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On a Nonlinear Bessel Equation*

By

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1. Introduction

In the theory of type II superconductivity A. A. Abrikosov ([1]) discovered in 1957 that the so-called Abrikosov's mixed state can be described as a special solution of Ginzburg-Landau equation the basic equation of the theory of superconductivity ([2]). Suppose that there exists a cylindrical superconductor of type II at temperature below its critical value T_c and there exists external magnetic field parallel to its axis of cylinder the strength of which is lower than upper critical field H_{c2}. Abrikosov's mixed state is the phenomenon that the magnetic flux penetrates the superconductor forming triangular lattices of flux lines and the fluxoid of each flux line is quantized. To describe one flux line Abrikosov derived from Ginzburg-Landau equation the singular boundary value problem for a nonlinear Bessel equation.

$$(1.1)\quad \begin{cases} -w''(r)-r^{-1}w'(r)+\nu^2r^{-2}w(r)=(1-w^2(r))w(r) & r\in(0,\infty) \\ w(0)=0, \quad w(\infty)=1\,. \end{cases}$$

Here r is the independent variable which means the distance from the center of flux line. $w(r)$ is the dependent variable called order parameter of Ginzburg-Landau which takes real value between 0 and 1 where $w(r)=0$ the sample is in normal state and where $w(r)=1$ the sample is in full superconducting state. $w'(r)$ means $\frac{d}{dr}w(r)$. In 1966 E. Abrahams and

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T. Tsuneto ([3]) derived simple nonlinear diffusion equation to describe space-time variation of Ginzburg-Landau order parameter $u(x, t)$ (complex valued) under suitable physical situation.

$$(1.2)\quad \frac{\partial}{\partial t}u(x, t)=\Delta u(x, t)+(1-|u(x, t)|^2)u(x, t)\quad (x, t)\in R^3\times(0, \infty).$$

(1.1) can be obtained from (1.2) through the separation of variable

$$(1.3)\qquad u(x, t)=u(x_1, x_2, x_3, t)=u(r\cos\theta, r\sin\theta, x_3, t)=w(r)e^{i\nu\theta}$$

by cylindrical coordinates in R^3. On the other hand in the theory of superfluidity in 1961 L.P. Pitaevskii ([4]) and in 1963 E. Gross ([5]) derived also (1.1) to describe vortex line in superfluid ${}_4He$. They start from the basic equation

$$(1.4)\quad i\frac{\partial}{\partial t}u(x, t)=-\Delta u(x, t)+|u(x, t)|^2u(x, t)\quad (x, t)\in R^3\times(0, \infty).$$

Through the separation of variable

$$(1.5)\qquad u(x_1, x_2, x_3, t)=u(r\cos\theta, r\sin\theta, x_3, t)=w(r)e^{i(\nu\theta-t)}.$$

(1.1) can be obtained from (1.4). From (1.3) or (1.5) $u(x, t)$ is single valued if and only if ν is an integer. This fact corresponds to the quantization of fluxoid (or circulation) of flux line (or vortex line). Through the numerical calculation of energy the fundamental mode corresponding to $\nu=1$ is most favorable to exist in nature. The numerical integration of (1.1) in the case of $\nu=1$ was performed by V.L. Ginzburg and L.P. Pitaevskii in 1958 ([6]). Their results can be seen in figure 1.

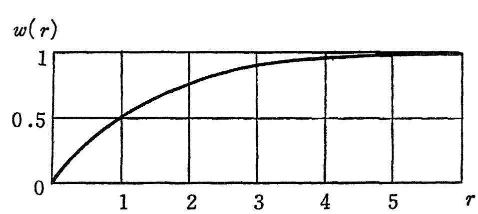

Fig. 1 Numerical solution of (1.1) in the case of $\nu=1$ due to V.L. Ginzburg and L.P. Pitaevskii ([6])

As shall be stated precisely in section 2 the nonlinearity appeared in (1.1) can be generalized slightly. But to see what happens it is sufficient to observe the special case (1.1). After some preparations (section 2~4) we can give our results in section 5. Theorem 1~4 are concerned with the boundary value problem (1.1) replacing the boundary condition at infinity by the boundary condition at finite point $r=L$. If L is smaller than or equal to some critical value L_0 then the interval $[0, L]$ is subcritical, that is, there exists no nontrivial solution (theorem 1). If L is larger than L_0 then the interval $[0, L]$ is supercritical, that is, there exists uniquely the nontrivial solution (theorem 2). Theorem 3 asserts that if we put Dirichlet boundary condition at $r=L$ then the nontrivial solution is the strictly increasing function of L. This fact plays an important role in the proof of theorem 6~12. Theorem 5 asserts that there exists a family of formal power series solution in powers of r of (1.1) in the case of $\nu=1, 2, 3, \cdots$. Theorem 6 and 7 assert that there exists uniquely the solution of (1.1). This solution is obtained as the limit function of sequences of approximate solutions from above and below (theorem 8). As can be seen in figure 1 the solution of (1.1) is strictly increasing function of r (theorem 10) and strictly concave function of r if ν satisfies $0<\nu\leqq 1$ (theorem 11). For $\nu=1, 2, 3, \cdots$ if we choose suitable value as the first coefficient of the formal power series solution in theorem 5 (approximate value for this can be found through theorem 9) then this formal power series is the asymptotic series of the true solution of (1.1) as r approaches zero on the real positive axis (theorem 12). So we can investigate precisely the influence of the nonlinearity on the modification of true solution from the Bessel function of order ν which is the exact solution of unperturbed liner equation obtained from (1.1) neglecting the nonlinear perturbation $w^3(r)$. Concerning the relation between the true solution of (1.1) and the formal power series solution in powers of r^{-1} of (1.1) near $r=\infty$ we can only assert the property (iii) of theorem 6. This means that the first two terms of the formal power series solution is asymptotically equal to the true solution of (1.1) as r tends to infinity.

2. Problems and Assumptions

We consider the boundary value problem BVP for the nonlinear Bessel equation on the infinite interval $(0, \infty)$.

$$(2.1)\qquad \begin{cases} \mathscr{L}[w(\ );\nu](r)=f(w(r)) & r\in(0,\infty) \\ w(0)=0, \quad w(\infty)=1. \end{cases}$$

Here

$$\mathscr{L}[w(\ );\nu](r)=-r^{-1}(rw'(r))'+\nu^2r^{-2}w(r).$$

r is the independent variable. $w(r)$ is the dependent variable. $w'(r)$ means $\frac{d}{dr}w(r)$. ν is a real positive parameter. $f(w)$ is a nonlinear function of w which will be specified later. If $f(w)=w$ then the equation (2.1) reduces to the Bessel equation of order ν. So we call the equation (2.1) nonlinear Bessel equation. At the same time we consider also the boundary value problem for the nonlinear Bessel equation on a finite interval $(0, L)$ $(L>0)$.

$$(2.2)\qquad \begin{cases} \mathscr{L}[w(\ );\nu](r)=f(w(r)) & r\in(0,L) \\ w(0)=0, \quad (1-\alpha)Lw'(L)+\alpha w(L)=0. \end{cases}$$

α is a parameter which varies in the interval $[0, 1]$. We assume that the nonlinearity $f(w)$ possesses following properties which are satisfied trivially by the special function $f(w)=(1-w^2)w$ which appeared in section 1.

Assumption 1. (i) *$f(w)\in C^2[0, 1]$, that is, $f(w)$, $f'(w)$ and $f''(w)$ are continuous functions of w on the interval $[0, 1]$.*

(ii) $f(0)=f(1)=0$.

(iii) $f''(w)<0$ *for* $w\in(0, 1)$.

Assumption 2. *$f(w)$ possesses the asymptotic series representation $\sum_{j=1}^{\infty} f_j w^j$ as w approaches zero on the real positive axis.*

Assumption 2′. *$f(w)$ is holomorphic in a neighborhood of the origin on the complex w-plane. It has the Taylor series expansion $f(w)=\sum_{j=1}^{\infty} f_j w^j$ which has a positive radius of convergence.*

By assumption 1 we can find the positive constant c_0 such that for any positive constant $c \geqq c_0$ the function $F(w; c)=f(w)+c^2 w$ satisfies

Assumption 1′. (i) $F(w; c) \in C^2[0, 1]$.
(ii) $F(0; c)=0, \quad F(1; c)=c^2$.
(iii) $F''(w; c)<0 \qquad$ for $\quad w \in (0, 1)$.
(iv) $F'(w; c)>0 \qquad$ for $\quad w \in [0, 1)$.

For the special case $f(w)=(1-w^2)w$ the smallest possible c_0 is $\sqrt{2}$. BVP (2.1) is equivalent to

$$
\begin{cases} \mathscr{L}[w(\ ); \nu, c](r)=F(w(r); c) & r \in (0, \infty) \\ w(0)=0, \qquad w(\infty)=1. \end{cases} \tag{2.1′}
$$

Here

$$
\mathscr{L}[w(\ ); \nu, c](r)=-r^{-1}(rw'(r))'+(c^2+\nu^2 r^{-2})w(r).
$$

BVP (2.2) is equivalent to

$$
\begin{cases} \mathscr{L}[w(\ ); \nu, c](r)=F(w(r); c) & r \in (0, L) \\ w(0)=0, \qquad (1-\alpha)Lw'(L)+\alpha w(L)=0. \end{cases} \tag{2.2′}
$$

3. Associated Linear Eigenvalue Problem

As a linearized problem for BVP (2.2′) we consider the following linear eigenvalue problem EVP.

$$
\begin{cases} \mathscr{L}[\varphi(\ ); \nu, c](r)=\mu\varphi(r) & r \in (0, L) \\ \varphi(0)=0, \qquad (1-\alpha)L\varphi'(L)+\alpha\varphi(L)=0 \end{cases} \tag{3.1}
$$

or equivalently

$$(3.1')\quad \begin{cases} \mathscr{L}[\varphi(\ );\nu](r)=\lambda\varphi(r) & r\in(0, L) \\ \varphi(0)=0, & (1-\alpha)L\varphi'(L)+\alpha\varphi(L)=0. \end{cases}$$

Here μ or $\lambda=\mu-c^2$ is the eigenvalue parameter. EVP (3.1) has the least eigenvalue $\mu_0(\nu, c, L, \alpha)=c^2+\lambda_0(\nu, L, \alpha)$. Here $\lambda_0(\nu, L, \alpha)=j_1^2(\nu, \alpha)L^{-2}$ is the least eigenvalue of EVP (3.1′). $j_1(\nu, \alpha)$ is the first positive zero of the function $(1-\alpha)zJ_\nu'(z)+\alpha J_\nu(z)$ (as usual $J_\nu(z)$ represents the Bessel function of order ν). Corresponding to this least eigenvalue EVP (3.1) has the unique normalized eigenfunction $\varphi_0(r;\nu, L, \alpha)=J_\nu^{-1}(j_1(\nu, 0))J_\nu(j_1(\nu, \alpha)L^{-1}r)$. Notice that $\varphi_0(r;\nu, L, \alpha)$ is also the eigenfunction of EVP (3.1′) corresponding to the least eigenvalue $\lambda_0(\nu, L, \alpha)$ so it does not depend on the constant c. Normalization of this eigenfunction is as follows

$$\max_{r\in[0,L]}\varphi_0(r;\nu, L, \alpha)=1=\varphi_0(r_{\max}(\nu, L, \alpha);\nu, L, \alpha).$$

Here $r_{\max}(\nu, L, \alpha)=j_1(\nu, 0)j_1^{-1}(\nu, \alpha)L$. Therefore $\varphi_0(r;\nu, L, \alpha)>0$ for $r\in(0, L)$.

4. Iteration Schemes

To construct approximate solutions we consider the following iteration schemes IS:

$$(4.1)\quad \begin{cases} \begin{cases} \mathscr{L}[\bar{w}_j(\ ;\nu, c);\nu, c](r)=F(\bar{w}_{j-1}(r;\nu;c);c) & r\in(0, \infty) \\ \bar{w}_j(0;\nu, c)=0, \quad \bar{w}_j(\infty;\nu, c)=1 & j=1, 2, 3, \cdots \end{cases} \\ \bar{w}_0(r;\nu, c)\equiv 1 \end{cases}$$

$$(4.2)\quad \begin{cases} \begin{cases} \mathscr{L}[\bar{w}_j(\ ;\nu, c, L, \alpha);\nu, c](r)=F(\bar{w}_{j-1}(r;\nu, c, L, \alpha);c) & r\in(0, L) \\ \bar{w}_j(0;\nu, c, L, \alpha)=0, & \\ (1-\alpha)L\bar{w}_j'(L;\nu, c, L, \alpha)+\alpha\bar{w}_j(L;\nu, c, L, \alpha)=0 & j=1, 2, 3, \cdots \end{cases} \\ \bar{w}_0(r;\nu, c, L, \alpha)\equiv 1 \end{cases}$$

$$
(4.3)\quad \begin{cases} \mathscr{L}[\underline{w}_j(\ ;\nu, c, L, \alpha, \delta);\nu, c](r)=F(\underline{w}_{j-1}(r;\nu, c, L, \alpha, \delta);c) \\ r\in(0, L) \qquad \underline{w}_j(0;\nu, c, L, \alpha, \delta)=0 \\ (1-\alpha)L\underline{w}_j'(L;\nu, c, L, \alpha, \delta)+\alpha\underline{w}_j(L;\nu, c, L, \alpha, \delta)=0 \\ \hfill j=1, 2, 3, \cdots \\ \underline{w}_0(r;\nu, c, L, \alpha, \delta)=\delta\varphi_0(r;\nu, L, \alpha) \quad 0<\delta\leqq\delta_0(\lambda_0(\nu, L, \alpha)) \end{cases}
$$

Here $\delta_0(\lambda)$ is the function of $\lambda\in[0, f'(0)]$ defined as the ξ-coordinate of the cross point of two curves $\eta=f(\xi)$ and $\eta=\lambda\xi$ on the (ξ, η)-plane which is different from the origin for $\lambda<f'(0)$. That is, the function $\delta_0(\lambda)$ is defined by the implicit relation:

$$
(4.4)\quad \begin{cases} f(\delta_0(\lambda))=\lambda\delta_0(\lambda), \quad \delta_0(\lambda)\neq 0 \qquad \text{for} \quad \lambda\in[0, f'(0)) \\ \delta_0(f'(0))=0 \end{cases}
$$

It follows from (4.4) that:

$$
(4.5)\qquad 0<1-\delta_0(\lambda)=-\{f'(1)\}^{-1}\lambda+O(\lambda^2) \qquad \text{as } \lambda\to 0\,.
$$

Let fix a positive constant $\underline{\delta}$ satisfying $0<\underline{\delta}<1$. We define the function $\underline{\delta}_0(\lambda)$ as the ξ-coordinate of the cross point of two direct lines $\eta=(1-\xi)f(\underline{\delta})(1-\underline{\delta})^{-1}$ and $\eta=\lambda\xi$. That is

$$
(4.6)\qquad \underline{\delta}_0(\lambda)=1-\{\lambda+f(\underline{\delta})(1-\underline{\delta})^{-1}\}^{-1}\lambda\,.
$$

It follows from (4.4) and (4.6) that:

$$
(4.7)\qquad \delta_0(\lambda)\geqq\underline{\delta}_0(\lambda) \qquad \text{for} \quad \lambda\in[0, f(\underline{\delta})\underline{\delta}^{-1}]\,.
$$

We call the interval $[0, L]$ is subcritical (supercritical) if $\lambda_0(\nu, L, \alpha)\geqq f'(0)$ ($<f'(0)$).

5. Results

Now we are ready to state our results. Under the assumption 1 we have

Theorem 1. *If the interval* $[0, L]$ *is subcritical then* BVP (2.2) *has no nontrivial solution* $w(r)$ *satisfying* $0 \leqq w(r) \leqq 1$ *for* $r \in [0, L]$.

Theorem 2. *If the interval* $[0, L]$ *is supercritical then* BVP (2.2) *has the unique nontrivial solution* $w(r; \nu, L, \alpha)$ *in the class of functions* $\{w(r) \in C^2(0, L);\ 0 \leqq w(r) \leqq 1$ *for* $r \in [0, L]\}$ *with following properties*:

(i) $w(r; \nu, L, \alpha) \in C^2(0, L]$.

(ii) $w(r; \nu, L, \alpha) = u_0(\nu, L, \alpha) r^\nu + O(r^{\nu+2})$ *as* $r \to 0$.

Here

$$(5.1) \qquad u_0(\nu, L, \alpha) = \int_0^L f(w(s; \nu, L, \alpha)) \{(2\nu)^{-1} s^{1-\nu} - g(\nu, L, \alpha) s^{1+\nu}\} ds$$

$g(\nu, L, \alpha)$ *is given by* (7.3).

(iii) $0 < w(r; \nu, L, \alpha) < 1$ *for* $r \in (0, L)$.

(iv) *The solution* $\bar{w}_j(r; \nu, c, L, \alpha)$ *of* IS (4.2) *tends to* $w(r; \nu, L, \alpha)$ *monotone decreasingly and uniformly with respect to* $r \in [0, L]$ *as* j *tends to infinity.*

(v) *The solution* $\underline{w}_j(r; \nu, c, L, \alpha, \delta)$ *of* IS (4.3) *tends to* $w(r; \nu, L, \alpha)$ *monotone increasingly and uniformly with respect to* $r \in [0, L]$ *as* j *tends to infinity.*

(vi) *For* $\nu = 1, 2, 3, \cdots$ $\tilde{w}(r; \nu, u_0(\nu, L, \alpha))$ *is the asymptotic series of* $w(r; \nu, L, \alpha)$ *as* r *approaches zero on the real positive axis. Here* $\tilde{w}(r; \nu, u_0)$ *is the formal power series solution appearing in theorem* 5. (*So this statement requires assumption* 2 *or* 2'.)

Theorem 3. *Let fix* $\alpha = 1$. *If the interval* $[0, L_1]$ *is supercritical then for any* $L_2 > L_1$ *the interval* $[0, L_2]$ *is also supercritical. Corresponding nonlinear eigenfunctions* $w(r; \nu, L_1, 1)$ *and* $w(r; \nu, L_2, 1)$ *satisfy the inequality*:

$$(5.2) \qquad w(r; \nu, L_1, 1) < w(r; \nu, L_2, 1) \quad \text{for} \quad r \in (0, L_1].$$

Theorem 4. *For any fixed* $\alpha_1 \in (0, 1]$ *if the interval* $[0, L]$ *is supercritical for* $\alpha = \alpha_1$ *then for any* $\alpha_2 \in [0, 1)$ *satisfying* $\alpha_2 < \alpha_1$ *the interval* $[0, L]$ *is also supercritical for* $\alpha = \alpha_2$. *Corresponding nonlinear* *eigenfunctions* $w(r;\nu, L, \alpha_1)$ *and* $w(r;\nu, L, \alpha_2)$ *satisfy the inequality*:

$$w(r;\nu, L, \alpha_1)<w(r;\nu, L, \alpha_2) \quad \textit{for} \quad r\in(0, L]. \tag{5.3}$$

Under the assumption 2 or 2′ we have

Theorem 5. *For* $\nu=1, 2, 3, \cdots$ *the nonlinear Bessel equation* $\mathscr{L}[w(\ );\nu](r)=f(w(r))$ *has a family of formal power series solution* $\tilde{w}(r;\nu, u_0)=r^\nu u(r;\nu, u_0)=\sum_{l=0}^{\infty} u_l r^{l+\nu}$. *Here* u_0 *is any positive number (any complex number in the case of assumption* 2′). u_l $(l=1, 2, 3, \cdots)$ *are given by*

$$\begin{cases} u_1=0 \\ u_l=-l^{-1}(l+2\nu)^{-1} \displaystyle\sum_{\substack{(j-1)\nu+k=l-2 \\ j\geq 1, k\geq 0}} f_j \sum_{k_1+\cdots+k_j=k} u_{k_1}\cdots u_{k_j} \quad l=2, 3, 4, \cdots. \end{cases} \tag{5.4}$$

Especially under the assumption 2′ $\tilde{w}(r;\nu, u_0)$ is a holomorphic true solution of the nonlinear Bessel equation $\mathscr{L}[w(\ );\nu](r)=f(w(r))$ in some complex neighborhood of $r=0$.

Under the assumption 1 we have following theorems:

Theorem 6 (Existence). BVP (2.1) *has the solution* $w(r;\nu)$ *with following properties*:

(i) $w(r;\nu)\in C^2(0, \infty)$.

(ii) $0<w(r;\nu)<1$ *for* $r\in(0, \infty)$.

(iii) $1-w(r;\nu)=O(r^{-2})$ *as* $r\to\infty$.

(iv) $w(r;\nu)=u_0(\nu)r^\nu+O(r^{\nu+2})$ *as* $r\to 0$.

Here

$$u_0(\nu)=(2\nu)^{-1}\int_0^\infty f(w(s;\nu))s^{1-\nu}ds. \tag{5.5}$$

Theorem 7 (Uniqueness). *Any nontrivial solution* $w(r)\in C^2(0, \infty)$

of the nonlinear Bessel equation $\mathscr{L}[w(\ );\nu](r)=f(w(r))$ *satisfying* $0\leqq w(r)\leqq 1$, $w(r)\not\equiv 0$ *for* $r\in(0,\infty)$ *coincides identically with* $w(r;\nu)$.

Theorem 8 (Convergence of approximate solutions). (i) *The solution* $w(r;\nu,L,1)$ *of* BVP (2.2) *tends to* $w(r;\nu)$ *monotone increasingly and compact uniformly with respect to* $r\in[0,\infty)$ *as* L *tends to infinity.*

(ii) *The solution* $\bar{w}_j(r;\nu,c)$ *of* IS (4.1) *tends to* $w(r;\nu)$ *monotone decreasingly and uniformly with respect to* $r\in[0,\infty)$ *as* j *tends to infinity.*

(iii)

$$\lim_{j\to\infty}\|(r^{-\nu}+1)\{w(r;\nu)-\bar{w}_j(r;\nu,c)\}\|=0. \tag{5.6}$$

$$\lim_{j\to\infty}\|(r^{-\nu}+1)r(r+1)^{-1}\{w'(r;\nu)-\bar{w}_j'(r;\nu,c)\}\|=0. \tag{5.7}$$

$$\lim_{j\to\infty}\|(r^{-\nu}+1)r^2(r^2+1)^{-1}\{w''(r;\nu)-\bar{w}_j''(r;\nu,c)\}\|=0. \tag{5.8}$$

(iv) *Especially for* $\nu=1$ *we have*

$$\lim_{j\to\infty}\|(r^{-1}+1)[r^{-1}\{w(r;1)-\bar{w}_j(r;1,c)\}]'\|=0. \tag{5.9}$$

$$\lim_{j\to\infty}\|(r^{-1}+1)\{w''(r;1)-\bar{w}_j''(r;1,c)\}\|=0. \tag{5.10}$$

Here we use the notation $\|w(r)\|=\sup_{r\in(0,\infty)}|w(r)|$.

Theorem 9 (Convergence of approximate solutions at $r=0$**).**

$$\lim_{j\to\infty}|u_0(\nu)-u_{0j}(\nu,c)|=0. \tag{5.11}$$

Here

$$u_{0j}(\nu,c)=\lim_{r\to 0}r^{-\nu}\bar{w}_j(r;\nu,c)= \tag{5.12}$$

$$=(2\nu)^{-1}\int_0^\infty[f(\bar{w}_{j-1}(s;\nu,c))+c^2\{\bar{w}_{j-1}(s;\nu,c)-\bar{w}_j(s;\nu,c)\}]s^{1-\nu}ds$$

$$j\geqq[\nu/2]+2.$$

Theorem 10 (Monotonicity).

$$w'(r;\nu)>0 \qquad \text{for} \quad r\in(0,\infty). \tag{5.13}$$

Theorem 11 (Concavity). *If ν satisfies $0<\nu\leqq 1$ then we have*

$$w''(r;\nu)<0 \qquad \text{for} \quad r\in(0,\infty). \tag{5.14}$$

Under the assumption 1 and 2 (or 2′) we have

Theorem 12 (Asymptotic expansion at $r=0$). *For $\nu=1, 2, 3, \cdots$ the formal power series solution $\tilde{w}(r;\nu, u_0(\nu))$ (appeared in theorem 5) is the asymptotic series of $w(r;\nu)$ as r approaches zero on the real positive axis.*

Remark 1. Under the assumption 2′ theorem 5 asserts that $\tilde{w}(r;\nu, u_0(\nu))$ $(\nu=1, 2, 3, \cdots)$ is also the true solution of the nonlinear Bessel equation moreover by theorem 12 its behavior near $r=0$ is asymptotically equal to that of the global solution $w(r;\nu)$. So it arises the question that whether $w(r;\nu, u_0(\nu))$ coincides identically with $w(r;\nu)$ or not. We could not answer this question in this paper.

Remark 2. If $f(1-v)=\sum_{j=1}^{\infty}\tilde{f}_j v^j$ is holomorphic in some neighborhood of $v=0$ then the nonlinear Bessel equation $\mathscr{L}[w(\ );\nu](r)=f(w(r))$ has a formal power series solution in powers of r^{-1}:

$$\tilde{\tilde{w}}(r;\nu)=1-\nu^2\tilde{f}_1^{-1}r^{-2}+\cdots.$$

Theorem 6 (iii) asserts that $w(r;\nu)$ is asymptotically equal to the first two terms of the formal power series solution $\tilde{\tilde{w}}(r;\nu)$ as r tends to infinity. It arises the question that whether $\tilde{\tilde{w}}(r;\nu)$ can be the asymptotic series of the true solution $w(r;\nu)$ or not. In this paper we could not answer this question. Especially we could not show the existence of $\lim_{r\to\infty} r^2\{1-w(r;\nu)\}$.

6. Preliminary Lemmas

Following lemmas are needed in the proof of lemmas in section 7.

Lemma 6.1. *The modified Bessel equation* $\mathscr{L}[w(\ );\nu,1](r)=0$ *has the fundamental system of solutions* $I_\nu(r)$ *and* $K_\nu(r)$ *which have following properties*:

(i) $r\{I_\nu'(r)K_\nu(r)-I_\nu(r)K_\nu'(r)\}=1$ *for* $r\in(0,\infty)$.

(ii) $I_\nu(r)=\sum_{n=0}^{\infty}\{n!\Gamma(\nu+n+1)\}^{-1}(r/2)^{2n+\nu}$

$$=\Gamma^{-1}(\nu+1)(r/2)^\nu+O(r^{\nu+2}) \quad \text{as} \quad r\to 0.$$

(iii) $\left(\frac{d}{dr}\right)^k I_\nu(r)>0$ *for* $r\in(0,\infty)$ $k=0,1,2,\cdots$.

(iv) $I_\nu'(r)=2^{-1}\{I_{\nu-1}(r)+I_{\nu+1}(r)\}=2\Gamma^{-1}(\nu)(r/2)^{\nu-1}+O(r^{\nu+1})$ *as* $r\to 0$.

(v) $I_\nu(r)=(2\pi r)^{-1/2}e^r\{1+O(r^{-1})\}$ *as* $r\to\infty$.

(vi) $I_\nu'(r)=(2\pi r)^{-1/2}e^r\{1+O(r^{-1})\}$ *as* $r\to\infty$.

(vii) $\frac{\partial}{\partial\nu}I_\nu(r)<0$ *for* $r\in(0,\infty)$.

(viii) $I_\nu'(r)=\nu r^{-1}I_\nu(r)+I_{\nu+1}(r)<(\nu r^{-1}+1)I_\nu(r)$ *for* $r\in(0,\infty)$

(ix) $K_\nu(r)=\pi\{2\sin\nu\pi\}^{-1}\{I_{-\nu}(r)-I_\nu(r)\}=$

$$=2^{-1}\sum_{j=0}^{\nu-1}(-1)^j(\nu-j-1)!(j!)^{-1}(r/2)^{2j-\nu}+(-1)^{\nu+1}I_\nu(r)\{\gamma+$$

$$+\log(r/2)\}+(-1)^\nu 2^{-1}\sum_{k=0}^{\infty}\{k!(\nu+k)!\}^{-1}\{\sum_{l=1}^{k}l^{-1}+$$

$$+\sum_{l=1}^{k+\nu}l^{-1}\}(r/2)^{\nu+2k} \qquad (\nu=0,1,2,\cdots).$$

Here γ *is the Euler constant.*

(x) $K_\nu(r)=2^{-1}\Gamma(\nu)(r/2)^{-\nu}+O(r^{-\nu+2})$ *as* $r\to 0$ (*if* $\nu\neq 1$)

$K_1(r)=r^{-1}+O(r|\log r|)$ *as* $r\to 0$.

(xi) $(-1)^k\left(\frac{d}{dr}\right)^k K_\nu(r)>0$ *for* $r\in(0,\infty)$ $k=0, 1, 2, \cdots$.

(xii) $K'_\nu(r)=-2^{-1}\{K_{\nu-1}(r)+K_{\nu+1}(r)\}=$

$=-2^{-2}\Gamma(\nu+1)(r/2)^{-\nu-1}+O(r^{-\nu+1})$ *as* $r\to 0$.

(xiii) $K_\nu(r)=\pi^{1/2}(2r)^{-1/2}e^{-r}\{1+O(r^{-1})\}$ *as* $r\to\infty$.

(xiv) $K'_\nu(r)=-\pi^{1/2}(2r)^{-1/2}e^{-r}\{1+O(r^{-1})\}$ *as* $r\to\infty$.

(xv) $\frac{\partial}{\partial\nu}K_\nu(r)>0$ *for* $r\in(0,\infty)$.

(xvi) $-K'_\nu(r)=K_{\nu-1}(r)+\nu r^{-1}K_\nu(r)\leqq$

$\leqq(\nu r^{-1}+1)K_\nu(r)$ *for* $r\in(0,\infty)$ (*if* $\nu\geqq 1/2$)

$-K'_\nu(r)\leqq \text{const.}(r^{-1}+1)K_\nu(r)$ *for* $r\in(0,\infty)$ (*if* $0<\nu<1/2$).

Here const. *dose not depend on* r.

These properties of the modified Bessel function are well known so we omit the proof of lemma 6.1.

Lemma 6.2. *If the function* $u(r)\in C^2(0,\infty)$ (*or* $C^2(0, L]$) *satisfies the equation*

$$\mathscr{L}[u(\ );\nu, c](r)=0 \quad \textit{for} \quad r\in(0,\infty) \quad (\textit{or}\ (0, L))$$

for $c\geqq 0$ *and behaves as* $u(r)=o(r^{-\nu})$ *as* $r\to 0$, $u(r)=O(1)$ *as* $r\to\infty$ (*or* $(1-\alpha)Lu'(L)+\alpha u(L)=0$) *then* $u(r)\equiv 0$ *for* $r\in(0,\infty)$ (*or* $(0, L)$).

7. Green's Functions

We define the Green's functions $G(r, s;\nu)$, $G(r, s;\nu, L, \alpha)$, $G(r, s;\nu, c)$ and $G(r, s;\nu, c, L, \alpha)$ for BVP (2.1), (2.2), (2.1′) and (2.2′) respectively regarding right hand sides of respective equations as given functions.

Definition 7.1.

$$G(r, s; \nu)=\begin{cases}(2\nu)^{-1}s^{\nu}r^{-\nu} & 0<s<r<\infty \\ (2\nu)^{-\nu}r^{\nu}s^{-\nu} & 0<r<s<\infty.\end{cases} \tag{7.1}$$

Definition 7.2.

$$G(r, s; \nu, L, \alpha)=G(r, s; \nu)-g(\nu, L, \alpha)r^{\nu}s^{\nu} \quad \text{for} \quad (r, s)\in[0, L]\times[0, L]. \tag{7.2}$$

Here

$$g(\nu, L, \alpha)=(2\nu)^{-1}\{\alpha-(1-\alpha)\nu\}\{\alpha+(1-\alpha)\nu\}^{-1}L^{-2\nu}. \tag{7.3}$$

Definition 7.3.

$$G(r, s; \nu, c)=\begin{cases}I_{\nu}(cs)K_{\nu}(cr) & 0<s<r<\infty \\ I_{\nu}(cr)K_{\nu}(cs) & 0<r<s<\infty.\end{cases} \tag{7.4}$$

Definition 7.4.

$$G(r, s; \nu, c, L, \alpha)=G(r, s; \nu, c)-g(\nu, c, L, \alpha)I_{\nu}(cr)I_{\nu}(cs) \quad \text{for} \quad (r, s)\in[0, L]\times[0, L]. \tag{7.5}$$

Here

$$g(\nu, c, L, \alpha)= =\{(1-\alpha)cLK_{\nu}'(cL)+\alpha K_{\nu}(cL)\}\{(1-\alpha)cLI_{\nu}'(cL)+\alpha I_{\nu}(cL)\}^{-1}. \tag{7.6}$$

For the sake of abbreviation we use following notations when the integrals of the right hand sides have the meaning.

Definition 7.5.

$$\mathscr{G}[f(\ ); \nu](r)=\int_{0}^{\infty}G(r, s; \nu)f(s)sds. \tag{7.7}$$

Definition 7.6.

(7.8) $$\mathscr{G}[f(\ );\nu,L,\alpha](r)=\int_0^L G(r,s;\nu,L,\alpha)f(s)sds\,.$$

Definition 7.7.

(7.9) $$\mathscr{G}[F(\ );\nu,c](r)=\int_0^\infty G(r,s;\nu,c)F(s)sds\,.$$

Definition 7.8.

(7.10) $$\mathscr{G}[F(\ );\nu,c,L,\alpha](r)=\int_0^L G(r,s;\nu,c,L,\alpha)F(s)sds\,.$$

Green's functions defined above have following properties:

Lemma 7.1.

(i) $G(r,s;\nu)\in C^0([0,\infty)\times[0,\infty)-\{(0,0)\})$.

(ii) $G(r,s;\nu)=G(s,r;\nu)$ *for* $(r,s)\in(0,\infty)\times(0,\infty)$.

(iii) $G(r,s;\nu)>0$ *for* $(r,s)\in(0,\infty)\times(0,\infty)$.

(iv) $\frac{\partial}{\partial r}G(r,s;\nu),\ \left(\frac{\partial}{\partial r}\right)^2 G(r,s;\nu)\in$

$\in C^0((0,\infty)\times(0,\infty)-\{(r,s)\in R^2;r=s\})$.

Here R^2 is the 2-dimensional Euclidean space.

(v) *For any fixed* $r_0\in(0,\infty)$

$$\lim_{\substack{(r,s)\to(r_0,r_0)\\ r<s}}\frac{\partial}{\partial r}G(r,s;\nu)-\lim_{\substack{(r,s)\to(r_0,r_0)\\ r>s}}\frac{\partial}{\partial r}G(r,s;\nu)=r_0^{-1}.$$

(vi) *For any fixed* $s\in(0,\infty)$

$\mathscr{L}[G(\ ,s;\nu);\nu](r)=0$ *for* $r\in(0,\infty)-\{s\}$.

(vii) $\mathscr{G}[\nu^2r^{-2};\nu](r)=1$ *for* $r\in(0,\infty)$.

Lemma 7.2.

(i) $G(r,s;\nu,L,\alpha)\in C^0([0,L]\times[0,L]-\{(0,0)\})$.

(ii) $G(r, s; \nu, L, \alpha)=G(s, r; \nu, L, \alpha)$ *for* $(r, s)\in(0, L]\times(0, L]$.

(iii) $G(r, s; \nu, L, \alpha)>0$ *for* $(r, s)\in(0, L)\times(0, L)$.

(iv) $\frac{\partial}{\partial r}G(r, s; \nu, L, \alpha)$, $\left(\frac{\partial}{\partial r}\right)^2 G(r, s; \nu, L, \alpha)\in$

$\in C^0((0, L]\times(0, L]-\{(r, s)\in R^2;\ r=s\})$.

(v) *For any fixed* $r_0\in(0, L)$

$$\lim_{\substack{(r,s)\to(r_0,r_0)\\ r<s}}\frac{\partial}{\partial r}G(r, s; \nu, L, \alpha)-\lim_{\substack{(r,s)\to(r_0,r_0)\\ r>s}}\frac{\partial}{\partial r}G(r, s; \nu, L, \alpha)=r_0^{-1}.$$

(vi) *For any fixed* $s\in(0, L)$

$\mathscr{L}[G(\ , s; \nu, L, \alpha); \nu](r)=0$ *for* $r\in(0, L)-\{s\}$.

(vii) $(1-\alpha)L\frac{\partial}{\partial r}G(r, s; \nu, L, \alpha)\Big|_{r=L}+\alpha G(L, s; \nu, L, \alpha)=0$

for $s\in(0, L)$.

(viii) $\mathscr{G}[\nu^2 r^{-2}; \nu, L, \alpha](r)=1-\{2^{-1}L^{-\nu}+\nu g(\nu, L, \alpha)L^{\nu}\}r^{\nu}\leqq 1$

for $r\in(0, L)$.

Lemma 7.3.

(i) $G(r, s; \nu, c)\in C^0([0, \infty)\times[0, \infty)-\{(0, 0)\})$.

(ii) $G(r, s; \nu, c)=G(s, r; \nu, c)$ *for* $(r, s)\in(0, \infty)\times(0, \infty)$.

(iii) $G(r, s; \nu, c)>0$ *for* $(r, s)\in(0, \infty)\times(0, \infty)$.

(iv) $\frac{\partial}{\partial r}G(r, s; \nu, c)$, $\left(\frac{\partial}{\partial r}\right)^2 G(r, s; \nu, c)\in$

$\in C^0((0, \infty)\times(0, \infty)-\{(r, s)\in R^2; r=s\})$.

(v) *For any fixed* $r_0\in(0, \infty)$

$$\lim_{\substack{(r,s)\to(r_0,r_0)\\ r<s}}\frac{\partial}{\partial r}G(r, s; \nu, c)-\lim_{\substack{(r,s)\to(r_0,r_0)\\ r>s}}\frac{\partial}{\partial r}G(r, s; \nu, c)=r_0^{-1}.$$

(vi) *For any fixed* $s \in (0, \infty)$

$$\mathscr{L}[G(\ , s; \nu, c); \nu, c](r)=0 \qquad \text{for} \quad r \in (0, \infty)-\{s\}.$$

(vii) $\mathscr{G}[c^2+\nu^2 r^{-2}; \nu, c](r)=1 \qquad \text{for} \quad r \in (0, \infty).$

(viii) $$\left|\frac{\partial}{\partial r} G(r, s; \nu, c)\right| \leqq C(\nu, c)(r^{-1}+1) G(r, s; \nu, c)$$

$$\text{for} \quad (r, s) \in (0, \infty) \times (0, \infty).$$

Here $C(\nu, c)$ *is a constant which depends on* ν *and* c *but is independent of* r *and* s.

Lemma 7.4.

(i) $G(r, s; \nu, c, L, \alpha) \in C^0([0, L] \times [0, L]-\{(0, 0)\}).$

(ii) $G(r, s; \nu, c, L, \alpha)=G(s, r; \nu, c, L, \alpha)$

$$\text{for} \ (r, s) \in (0, L) \times (0, L).$$

(iii) $G(r, s; \nu, c, L, \alpha)>0 \qquad \text{for} \quad (r, s) \in (0, L) \times (0, L).$

(iv) $$\frac{\partial}{\partial r} G(r, s; \nu, c, L, \alpha), \ \left(\frac{\partial}{\partial r}\right)^2 G(r, s; \nu, c, L, \alpha) \in$$

$$\in C^0((0, L] \times (0, L]-\{(r, s) \in R^2; r=s\}).$$

(v) *For any fixed* $r_0 \in (0, L)$

$$\lim_{\substack{(r,s) \to (r_0, r_0) \\ r<s}} \frac{\partial}{\partial r} G(r, s; \nu, c, L, \alpha)-\lim_{\substack{(r,s) \to (r_0, r_0) \\ r>s}} \frac{\partial}{\partial r} G(r, s; \nu, c, L, \alpha)=$$

$$=r_0^{-1}.$$

(vi) *For any fixed* $s \in (0, L)$

$$\mathscr{L}[G(\ , s; \nu, c, L, \alpha); \nu, c](r)=0 \qquad \text{for} \quad r \in (0, L)-\{s\}.$$

(vii) $$(1-\alpha) L \frac{\partial}{\partial r} G(r, s; \nu, c, L, \alpha)\bigg|_{r=L}+\alpha G(L, s; \nu, c, L, \alpha)=0$$

$$\text{for} \quad s \in (0, L).$$

(viii) $\mathscr{G}[c^2+\nu^2 r^{-2}; \nu, c, L, \alpha](r)=$

$$=1-cL\{g(\nu,\ c,\ L,\ \alpha)I_{\nu}'(cL)-K_{\nu}'(cL)\}I_{\nu}(cr)\leqq 1$$

$$\mathit{for}\quad r\in[0,\ L].$$

Lemma 7.1 and 7.2 can be obtained by standard arguments noticing that the differential equation $\mathscr{L}[u(\ );\ \nu](r)=0$ has the fundamental system of solutions $u(r)=r^{\nu}$ and $r^{-\nu}$. So we omit the detailed proof. Lemma 7.3 and 7.4 follow from the fact that the differential equation $\mathscr{L}[u(\ );\ \nu,\ c](r)=0$ has the fundamental system of solutions $u(r)=I_{\nu}(cr)$ and $K_{\nu}(cr)$ and from their properties stated in lemma 6.1. We shall only show the proof of lemma 7.4 (viii).

Proof of Lemma 7.4 (viii). Let $r\in(0,\ L)$. we have by definition 7.8 and definition 7.4 that:

$$(7.11)\quad \mathscr{G}[c^{2}+\nu^{2}r^{-2};\ \nu,\ c,\ L,\ \alpha](r)=$$

$$=\int_{0}^{L}\{G(r,\ s;\ \nu,\ c)-g(\nu,\ c,\ L,\ \alpha)I_{\nu}(cr)I_{\nu}(cs)\}(c^{2}+\nu^{2}s^{-2})sds=$$

$$=\int_{0}^{\infty}G(r,\ s;\ \nu,\ c)(c^{2}+\nu^{2}s^{-2})sds-\int_{L}^{\infty}G(r,\ s;\ \nu,\ c)(c^{2}+\nu^{2}s^{-2})sds-$$

$$-g(\nu,\ c,\ L,\ \alpha)I_{\nu}(cr)\int_{0}^{L}I_{\nu}(cs)(c^{2}+\nu^{2}s^{-2})sds\ .$$

By lemma 7.3 (vii) we have

$$(7.12)\qquad \int_{0}^{\infty}G(r,\ s;\ \nu,\ c)(c^{2}+\nu^{2}s^{-2})sds=1\qquad \text{for}\quad r\in(0,\ L).$$

From (7.11) and (7.12) we have

$$\mathscr{G}[c^{2}+\nu^{2}r^{-2};\ \nu,\ c,\ L,\ \alpha](r)=$$

$$=1-I_{\nu}(cr)\int_{L}^{\infty}K_{\nu}(cs)(c^{2}+\nu^{2}s^{-2})sds-$$

$$-g(\nu,\ c,\ L,\ \alpha)I_{\nu}(cr)\int_{0}^{L}I_{\nu}(cs)(c^{2}+\nu^{2}s^{-2})sds=$$

$$=1-I_{\nu}(cr)\int_{L}^{\infty}\frac{d}{ds}\{csK_{\nu}'(cs)\}\,ds-$$

$$-g(\nu, c, L, \alpha)I_\nu(cr)\int_0^L \frac{d}{ds}\{csI'_\nu(cs)\}ds=$$

$$=1-cL\{g(\nu, c, L, \alpha)I'_\nu(cL)-K'_\nu(cL)\}I_\nu(cr).$$

This proves the first equality in lemma 7.4 (viii). Here we use following relations:

$$\frac{d}{ds}\{csK'_\nu(cs)\}=K_\nu(cs)(c^2+\nu^2 s^{-2})s,$$

$$\frac{d}{ds}\{csI'_\nu(cs)\}=I_\nu(cs)(c^2+\nu^2 s^{-2})s,$$

$$\{csK'_\nu(cs)\}\Big|_{s=\infty}=0 \quad \text{and} \quad \{csI'_\nu(cs)\}\Big|_{s=0}=0.$$

From lemma 6.1 (i) we have

$$\text{(7.13)} \qquad cL\{K_\nu(cL)I'_\nu(cL)-K'_\nu(cL)I_\nu(cL)\}=1 \qquad \text{for} \quad L>0.$$

Using this relation (7.13) direct calculation leads to

$$\text{(7.14)} \qquad cL\{g(\nu, c, L, \alpha)I'_\nu(cL)-K'_\nu(cL)\}=$$

$$=\alpha\{(1-\alpha)cLI'_\nu(cL)+\alpha I_\nu(cL)\}^{-1}\geqq 0.$$

(7.14) shows the last inequality in lemma 7.4 (viii).

Lemma 7.5. *If $f(r)\in C^0(0, \infty)$ satisfies $f(r)=O(r^\nu)$ as $r\to 0$ and $f(r)=O(r^{-2})$ as $r\to\infty$ then following two statements are equivalent.*

$$\text{(i)} \quad \begin{cases} u(r)\in C^2(0, \infty), \\ \mathscr{L}[u(\ );\nu](r)=f(r) \qquad \textit{for} \quad r\in(0, \infty), \\ u(r)=O(r^\nu) \quad \textit{as } r\to 0, \qquad u(r)=O(1) \quad \textit{as } r\to\infty. \end{cases}$$

(ii) $u(r)=\mathscr{G}[f(\ );\nu](r)$ *for* $r\in(0, \infty)$.

Lemma 7.6. *If $f(r)\in C^0(0, L]$ satisfies $f(r)=O(r^\nu)$ as $r\to 0$ then following two statements are equivalent.*

$$\text{(i)}\quad \begin{cases} u(r)\in C^2(0, L], \\ \mathscr{L}[u(\ );\nu](r)=0 \qquad \mathit{for}\quad r\in(0, L), \\ u(r)=O(r^{\nu}) \quad \mathit{as}\ r\to 0, \qquad (1-\alpha)Lu'(L)+\alpha u(L)=0. \end{cases}$$

$$\text{(ii)}\quad u(r)=\mathscr{G}[f(\ );\nu, L, \alpha](r) \qquad \mathit{for}\quad r\in(0, \infty).$$

Lemma 7.7. *If* $F(r)\in C^0(0, \infty)$ *satisfies* $F(r)=O(r^p)$ $(p>-\nu-2)$ *as* $r\to 0$ *and* $F(r)=O(1)$ *as* $r\to\infty$ *then following two statements are equivalent.*

$$\text{(i)}\quad \begin{cases} u(r)\in C^2(0, \infty), \\ \mathscr{L}[u(\ );\nu, c](r)=F(r) \qquad \mathit{for}\quad r\in(0, \infty), \\ u(r)=o(r^{-\nu}) \quad \mathit{as}\ r\to 0, \qquad u(r)=O(1) \quad \mathit{as}\ r\to\infty. \end{cases}$$

$$\text{(ii)}\quad u(r)=\mathscr{G}[F(\ );\nu, c](r) \qquad \mathit{for}\quad r\in(0, \infty).$$

Moreover $u(r)$ *given by* (i) *or* (ii) *satisfies*

$$\text{(iii)}\quad u(r), ru'(r), r^2u''(r)=\begin{cases} O(r^{\nu}) & (\mathit{if}\ \nu<p+2) \\ O(r^{p+2}|\log r|) & (\mathit{if}\ \nu=p+2) \\ O(r^{p+2}) & (\mathit{if}\ \nu>p+2) \quad \mathit{as}\ r\to 0. \end{cases}$$

$$\text{(iv)}\quad u(r), u'(r), u''(r)=O(1) \qquad \mathit{as}\quad r\to\infty.$$

Lemma 7.8. *If* $F(r)\in C^0(0, L]$ *satisfies* $F(r)=O(r^p)$ $(p>-\nu-2)$ *as* $r\to 0$ *then following two statements are equivalent.*

$$\text{(i)}\quad \begin{cases} u(r)\in C^2(0, L], \\ \mathscr{L}[u(\ );\nu, c](r)=F(r) \qquad \mathit{for}\quad r\in(0, L), \\ u(r)=o(r^{-\nu}) \quad \mathit{as}\ r\to 0, \qquad (1-\alpha)Lu'(L)+\alpha u(L)=0 \end{cases}$$

$$\text{(ii)}\quad u(r)=\mathscr{G}[F(\ );\nu, c, L, \alpha](r) \qquad \mathit{for}\quad r\in(0, L).$$

Moreover $u(r)$ *given by* (i) *or* (ii) *satisfies* (iii) *in lemma* 7.7.

We shall only prove lemma 7.7. Proofs of lemma 7.5, 7.6 and 7.8

are not essentially different from that of lemma 7.7. So we omit it.

Proof of Lemma 7.7. First we show that (ii) implies (i) and (iii). By direct differentiation of the expression (ii) it is easy to see from lemma 7.3 that $u(r)$ satisfies the equation

$$\mathscr{L}[u(\ );\nu, c](r)=F(r) \qquad \text{for} \quad r\in(0, \infty). \tag{7.15}$$

Let $\nu<p+2$. We have from (ii) that:

$$u(r)=I_\nu(cr)\int_0^\infty K_\nu(cs)F(s)s\,ds-$$

$$-\int_0^r\{I_\nu(cr)K_\nu(cs)-I_\nu(cs)K_\nu(cr)\}F(s)s\,ds=$$

$$=O(r^\nu)+O(r^{p+2})=O(r^\nu) \qquad \text{as} \quad r\to 0.$$

This proves (iii) in the case of $\nu<p+2$. Let $\nu\geqq p+2$. We have from (ii) that:

$$u(r)=K_\nu(cr)\int_0^r I_\nu(cs)F(s)s\,ds+$$

$$+I_\nu(cr)\int_r^1 K_\nu(cs)F(s)s\,ds+I_\nu(cr)\int_1^\infty K_\nu(cs)F(s)s\,ds=$$

$$=O(r^{p+2})+\left\{\begin{array}{ll} O(r^{p+2}|\log r|) & (\text{if } \nu=p+2) \\ O(r^{p+2}) & (\text{if } \nu>p+2) \end{array}\right\}+O(r^\nu)=$$

$$=\left\{\begin{array}{ll} O(r^{p+2}|\log r|) & (\text{if } \nu=p+2) \\ O(r^{p+2}) & (\text{if } \nu>p+2) \end{array}\right. \qquad \text{as} \quad r\to 0.$$

This proves (iii) in the case of $\nu\geqq p+2$. Let $r\geqq 1$. We have from (ii) that:

$$|u(r)|\leqq K_\nu(cr)\int_0^1 I_\nu(cs)F(s)s\,ds+ \tag{7.16}$$

$$+\sup_{r\in[1,\infty)}|F(r)|\int_1^\infty G(r, s;\nu, c)s\,ds\,.$$

By lemma 7.3 (vii) we have

$$(7.17) \qquad \int_1^\infty G(r, s; \nu, c)c^2 s ds < \int_0^\infty G(r, s; \nu, c)(c^2+\nu^2 s^{-2}) s ds = 1.$$

From (7.16) and (7.17) we have $u(r)=O(1)$ as $r\to\infty$. The part in (iii) and (iv) concerning the behavior of $u'(r)$ is obtained as follows: first differentiate (ii), second use the estimate of the first derivative $\frac{\partial}{\partial r}G(r, s; \nu, c)$ of the Green's function in lemma 7.3 (viii), then repeat the same arguments as that for $u(r)$. The part concerning the behavior of $u''(r)$ is obtained from the equation (7.15). Thus we proved that (ii) implies (iii). Next we show that (i) implies (ii). By the same arguments as above

$$\hat{u}(r)=\mathscr{G}[F(\ ); \nu, c](r) \qquad \text{for} \quad r\in(0, \infty)$$

satisfies (i) replacing $u(r)$ by $\hat{u}(r)$. Applying lemma 6.2 to the function $u(r)-\hat{u}(r)$ we can conclude that $u(r)-\hat{u}(r)\equiv 0$ for $r\in(0, \infty)$. This shows that $u(r)$ satisfies (ii).

Lemma 7.9. *If $F(r)$ is a bounded continuous nonnegative function defined on $(0, \infty)$ and does not vanish identically then we can find a positive constant δ for any positive number L such that*

$$(7.18) \quad u(r)=\mathscr{G}[F(\ ); \nu, c](r)\geqq \delta\varphi_0(r; \nu, L, \alpha) \qquad \textit{for} \quad r\in[0, L].$$

Here $\varphi_0(r; \nu, L, \alpha)$ is the eigenfunction appeared in section 3.

Lemma 7.10. *If $F(r)$ is a bounded continuous nonnegative function defined on $(0, L)$ and does not vanish identically then we can find a positive constant δ such that*

$$(7.19) \quad u(r)=\mathscr{G}[F(\ ); \nu, c, L, \alpha](r)\geqq \delta\varphi_0(r; \nu, L, \alpha) \quad \textit{for } r\in[0, L].$$

We shall only prove lemma 7.10. Proof of lemma 7.9 is not essentially different from that of lemma 7.10. So we omit it.

Proof of Lemma 7.10. From the assumptions on the function $F(r)$ we can find suitable positive numbers F_0, L_1 and L_2 $(0<L_1<L_2<L)$ such

that

$$F(r) \geqq F_0 \qquad \text{for} \quad r \in [L_1, L_2]. \tag{7.20}$$

From (7.20) we have

$$u(r) = \int_0^L G(r, s; \nu, c, L, \alpha) F(s) s ds \geqq \tag{7.21}$$

$$\geqq F_0 \int_{L_1}^{L_2} G(r, s; \nu, c, L, \alpha) s ds = F_0 \underline{u}(r).$$

Here

$$\underline{u}(r) = \int_{L_1}^{L_2} G(r, s; \nu, c, L, \alpha) s ds . \tag{7.22}$$

It is easy to see that $\varphi_0(r; \nu, L, \alpha)$ satisfies that:

$$\varphi_0(r; \nu, L, \alpha) \in C^0[0, L],$$

$$\varphi_0(r; \nu, L, \alpha) = O(r^\nu) \qquad \text{as} \quad r \to 0,$$

$$0 < \varphi_0(L; \nu, L, \alpha) \leqq 1 \quad (\text{if } 0 \leqq \alpha < 1),$$

$$\varphi_0(L; \nu, L, 1) = 0, \qquad 0 < -\varphi_0'(L; \nu, L, 1) < \infty \quad (\text{if } \alpha = 1).$$

So to show lemma 7.10 it is sufficient to show that:

$$\lim_{r \to 0} r^{-\nu} \underline{u}(r) > 0, \tag{7.23}$$

$$\underline{u}(L) > 0 \qquad (\text{if } 0 \leqq \alpha < 1), \tag{7.24}$$

and

$$\underline{u}(L) = 0, \qquad \underline{u}'(L) < 0 \quad (\text{if } \alpha = 1). \tag{7.25}$$

Let $r \in (0, L_1)$. We have from (7.22)

$$\underline{u}(r) = I_\nu(cr) \int_{L_1}^{L_2} \{K_\nu(cs) - g(\nu, c, L, \alpha) I_\nu(cs)\} s ds . \tag{7.26}$$

From (7.26) and lemma 6.1 (ii) we have

(7.27) $$\lim_{r\to 0} r^{-\nu}\underline{u}(r)=$$

$$=\Gamma^{-1}(\nu+1)(c/2)^{\nu}\int_{L_1}^{L_2}\{K_\nu(cs)-g(\nu,\ c,\ L,\ \alpha)I_\nu(cs)\}sds>0.$$

(7.27) shows (7.23). From (7.22) we have

(7.28) $$\underline{u}(L)=\{K_\nu(cL)-g(\nu,\ c,\ L,\ \alpha)I_\nu(cL)\}\int_{L_1}^{L_2}I_\nu(cs)sds\,.$$

(7.29) $$\underline{u}'(L)=c\{K_\nu'(cL)-g(\nu,\ c,\ L,\ \alpha)I_\nu'(cL)\}\int_{L_1}^{L_2}I_\nu(cs)sds\,.$$

From (7.6) and lemma 6.1 (i) we have

(7.30) $$\underline{u}(L)=(1-\alpha)\{(1-\alpha)cLI_\nu'(cL)+\alpha I_\nu(cL)\}^{-1}\int_{L_1}^{L_2}I_\nu(cs)sds \begin{cases} >0 & (\text{if } 0\leqq\alpha<1) \\ =0 & (\text{if } \alpha=1). \end{cases}$$

(7.31) $$\underline{u}'(L)=-L^{-1}I_\nu^{-1}(cL)\int_{L_1}^{L_2}I_\nu(cs)sds<0 \qquad (\text{if } \alpha=1).$$

(7.30) and (7.31) show (7.24) and (7.25). This completes the proof of lemma 7.10.

Lemma 7.11.

(i) $\frac{\partial}{\partial L}G(r,\ s;\ \nu,\ c,\ L,\ 1)<0$ for $(r,\ s)\in(0,\ L)\times(0,\ L)$.

(ii) If $0<L_1<L_2$ then we have

(7.32) $$G(r,\ s;\ \nu,\ c,\ L_1,\ 1)<G(r,\ s;\ \nu,\ c,\ L_2,\ 1)<G(r,\ s;\ \nu,\ c)$$

for $(r,\ s)\in(0,\ L_1)\times(0,\ L_1)$.

Proof of Lemma 7.11. The first inequality in (7.32) follows from (i). The second inequality in (7.32) follows from (7.5) and (7.6). From (7.5), (7.6) and lemma 6.1 (i) we have

$$(7.33)\quad \frac{\partial}{\partial L}G(r, s; \nu, c, L, 1) = -\frac{\partial}{\partial L}g(\nu, c, L, 1)I_\nu(cr)I_\nu(cs) =$$

$$= L^{-1}I_\nu^{-2}(cL)I_\nu(cr)I_\nu(cs) > 0 \qquad \text{for} \quad (r, s) \in (0, L) \times (0, L).$$

(7.33) shows (i).

Lemma 7.12.

(i) $\frac{\partial}{\partial \alpha}G(r, s; \nu, c, L, \alpha) < 0$ *for* $(r, s) \in (0, L] \times (0, L]$.

(ii) If $1 \geqq \alpha_1 > \alpha_2 \geqq 0$ *we have*

$$(7.34)\quad G(r, s; \nu, c, L, \alpha_1) < G(r, s; \nu, c, L, \alpha_2) \quad \textit{for} \ (r, s) \in (0, L] \times (0, L].$$

Proof of Lemma 7.12. From (7.5), (7.6) and lemma 6.1 (i) we have

$$(7.35)\quad \frac{\partial}{\partial \alpha}G(r, s; \nu, c, L, \alpha) = -\frac{\partial}{\partial \alpha}g(\nu, c, L, \alpha)I_\nu(cr)I_\nu(cs) =$$

$$= -\{(1-\alpha)cLI_\nu'(cL) + \alpha I_\nu(cL)\}^{-2}I_\nu(cr)I_\nu(cs) < 0$$

$$\text{for} \quad (r, s) \in (0, L] \times (0, L].$$

(7.35) shows (i). (7.34) follows from (i).

8. Proof of Theorem 1

Suppose that BVP (2.2) has a nontrivial solution $w(r)$ which satisfies $0 \leqq w(r) \leqq 1$ for $r \in [0, L]$. By lemma 7.8 we have

$$(8.1)\qquad w(r) = \int_0^L G(r, s; \nu, c, L, \alpha)F(w(s); c)s\,ds$$

and

$$(8.2)\quad \varphi_0(r; \nu, L, \alpha) = \int_0^L G(r, s; \nu, c, L, \alpha)\mu_0(\nu, c, L, \alpha)\varphi_0(s; \nu, L, \alpha)s\,ds\,.$$

It follows by integration applied to equation (8.1) multiplied with $\mu_0(\nu, c, L, \alpha)\varphi_0(r; \nu, L, \alpha)r$

$$(8.3)\quad \int_0^L \mu_0(\nu, c, L, \alpha)\varphi_0(r; \nu, L, \alpha)w(r)r\,dr=$$

$$=\int_0^L\int_0^L G(r, s; \nu, c, L, \alpha)F(w(s); c)\mu_0(\nu, c, L, \alpha)\varphi_0(r; \nu, L, \alpha)s\,ds\,r\,dr.$$

It follows by integration applied to equation (8.2) multiplied with $F(w(r); c)r$

$$(8.4)\quad \int_0^L \varphi_0(r; \nu, L, \alpha)F(w(r); c)r\,dr=$$

$$=\int_0^L\int_0^L G(r, s; \nu, c, L, \alpha)\mu_0(\nu, c, L, \alpha)\varphi_0(s; \nu, L, \alpha)F(w(r); c)s\,ds\,r\,dr.$$

By the symmetricity of the Green's function (lemma 7.4 (ii)) the right hand sides of both (8.3) and (8.4) are identical. By subtraction we have

$$(8.5)\qquad \int_0^L \varphi_0(r; \nu, L, \alpha)\{\mu_0(\nu, c, L, \alpha)w(r)-F(w(r); c)\}r\,dr=0.$$

Now the interval $[0, L]$ is subcritical, that is, $\mu_0(\nu, c, L, \alpha)\geqq F'(0; c)$. Since $F''(w; c)<0$ for $w\in(0, 1)$ then the function $\mu_0(\nu, c, L, \alpha)w(r)-F(w(r); c)$ is continuous nonnegative and is not identically zero. Since $\varphi_0(r; \nu, L, \alpha)>0$ for $r\in(0, L)$ then the value of the left hand side of (8.5) must be strictly positive. This is a contradiction. This completes the proof of theorem 1.

9. Proof of Theorem 2

Theorem 2 is obtained by almost direct application of the results obtained by A. Pazy and P. Rabinowitz ([7]). For the sake of self-consistency we show the outline of the proof of theorem 2.

Proposition 9.1.

(i) $1\equiv\bar{w}_0(r; \nu, c, L, \alpha)>\bar{w}_1(r; \nu, c, L, \alpha)\geqq\cdots\geqq$

$\geqq\bar{w}_{j-1}(r; \nu, c, L, \alpha)\geqq\bar{w}_j(r; \nu, c, L, \alpha)\geqq 0$ *for* $r\in[0, L]$

$j=1, 2, 3, \cdots.$

(ii) $\bar{w}(r; \nu, c, L, \alpha)=\lim_{j\to\infty} \bar{w}_j(r; \nu, c, L, \alpha)$ *is the solution of* BVP (2.2) *and it satisfies*

$$0 \leqq \bar{w}(r; \nu, c, L, \alpha) \leqq 1 \qquad \textit{for} \quad r \in [0, L].$$

Proof of Proposition 9.1. By lemma 7.8 it follows from IS (4.2)

$$(9.1) \qquad \bar{w}_j(r; \nu, c, L, \alpha)=\mathscr{G}[F(\bar{w}_{j-1}(\ ; \nu, c, L, \alpha); c); \nu, c, L, \alpha](r)$$
$$\text{for} \quad r \in (0, L) \quad j=1, 2, 3, \cdots.$$

By lemma 7.4 (viii) we have

$$\bar{w}_1(r; \nu, c, L, \alpha)=\mathscr{G}[F(1; c); \nu, c, L, \alpha](r)<$$
$$<\mathscr{G}[c^2+\nu^2 r^{-2}; \nu, c, L, \alpha](r) \leqq 1 \equiv \bar{w}_0(r; \nu, c, L, \alpha) \quad \text{for} \quad r \in (0, L).$$

Therefore by the monotonicity of the nonlinearity $F(w; c)$ (assumption $1'$ (iv)) and by the positivity of the Green's function $G(r, s; \nu, c, L, \alpha)$ (lemma 7.4 (iii)) we have step-by-step the required inequality of proposition 9.1 (i). If follows that $\bar{w}_j(r; \nu, c, L, \alpha)$ are uniformly bounded with respect to j and converge to the limit function $\bar{w}(r; \nu, c, L, \alpha)$ for every fixed $r \in [0, L]$ as j tends to infinity. Letting j tend to infinity in (9.1) the limit function $\bar{w}(r; \nu, c, L, \alpha)$ satisfies the integral equality

$$(9.2) \qquad \bar{w}(r; \nu, c, L, \alpha)=\mathscr{G}[F(\bar{w}(\ ; \nu, c, L, \alpha); c); \nu, c, L, \alpha](r)$$
$$\text{for} \quad r \in (0, L).$$

By lemma 7.8 it follows from (9.2) that $\bar{w}(r; \nu, c, L, \alpha)$ satisfies BVP (2.2′) or equivalently BVP (2.2). This completes the proof of proposition 9.1.

Proposition 9.2.

(i) If $0<\delta \leqq \delta_0(\lambda_0(\nu, L, \alpha))$ *we have*

$$\delta\varphi_0(r; \nu, L, \alpha) \equiv w_0(r; \nu, c, \alpha, \delta) \leqq w_1(r; \nu, c, L, \alpha, \delta) \leqq \cdots \leqq$$
$$\leqq \underline{w}_{j-1}(r; \nu, c, L, \alpha, \delta) \leqq \underline{w}_j(r; \nu, c, L, \alpha, \delta) \leqq 1 \quad \text{for } r \in (0, L).$$

(ii) $\underline{w}(r;\nu,c,L,\alpha,\delta)=\lim_{j\to\infty}\underline{w}_j(r;\nu,c,L,\alpha,\delta)$ *is the nontrivial solution of* BVP (2.2) *satisfying*

$$0\leqq\underline{w}(r;\nu,c,L,\alpha,\delta)\leqq 1 \qquad for \quad r\in[0,L].$$

Proof of Proposition 9.2. By lemma 7.8 it follows from IS (4.3)

$$(9.3)\quad \underline{w}_j(r;\nu,c,L,\alpha,\delta)=\mathscr{G}[F(\underline{w}_{j-1}(\ ;\nu,c,L,\alpha,\delta);c);\nu,c,L,\alpha](r)$$
$$\text{for}\quad r\in(0,L)\quad j=1,2,3,\cdots.$$

Since the interval $[0, L]$ is supercritical, that is, $\mu_0(\nu,c,L,\alpha)<F'(0;c)$ then we have

$$(9.4)\qquad F(\delta\varphi_0(r;\nu,L,\alpha);c)\geqq\mu_0(\nu,c,L,\alpha)\delta\varphi_0(r;\nu,L,\alpha)$$
$$\text{for } r\in[0,L].$$

From (9.3) and (9.4) we have

$$(9.5)\quad \underline{w}_1(r;\nu,c,L,\alpha,\delta)=\mathscr{G}[F(\delta\varphi_0(\ ;\nu,L,\alpha);c);\nu,c,L,\alpha](r)\geqq$$
$$\geqq\mathscr{G}[\mu_0(\nu,c,L,\alpha)\delta\varphi_0(\ ;\nu,L,\alpha);\nu,c,L,\alpha](r)=$$
$$=\delta\varphi_0(r;\nu,L,\alpha)\equiv\underline{w}_0(r;\nu,c,L,\alpha,\delta)\qquad\text{for}\quad r\in(0,L).$$

By the monotonicity of the nonlinearity and the positivity of the Green's function starting from (9.5) we have step-by-step the required inequality of lemma 9.2 (i). It follows that $\underline{w}_j(r;\nu,c,L,\alpha,\delta)$ are uniformly bounded with respect to j and converge to the limit function $\underline{w}(r;\nu,c,L,\alpha,\delta)$ for every fixed $r\in[0,L]$ as j tends to infinity. Letting j tend to infinity in (9.3) we have

$$(9.6)\quad \underline{w}(r;\nu,c,L,\alpha,\delta)=\mathscr{G}[F(\underline{w}(\ ;\nu,c,L,\alpha,\delta);c);\nu,c,L,\alpha](r)$$
$$r\in(0,L).$$

By lemma 7.8 it follows from (9.5) that $\underline{w}(r;\nu,c,L,\alpha,\delta)$ satisfies BVP (2.2). This completes the proof of proposition 9.2.

Proposition 9.3.

(i) *For any* δ *satisfying* $0<\delta\leqq\delta_0(\lambda_0(\nu, L, \alpha))$ *we have*

$$\bar{w}(r; \nu, c, L, \alpha)\geqq\underline{w}(r; \nu, c, L, \alpha, \delta) \qquad \textit{for } r\in[0, L].$$

(ii) *For any nontrivial solution* $w(r)$ *of* BVP (2.2) *satisfying* $0\leqq w(r)\leqq 1$ *for* $r\in[0, L]$ *we can find a constant* δ $(0<\delta\leqq\delta_0(\lambda_0(\nu, L, \alpha)))$ *such that*

$$\bar{w}(r: \nu, c, L, \alpha)\geqq w(r)\geqq\underline{w}(r; \nu, c, L, \alpha, \delta) \qquad \textit{for } r\in[0, L].$$

Proof of Proposition 9.3. Since we have

$$(9.7) \qquad \bar{w}_0(r; \nu, c, L, \alpha)\equiv 1>\delta\varphi_0(r; \nu, L, \alpha)\equiv\underline{w}_0(r; \nu, c, L, \alpha, \delta)$$
$$\text{for } r\in(0, L),$$

it follows

$$(9.8) \qquad \bar{w}_1(r; \nu, c, L, \alpha)=\mathscr{G}[F(\bar{w}_0(\ ; \nu, c, L, \alpha); c); \nu, c, L, \alpha](r)\geqq$$
$$\geqq\mathscr{G}[F(\underline{w}_0(\ ; \nu, c, L, \alpha, \delta); c); \nu, c, L, \alpha](r)=\underline{w}_1(r; \nu, c, L, \alpha, \delta)$$
$$\text{for } r\in(0, L).$$

From (9.1), (9.3) and (9.8) it follows step-by-step

$$(9.9) \quad \bar{w}_j(r; \nu, c, L, \alpha)\geqq\underline{w}_j(r; \nu, c, L, \alpha, \delta) \qquad \text{for } r\in(0, L)\ j=1, 2, 3, \cdots.$$

Letting j tend to infinity in (9.9) we have the required inequality in (i). By lemma 7.8 the nontrivial solution $w(r)$ of BVP (2.2) satisfies the integral equality

$$(9.10) \qquad w(r)=\mathscr{G}[F(w(\ ); c); \nu, c, L, \alpha](r) \qquad \text{for } r\in(0, L).$$

By lemma 7.10 we can find a positive constant δ $(0<\delta\leqq\delta_0(\lambda_0(\nu, L, \alpha)))$ such that

$$(9.11) \qquad w(r)\geqq\delta\varphi_0(r; \nu, L, \alpha)\equiv\underline{w}_0(r; \nu, c, L, \alpha, \delta) \qquad \text{for } r\in(0, L).$$

By the same reasoning as above we have from (9.11) that:

(9.12) $\bar{w}_1(r; \nu, c, L, \alpha) \geqq w(r) \geqq \underline{w}_1(r; \nu, c, L, \alpha, \delta)$ for $r \in (0, L)$.

From (9.12) we have step-by-step

(9.13) $\bar{w}_j(r; \nu, c, L, \alpha) \geqq w(r) \geqq \underline{w}_j(r; \nu, c, L, \alpha, \delta)$ for $r \in (0, L)$

$j = 1, 2, 3, \cdots$.

Letting j tend to infinity in (9.13) we obtain the required inequality in (ii). This completes the proof of proposition 9.3.

Proposition 9.4. *For any* δ *satisfying* $0 < \delta \leqq \delta_0(\lambda_0(\nu, L, \alpha))$ *we have*

(9.14) $\bar{w}(r; \nu, c, L, \alpha) \equiv \underline{w}(r; \nu, c, L, \alpha, \delta)$ *for* $r \in [0, L]$.

Proof of Proposition 9.4. By the symmetricity of the Green's function $G(r, s; \nu, c, L, \alpha)$ (lemma 7.4 (ii)) it follows from (9.2) and (9.6)

(9.15)
$$\int_0^L \{F(\underline{w}(r; \nu, c, L, \alpha, \delta); c)\underline{w}^{-1}(r; \nu\ c, L, \alpha, \delta) -$$
$$- F(\bar{w}(r; \nu, c, L, \alpha); c)\bar{w}^{-1}(r; \nu, c, L, \alpha)\}\bar{w}(r; \nu, c, L, \alpha) \times$$
$$\times \underline{w}(r; \nu, c, L, \alpha, \delta) r\,dr = 0.$$

Suppose that $\bar{w}(r; \nu, c, L, \alpha) - \underline{w}(r; \nu, c, L, \alpha, \delta)$ do not vanish identically. Since $F''(w; c) < 0$ for $w \in (0, 1)$ (assumption 1′ (iii)) then the integrand of above integral in (9.15) is a nonnegative continuous function of r which does not vanish identically. So the value of the above integral in (9.15) is strictly positive. This is a contradiction. This proves proposition 9.4.

Proposition 9.3 and 9.4 show that the limit function $\bar{w}(r; \nu, c, L, \alpha) \equiv \underline{w}(r; \nu, c, L, \alpha, \delta)$ dose not depend on the special choice of the constants c and δ. So we write this limit function as $w(r; \nu, L, \alpha)$. Thus we obtained the nontrivial solution $w(r; \nu, L, \alpha)$ of BVP (2.2). Proposition 9.4 together with Proposition 9.3 asserts the uniqueness of the nontrivial solution of BVP (2.2) in the class of functions mentioned in theorem 2. Theorem 2 (iii) follows from

$$0<\delta_0(\lambda_0(\nu, L, \alpha))\varphi_0(r; \nu, L, \alpha)\leqq w(r; \nu, L, \alpha)\leqq \bar{w}_1(r; \nu, c, L, \alpha)<1 \quad \text{for } r\in(0, L).$$

Theorem 2 (ii) and (vi) follows from the expression

$$(9.16)\qquad w(r; \nu, L, \alpha)=u_0(\nu, L, \alpha)r^{\nu}-$$

$$-r^{\nu}(2\nu)^{-1}\int_0^r f(w(s; \nu, L, \alpha))(s^{1-\nu}-r^{-2\nu}s^{1+\nu})ds \qquad \text{for } r\in(0, L).$$

Here $u_0(\nu, L, \alpha)$ is given by (5.1). (9.16) follows from lemma 7.6. We omit the detailed discussion because which is not essentially different from that for theorem 6 (iv) and theorem 12.

10. Proof of Theorem 3 and 4

By lemma 7.8 $w(r; \nu, L_1, 1)$ and $w(r; \nu, L_2, 1)$ satisfy following integral equalities respectively.

$$(10.1)\qquad w(r; \nu, L_1, 1)=\mathscr{G}[F(w(\ ; \nu, L_1, 1); c); \nu, c, L_1, 1](r) \quad \text{for } r\in(0, L_1).$$

$$(10.2)\qquad w(r; \nu, L_2, 1)=\mathscr{G}[F(w(\ ; \nu, L_2, 1); c); \nu, c, L_2, 1](r) \quad \text{for } r\in(0, L_2).$$

From theorem 2 (iii) we have

$$(10.3)\qquad w(r; \nu, L_1, 1)<1 \qquad \text{for } r\in(0, L_1).$$

By lemma 7.11 it follows from (9.1) in which L is replaced by L_2 and from (10.3)

$$\bar{w}_1(r; \nu, c, L_2, 1)=\mathscr{G}[F(1; c); \nu, c, L_2, 1](r)$$

$$\geqq \mathscr{G}[F(w(\ ; \nu, L_1, 1); c); \nu, c, L_1, 1](r)=w(r; \nu, L_1, 1) \quad \text{for } r\in(0, L_1).$$

Repeating these arguments we have

(10.4) $\bar{w}_j(r;\nu,c,L_2,1)>w(r;\nu,L_1,1)$ for $r\in(0,L_1)$.

Letting j tend to infinity in (10.4) we have

(10.5) $w(r;\nu,L_2,1)\geqq w(r;\nu,L_1,1)$ for $r\in(0,L_1)$.

By lemma 7.11 it follows from (10.1), (10.2) and (10.5)

$$w(r;\nu,L_2,1)>w(r;\nu,L_1,1) \quad \text{for } r\in(0,L_1).$$

This proves theorem 3. Proof of theorem 4 is not essentially different from that of theorem 3. The part of lemma 7.11 must be replaced by lemma 7.12. We omit the detailed discussion.

11. Proof of Theorem 5

Replacing dependent variable $w(r)$ by $r^\nu u(r)$ the nonlinear Bessel equation $\mathscr{L}[w(\ );\nu](r)=f(w(r))$ reduces to

(11.1) $$r^2u''(r)+(2\nu+1)ru'(r)+f(r^\nu u(r))r^{2-\nu}=0.$$

Inserting formal power series $u(r;\nu,u_0)=\sum_{k=0}^{\infty}u_kr^k$ into $u(r)$ in (11.1) we obtain the formal relation

(11.2) $$(2\nu+1)u_1r+\sum_{l=2}^{\infty}r^l\{l(l+2\nu)u_l+\sum_{\substack{(j-1)\nu+k=l-2\\ j\geqq1,\,k\geqq0}}f_j\sum_{k_1+\cdots+k_j=k}u_{k_1}\cdots u_{k_j}\}=0.$$

Equating the coefficient of r^l of the left hand side of (11.2) to zero for $l=1,2,3,\cdots$ we obtain the recurrence formula which determine the coefficients u_l recurrently and uniquely.

(11.3) $$\begin{cases} u_1=0, \\ u_l=-l^{-1}(l+2\nu)^{-1}\sum_{\substack{(j-1)\nu+k=l-2\\ j\geqq1,\,k\geqq0}}f_j\sum_{k_1+\cdots+k_j=k}u_{k_1}\cdots u_{k_j} \end{cases}$$

$$l=2,3,4,\cdots.$$

u_0 may be any positive constant in the case of assumption 2 and may be any complex constant in the case of assumption 2′. Under the assumption 2′ $\hat{f}(w)=\sum_{j=1}^{\infty}|f_j|w^j$ has the same radius of convergence as $f(w)$. So $h(w)=w^{-1}\hat{f}(w)$ is a holomorphic function of w in a complex neighborhood of $w=0$. By the implicit function theorem we have the unique holomorphic solution $U(r)$ of the equation

$$(11.4)\qquad \begin{cases} U(r)-|u_0|-r^2U(r)h(r^\nu U(r))=0 \\ U(0)=|u_0|. \end{cases}$$

Replacing $U(r)$ in (11.4) by its Taylor series expansion $\sum_{l=0}^{\infty} U_l r^l$ we have the equation:

$$(11.5)\qquad \sum_{l=0}^{\infty} U_l r^l - |u_0| - \sum_{l=2}^{\infty} \sum_{\substack{(j-1)\nu+k=l-2 \\ j\geqq 1, k\geqq 0}} |f_j| \sum_{k_1+\cdots+k_j=k} U_{k_1}\cdots U_{k_j} r^l = 0.$$

It follows from (11.5) the recurrence formula:

$$(11.6)\qquad U_0=|u_0|,\ U_1=0,$$

$$U_l=\sum_{\substack{(j-1)\nu+k=l-2 \\ j\geqq 1, k\geqq 0}} |f_j| \sum_{k_1+\cdots+k_j=k} U_{k_1}\cdots U_{k_j} \qquad l=2, 3, 4, \cdots.$$

From (11.3) and (11.6) we have $|u_l|\leqq U_l$ for $l=0, 1, 2, \cdots$ This means the convergent power series $U(r)=\sum_{l=0}^{\infty} U_l r^l$ is a majorant series of the formal power series $u(r; \nu, u_0)=\sum_{l=0}^{\infty} u_l r^l$. Thus we conclude that the formal power series solution $u(r; \nu, u_0)$ is a holomorphic function of r in a complex neighborhood of $r=0$. Proof of theorem 5 is completed.

12. Proof of Theorem 6 and 7

Proposition 12.1.

(i) $1\equiv\bar{w}_0(r; \nu, c)>\bar{w}_1(r; \nu, c)\geqq\cdots\geqq$

$\geqq\bar{w}_{j-1}(r; \nu, c)\geqq\bar{w}_j(r; \nu, c)\geqq 0$ *for* $r\in[0, \infty)$ $j=1, 2, 3, \cdots$.

(ii) $\bar{w}(r; \nu, c)=\lim_{j\to\infty} \bar{w}_j(r; \nu, c)$ *satisfies the integral equality*:

$$(12.1)\qquad \bar{w}(r; \nu, c)=\mathscr{G}[F(\bar{w}(\ ; \nu, c); c); \nu, c](r) \qquad \textit{for } r\in(0, \infty).$$

Proof of Proposition 12.1. By lemma 7.7 we have

(12.2) $\bar{w}_j(r;\nu,c)=\mathscr{G}[F(\bar{w}_{j-1}(\ ;\nu,c);c);\nu,c](r)$ for $r\in(0,\infty)$.

By lemma 7.3 (vii) we have

(12.3) $\bar{w}_1(r;\nu,c)=\mathscr{G}[F(1;c);\nu,c](r)<$

$$<\mathscr{G}[c^2+\nu^2r^{-2};\nu,c](r)\equiv 1\equiv\bar{w}_0(r;\nu,c) \quad \text{for } r\in(0,\infty).$$

From (12.2) and (12.3) we have

$$\bar{w}_2(r;\nu,c)=\mathscr{G}[F(\bar{w}_1(\ ;\nu,c);c);\nu,c](r)\leqq$$

$$\leqq\mathscr{G}[F(1;c);\nu,c](r)=\bar{w}_1(r;\nu,c) \quad \text{for } r\in(0,\infty).$$

Repeating these arguments we have

(12.4) $\bar{w}_j(r;\nu,c)\leqq\bar{w}_{j-1}(r;\nu,c)$ for $r\in(0,\infty)$ $j=1,2,3,\cdots$.

From (12.4) it follows that $\bar{w}_j(r;\nu,c)$ tends to the limit function $\bar{w}(r;\nu,c)$ monotone decreasingly as j tends to infinity. Letting j tend to infinity in (12.2) we have (12.1). This completes the proof of proposition 12.1.

Proposition 12.2. *The solution $w(r;\nu,L,1)$ of* BVP (2.2) *with $\alpha=1$ converges to the limit function $\underline{w}(r;\nu)$ monotone increasingly and compact uniformly with respect to $r\in[0,\infty)$ as L tends to infinity. $\underline{w}(r;\nu)$ satisfies the integral equality*:

(12.5) $\underline{w}(r;\nu)=\mathscr{G}[F(\underline{w}(\ ;\nu);c);\nu,c](r)$ *for* $r\in(0,\infty)$.

Proof of Proposition 12.2. By lemma 7.8 $w(r;\nu,L,1)$ satisfies the integral equality:

(12.6) $w(r;\nu,L,1)=\mathscr{G}[F(w(\ ;\nu,L,1);c);\nu,c,L,1](r)$ for $r\in(0,\infty)$.

By theorem 3 $w(r;\nu,L,1)$ converges to the limit function $\underline{w}(r;\nu)$ monotone increasingly as L tends to infinity. Letting L tend to infinity in (12.6) we have (12.5). It follows that $\underline{w}(r;\nu)$ is continuous with respect to r. By Dini's theorem. $w(r;\nu,L,1)$ converge to $\underline{w}(r;\nu)$ monotone increasingly and compact uniformly with respect to $r\in[0,\infty)$ as L tends to infinity.

Proposition 12.3. *If the function* $w(r)$ *belongs to the class of functions* $\{w(r)\in C^2(0,\infty);\ 0\leqq w(r)\leqq 1,\ w(r)\not\equiv 0$ *for* $r\in(0,\infty)\}$ *and satisfies the nonlinear Bessel equation* $\mathscr{L}[w(\ );\nu](r)=f(w(r))$ *for* $r\in(0,\infty)$ *then it satisfies the inequality*

$$(12.7)\qquad \underline{w}(r;\nu)\leqq w(r)\leqq \bar{w}(r;\nu,c)\qquad \textit{for } r\in(0,\infty).$$

Proof of Proposition 12.3. By lemma 7.7 $w(r)$ satisfies the integral equality:

$$(12.8)\qquad w(r)=\mathscr{G}[F(w(\ );c);\nu,c](r)\qquad \text{for } r\in(0,\infty).$$

For any positive number L (for which the interval $[0,L]$ is supercritical) by lemma 7.9 we can find a constant δ $(0<\delta\leqq\delta_0(\lambda_0(\nu,L,1)))$ such that

$$(12.9)\qquad w(r)\geqq\delta\varphi_0(r;\nu,L,1)\ \equiv\underline{w}_0(r;\nu,c,L,1,\delta)\qquad \text{for } r\in(0,L).$$

By lemma 7.11 it follows from (12.9)

$$(12.10)\qquad w(r)=\mathscr{G}[F(w(\ );c);\nu,c](r)\geqq$$
$$\geqq\mathscr{G}[F(\underline{w}_0(;\nu,c,L,1,\delta);c);\nu,c,L,1](r)=\underline{w}_1(r;\nu,c,L,1,\delta)$$
$$\text{for } r\in(0,L).$$

It follows from (9.3), (12.8) and (12.10)

$$(12.11)\qquad w(r)\geqq\underline{w}_j(r;\nu,c,L,1,\delta)\qquad \text{for } r\in(0,L)\ j=1,2,3,\cdots.$$

Letting j tend to infinity in (12.11) we have

$$(12.12)\qquad w(r)\geqq\underline{w}(r;\nu,L,1)\qquad \text{for } r\in(0,L).$$

Letting L tend to infinity we have

$$w(r)\geqq\underline{w}(r;\nu)\qquad \text{for } r\in(0,\infty).$$

Starting from the inequality

$$w(r)\leqq 1\qquad \text{for } r\in(0,\infty),$$

we have by the similar arguments as above

$$(12.13)\qquad w(r)\leqq\bar{w}_j(r;\nu,c)\qquad \text{for } r\in(0,\infty)\quad j=1,2,3,\cdots.$$

Letting j tend to infinity in (12.13) we have

$$w(r) \leqq \bar{w}(r; \nu, c) \qquad \text{for } r \in (0, \infty).$$

This completes the proof of proposition 12.3.

Proposition 12.4.

$$(12.14) \qquad 0 < 1 - \bar{w}(r; \nu, c) \leqq 1 - \underline{w}(r; \nu) \leqq \text{const. } r^{-2} \qquad \textit{for } r \in (0, \infty).$$

Proof of Proposition 12.4. Suppose that two positive numbers r and L are connected by the relation

$$(12.15) \qquad r = r_{\max}(\nu, L, 1) = j_1(\nu, 0) j_1^{-1}(\nu, 1) L.$$

By the normalization of the eigenfunction we have from (12.15)

$$(12.16) \qquad \varphi_0(r; \nu, L, 1) = 1.$$

For sufficiently large r (for which the corresponding L satisfies $\lambda_0(\nu, L, 1) < f(\underline{\delta})\underline{\delta}^{-1}$) we have by proposition 12.2

$$(12.17) \qquad \underline{w}(r; \nu) \geqq w(r; \nu, L, 1) \geqq \delta_0(\lambda_0(\nu, L, 1)) \varphi_0(r; \nu, L, 1).$$

From (4.6), (4.7), (12.16) and (12.17) we have

$$1 - \underline{w}(r; \nu) \leqq 1 - \delta_0(\lambda_0(\nu, L, 1)) \leqq 1 - \underline{\delta}_0(\lambda_0(\nu, L, 1)) =$$

$$= \lambda_0(\nu, L, 1) \{\lambda_0(\nu, L, 1) + f(\underline{\delta})(1 - \underline{\delta})^{-1}\}^{-1} \leqq \text{const. } L^{-2} \leqq \text{const. } r^{-2}.$$

This completes the proof of proposition 12.4.

Proposition 12.5.

(i) $\bar{w}_j(r; \nu, c),\ r\bar{w}_j'(r; \nu, c),\ r^2 \bar{w}_j''(r; \nu, c) =$

$$= \begin{cases} O(r^\nu) & (\textit{if } \nu < 2j) \\ O(r^{2j} |\log r|) & (\textit{if } \nu = 2j) \\ O(r^{2j}) & (\textit{if } \nu > 2j) \quad \textit{as } r \to 0. \end{cases}$$

(ii) $\bar{w}(r; \nu, c),\ r\bar{w}'(r; \nu, c),\ r^2 \bar{w}''(r; \nu, c) = O(r^\nu) \qquad \textit{as } r \to 0.$

(iii) $\underline{w}(r;\nu),\ r\underline{w}'(r;\nu),\ r^2\underline{w}''(r;\nu)=O(r^{\nu})$ *as* $r\to 0$.

(iv) $\bar{w}_j'(r;\nu,c),\ \bar{w}'(r;\nu,c),\ \underline{w}'(r;\nu)=O(1)$ *as* $r\to\infty$ $j=1,2,3,\cdots$.

(v) $\bar{w}_j''(r;\nu,c),\ \bar{w}''(r;\nu,c),\ \underline{w}''(r;\nu)=O(1)$ *as* $r\to\infty$

$j=1,2,3,\cdots$.

Proof of Proposition 12.5. Proposition 12.5 is easily obtained by repeated applications of lemma 7.7 to (12.1), (12.2) and (12.5). So we omit the detailed calculation.

Proposition 12.6.

$$\bar{w}(r;\nu,c)\equiv\underline{w}(r;\nu)\qquad \textit{for } r\in(0,\infty). \tag{12.18}$$

Proof of Proposition 12.6. By lemma 7.7 it follows from (12.1) and (12.5) that $\bar{w}(r;\nu,c)$ and $\underline{w}(r;\nu)$ satisfy following equations respectively

$$-\{r\bar{w}'(r;\nu,c)\}'+r(c^2+\nu^2r^{-2})\bar{w}(r;\nu,c)=F(\bar{w}(r;\nu,c);c)r \quad \text{for } r\in(0,\infty). \tag{12.19}$$

$$-\{r\underline{w}'(r;\nu)\}'+r(c^2+\nu^2r^{-2})\underline{w}(r;\nu)=F(\underline{w}(r;\nu);c)r \quad \text{for } r\in(0,\infty). \tag{12.20}$$

From (12.19) and (12.20) we have

$$\begin{aligned}&[r\{\bar{w}'(r;\nu,c)\underline{w}(r;\nu)-\bar{w}(r;\nu,c)\underline{w}'(r;\nu)\}]'=\\ &=\{F(\underline{w}(r;\nu);c)\underline{w}^{-1}(r;\nu)-F(\bar{w}(r;\nu,c);c)\bar{w}^{-1}(r;\nu,c)\}\times\\ &\times\underline{w}(r;\nu)\bar{w}(r;\nu,c)r \qquad \text{for } r\in(0,\infty).\end{aligned} \tag{12.21}$$

By proposition 12.5 (ii) and (iii) we have

$$r\{\bar{w}'(r;\nu,c)\underline{w}(r;\nu)-\bar{w}(r;\nu,c)\underline{w}'(r;\nu)\}=O(r^{2\nu}) \qquad \text{as } r\to 0. \tag{12.22}$$

Integrating both sides of the identity (12.21) over the interval $[0, r]$ and using (12.22) we have

$$(12.23)\qquad r\underline{w}(r;\nu)\bar{w}(r;\nu,c)\{\bar{w}'(r;\nu,c)\,\bar{w}^{-1}(r;\nu,c)-\underline{w}'(r;\nu)\underline{w}^{-1}(r;\nu)\}=$$

$$=\int_0^r\{F(\underline{w}(s;\nu);c)\underline{w}^{-1}(s;\nu)-F(\bar{w}(s;\nu,c);c)\bar{w}^{-1}(s;\nu\ c)\}\underline{w}(s;\nu)\times$$

$$\times\bar{w}(s;\nu,c))sds\quad\text{for } r\in(0,\infty).$$

Suppose that (12.18) do not hold. By assumption 1′ the integrand of the right hand side of (12.23) is continuous, nonnegative and not identically zero. So the value of the right hand side of (12.23) is greater than some positive constant β' for sufficiently large positive r. Thus we obtain the inequality:

$$(12.24)\qquad r\underline{w}(r;\nu)\bar{w}(r;\nu,c)\,[\log\{\bar{w}(r;\nu\ c)\underline{w}^{-1}(r;\nu)\}]'\geqq\beta'>0$$

$$\text{for sufficiently large } r.$$

By proposition 12.4 we can find another suitable positive constant β such that

$$(12.\ 25)\qquad [\log\{\bar{w}(r;\nu,c)\underline{w}^{-1}(r;\nu)\}]'\geqq\{\log r^{\beta}\}'$$

$$\text{for } r\geqq r_1\ (r_1 \text{ is sufficiently large}).$$

Integrating (12.25) over the interval $[r_1, r]$ we have

$$(12.26)\qquad \log\{\bar{w}(r;\nu,c)\underline{w}^{-1}(r;\nu)\}\geqq$$

$$\geqq\log r^{\beta}-\log r_1^{\beta}+\log\{\bar{w}(r_1;\nu,c)\underline{w}^{-1}(r_1;\nu)\}\qquad\text{for } r\geqq r_1.$$

As r tends to infinity the left hand side of (12.26) approaches zero by proposition 12.4, on the other hand the right hand side of (12.26) tends to infinity. This is a contradiction. This proves proposition 12.6.

By proposition 12.6 the limit function $\bar{w}(r;\nu,c)=\underline{w}(r;\nu)$ does not depend on the special choice of the constant c. So we write this limit function as $w(r;\nu)$. By (12.1) or (12.5) $w(r;\nu)$ satisfies the integral equality:

$$(12.27)\qquad w(r;\nu)=\mathscr{G}[F(w(\ ;\nu);c);\nu,c](r)\qquad\text{for } r\in(0,\infty).$$

By proposition 12.5 (ii) or (iii) we have

$$(12.28) \qquad w(r;\nu)=O(r^{\nu}) \qquad \text{as } r\to 0.$$

By proposition 12.4 we have

$$(12.29) \qquad 1-w(r;\nu)=O(r^{-2}) \qquad \text{as } r\to\infty.$$

By lemma 7.7 it follows from (12.27), (12.28) and (12.29) that $w(r;\nu)$ is the solution of BVP (2.1). Theorem 6 (ii) follows from proposition 12.1 and 12.2. Theorem 6 (iii) follows from (12.29). Theorem 7 follows from proposition 12.3 and 12.6. By lemma 7.5 $w(r;\nu)$ satisfies the integral equality:

$$(12.30) \qquad w(r;\nu)=\mathscr{G}[f(w(\ ;\nu));\nu](r) \qquad \text{for } r\in(0,\infty).$$

From (12.30) we have

$$(12.31) \qquad w(r;\nu)=u_0(\nu)r^{\nu}+(2\nu)^{-1}r^{\nu}\int_0^r f(w(s;\nu))\{r^{-2\nu}s^{1+\nu}-s^{1-\nu}\}ds$$
$$\text{for } r\in(0,\infty).$$

Here $u_0(\nu)$ is given by (5.5). By (12.28) it follows easily that the second term of the right hand side of (12.31) is $O(r^{\nu+2})$ as $r\to 0$. This shows theorem 6 (iv). This completes the proof of theorem 6 and 7.

13. Proof of Theorem 8

Theorem 8 (i) and (ii) follows from proposition 12.1, 12.2 and Dini's theorem. From theorem 8 (ii) it follows

$$(13.1) \qquad \lim_{j\to\infty}\|w(r;\nu)-\bar{w}_j(r;\nu,c)\|=0.$$

(12.27) and (12.1) can be rewritten as

$$(13.2) \qquad w(r;\nu)=\int_0^{\infty}G(r,s;\nu,c)F(w(s;\nu);c)sds \qquad \text{for } r\in(0,\infty).$$

$$(13.3) \qquad \bar{w}_j(r;\nu,c)=\int_0^{\infty}G(r,s;\nu,c)F(\bar{w}_{j-1}(s;\nu,c);c)sds$$

for $r\in(0,\infty)\quad j=1,2,3,\cdots$.

From (13.2) and (13.3) we have

$$(13.4)\qquad |w(r;\nu)-\bar{w}_j(r;\nu,c)|\leqq$$

$$\leqq F'(0;c)c^{-2}\int_0^\infty G(r,s;\nu,c)|w(s;\nu)-\bar{w}_{j-1}(s;\nu,c)|c^2 s ds$$

for $r\in(0,\infty)$.

It follows from (13.4)

$$(13.5)\qquad |w(r;\nu)-\bar{w}_j(r;\nu,c)|\leqq$$

$$\leqq F'(0;c)c^{-2}\|w(r;\nu)-\bar{w}_{j-1}(r;\nu,c)\|\int_0^\infty G(r,s;\nu,c)c^2 s ds=$$

$$=F'(0;c)c^{-2}\|w(r;\nu)-\bar{w}_{j-1}(r;\nu,c)\|\bar{w}_1(r;\nu,c)$$

for $r\in(0,\infty)$.

Inserting the inequality (13.5) in which j is replaced by $j-1$ into the right hand side of (13.4) we have

$$(13.6)\quad |w(r;\nu)-\bar{w}_j(r;\nu,c)|\leqq$$

$$\leqq\{F'(0;c)c^{-2}\}^2\|w(r;\nu)-\bar{w}_{j-2}(r;\nu,c)\|\int_0^\infty G(r,s;\nu,c)c^2\bar{w}_1(s;\nu,c)s ds\leqq$$

$$\leqq\{F'(0;c)c^{-2}\}^2\|w(r;\nu)-\bar{w}_{j-2}(r;\nu,c)\|\int_0^\infty G(r,s;\nu,c)F(\bar{w}_1(s;\nu,c);c)s ds=$$

$$=\{F'(0;c)c^{-2}\}^2\|w(r;\nu)-\bar{w}_{j-2}(r;\nu,c)\|\bar{w}_2(r;\nu,c)\qquad \text{for } r\in(0,\infty).$$

Repeating these arguments finally we have

$$(13.7)\qquad |w(r;\nu)-\bar{w}_j(r;\nu,c)|\leqq$$

$$\leqq\{F'(0;c)c^{-2}\}^{[\nu/2]+1}\bar{w}_{[\nu/2]+1}(r;\nu,c)\|w(r;\nu)-\bar{w}_{j-[\nu/2]-1}(r;\nu,c)\|$$

for $r\in(0,\infty)$.

By proposition 12.5 (i)

$$\|(r^{-\nu}+1)\bar{w}_{[\nu/2]+1}(r;\nu,c)\|$$

is a finite number therefore we have from (13.7)

$$\begin{aligned}(13.8)\qquad &\|(r^{-\nu}+1)\{w(r;\nu)-\bar{w}_j(r;\nu,c)\}\|\leqq\\ &\leqq\{F'(0;c)c^{-2}\}^{[\nu/2]+1}\|(r^{-\nu}+1)\bar{w}_{[\nu/2]+1}(r;\nu,c)\|\times\\ &\times\|w(r;\nu)-\bar{w}_{j-[\nu/2]-1}(r;\nu,c)\|.\end{aligned}$$

From (13.8) and (13.1) we have (5.6). Differentiating (13.2) and (13.3) with respect to r we have

$$(13.9)\qquad w'(r;\nu)=\int_0^\infty\frac{\partial}{\partial r}G(r,s;\nu,c)F(w(s;\nu);c)sds\qquad\text{for } r\in(0,\infty).$$

$$(13.10)\qquad \bar{w}_j'(r;\nu,c)=\int_0^\infty\frac{\partial}{\partial r}G(r,s;\nu,c)\,F(\bar{w}_{j-1}(s;\nu,c);c)sds$$

$$\text{for } r\in(0,\infty)\quad j=1,2,3,\cdots.$$

From (13.9) and (13.10) we have

$$\begin{aligned}(13.11)\qquad &|w'(r;\nu)-\bar{w}_j'(r;\nu,c)|\leqq\\ &\leqq F'(0;c)c^{-2}\int_0^\infty\left|\frac{\partial}{\partial r}G(r,s;\nu,c)\right||w(s;\nu)-\bar{w}_{j-1}(s;\nu,c)|c^2sds\leqq\\ &\leqq C(\nu,c)(r^{-1}+1)F'(0;c)c^{-2}\int_0^\infty G(r,s;\nu,c)|w(s;\nu)-\bar{w}_{j-1}(s;\nu,c)|c^2sds\\ &\qquad\text{for } r\in(0,\infty).\end{aligned}$$

By the same arguments as was used to obtain (13.7) from (13.4) we have from (13.11)

$$\begin{aligned}(13.12)\qquad &|w'(r;\nu)-\bar{w}_j'(r;\nu,c)|\leqq C(\nu,c)(r^{-1}+1)\{F'(0;c)c^{-2}\}^{[\nu/2]+1}\times\\ &\times\bar{w}_{[\nu/2]+1}(r;\nu,c)\|\|w(r;\nu)-\bar{w}_{j-[\nu/2]-1}(r;\nu,c)\|\quad\text{for } r\in(0,\infty).\end{aligned}$$

From (13.12) we have

$$(13.13)\qquad \|(r^{-\nu}+1)r(r+1)^{-1}\{w'(r;\nu)-\bar{w}_j'(r;\nu,c)\}\|\leqq$$

$$\leqq C(\nu, c)\{F'(0; c)c^{-2}\}^{[\nu/2]+1}\|(r^{-\nu}+1)\bar{w}_{[\nu/2]+1}(r; \nu, c)\| \times$$

$$\times \|w(r; \nu)-\bar{w}_{j-[\nu/2]-1}(r; \nu, c)\| \qquad j \geqq [\nu/2]+1.$$

From (13.13) it follows (5.7). $w(r; \nu)$ and $\bar{w}_j(r; \nu, c)$ satisfy following equations:

$$(13.14) \qquad -w''(r; \nu)-r^{-1}w'(r; \nu)+(c^2+\nu^2 r^{-2})w(r; \nu)=F(w(r; \nu); c)$$

$$\text{for } r \in (0, \infty).$$

$$(13.15) \qquad -\bar{w}_j''(r; \nu, c)-r^{-1}\bar{w}_j'(r; \nu, c)+(c^2+\nu^2 r^{-2})\bar{w}_j(r; \nu, c)=$$

$$=F(\bar{w}_{j-1}(r; \nu, c); c) \qquad \text{for } r \in (0, \infty).$$

From (13.14) and (13.15) we have

$$(13.16) \qquad \|(r^{-\nu}+1)r^2(r^2+1)^{-1}\{w''(r; \nu)-\bar{w}_j''(r; \nu, c)\}\| \leqq$$

$$\leqq \|(r^{-\nu}+1)r^2(r^2+1)^{-1}r^{-1}\{w'(r; \nu)-\bar{w}_j'(r; \nu, c)\}\|+$$

$$+\|(r^{-\nu}+1)r^2(r^2+1)^{-1}(c^2+\nu^2 r^{-2})\{w(r; \nu)-\bar{w}_j(r; \nu, c)\}\|+$$

$$+\|(r^{-\nu}+1)r^2(r^2+1)^{-1}\{F(w(r; \nu); c)-F(\bar{w}_{j-1}(r; \nu, c); c)\}\| \leqq$$

$$\leqq \text{const.}\ [\|(r^{-\nu}+1)r(r+1)^{-1}\{w'(r; \nu)-\bar{w}_j'(r; \nu, c)\}\|+$$

$$+\|(r^{-\nu}+1)\{w(r; \nu)-\bar{w}_j(r; \nu, c)\}\|+$$

$$+\|(r^{-\nu}+1)\{w(r; \nu)-\bar{w}_{j-1}(r; \nu, c)\}\|].$$

From (13.16), (5.6) and (5.7) we have (5.8). Thus we proved theorem 8 (iii). Next we consider the special case of $\nu=1$.

Proposition 13.1. *For $\nu=1$ we have*

(i) $\{r^{-1}\bar{w}_j(r; 1, c)\}', \bar{w}_j''(r; 1, c)=O(r)$ *as* $r \to 0$ $j=2, 3, 4, \cdots$

(ii) $\{r^{-1}w(r; 1)\}', w''(r; 1)=O(r)$ *as* $r \to 0$.

Proof of Proposition 13.1. From (13.14) and (13.15) it follows that $v(r)=\{r^{-1}w(r; 1)\}'$ and $v_j(r)=\{r^{-1}\bar{w}_j(r; 1, c)\}'$ satisfy following equations:

(13.17) $\{r^3 v(r)\}' = r^2\{c^2 w(r; 1) - F(w(r; 1); c)\}$ for $r \in (0, \infty)$.

(13.18) $\{r^3 v_j(r)\}' = r^2\{c^2 \bar{w}_j(r; 1, c) - F(\bar{w}_{j-1}(r; 1, c); c)\}$ for $r \in (0, \infty)$.

By proposition 12.5 (i) and (ii) we have

(13.19) $r^3 v(r) = r\{rw'(r; 1) - w(r; 1)\} = O(r^2)$ as $r \to 0$.

(13.20) $r^3 v_j(r) = r\{r\bar{w}_j'(r; 1, c) - \bar{w}_j(r; 1, c)\} = O(r^2)$

as $r \to 0$ $j = 2, 3, 4, \cdots$.

Integrating (13.17) and (13.18) over the interval $[0, r]$ and using (13.19) and (13.20) we have

(13.21) $r^3 v(r) = \int_0^r s^2\{c^2 w(s; 1) - F(w(s; 1); c)\} ds$ for $r \in (0, \infty)$.

(13.22) $r^3 v_j(r) = \int_0^r s^2\{c^2 \bar{w}_j(s; 1, c) - F(\bar{w}_{j-1}(s; 1, c); c)\} ds$

for $r \in (0, \infty)$ $j = 2, 3, 4, \cdots$.

By proposition 12.5 (i) and (ii) it follows from (13.21) and (13.22)

(13.23) $v(r),\ v_j(r) = O(r)$ as $r \to 0$ $j = 2, 3, 4, \cdots$.

From (13.14), (13.15), proposition 12.5 (i), (ii) and (13.23) we have

(13.24) $w''(r; 1),\ \bar{w}_j''(r; 1, c) = O(r)$ as $r \to 0$ $j = 2, 3, 4, \cdots$.

This proves proposition 13.1.

From (13.21) and (13.22) we have

(13.25) $|v(r) - v_j(r)| \leqq r^{-3}\int_0^r s^3(s+1)^{-1} ds \times$

$\times [c^2 \|(r^{-1}+1)\{w(r; 1) - \bar{w}_j(r; 1, c)\}\| +$

$+ F'(0; c)\|(r^{-1}+1)\{w(r; 1) - \bar{w}_{j-1}(r; 1, c)\}\|] \leqq$

$\leqq \text{const. } r(r+1)^{-1}[c^2\|(r^{-1}+1)\{w(r; 1) - \bar{w}_j(r; 1, c)\}\| +$

$$+F'(0;c)\|(r^{-1}+1)\{w(r;1)-\bar{w}_{j-1}(r;1,c)\}\|] \qquad \text{for } r\in(0;\infty)$$
$$j=2,3,4,\cdots.$$

From (13.25) we have

(13.26) $$\|(r^{-1}+1)\{v(r)-v_j(r)\}\|\leqq$$
$$\leqq \text{const.}[c^2\|(r^{-1}+1)\{w(r;1)-\bar{w}_j(r;1,c)\}\|+$$
$$+F'(0;c)\|(r^{-1}+1)\{w(r;1)-\bar{w}_{j-1}(r;1,c)\}\|] \qquad j=2,3,4,\cdots.$$

(5.9) follows from (13.26). (5.10) follows from (13.14), (13.15), (5.6) and (5.9). This completes the proof of theorem 8.

14. Proof of Theorem 9

By lemma 7.5 it follows from IS (4.1) and proposition 12.5 (i)

(14.1) $$\bar{w}_j(r;\nu,c)=\mathscr{G}[f(\bar{w}_{j-1}(\ ;\nu,c)+c^2\{\bar{w}_{j-1}(\ ;\nu,c)-\bar{w}_j(\ ;\nu,c)\};\nu](r)$$
$$\text{for } r\in(0,\infty) \quad j\geqq[\nu/2]+2.$$

(14.1) can be rewritten as

(14.2) $$\bar{w}_j(r;\nu,c)=u_{oj}(\nu,c)r^\nu-$$
$$-r^\nu(2\nu)^{-1}\int_0^r[f(\bar{w}_{j-1}(s;\nu,c))+c^2\{\bar{w}_{j-1}(s;\nu,c)-\bar{w}_j(s;\nu,c)\}]\times$$
$$\times\{s^{1-\nu}-r^{-2\nu}s^{1+\nu}\}ds \qquad \text{for } r\in(0,\infty) \qquad j\geqq[\nu/2]+2.$$

Here

(14.3) $$u_{oj}(\nu,c)=$$
$$=(2\nu)^{-1}\int_0^\infty[f(\bar{w}_{j-1}(s;\nu,c))+c^2\{\bar{w}_{j-1}(s;\nu,c)-\bar{w}_i(s;\nu,c)\}]s^{1-\nu}ds$$
$$j\geqq[\nu/2]+2.$$

From (14.2) it follows easily

(14.4) $$u_{oj}(\nu,c)=\lim_{r\to0}r^{-\nu}\bar{w}_j(r;\nu,c) \qquad j\geqq[\nu/2]+2.$$

From (5.5) and (14.3) it follows

$$(14.5)\qquad |u_0(\nu)-u_{oj}(\nu,c)|=$$

$$=|(2\nu)^{-1}\int_0^\infty [F(w(s;\nu);c)-F(\bar{w}_{j-1}(s;\nu,c);c)-$$

$$-c^2\{w(s;\nu)-\bar{w}_j(s;\nu,c)\}]s^{1-\nu}ds\leqq$$

$$\leqq(2\nu)^{-1}\int_0^\infty [F'(0;c)|w(s;\nu)-\bar{w}_{j-1}(s;\nu,c)|+$$

$$+c^2|w(s;\nu)-\bar{w}_j(s;\nu,c)|]s^{1-\nu}ds\leqq$$

$$\leqq(2\nu)^{-1}\{F'(0;c)+c^2\}\int_0^\infty\{|w(s;\nu)-\bar{w}_{j-1}(s;\nu,c)|(s^2+s^{-\nu})\}\times$$

$$\times s(s^{\nu+2}+1)^{-1}ds\qquad j\geqq[\nu/2]+2.$$

By proposition 12.4 and 12.5 (i) we have

$$(14.6)\qquad |w(s;\nu)-\bar{w}_{j-1}(s;\nu,c)|(s^2+s^{-\nu})\leqq\text{const. independent of } s \text{ and } j$$

$$\text{for } s\in(0,\infty)\quad j\geqq[\nu/2]+2.$$

By (5.6) we have

$$(14.7)\qquad \lim_{j\to\infty}|w(s;\nu)-\bar{w}_{j-1}(s;\nu,c)|(s^2+s^{-\nu})=0\qquad \text{for } s\in(0,\infty).$$

Since $s(s^{\nu+2}+1)^{-1}$ is integrable on the interval $[0,\infty)$ from (14.5), (14.6) and (14.7) it follows (5.10) by Lebesgue's bounded convergence theorem. This proves theorem 9.

15. Proof of Theorem 10

$w(r;\nu)$ and $\bar{w}_j(r;\nu,c)$ satisfy the differential equations

$$(15.1)\qquad \mathscr{L}[w(\ ;\nu);\nu,c](r)=F(w(r;\nu);c)\qquad \text{for } r\in(0,\infty).$$

$$(15.2)\qquad \mathscr{L}[\bar{w}_j(\ ;\nu,c);\nu,c](r)=F(\bar{w}_{j-1}(r;\nu,c);c)\qquad \text{for } r\in(0,\infty)$$

$$j=1,2,3,\dots.$$

Differentiating both (15.1) and (15.2) with respect to r we have the differential equations which are satisfied by $w'(r;\nu)$ and $\bar{w}_j'(r;\nu, c)$ respectively.

(15.3)
$$\mathscr{L}[w'(\ ;\nu);\sqrt{\nu^2+1}, c](r)=$$
$$=F'(w(r;\nu);c)w'(r;\nu)+2\nu^2r^{-3}w(r;\nu)\equiv F_1(r;\nu, c)$$
for $r\in(0,\infty)$.

(15.4)
$$\mathscr{L}[\bar{w}_j'(\ ;\nu, c);\sqrt{\nu^2+1}, c](r)=$$
$$=F'(\bar{w}_{j-1}(r;\nu, c);c)\bar{w}_{j-1}'(r;\nu, c)+2\nu^2r^{-3}\bar{w}_j(r;\nu, c)\equiv F_{1,j}(r;\nu, c)$$
for $r\in(0,\infty)$ $j=1, 2, 3, \cdots$.

By lemma 7.7 we have from (15.3) and (15.4)

(15.5) $w'(r;\nu)=\mathscr{G}[F_1(\ ;\nu, c);\sqrt{\nu^2+1}, c](r)$ for $r\in(0,\infty)$.

(15.6) $\bar{w}_j'(r;\nu, c)=\mathscr{G}[F_{1,j}(\ ;\nu, c);\sqrt{\nu^2+1}, c](r)$

for $r\in(0,\infty)$ $j=1, 2, 3, \cdots$.

Since $\bar{w}_0(r;\nu, c)\equiv 1$ it follows

(15.7) $F_{1,1}(r;\nu, c)>0$ for $r\in(0,\infty)$.

By the positivity of the Green's function $G(r, s;\sqrt{\nu^2+1}, c)$ (lemma 7.3 (iii)) it follows from (15.6)

(15.8) $\bar{w}_1'(r;\nu, c)>0$ for $r\in(0,\infty)$.

From (15.8) it follows

(15.9) $F_{1,2}(r;\nu, c)>0$ for $r\in(0,\infty)$.

From (15.6) and (15.9) it follows

(15.10) $\bar{w}_2'(r;\nu, c)>0$ for $r\in(0,\infty)$.

Repeating these arguments we have

$$(15.11) \qquad \bar{w}_j'(r; \nu, c) > 0 \qquad \text{for } r \in (0, \infty) \quad j = 1, 2, 3, \cdots.$$

Letting j tend to infinity in (15.11) we have

$$(15.12) \qquad w'(r; \nu) \geqq 0 \qquad \text{for } r \in (0, \infty).$$

From (15.12) it follows

$$(15.13) \qquad F_1(r; \nu, c) > 0 \qquad \text{for } r \in (0, \infty).$$

From (15.5) and (15.13) it follows (5.13). This proves theorem 10.

16. Proof of Theorem 11

Differentiating (15.3) and (15.4) with respect to r we have

$$(16.1) \quad \mathscr{L}[w''(\ ; \nu); \sqrt{\nu^2+2}, c](r) = 2(2\nu^2+1)r^{-3}w'(r; \nu) - \\ -6\nu^2 r^{-4} w(r; \nu) + F''(w(r; \nu); c)\{w'(r; \nu)\}^2 + F'(w(r; \nu); c)w''(r; \nu) \\ \text{for } r \in (0, \infty).$$

$$(16.2) \quad \mathscr{L}[\bar{w}_j''(\ ; \nu, c); \sqrt{\nu^2+2}, c](r) = 2(2\nu^2+1)r^{-3}\bar{w}_j'(r; \nu, c) - \\ -6\nu^2 r^{-4}\bar{w}_j(r; \nu, c) + F''(\bar{w}_{j-1}(r; \nu, c); c)\{\bar{w}_{j-1}'(r; \nu, c)\}^2 + \\ + F'(\bar{w}_{j-1}(r; \nu, c); c)\bar{w}_{j-1}''(r; \nu, c) \qquad \text{for } r \in (0, \infty) \\ j = 1, 2, 3, \cdots.$$

Inserting the relations

$$(16.3) \qquad r^{-1}w'(r; \nu) = -w''(r; \nu) + (c^2 + \nu^2 r^{-2})w(r; \nu) - \\ -F(w(r; \nu); c) \qquad \text{for } r \in (0, \infty)$$

$$(16.4) \quad r^{-1}\bar{w}_j'(r; \nu, c) = -\bar{w}_j''(r; \nu, c) + (c^2 + \nu^2 r^{-2})\bar{w}_j(r; \nu, c) - \\ -F(\bar{w}_{j-1}(r; \nu, c); c) \qquad \text{for } r \in (0, \infty) \\ j = 1, 2, 3, \cdots$$

into the first term of the right hand sides of (16.1) and (16.2) respectively we have

(16.5) $$\mathscr{L}[-w''(\ ;\nu);\sqrt{5\nu^2+4},c](r)=4\nu^2(1-\nu^2)r^{-4}\,w(r;\nu)+ +2(2\nu^2+1)r^{-2}f(w(r;\nu))-F''(w(r;\nu);c)\{w'(r;\nu)\}^2- -F'(w(r;\nu);c)w''(r;\nu)\equiv F_2(r;\nu,c) \qquad \text{for } r\in(0,\infty).$$

(16.6) $$\mathscr{L}[-\bar{w}_j''(\ ;\nu,c);\sqrt{5\nu^2+4},c](r)=4\nu^2(1-\nu^2)r^{-4}\bar{w}_j(r;\nu,c)+ +2(2\nu^2+1)r^{-2}[f(\bar{w}_{j-1}(r;\nu,c))+c^2\{\bar{w}_{j-1}(r;\nu,c)-\bar{w}_j(r;\nu,c)\}]- -F''(\bar{w}_{j-1}(r;\nu,c);c)\{\bar{w}_{j-1}'(r;\nu,c)\}^2-F'(\bar{w}_{j-1}(r;\nu,c);c)\bar{w}_{j-1}''(r;\nu,c)\equiv \equiv F_{2,j}(r;\nu,c) \qquad \text{for } r\in(0,\infty) \quad j=1,2,3,\cdots.$$

By lemma 7.7 we have from (16.5) and (16.6)

(16.7) $$-w''(r;\nu)=\mathscr{G}[F_2(\ ;\nu,c);\sqrt{5\nu^2+4},c](r) \qquad \text{for } r\in(0,\infty).$$

(16.8) $$-\bar{w}_j''(r;\nu,c)=\mathscr{G}[F_{2,j}(\ ;\nu,c);\sqrt{5\nu^2+4},c](r) \qquad \text{for } r\in(0,\infty) \quad j=1,2,3,\cdots.$$

Since ν is assumed to satisfy $0<\nu\leqq 1$ then we have

(16.9) $$F_{2,1}(r;\nu,c)>0 \qquad \text{for } r\in(0,\infty).$$

By the positivity of the Green's function $G(r,s;\sqrt{5\nu^2+4},c)$ (lemma 7.3 (iii)) we have from (16.8) and (16.9)

(16.10) $$-\bar{w}_1''(r;\nu,c)>0 \qquad \text{for } r\in(0,\infty).$$

From (16.10) we have

(16.11) $$F_{2,2}(r;\nu,c)>0 \qquad \text{for } r\in(0,\infty).$$

It follows from (16.8) and (16.11)

(16.12) $$-\bar{w}_2''(r;\nu,c)>0 \qquad \text{for } r\in(0,\infty).$$

Repeating these arguments we have

(16.13) $$-\bar{w}_j''(r;\nu,c)>0 \qquad \text{for } r\in(0,\infty) \quad j=1,2,3,\cdots.$$

Letting j tend to infinity in (16.13) we have

$$-\bar{w}''(r;\nu)\geqq 0 \qquad \text{for } r\in(0,\infty). \tag{16.14}$$

From (16.14) we have

$$F_2(r;\nu,c)>0 \qquad \text{for } r\in(0,\infty). \tag{16.15}$$

From (16.7) and (16.15) we have (5.14). This proves theorem 11.

17. Proof of Theorem 12

From (12.31) $u(r;\nu)=r^{-\nu}w(r;\nu)$ satisfies

$$u(r;\nu)=u_0(\nu)-(2\nu)^{-1}\int_0^r f(s^\nu u\,(s;\nu))\{s^{1-\nu}-r^{-2\nu}s^{1+\nu}\}\,ds= \tag{17.1}$$

$$=u_0(\nu)+u_2(r;\nu) \qquad \text{for } r\in(0,\infty).$$

Here $u_0(\nu)$ is given by (5.5). It is easy to see

$$u_2(r;\nu)=O(r^2) \qquad \text{as } r\to 0. \tag{17.2}$$

Suppose that it has been proved that:

$$u_m(r;\nu)=u(r;\nu)-\sum_{l=0}^{m-1}u_l r^l=O(r^m) \qquad \text{as } r\to 0. \tag{17.3}$$

Here coefficients u_l are determined by the recurrence formula (5.4) with $u_0=u_0(\nu)$. From (17.3) and assumption 2 or 2′ we have

$$f(s^\nu u(s;\nu))=\sum_{l=2}^{m}\sum_{\substack{(j-1)\nu+k=l-2\\ j\geqq 1,\,k\geqq 0}}f_j\sum_{k_1+\cdots+k_j=k}u_{k_1}\cdots u_{k_j}s^{l+\nu-2}+O(s^{m+\nu-1}) \tag{17.4}$$

$$\text{as } s\to 0.$$

If we introduce the coefficient u_m by the recurrence formula (5.4) then we can rewrite (17.4) as

$$f(s^\nu u(s;\nu))=-\sum_{l=2}^{m}l(l+2\nu)u_l s^{l+\nu-2}+O(s^{m+\nu-1}) \qquad \text{as } s\to 0. \tag{17.5}$$

Inserting (17.5) into (17.1) we have

$$(17.6)\qquad u(r;\nu)=u_0(\nu)+$$

$$+\sum_{l=2}^{m} u_l\int_0^r (2\nu)^{-1}l(l+2\nu)s^{l+\nu-2}\{s^{1-\nu}-r^{-2\nu}s^{1+\nu}\}\,ds+O(\int_0^r s^m ds)=$$

$$=\sum_{l=0}^{m} u_l r^l+O(r^{m+1})\qquad \text{as } r\to 0.$$

From (17.6) we have

$$(17.7)\qquad u_{m+1}(r;\nu)=u(r;\nu)-\sum_{l=0}^{m} u_l r^l=O(r^{m+1})\qquad \text{as } r\to 0.$$

This proves that the formal power series $\sum_{l=0}^{\infty} u_l r^l$ is the asymptotic series of $u(r;\nu)$ as r approaches zero on the real positive axis. This completes the proof of theorem 12.

References

[1] Abrikosov, A. A., On the magnetic properties of superconductors of the second group, *Soviet Physics JETP* **5** (1957), 1174-1182.

[2] Ginzburg, V. L., and L. D. Landau, On the theory of superconductivity (in Russian), *Zh. Exsper. Teor. Fiz.* **20** (1950), 1064-1082.

[3] Abrahams, E. and T. Tsuneto, Time variation of the Ginzburg-Landau order parameter, *Phys. Rev.* **152** (1966), 416-432.

[4] Pitaevskii, L. P., Vortex lines in an imperfect Bose gas, *Soviet Physics JETP* **13** (1961), 451-454.

[5] Gross, E. P., Hydrodynamics of a superfluid condensate, *J. Math. Phys.* **4** (1963), 195-207.

[6] Ginzburg V. L. and L. P. Pitaevskii, On the theory of superfluidity, *Soviet Physics JETP* **7** (1958), 858-861.

[7] Pazy, A. and P. H. Rabinowitz, A nonlinear integral equation with applications to neutron transport theory, *Arch. Rat. Mech. Anal.* **32** (1969), 226-246.