In this paper, we shall show that the Gaussian measure [1] on $R^{\infty\infty}$ is obtained as the projective limit of Haar measures on $O(n)$. This is a natural extension of the fact [2] [3]: the Gaussian measure on R^{∞} is obtained as the projective limit of the uniform measures on the n-dimensional spheres.

D. Shale [4] considered the family of Haar measures on $O(n)$ to construct a finitely additive measure on $O(\infty)$. But he did not treat the projective limit. Also a report by H. Shimomura [5] is useful for the information on this topic.

§1. Orthogonal Group $O(n)$

The n-dimensional orthogonal group $O(n)$ is the group of all orthogonal transformations of R^n. If we fix a C.O.N.S. (=complete orthonormal system) of R^n, it is identified with the group of all matrices (u_{ij}) which satisfy the orthogonality relations:

$$(1.1) \qquad \sum_{k=1}^{n} u_{ik} u_{jk} = \delta_{ij} \qquad (1 \leqq i, j \leqq n).$$

Because of (1.1), only $n(n-1)/2$ matrix elements are independent, and the other $n(n+1)/2$ matrix elements can be considered as functions of the formers.

Usually, $n(n-1)/2$ Euler angles are used as independent variables of

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$O(n)$, but in this paper, for the convenience of later analysis, we shall use another system of independent variables which we shall explain in the last part of this section.

Let S_{n-1} be the unit sphere of R^n;

$$S_{n-1}=\{(x_1, x_2, \cdots, x_n);\ x_1^2+x_2^2+\cdots+x_n^2=1\}. \tag{1.2}$$

The group $O(n)$ can be regarded as a transformation group of S_{n-1}. The group of all orthogonal transformations which keep the vector $(0, 0, \cdots, 0, 1)$ invariant, is isomorphic with $O(n-1)$, so we identify them. For any $U, V\in O(n)$, we have $UV^{-1}\in O(n-1)$, if and only if the last row vector of U is equal with that of V, namely

$$u_{nj}=v_{nj} \qquad (1\leqq j\leqq n). \tag{1.3}$$

Therefore, the coset space $O(n-1)\backslash O(n)$ is identified with S_{n-1}. Suppose that a mapping $S_{n-1}\ni x\rightarrow U_x\in O(n)$ is given such that the last row vector of U_x is just x. In other words, each U_x is a representative of the coset which corresponds to x. Then, any $U\in O(n)$ is written uniquely in the form:

$$U=\begin{pmatrix} U_1 & 0 \\ 0 & 1 \end{pmatrix} U_x \qquad U_1\in O(n-1),\ x\in S_{n-1}. \tag{1.4}$$

For $V\in O(n)$, the last row vector of U_xV is xV. So, if U is represented as (1.4), we have

$$\begin{aligned} UV&=\begin{pmatrix} U_1 & 0 \\ 0 & 1 \end{pmatrix} U_x V=\begin{pmatrix} U_1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} W & 0 \\ 0 & 1 \end{pmatrix} U_{xV} \\ &=\begin{pmatrix} U_1W & 0 \\ 0 & 1 \end{pmatrix} U_{xV} \qquad \text{for some } W\in O(n-1). \end{aligned} \tag{1.5}$$

Therefore, any multiplication from right on $O(n)$ induces (1) a multiplication from right on $O(n-1)$, and (2) an orthogonal transformation on S_{n-1}.

Consider the uniform measure on S_{n-1} and the Haar measure on $O(n-1)$. From the above discussion, we see that their product measure is just the Haar measure on $O(n)$, if we identify $U\in O(n)$ with (U_1, x) $\in O(n-1)\times S_{n-1}$. Here we assume that the mapping $x\to U_x$ is measurable, but this assumption is satisfied if U_x is defined and continuous except on some closed null set of S_{n-1}.

Now, we shall define concretely the mapping $x\to U_x$. Let $\{e_1, e_2, \cdots, e_n\}$ be the C.O.N.S. of R^n. If $x_n=<x, e_n>\neq 0$, the vectors $x, e_1, e_2, \cdots, e_{n-1}$ are linearly independent. Then, we adopt the Schmidt's orthonormalization of them as row vectors of U_x.

Explicitly writing, the matrix elements of U_x is as follows;

$$
(1.6)\quad \begin{cases} U_x=(u_{ij}) \\ \quad u_{nj}=x_j \\ \quad u_{ij}=0 \qquad \text{if} \quad j<i\leqq n-1 \\ \quad u_{ii}=\sqrt{x_{i+1}^2+\cdots+x_n^2}\,/\,\sqrt{x_i^2+\cdots+x_n^2} \qquad \text{if} \quad i\leqq n-1 \\ \quad u_{ij}=-x_ix_j/\sqrt{x_i^2+\cdots+x_n^2}\sqrt{x_{i+1}^2+\cdots+x_n^2} \qquad \text{if } i<j. \end{cases}
$$

Hereafter, let the mapping $x\to U_x$ be always the above one.

Since $O(n)\simeq O(n-1)\times S_{n-1}$, repeating the similar procedure, we have $O(n)\simeq S_1\times S_2\times\cdots\times S_{n-1}$. Then, the Haar measure on $O(n)$ is the product measure of uniform measures on S_k $(1\leqq k\leqq n-1)$. More exactly speaking, we can formulate as follows. Let φ_n be the mapping $O(n)\to O(n-1)$ such that

$$
(1.7)\qquad \varphi_n(U)=U_1 \qquad \text{where } U=\begin{pmatrix} U_1 & 0 \\ 0 & 1 \end{pmatrix} U_x.
$$

Then the mapping $\varphi_{k+1}\circ\varphi_{k+2}\circ\cdots\circ\varphi_n$ maps $O(n)$ to $O(k)$. Denote the matrix elements of the image matrix as $u_{ij}^{(k)}$ $(1\leqq i, j\leqq k)$. They are functions on $O(n)$.

If we adopt $u_{kj}^{(k)}$ $(1\leqq j\leqq k-1, 2\leqq k\leqq n)$ as $n(n-1)/2$ independent variables of $O(n)$, the Haar measure μ_n on $O(n)$ is represented as follows (except normalization constant):

(1.8) $$d\mu_n=\prod_{k=2}^{n}[\{1-\sum_{j=1}^{k-1}u_{kj}^{(k)2}\}^{-\frac{1}{2}}\prod_{j=1}^{k-1}du_{kj}^{(k)}],$$

the content of [ ] being the uniform measure on S_{k-1}.

§2. Projective Limit

Since the Haar measures μ_n satisfy

(2.1) $\mu_{n-1}(A)=\mu_n(\varphi_n^{-1}(A))$ for a Borel subset A of $O(n-1)$,

according to a theorem due to Bochner, we can construct the projective limit probability space $(\Omega, \mathscr{B}, \mu)$. It satisfies the following properties:

P1) $\Omega\subset\prod_{n=1}^{\infty}O(n)$

P2) $f_{n-1}=\varphi_n\circ f_n$. Here, f_n is the restriction of π_n on Ω, where π_n is the projection from $\prod_{n=1}^{\infty}O(n)$ onto $O(n)$.

P3) $\mathscr{B}$ is generated by $\bigcup_{n=1}^{\infty}f_n^{-1}(\mathscr{B}_n)$, where $\mathscr{B}_n$ is the whole of Borel subsets of $O(n)$.

P4) $\mu(f_n^{-1}(A))=\mu_n(A)$ for $A\in\mathscr{B}_n$.

The matrix elements $u_{ij}^{(k)}$ $(1\leqq i, j\leqq k)$ can be regarded as functions on Ω, as well as on $O(n)$ for $n\geqq k$. Then, for any $\omega\in\Omega$,

(2.2) $$f_k(\omega)=(u_{ij}^{(k)}(\omega))\in O(k).$$

$u_{ij}^{(k)}(\omega)$ is a measurable function on Ω because of P3).

Lemma 1.

(2.3) $$\int_\Omega u_{ij}^{(n)}(\omega)d\mu=0$$

(2.4) $$\int_\Omega u_{ij}^{(n)}(\omega)u_{pq}^{(m)}(\omega)d\mu=\delta_{ip}\delta_{jq}\frac{1}{m}\frac{\Gamma\left(\frac{n-j+1}{2}\right)\Gamma\left(\frac{m-j+2}{2}\right)}{\Gamma\left(\frac{n-j+2}{2}\right)\Gamma\left(\frac{m-j+1}{2}\right)}$$

for $m\leqq n$, where Γ is Gamma function.

Proof. From P4), we have

$$\int_{\Omega} u_{ij}^{(n)}(\omega)d\mu=\int_{O(n)} u_{ij}^{(n)}\, d\mu_n=0.$$

Similarly, using 2) also, we see that

$$\int_{\Omega} u_{ij}^{(n)}(\omega)u_{pq}^{(m)}(\omega)d\mu=\int_{O(n)} u_{ij}^{(n)} u_{pq}^{(m)}\, d\mu_n.$$

Substituting (1.8),

$$=\int_{O(m)} I_{ij}u_{pq}^{(m)}\, d\mu_m \tag{2.5}$$

where I_{ij}=integration of $u_{ij}^{(n)}$ by $\prod_{k=m}^{n-1}\int_{O(k)\setminus O(k+1)} dm_k$, m_k being the uniform measure on S_k.

Now, we shall calculate

$$I^{(n-1)}=\int_{O(n-1)\setminus O(n)} U^{(n)} dm_{n-1}.$$

Since $U^{(n)}=\begin{bmatrix} U^{(n-1)} & 0 \\ 0 & 1\end{bmatrix}U_x$, we have

$$I^{(n-1)}=\int_{S_{n-1}}\begin{bmatrix} U^{(n-1)} & 0 \\ 0 & 1\end{bmatrix}U_x\, dm_{n-1}=\begin{bmatrix} U^{(n-1)} & 0 \\ 0 & 1\end{bmatrix}\int_{S_{n-1}} U_x dm_{n-1}.$$

Using (1.6), we have

$$\int_{S_{n-1}} U_x dm_{n-1}=\begin{bmatrix} J_1^{(n-1)} & & & 0 \\ & J_2^{(n-1)} & & \\ & & \ddots & \\ & & & J_{n-1}^{(n-1)} & \\ 0 & & & & 0\end{bmatrix}$$

where $J_i^{(n-1)}=\int_{S_{n-1}}\frac{r_{n-i}}{r_{n-i+1}}dm_{n-1}=\int_0^{\pi}\sin^{n-i}\theta\, d\theta\Big/\int_0^{\pi}\sin^{n-i-1}\theta\, d\theta.$ Thus we have

$$I_{ij}=u_{ij}^{(m)}\prod_{k=m}^{n-1} J_j^{(k)}=u_{ij}^{(m)}\int_0^{\pi}\sin^{n-j}\theta\, d\theta\Big/\int_0^{\pi}\sin^{m-j}\theta\, d\theta \tag{2.6}$$

$$=u_{ij}^{(m)}\frac{\Gamma\left(\frac{n-j+1}{2}\right)}{\Gamma\left(\frac{n-j+2}{2}\right)}\frac{\Gamma\left(\frac{m-j+2}{2}\right)}{\Gamma\left(\frac{m-j+1}{2}\right)}.$$

On the other hand, since

$$\int_{O(m)} u_{ij}^{(m)} u_{pq}^{(m)} d\mu_m = \frac{1}{m}\delta_{ip}\delta_{jq},$$

we have (2.4) from (2.5) and (2.6).

Lemma 2. $\{\sqrt{n}\, u_{ij}^{(n)}(\omega);\ n \geqq \max(i, j)\}$ *forms a Cauchy sequence in* $L^2(\Omega, \mu)$. *The speed of convergence is dependent on* j, *but uniform in* i.

Proof. From (2.4), we have $\|\sqrt{n}\, u_{ij}^{(n)}\| = 1$, and

$$<\sqrt{n}\, u_{ij}^{(n)}, \sqrt{m}\, u_{ij}^{(m)}> = \sqrt{\frac{n}{m}}\,\frac{\Gamma\left(\frac{n-j+1}{2}\right)}{\Gamma\left(\frac{n-j+2}{2}\right)}\frac{\Gamma\left(\frac{m-j+2}{2}\right)}{\Gamma\left(\frac{m-j+1}{2}\right)}.$$

The latter tends to 1 as $n, m \to \infty$ because we have asymptotically

$$\frac{\Gamma\left(t+\frac{1}{2}\right)}{\Gamma(t)} \sim \sqrt{t}\,.$$

From Lemma 2, $\sqrt{n}\, u_{ij}^{(n)}$ converges to a function X_{ij} in $L^2(\Omega, \mu)$. Then, X_{ij} is defined for almost all ω, and some suitable subsequence of $\{\sqrt{n}\, u_{ij}^{(n)}(\omega)\}$ converges to $X_{ij}(\omega)$ almost everywhere. Evidently, we have

$$\text{(2.7)} \qquad \int_\Omega X_{ij}(\omega) d\mu = 0 \quad \text{and} \quad \int_\Omega X_{ij}(\omega) X_{pq}(\omega) d\mu = \delta_{ip}\delta_{jq}.$$

§3. Identification with the Gaussian Measure

Proposition 1. *For almost all* $\omega, \omega' \in \Omega$, *the following* (1) *and* (2) *are equivalent.*

(1) $X_{ij}(\omega)=X_{ij}(\omega')$ for any i, j.

(2) $u_{ij}^{(n)}(\omega)=u_{ij}^{(n)}(\omega')$ for any i, j, n where $n \geqq \max(i, j)$.

Proof. (2)⇒(1) is evident, because $X_{ij}(\omega)$ is the limit of some subsequence of $\{\sqrt{n}\, u_{ij}^{(n)}(\omega)\}$.

On the other hand, from the definition (1.6) of U_x, we see that for $U \in O(n)$, if $U=\begin{bmatrix} U_1 & 0 \\ 0 & 1 \end{bmatrix} U_x$, then the column vectors of U_1 is obtained by the Schmidt's orthonormalization of the projections to R^{n-1} of the column vectors of U. Therefore, if $m \leqq n$, the column vectors of $U^{(m)}$ is the Schmidt's orthonormalization of the projections to R^m of the column vectors of $U^{(n)}$, and in the limit of $n \to \infty$ (fixing m), $\sqrt{n}$ times of the matrix elements of $U^{(n)}$ tend to X_{ij}. So, $u_{ij}^{(m)}$ $(1 \leqq i, j \leqq m)$ is obtained by the Schmidt's procedure from X_{ij}. Q.E.D.

Consider the mapping ψ: $\omega \in \Omega \to (X_{ij}(\omega)) \in R^{\infty\infty}$, where $R^{\infty\infty}$ is the space of all double sequences. ψ is one-to-one excpet on a suitable null set of Ω, because $X_{ij}(\omega)=X_{ij}(\omega')$ for any i, j implies $u_{ij}^{(n)}(\omega)=u_{ij}^{(n)}(\omega')$ for any i, j, n, therefore $f_n(\omega)=f_n(\omega')$ for any n, so that $\omega=\omega'$.

Next, we shall discuss the measurability. From P3) of §2, the probability measure μ is defined on the smallest σ-ring $\mathscr{B}$ which makes all $u_{ij}^{(n)}(\omega)$ measurable. This is equivalent to say $\mathscr{B}$ is the smallest σ-ring which makes $X_{ij}(\omega)$ measurable as seen from the proof of the proposition 1. Therefore the image $\psi(\mathscr{B})$ is the smallest σ-ring which makes all projections $\alpha=(\alpha_{ij}) \in R^{\infty\infty} \to \alpha_{ij} \in R^1$ measurable. In other words, $\psi(\mathscr{B})$ is the smallest σ-ring which makes all Borel cylinder sets with the base in $R_0^{\infty\infty}$ measurable, where $R_0^{\infty\infty}$ is the space of all double sequences which vanish except for finite number of (i, j).

Finally, we shall show that the measure μ on Ω is mapped to the Gaussian measure g on $R^{\infty\infty}$.

For this purpose, we shall prove that

$$\int_{\Omega} \exp\Big[\sqrt{-1} \sum_{i,j} t_{ij} X_{ij}(\omega)\Big] d\mu = \exp\Big[-\frac{1}{2} \sum_{i,j} t_{ij}^2\Big], \tag{3.1}$$

where t_{ij} are arbitrary real numbers and the summation is carried out for $i+j\leqq k$.

Since $\sqrt{n}u_{ij}^{(n)}(\omega)$ tends to $X_{ij}(\omega)$ in $L^2(\Omega, \mu)$, the left side of (3.1) is approximated by

$$\int_{O(n)}\exp[\sqrt{-1}\sum_{i,j} t_{ij}\sqrt{n}\, u_{ij}^{(n)}]d\mu_n, \tag{3.2}$$

the error tending to 0 as $n\to\infty$.

The integral (3.2) is equal with

$$\int_{O(n)}\exp[\sqrt{-1}\sum_{i,j} t_{ij}\sqrt{n}\, u_{n-i,j}^{(n)}]d\mu_n, \tag{3.3}$$

because μ_n is the Haar measure on $O(n)$.

Since the convergence $\sqrt{n}\,u_{ij}^{(n)}\to X_{ij}$ is uniform in i, $\sqrt{n}\,u_{n-i,j}^{(n)}$ in the integrand of (3.3) can be replaced by $\sqrt{n-i}\,u_{n-i,j}^{(n-i)}$ with good approximation, if n is large enough and $i+j\leqq k$ for some fixed k. Namely, with small error the left side of (3.1) is approximated by

$$\int_{O(n)}\exp[\sqrt{-1}\sum_{i,j} t_{ij}\sqrt{n-i}\, u_{n-i,j}^{(n-i)}]d\mu_n.$$

Substituting (1.8), this quantity is equal except the normalization constant of μ_n with

$$\int\exp[\sqrt{-1}\sum_{i,j} t_{ij}\sqrt{n-i}\, u_{n-i,j}^{(n-i)}]\prod_{i=1}^{k}\left[\left\{1-\sum_{j=1}^{k-i}u_{n-i,j}^{(n-i)2}\right\}^{\frac{n-k-2}{2}}\prod_{j=1}^{k-i}du_{n-i,j}^{(n-i)}\right] \tag{3.4}$$

$$=\int\exp[\sqrt{-1}\sum_{i,j} t_{ij}\lambda_{ij}]\prod_{i=1}^{k}\left[\left\{1-\sum_{j=1}^{k-i}\frac{\lambda_{ij}^2}{n-i}\right\}^{\frac{n-k-2}{2}}\prod_{j=1}^{k-i}\frac{d\lambda_{ij}}{\sqrt{n-i}}\right].$$

However $\left\{1-\sum_{j=1}^{k-i}\frac{\lambda_{ij}^2}{n-i}\right\}^{\frac{n-k-2}{2}}$ converges to $\exp\left[-\frac{1}{2}\sum_{j=1}^{k-i}\lambda_{ij}^2\right]$ uniformly in λ_{ij} as $n\to\infty$. Thus, the integral (3.4) including the normalization constant converges to $\exp\left[-\frac{1}{2}\sum_{i,j} t_{ij}^2\right]$ as $n\to\infty$.

Thus, we have proved

Theorem. *The projective limit space* $(\Omega, \mathcal{B}, \mu)$ *is isomorphic with the Gaussian measure* g *on* $R^{\infty\infty}$. *Namely, there exists a measure-preserving one-to-one mapping* ψ *from a suitable subset* $\tilde{\Omega}$ *of* Ω *onto a suitable subset* $\tilde{R}^{\infty\infty}$ *of* $R^{\infty\infty}$ *where* $\mu(\tilde{\Omega})=g(\tilde{R}^{\infty\infty})=1$, *and* ψ *preserves matrix elements in some sense.*

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