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Automorphic Systems and Lie-Vessiot Systems

By

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Introduction

In the theory of systems of partial differential equations, problems of integration of a given system have long occupied an important position.

Such are the problem of deciding whether integration of a given system could be reduced to those of several systems of ordinary differential equations or not and the problem of achieving integration of a given system, provided that the above reduction is possible.

S. Lie studied such a system of partial differential equations that a general solution of the system depends on a finite number of constant parameters. He reduced there the problem to the case of an involutive distribution. But his explanation of the method of the reduction is quite ambiguous (p. 115 in [2]). He carried out further reduction of integration of the involutive distribution to that of a 1-dimensional distribution according to Mayer's method.

Regarding these reductions as a fait accompli, he studied in [2] mainly integration of a 1-dimensioal distribution, which contains the study of integration of a non-linear ordinary differential equation of any order. From a standpoint of the theory of integration, he tried to classify ordinary differential equations in another paper. In [2] he studied the case that a 1-dimensional distribution has some connection with a continuous transformation group of finite type. The case that a group is simple and, in particular, isomorphic to the projective transformation group was investigated in detail by him. In the case that a group is solvable, integration of the distribution is deeply connected with quadrature, as is suggested by

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E. Cartan in [1].

On the other hand, E. Vessiot studies the method of finding out integral curves of a given 1-dimensional distribution (which is, in a sense, equivalent to finding out first integrals of a given 1-dimensional distribution) in [4]. In other words, with respect to a certain kind of systems of ordinary differential equations of 1-st order, he attempted to develop the theory analogous to Galois theory of algebraic equations. Vessiot considered transformation groups which, roughly speaking, act on unknown functions of a given system of differential equations.

Now we shall go back to such a system of partial differential equations of any order that a general solution depends on a finite number of constant parameters.

Considering a continuous group of finite type acting on unknown functions of a given system and strengthening the condition that a general solution depends on a finite number of constant parameters, we put the assumption that, by the action of the group on a special solution, we can obtain a general solution of a given system of partial differential equations, which Vessiot called an autmorphic system with respect to the group.

Our main purpose is first to make clear the obscure point in the treatment in [2] of reducing integration of a given autmorphic system to that of an involutive distribution and secondly to give, in a form of a necessary and sufficient condition, an interpretation of the solvability of G by properties of integration of an automorphic system with respect to G.

In §1 we give the definition of a Lie-Vessiot system D on a principal fiber bundle $P(M, G, \pi)$ (Definition 1.1) and then we prove, in a strict form, the theorem stated in [1], which means that, if a Lie-Vessiot system is solvable, then it is integrated by quadratures (Theorem 1.1).

In §2 we define at first a G-autmorphic system (Definition 2.3). Under some general conditions we can induce a G-automorphic system with desirable properties from the given G-automorphic system such that the former is equivalent to the latter (Proposition 2.1, Proposition 2.4). We define the solvability of such a G-autmorphic system from a standpoint of integration (Definition 2.8). For such a G-automorphic system $(A)_l$ on $J^l(N, Q)$, we give a reduction theorem which reduce the integration of $(A)_l$ to that of a Lie-Vessiot system D on $P(N, G, \pi)$ (Theorem 2.1). Using this theorem, we obtain the main theorem (Theorem 2.2) which gives an interpretation of the solvability of G by that of a G-autmorphic system.

As for integration problems, not the existence or the property of solutions but the method of obtaining a solution is a question, though of course we need to certify the existence of a solution. S. Lie fixed his eyes upon a continuous transformation group as one of languages which express the necessary method to obtain a solution. It is also the origin of the notion of a Lie group. This language is very much available in some cases and express the properties of integration briefly. But of course this language is not all mighty.

We have written this note, taking his great thought as a starting point of our studies.

Finally, we should like to thank Professor N. Tanaka for his many valuable suggestions by reading our manuscript carefully.

§ 1. Lie-Vessiot Systems

We assume that the differentiability is the class C^∞ and "a Lie group" always means "a connected Lie group" through this paper unless otherwise stated.

We denote by $P(M, G, \pi)$ a principal fiber bundle over the base space M, with the total space P, the structure group G and the projection π.

Let $P(M, G, \pi)$ be a principal fiber bundle and let $\mathfrak{g}$ be the Lie algebra of G. Then for each $X \in \mathfrak{g}$, $\exp tX$ induces a vector field X^* on P. We set $\mathfrak{g}^* = \{X^* \mid X \in \mathfrak{g}\}$. Clearly $\mathfrak{g}^*$ is a Lie algebra isomorphic to $\mathfrak{g}$.

Definition 1.1. Let $P(M, G, \pi)$ be a principal fiber bundle. A distribution D defined on a neighbourhood of $p \in P$ is called a Lie-Vessiot system at p on $P(M, G, \pi)$ if it satisfies the following conditions;

(1) D is an m-dimensional involutive distribution, $m = \dim M$.

(2) $\pi_* D_p = T_{\pi(p)}(M)$

(3) $[X, Y]$ is a cross-section of D for any cross-section X of D and any $Y \in \mathfrak{g}^*$.

Note that, for any Lie-Vessiot system D at p on $P(M, G, \pi)$, there exists a local basis $\{X_i, \cdots, X_m\}$ of D at p such that $[X_i, X_j]=0$ $(i, j= 1 \cdots m)$ and $[X_j, \mathfrak{g}^*]=0$ $(j=1, \cdots, m)$.

Definition 1.2. A Lie-Vessiot system D at p on $P(M, G, \pi)$ is said to be simple (resp. solvable) if G is simple (resp. solvable).

Definition 1.3. Let D be any distribution on a manifold S. A function φ locally defined at $p \in S$ is called a first integral of D at p if, for any local cross-section X of D at p, we have $X \cdot \varphi = 0$.

Definition 1.4. Let D be a distribution on S. By the integration of D at $p \in S$ we mean to find all first integrals of D at p.

Definition 1.5. Let D be an m-dimensional involutive distribution on S. A family $\{\varphi^j\}_{j=1}^r$ of first integrals of D at p is called a fundamental system of solutions of D at p if $d\varphi^1, \cdots, d\varphi^r$ are linearly independent at p and $r = \dim S - m$.

Let $\{a_h\}_{h=1}^{k-1}$ be a family of real numbers. If we are given a family of real-valued functions $\{f_h\}_{h=1}^k$ defined on an open set U of a manifold, we set $f=(f_1, \cdots, f_k)$, $U_f^l = \{q \in U_f^{l-1} \mid f_l(q)=a_l\}$ and $f_j^l = f_j^{l-1} \mid U_f^l$ $(l= 0, \cdots, k-1)$ where $U_f^0 = U$ and $f_j^0 = f_j$.

Theorem 1.1. *Let D be a solvable Lie-Vessiot system at p on $P(M, G, \pi)$, $\dim G = r$. Then there exist a fundamental system of solutions $\{\varphi_k\}_{k=1}^r$ of D at p defined on U and a basis $\{V_k\}_{k=1}^r$ of $\mathfrak{g}^*$ such that we have*

$$V_{l+1} \cdot \varphi_{l+1} = 1$$

$$V_{l+k} \cdot \varphi_{l+1} = 0 \qquad (2 \leqq k \leqq r-l)$$

for $0 \leqq l \leqq r-1$.

Proof. Since $\mathfrak{g}^*$ is solvable, we have a sequence of subalgebras $\mathfrak{g}^*=\mathfrak{g}_0^*\supset\mathfrak{g}_1^*\supset\cdots\mathfrak{g}_{r-1}^*\supset\mathfrak{g}_r^*=\{0\}$ where $\dim\mathfrak{g}_j^*-\dim\mathfrak{g}_{j+1}^*=1$ and $\mathfrak{g}_{j+1}^*$ is an ideal of $\mathfrak{g}_j^*$. Let $\{V_k\}_{k=1}^r$ be a basis of $\mathfrak{g}^*$ such that $\{V_k\}_{k=j}^r$ is a basis of $\mathfrak{g}_{j-1}^*$. Let $\{\psi_k\}_{k=1}^r$ be a fundamental system of solutions of D at p defined on U such that $X_h\cdot\psi_j=V_{j+k}\cdot\psi_j=0$ $(1\leqq j\leqq r,\ 1\leqq k\leqq r-j)$ where $\{X_h\}_{h=1}^m$ is any local basis of D at p. The existence of such $\{\psi_k\}_{k=1}^r$ is assured, for $X_1,\cdots,X_m,V_{j+1},\cdots,V_r$ generate an $(m+r-j)$-dimensional involutive distribution defined on a neighbourhood U of p. Using the existence of such $\{\psi_k\}_{k=1}^r$, we shall show that there exists a fundamental system of solutions $\{\varphi_k\}_{k=1}^r$ of D at p defined on U such that we have

$$\begin{cases} X_h\cdot\varphi_{l+1}=V_{l+k}\cdot\varphi_{l+1}=0 & (1\leqq h\leqq m,\ 2\leqq k\leqq r-l)\\ V_{l+1}\cdot\varphi_{l+1}=1 \end{cases}$$

for $0\leqq l\leqq r-1$. Since ψ_1 satisfies $X_h\cdot\psi_1=V_k\cdot\psi_1=0$ $(1\leqq h\leqq m,\ 2\leqq k\leqq r)$ and $\mathfrak{g}_1^*$ is an ideal of $\mathfrak{g}^*$, $V_1\cdot\psi_1$ also satisfies $X_h(V_1\cdot\varphi_1)=V_k(V_1\cdot\psi_1)=0$ $(1\leqq h\leqq m,\ 2\leqq k\leqq r)$. Therefore we have a function $K(t)$ of one variable such that $V_1\cdot\psi_1=K(\psi_1)$. We set $H(t)=\int_0^t K(t)^{-1}dt$. Then $H(\psi_1)$ also satisfies $X_h(H(\psi_1))=V_k(H(\psi_1))=0$ $(1\leqq h\leqq m,\ 2\leqq k\leqq r)$. Moreover we have $V_1(H(\psi_1))=\left(\frac{dH}{dt}\right)_{t=\psi_1}\circ V_1(\psi_1)=1$. Therefore we see that the system of partial differential equations

$$(*)_1^0\qquad \begin{cases} X_h\cdot f=V_k\cdot f=0 & (1\leqq h\leqq m,\ 2\leqq k\leqq r)\\ V_1\cdot f=1 \end{cases}$$

has a solution on U. We may assume that $\pi^{-1}(J)=U\approx J\times W$ and let $\{x_h\}_h^m$(resp. $\{w_k\}_{k=1}^r$) be a coordinate system on J (resp. W) Then $\{x_1,\cdots,x_m,w_1,\cdots,w_r\}$is a coordinate system on U. By using this coordinate system, the above $(*)_1^0$ is expressed as

$$(*)_2^0\qquad \begin{cases} \sum\alpha_h^i\dfrac{\partial f}{\partial x_1}+\sum\beta_h^j\dfrac{\partial f}{\partial w_j}=0 & (1\leqq h\leqq m)\\ \sum\gamma_k^j\dfrac{\partial f}{\partial w_j}=0 & (2\leqq k\leqq r)\\ \sum\gamma_1^j\dfrac{\partial f}{\partial w_j}=1. \end{cases}$$

Since $\{X_1, \cdots, X_m, V_1, \cdots, V_r\}$ are linearly independent on U, we can solve $(*)_2^0$ with respect to $\frac{\partial f}{\partial x_h}, \frac{\partial f}{\partial w_k}$ $(1 \leqq h \leqq m, 1 \leqq k \leqq r)$. We have

$$(*)_3^0 \qquad \begin{cases} \dfrac{\partial f}{\partial x_h} = \rho_h^0(x_1, \cdots, x_m, w_1, \cdots, w_r) & (1 \leqq h \leqq m) \\ \dfrac{\partial f}{\partial w_k} = \sigma_k^0(x_1, \cdots, x_m, w_1, \cdots, w_r) & (1 \leqq k \leqq r). \end{cases}$$

Now we assume that we have linearly independent functions $\varphi_1, \cdots, \varphi_l$ on U such that

$$\begin{cases} X_h \cdot \varphi_j = V_{j+k} \cdot \varphi_j = 0 & (1 \leqq h \leqq m, 1 \leqq k \leqq r-j) \\ V_j \cdot \varphi_j = 1 \end{cases}$$

for $1 \leqq j \leqq l$. Then we shall show that we can find a function φ_{l+1} on U which is independent of $\varphi_1, \cdots, \varphi_l$ and satisfies

$$\begin{cases} X_h \cdot \varphi_{l+1} = V_{l+k} \cdot \varphi_{l+1} = 0 & (1 \leqq h \leqq m, 2 \leqq k \leqq r-l) \\ V_{l+1} \cdot \varphi_{l+1} = 1. \end{cases}$$

Since $\varphi_j (1 \leqq j \leqq l)$ satisfies $X_h \cdot \varphi_j = V_{j+k} \varphi_j = 0$ $(1 \leqq h \leqq m, 1 \leqq k \leqq r-j)$, we can restrict $X_h (1 \leqq h \leqq m)$ (resp. $V_{l+k}(1 \leqq k \leqq r-l)$) to U_φ^l which we denote by X_h^l (resp. V_{l+k}^l). By the same reason as for the case $(*)_1^0$, we can induce a function φ_{l+1}^l on U_φ^l from the function $\psi_{l+1}^l (= \psi_{l+1} | U_\varphi^l)$ such that we have $X_h^l \cdot \varphi_{l+1}^l = V_{l+k}^l \cdot \varphi_{l+1}^l = 0$ $(1 \leqq h \leqq m, 2 \leqq k \leqq r-l)$ and $V_{l+1}^l \cdot \varphi_{l+1}^l = 1$, that is to say, we can see that the system of partial differential equations

$$(*)_1^l \qquad \begin{cases} X_h^l \cdot f = V_{l+k}^l \cdot f = 0 & (1 \leqq h \leqq m, 2 \leqq k \leqq r-l) \\ V_{l+1}^l \cdot f = 1 \end{cases}$$

has a solution on U_φ^l.

Now we may consider $\{x_1, \cdots, x_m, w_1, \cdots, w_{r-l}, \varphi_1, \cdots, \varphi_l\}$ as a coordinate system on U. Then $\{x_1, \cdots, x_m, w_1, \cdots, w_{r-l}\}$ is a coordinate system on U_φ^l. By using the coordinate system on U_φ^l, we have

$$(*)_2^l \qquad \begin{cases} \sum_{i=1}^{m} \alpha_h^{i,l} \dfrac{\partial f}{\partial x_i} + \sum_{j=1}^{r} \beta_h^{j,l} \dfrac{\partial f}{\partial w_j} = 0 & (1 \leqq h \leqq m) \\ \sum_{j=1}^{r} \gamma_k^{j,l} \dfrac{\partial f}{\partial w_j} = 0 & (l+2 \leqq k \leqq r) \\ \sum_{j=1}^{r} \gamma_{l+1}^{j,l} \dfrac{\partial f}{\partial w_j} = 1. \end{cases}$$

Since X_h^l, $V_{l+k}^l (1 \leqq h \leqq m,\ 1 \leqq k \leqq r-l)$ are linearly independent on U^l, we can solve $(*)_2^l$ with respect to $\dfrac{\partial f}{\partial x_h}, \dfrac{\partial f}{\partial w_k} (1 \leqq h \leqq m,\ 1 \leqq k \leqq r-l)$. We have then,

$$(*)_3^l \qquad \begin{cases} \dfrac{\partial f}{\partial x_h} = \rho_h^l(x_1, \cdots, x_m, w_1, \cdots, w_{r-l}, \varphi_1, \cdots, \varphi_l) & (1 \leqq h \leqq m) \\ \dfrac{\partial f}{\partial w_k} = \sigma_k^l(x_1, \cdots, x_m, w_1, \cdots, w_{r-l}, \varphi_1, \cdots, \varphi_l) & (1 \leqq k \leqq r-l), \end{cases}$$

ρ_h^l and σ_k^l are differentiable with respect to $x_1, \cdots, x_m, w_1, \cdots, w_{r-l}, \varphi_1, \cdots, \varphi_l$. For any family of real numbers $\{a_j\}_{j=1}^l$, $(*)_3^l$ has a solution on U_φ^l. Therefore $(*)_3^l$ has a solution φ_{l+1} on U. Clearly φ_{l+1} satisfies

$$\begin{cases} X_h \cdot \varphi_{l+1} = V_{l+k} \cdot \varphi_{l+1} = 0 & (1 \leqq h \leqq m,\ 2 \leqq k \leqq r-l) \\ V_{l+1} \cdot \varphi_{l+1} = 1. \end{cases}$$

Thus we get a fundamental system of solutions $\{\varphi_j\}_{j=1}^r$ of D at p such that we have

$$\begin{cases} V_{l+1} \cdot \varphi_{l+1} = 1 \\ V_{l+k} \cdot \varphi_{l+1} = 0 & (2 \leqq k \leqq r-l). \end{cases}$$

This completes the proof of Theorem 1.1.

Corollary 1.1. *Let D be a solvable Lie-Vessiot system at p on $P(M, G, \pi)$. Then we can find a fundamental system of solutions $\{\varphi_k\}_{k=1}^r$ of D at p by quadratures.*

Proof. By the proof of Theorem 1.1, there exists a fundamental system of solutions $\{\varphi_k\}_{k=1}^r$ of D at p such that we have

$$(*)_3^l \quad \begin{cases} \dfrac{\partial \varphi_{l+1}}{\partial x_h} = \rho_h^l(x_1, \cdots, x_m, w_1, \cdots, w_{r-l}, \varphi_1, \cdots, \varphi_l) & (1 \leqq h \leqq m) \\ \dfrac{\partial \varphi_{l+1}}{\partial w_k} = \sigma_k^l(x_1, \cdots, x_m, w_1, \cdots, w_{r-l}, \varphi_1, \cdots, \varphi_l) & (1 \leqq k \leqq r-l) \end{cases}$$

for $0 \leqq l \leqq r-1$, which we can integrate by quadratures according to the method due to Lagrange and Charpit.

§2. Solvability of Automorphic Systems of Finite Type

Let N and Q be manifolds. We denote by $J^l(N, Q)$ the space of l-jets of local maps of N to Q. Let s be any map of a neighbourhood U_{x_0} of $x_0 \in N$ to Q and set $j_x^l(s) = (x, s(x), s_{(1)}(x), \cdots, s_{(l)}(x))$ where $s_{(k)}$ is the set of partial derivatives of s of order k. If p is in $J^l(N, Q)$, we have $p = j_x^l(s)$. x (resp. $s(x)$) is called the source of p (resp. the target of p). For a map $\bar{s}$ of $U_{x_0} \times V_e \subset N \times G$ to Q where V_e is a neighbourhood of the unit element e of a Lie group G, we set $s_g(x) = \bar{s}(x, g)$ and define $j^l(\bar{s})$ by $j^l(\bar{s})(x, g) = j_x^l(s_g)$. Then $j^l(\bar{s})$ is a map of $U_{x_0} \times V_e$ to $J^l(N, Q)$.

In this section we assume that $N = \boldsymbol{R}^n$ and $Q = \boldsymbol{R}^q$. We denote by $x_1, \cdots, x_n$ the coordinate system of N, by $z_1, \cdots, z_q$ that of Q and by $x_i (1 \leqq i \leqq n)$, $z_j (1 \leqq j \leqq q)$, $p_{j_1 \cdots j_k}^{\lambda} (1 \leqq \lambda \leqq q, 1 \leqq j_1, \cdots, j_k \leqq n, 1 \leqq k \leqq l)$ that of $J^l(N, Q)$.

Definition 2.1. Let $\{F_j\}_{j=1}^{\alpha}$ be a family of functions defined on a neighbourhood of $p_0 \in J^k(N, Q)$. A system of equations

$$(A)_k: \quad F_1 = 0, \cdots, F_\alpha = 0$$

is called a system of partial differential equations at $p_0 \in J^k(N, Q)$.

We denote by $I(A)_k$ the set of points in $J^k(N, Q)$ satisfying $(A)_k$.

Definition 2.2. Let x_0 be the source of p_0. Any local map s of a neighbourhood U_{x_0} of x_0 to Q is called a solution of $(A)_k$ if $j^l(s)(U_{x_0}) = \{j_x^l(s) \mid x \in U_{x_0}\}$ is contained in $I(A)_k$.

Let $(A)_k$: $F_1 = 0, \cdots, F_\alpha = 0$ be a system of partial differential equations given on a neighbourhood U of $p_0 \in J^k(N, Q)$. We denote by $Q^k(U)$ the sheaf of all local functions on U and by $(A)_k^*$ the sheaf of ideals of $Q^k(U)$ generated by $F_1, \cdots, F_\alpha$. Moreover let ρ_k^{k+1} be the projection of $J^{k+1}(N, Q)$ onto $J^k(N, Q)$ and we denote by $p(A)_k^*$ the sheaf of ideals of $Q^{k+1}(\tilde{U})$, $\tilde{U}=(\rho_k^{k+1})^{-1}(U)$, generated by $(A)_k^*$ and $\partial_\#^i F_j (1 \leqq i \leqq n, 1 \leqq j \leqq \alpha)$ where $\partial_\#^i F_j$ is defined by

$$\partial_\#^i F_j = \frac{\partial F_j}{\partial x_i} + \sum_\lambda \frac{\partial F_j}{\partial z_\lambda} p_i^\lambda + \cdots + \sum_{\lambda, j_1, \cdots, j_k} \frac{\partial F_j}{\partial p_{j_1 \cdots j_k}^\lambda} p_{j_1 \cdots j_k i}^\lambda .$$

$p(A)_k^*$ is called the prolongation of $(A)_k^*$. We set $p_0(A)_k^* = Q^l(U) \cap p(A)_k^*$ and $\overline{(A)}_k^* = \bigcup_{n=1}^{\infty} p_0^n(A)_k^*$, $p_0^n(A)_k^* = p_0(p_0^{n-1}(A)_k^*)$. M. Matsuda called $\overline{(A)}_k^*$ the p-closure of $(A)_k^*$.

Now let G be a Lie transformation group acting effectively on Q. For any $p = j_x^k(s) \in J^k(N, Q)$ and $g \in G$, we set $g \cdot j_x^k(s) = j_x^k(g \cdot s)$. Then G acts on $J^k(N, Q)$ as an effective Lie transformation group.

Definition 2.3. Let G be a Lie transformation group acting effectively on Q. A system of partial differential equations $(A)_k$ at $p_0 \in J^k(N, Q)$ is said to be G-autmorphic if there exists a map $\bar{s}$ of $U_{x_0} \times V_e \subset N \times G$ to Q ($x_0 =$ the source of p_0) satisfying the following conditions;

(1) For any $g \in V_e$, s_g is a solution of $(A)_k$.

(2) Any solution of $(A)_k$ is uniquely expressed as s_g, $g \in V_e$.

(3) We have $\bar{s}(x, g) = g \cdot \bar{s}(x, e)$ for any $g \in V_e$.

We call such a map $\bar{s}$ a general solution of the G-autmorphic system $(A)_k$.

Remark 2.1. We have $g \cdot j^k(\bar{s})(x, e) = j^k(\bar{s})(x, g)$ for any $g \in V_e$.

We denote by $\binom{\lambda}{j_1 \cdots j_s}$ the pair of an integer λ and a family of integers $\{j_h\}_{h=1}^s$. We set $\tilde{I}^{(l)} = \{\binom{\lambda}{j_1 \cdots j_s} \mid 1 \leqq \lambda \leqq q, 1 \leqq s \leqq l, 1 \leqq j_h \leqq n\}$ where $q = \dim Q$, $n = \dim N$.

Definition 2.4. A system of partial differential equations $(A)_k$ at p in $J^k(N, Q)$ is said to be of normal form if $(A)_k$ possesses the form $p_{j_1 \cdots j_s}^\lambda = H_{j_1 \cdots j_s}^\lambda (x_1, \cdots, x_n, z_1, \cdots, z_q, p_1^1, \cdots)$, $\binom{\lambda}{j_1 \cdots j_s} \in I$ where I is a subset of

$\tilde{I}^{(k)}$ satisfying the following conditions;

(1) We set $I_k=\{\binom{\lambda}{j_1\cdots j_k}\mid 1\leqq\lambda\leqq q,\ 1\leqq j_h\leqq n\}$.

Then we have $I\supset I_k$.

(2) $H^{\lambda}_{j_1\cdots j_s}$ is a function on $J^{k-1}(N, Q)$ for any $\binom{\lambda}{j_1\cdots j_s}\in I$.

(3) For any $\binom{\lambda}{j_1\cdots j_s}\in I$, $H^{\lambda}_{j_1\cdots j_s}$ does not depend on $p^{\lambda}_{j_1\cdots j_s}$.

In order to emphasize that $(A)_k$ is of normal form, we denote by $\mathfrak{N}(A)_k$ in place of $(A)_k$.

Proposition 2.1. *Let $(A)_k$ be a system of partial differential equations at $p_0\in J^k(N, Q)$ satisfying the following conditions;*

(1) *$(A)_k$ is G-automorphic.*

(2) *There exists a general solution $\bar{s}$ of $(A)_k$ such that, for an integer $l\geqq k$, $j^l(\bar{s})$ is an embedding of a neighbourhood $U_{x_0}\times V_e$ of $(x_0, e)\in N\times G$ into $J^l(N, Q)$.*

Then there exists a system of partial differential equations $(A)_{l+1}$ at $\tilde{p}_0$ in $J^{l+1}(N, Q)$ with $\rho^{l+1}_k(\tilde{p}_0)=p_0$ (ρ^{l+1}_k is the projection of $J^{l+1}(N, Q)$ onto $J^k(N, Q)$) satisfying the following conditions;

(i) *$(A)_{l+1}$ is G-automorphic and has a general solution $\bar{\omega}$: $U_{x_0}\times V_e\supset U'_{x_0}\times V'_e\to Q$ with $\bar{\omega}=\bar{s}\mid U'_{x_0}\times V'_e$.*

(ii) *There exists a neighbourhood W of $\tilde{p}_0$ such that $I((A)_{l+1})\cap W=\tilde{S}\cap W$ where $\tilde{S}=j^{l+1}(\bar{s})(U_{x_0}\times V_e)$.*

(iii) *$(A)_{l+1}$ contains, as a subsystem, a system of partial differential equations $\mathfrak{N}(B)_l$ of normal form.*

Proof. We may consider that $\{x_1, \cdots, x_n, z_1, \cdots, z_t, w_1, \cdots, w_{r-t}\}$ is a local coordinate system at $\bar{p}_0$ in $S=j^l(s)(U_{x_0}\times V_e)$, $\rho^{l+1}_l(\tilde{p}_0)=\bar{p}_0$, where $r=\dim G$ and w_i $(1\leqq j\leqq r-t)$ is some $p^{\lambda}_{j_1\cdots j_s}$. Since $\tilde{S}$ is diffeomorphic to S by the projection ρ^{l+1}_l of $J^{l+1}(N, Q)$ onto $J^l(N, Q)$, we may also consider $\{x_1, \cdots, x_n, z_1, \cdots, z_t, w_1, \cdots, w_{r-t}\}$ as a local coordinate system at $\tilde{p}_0$ in $\tilde{S}$. Let $p^{\lambda}_{j_1\cdots j_s}(1\leqq s\leqq l+1)$ be any coordinate function on $J^{l+1}(N, Q)$ such that $p^{\lambda}_{j_1\cdots j_s}\neq w_j(j=1, \cdots, r-t)$. Then we have $p^{\lambda}_{j_1\cdots j_s}=H^{\lambda}_{j_1\cdots j_s}(x_1, \cdots, x_n, z_1, \cdots, z_t, w_1, \cdots, w_{r-t})$ on a neighbourhood $U_{\tilde{p}_0}$ of $\tilde{p}_0$ in $\tilde{S}$. Similarly if $z_i\neq z_h$ $(h=1, \cdots, t)$, then we have $z_i=H^i(x_1, \cdots, x_n, z_1,$

$\cdots, z_t, w_1, \cdots, w_{r-t})$. We denote by $(A)_{l+1}$ the system of partial differential equations consisting of all such $p^\lambda_{j_1\cdots j_s}=H^\lambda_{j_1\cdots j_s}$ and $z_i=H^i$. We denote also by $\mathfrak{N}(B)_{l+1}$ the system of partial differential equations consisting of all such $p^\lambda_{j_1\cdots j_s}=H^\lambda_{j_1\cdots j_s}$. Then $(A)_{l+1}$ clearly satisfies (ii) and (iii). We shall prove that $(A)_{l+1}$ satisfies (i).

First of all we shall show that if $s\colon U'_{x_0}\to Q$ is a solution of $(A)_{l+1}$ then s is a solution of $(A)_k$. Clearly $j^l(s)$ is a local cross-section of S. For each $x\in U'_{x_0}$, $j^l_x(s)\in S$. Therefore we have a solution $s_{g(x)}$, $g(x)\in V_e$, of $(A)_k$ such that $j^l_x(s)=j^l_x(s_{g(x)})$. In particular we have $j^l_x(s)=j^k_x(s_{g(x)})$ for $k\leqq l$. Therefore $j^k_x(s)\in I(A)_k$. Since x is any point in U'_{x_0}, s is a solution of $(A)_k$. Next assume that s is a solution of $(A)_k$. Then $j^l(s)$ is a local crosssection of S and therefore $j^{l+1}(s)$ is a local cross-section of $\tilde{S}$. This implies that s is a solution of $(A)_{l+1}$.

Therefore $(A)_{l+1}$ has a general solution $\tilde{s}\,|\,U'_{x_0}\times V'_e$. It is now clear that $(A)_{l+1}$ is G-automorphic. This completes the proof of Proposition 2.1.

Proposition 2.2. *Let $\mathfrak{N}(A)_l$ be a G-automorphic system at $p_0\in J^l(N, Q)$. We denote by $\overline{\mathfrak{N}}(A)^*_l$ the p-closure of $\mathfrak{N}(A)^*_l$. We assume that the point $p_0\in I(\overline{\mathfrak{N}}(A)^*_l)$ is an ordinary integral point of $\overline{\mathfrak{N}}(A)^*_l$. Then there exists a neighbourhood U_{p_0} of p_0 in $J^l(N, Q)$ such that $\overline{\mathfrak{N}}(A)^*_l$ is involutive at $p\in I(\overline{\mathfrak{N}}(A)^*_l)\cap U_{p_0}$. (As for the definition of "(quasi-) involutive" confer [3].)*

Proof. Since, for a suitable U_{p_0}, $p\in I(\overline{\mathfrak{N}}(A)^*_l)\cap U_{p_0}$ is an ordinary integral point of $\overline{\mathfrak{N}}(A)^*_l$ and $\overline{\mathfrak{N}}(A)^*_l$ is compatible at $p\in I(\overline{\mathfrak{N}}(A)^*_l)\cap U_{p_0}$, we have only to show that $C_p(\overline{\mathfrak{N}}(A)^*_l)$ is involutive and the dimension of $C_p(\overline{\mathfrak{N}}(A)^*_l)^{(1)}$, the first prolongation of $C_p(\overline{\mathfrak{N}}(A)^*_l)$, is locally constant at p. By definition we have $C_p(\overline{\mathfrak{N}}(A)^*_l)=\{\mathrm{X}\in T_p(J^l(N, Q))\,|\,(\rho^l_{l-1})_*\mathrm{X}=0$, $df_p(\mathrm{X})=0$, $^\forall f\in\overline{\mathfrak{N}}(A)^*\}=\left\{\sum_{\lambda,j_1,\cdots,j_l}\xi^\lambda_{j_1\cdots j_l}(p)\dfrac{\partial}{\partial p^\lambda_{j_1\cdots j_l}}\,\middle|\,\left(\sum_{\lambda,j_1,\cdots,j_l}\xi^\lambda_{j_1\cdots j_l}\dfrac{\partial}{\partial p^\lambda_{j_1\cdots j_l}}\right)_p f=0,\ ^\forall f\in\overline{\mathfrak{N}}(A)^*_l\right\}$. On the other hand $\overline{\mathfrak{N}}(A)^*_l$ contains $p_{j_1\cdots j_l}-H_{j_1\cdots j_l}$ for any $\binom{\lambda}{j_1\cdots j_l}\in I_l$. It follows immediately that $C_p(\overline{\mathfrak{N}}(A)^*_l)=0$. Therefore, in particular, dim $C_p(\overline{\mathfrak{N}}(A)^*_l)^{(1)}=$ constant and $C_p(\overline{\mathfrak{N}}(A)^*_l)$ is involutive. This completes the proof of Proposition 2.2.

Let $\mathfrak{R}(A)_l$ be a system at $p_0 \in J^l(N, Q)$. For any $H^\lambda_{j_1\cdots j_{s+1}}$ which appears in $\mathfrak{R}(A)_l$, we define a function ${}_iH^\lambda_{j_1\cdots j_{s+1}}$ on $J^l(N, Q)$ given in a neighbourhood of p_0 by the following way: For $\partial^i_\# H^\lambda_{j_1\cdots j_{s+1}} = \dfrac{\partial H^\lambda_{j_1\cdots j_{s+1}}}{\partial x_i} + \sum_\mu p^\mu_i \dfrac{H^\lambda_{j_1\cdots j_{s+1}}}{\partial z_\mu} + \cdots + \sum_{\mu, h_1, \dots, h_{l-1}} p^\mu_{h_1\cdots h_{l-1}} i \dfrac{\partial H^\lambda_{j_1\cdots j_{s+1}}}{\partial p^\mu_{h_1\cdots h_{l-1}}}$, replace the coefficients $p^\mu_{h_1\cdots h_k i}$ of $\partial^i_\# H^\lambda_{j_1\cdots j_{s+1}}$ $(0 \leqq k \leqq l-1)$ which appear in the left hand side of $\mathfrak{R}(A)_l$ by the right hand side of it, which we denote by ${}_iH^\lambda_{j_1\cdots j_{s+1}}$. If both $({}^\lambda_{\alpha j_1\cdots j_s})$ and $({}^\lambda_{\beta j_1\cdots j_s})$ are in I, we consider the function ${}_\alpha H^\lambda_{\beta j_1\cdots j_s} - {}_\beta H^\lambda_{\alpha j_1\cdots j_s}$ and if $({}^\lambda_{\beta j_1\cdots j_s}) \in I$, and $({}^\lambda_{\alpha j_1\cdots j_s}) \notin I$, we consider the function ${}_\alpha H^\lambda_{\beta j_1\cdots j_s} - P^\lambda_{j_1\cdots j_s \alpha\beta}$. We denote by $\mathfrak{F}\mathfrak{R}(A)_l$ the sheaf of rings of all such functions on a neighbourhood of p_0.

Proposition 2.3. *Let $\mathfrak{R}(A)_l$ be a system at $p_0 \in J^l(N, Q)$. We have, then, $I(\mathfrak{F}\mathfrak{R}(A)_l) \supset I(\overline{\mathfrak{R}}(A)^*_l)$.*

Proof. By definition, $\overline{\mathfrak{R}}(A)^*_l$ contains $\partial^\alpha_\# H^\lambda_{\beta j_1\cdots j_{l-1}} - \partial^\beta_\# H_{\alpha j_1\cdots j_{l-1}}$ for any $({}^\lambda_{\alpha\beta j_1\cdots j_{l-1}})$ and $p^\lambda_{j_1\cdots j_s} - H^\lambda_{j_1\cdots j_s}$ for any $({}^\lambda_{j_1\cdots j_s}) \in I$. Moreover if $({}^\lambda_{\beta j_1\cdots j_s}) \in I$ and $({}^\lambda_{\alpha j_1\cdots j_s}) \notin I$, then $\partial^\alpha_\# H_{j_1\cdots j_s} - p^\lambda_{j_1\cdots j_s \alpha\beta}$ is contained in $\overline{\mathfrak{R}}(A)^*_l$. Therefore we have $\mathfrak{F}\mathfrak{R}(A)_l \subset \overline{\mathfrak{R}}(A)^*_l$, that is, we get $I(\mathfrak{F}\mathfrak{R}(A)_l) \supset I(\overline{\mathfrak{R}}(A)^*_l)$.

Proposition 2.4. *Let $\mathfrak{R}(A)_l$ be a G-automorphic system at $p_0 \in J^l(N, Q)$ such that $j^l(\bar{s})$ is an embedding of $U_{x_0} \times V_e$ into $J^l(N, Q)$. We assume that the differentiability is the class C^ω and the point $p_0 \in I(\overline{\mathfrak{R}}(A)^*_l)$ is an ordinary integral point of $\overline{\mathfrak{R}}(A)^*_l$. Then there exists a neighbourhood U_{p_0} of p_0 in $J^l(N, Q)$ such that we have $I(\overline{\mathfrak{R}}(A)^*_l) \cap U_{p_0} = S \cap U_{p_0}$ where $S = j^l(\bar{s})(U_{x_0} \times V_e)$.*

Proof. By Proposition 2.2, we can choose a neighbourhood U_{p_0} of p_0 in $J^l(N, Q)$ such that $\overline{\mathfrak{R}}(A)^*_l$ is involutive at any $p \in I(\overline{\mathfrak{R}}(A)^*_l) \cap U_{p_0}$. Therefore we have a solution $s: U' \to Q$ of $\overline{\mathfrak{R}}(A)^*_l$ such that $j^l(s)(U')$ contains p. This implies $I(\overline{\mathfrak{R}}(A)^*_l) \cap U_{p_0} \subset S \cap U_{p_0}$. On the other hand we have clearly $S \cap U_{p_0} \subset I(\overline{\mathfrak{R}}(A)^*_l) \cap U_{p_0}$. Therefore we get $I(\overline{\mathfrak{R}}(A)^*_l) \cap U_{p_0} = S \cap U_{p_0}$.

Therefore from now on we shall deal with a system of partial differential equations $(A)_l$ at $p_0 \in J^l(N, Q)$ satisfying the following conditions;

[α_1] *$(A)_l$ is G-automorphic and $j^l(\bar{s})$ is an embedding of $U_{x_0} \times V_e$ into $J^l(N, Q)$ where $\bar{s}: U_{x_0} \times V_e \to Q$ is a general solution of $(A)_l$.*

[α_2] *There exists a neighbourhood W of p_0 such that $S \cap W = I(A)_l \cap W$ where $S = j^l(\bar{s})(U_{x_0} \times V_e)$.*

[α_3] *$(A)_l$ contains, as a subsystem, a system of partial differential equations $\mathfrak{N}(B)_l$ of normal form.*

Moreover we set the following assumption;

[β] *We know a diffeomorphism Δ of $U_{x_0} \times V_e$ onto S such that $g \cdot \Delta(x, e) = \Delta(x, g)$ for any $g \in V_e$ and $x \in U_{x_0}$.*

For such a system of partial differential equations, we have the following reduction theorem.

Theorem 2.1. *Let $(A)_l$ be a system of partial differential equations at p_0 satisfying [α_i] $(i=1, 2, 3)$ and [β]. Then we can induce from $(A)_l$ a Lie-Vessiot system D at $q_0 = \Delta^{-1}(p_0)$ on the trivial principal fiber bundle $(N \times G)(N, G, \pi)$ such that, for any first integral φ of D at q_0, $\varphi \circ \Delta^{-1}$ is constant on $S_\omega = \{j_x^l(\omega) \mid x \in U'_{x_0}\}$ for any solution $\omega: U'_{x_0} \to Q$ of $(A)_l$.*

Proof. We set $E_i = \dfrac{\partial}{\partial x_i} + \sum_\lambda p_i^\lambda \dfrac{\partial}{\partial z_\lambda} + \cdots + \sum_{\lambda, j_1, \dots, j_{l-1}} p_{j_1 \cdots j_{l-1} i}^\lambda \dfrac{\partial}{\partial p_{j_1 \cdots j_{l-1}}^\lambda}$. Then E_i is a vector field on $J^l(N, Q)$. We replace the coefficients $p_{j_1 \cdots j_k i}^\lambda$ of E_i which appear in the left hand side of $\mathfrak{N}(B)_l$ by the right hand side of it. Then we obtain a new vector field A_i on $J^{l-1}(N, Q)$ which can be regarded naturally as a vector field on $J^l(N, Q)$, for we have the assumption $N = \boldsymbol{R}^n$ and $Q = \boldsymbol{R}^q$. Moreover $A_1, \cdots, A_n$ are linearly independent at any point in $J^l(N, Q)$. Therefore they generate an $n(=\dim N)$-dimensional distribution $\tilde{D}$ on $J(N, Q)$. Let $s: U_{x_0} \times V_e \to Q$ be a general solution of $(A)_l$ and set $S_g = \{j_x^l(s_g) \mid x U_{x_0}\}$, $g \in V_e$. By the construction of A_i, for a map $\omega: U'_{x_0} \to Q$, A_i is tangent to $S_\omega = \{j_x^l(\omega) \mid x U'_{x_0}\}$ if ω is a solution of $(A)_l$. Therefore A_i is tangent to S_g. Since we have $S = \bigcup_{g \in V_e} S_g$, $\tilde{D}$ is tangent to S at any point. By calculating $[A_i, A_j]$, it follows that $[A_i, A_j]_p = 0$ if and only if $p \in I(\mathfrak{F}\mathfrak{N}(B)_l)$. Therefore by Proposition 2.3 and $I(\overline{\mathfrak{N}}(B)_l^*)\supset S$ the restriction D^S of $\tilde{D}$ to S is involutive and S_g, $g\in V_e$, is a maximal integral manifold of D^S and vice versa. We set $(\Delta^{-1})_*D^S=D$. Then D is an involutive distribution on $U_{x_0}\times V_e$. For each $g\in V_e$, g transforms any maximal integral manifold S_h of D^S to another maximal integral manifold S_{gh} if $gh\in V_e$, which implies that, for any cross-section X of D^S and any $Y\in\mathfrak{g}_s^*$, we have also a cross-section $[\mathrm{X}, Y]$ of D^S where $\mathfrak{g}_s^*$ is the Lie algebra induced from the action of G on S. Therefore by the property $[\beta]$ of Δ, we can also see that, for any cross-section X of D and any $Y\in\mathfrak{g}^*$, we have a cross-section $[\mathrm{X}, Y]$ of D where $\mathfrak{g}^*$ is a Lie algebra induced from the action of G on $N\times G$ as a principal fiber bundle $(N\times G)(N, G, \pi)$. We set $q_0=\Delta^{-1}(p_0)$. Then D is a Lie-Vessiot system at q_0 on $(N\times G)(N, G, \pi)$. Any first integral ψ of D at q_0 induces a first integral $\varphi=\psi\circ\Delta^{-1}$ of D^S at p_0. Since φ is constant on S_ω for any solution ω: $U'_{x_0}\to Q$ of $(A)_l$, this completes the proof of Theorem 2.1.

Definition 2.5. Let ρ^l be the projection of $J^l(N, Q)$ onto $N\times Q$. For a submanifold S of $J^l(N, Q)$ we set $\rho_S^l=\rho^l\,|\,S$, the restriction of ρ^l to S. A point $p\in S$ is said to be of maximal rank in S if $(d\rho_S^l)_p$ is of maximal rank. S is said to be of maximal rank if each point of S is of maximal rank.

Corollary 2.1. *Let $(A)_l$ be a system at p_0 satisfying $[\alpha_i]$ $(i=1, 2, 3)$ and $[\beta]$. We assume that S is of maximal rank. Then we can induce from $(A)_l$ a Lie-Vessiot system D at $q_0=\Delta^{-1}(p_0)$ on the trivial principal fiber bundle $(N\times G)(N, G, \pi)$ such that we can integrate $(A)_l$ at p_0 by seeking for an arbitrary fundamental system of solutions of D at q_0.*

Proof. Let $\varphi_1, \cdots, \varphi_r$ be any fundamental system of solutions of D^S at p_0. Then we have the functional determinant $D(\varphi_1, \cdots, \varphi_r)/D(z_1, \cdots, z_q, w_1, \cdots, w_{r-q})\neq 0$ on a neighbourhood $U_{p_0}^S$ of p_0 where $\{x_1, \cdots, x_n, z_1, \cdots, z_q, w_1, \cdots, w_{r-q}\}$ is the coordinate system on $U_{p_0}^S$ given in the proof of Proposition 2.1. We set $x_i^0=x_i(p_0)$ $(1\leqq i\leqq n)$, $z_j^0=z_j(p_0)$ $(1\leqq j\leqq q)$, $w_k^0=w_k(p_0)$ $(1\leqq k\leqq r-q)$ and $\varphi_j(x_1^0, \cdots, x_n^0, z_1^0, \cdots, z_q^0, w_1^0, \cdots, w_{r-q}^0)=c_j$

$(1 \leqq j \leqq r)$. Then by the implicit function theorem we have locally a unique system of functions $f_{z_j}(1 \leqq j \leqq q)$, $f_{w_k}(1 \leqq k \leqq r-q)$ of $x_1, \cdots, x_n$ such that $\varphi_j(x_1, \cdots, x_n, f_{z_1}(x), \cdots, f_{z_q}(x), f_{w_1}(x), \cdots, f_{w_{r-q}}(x)) = c_j(1 \leqq j \leqq r)$ and $f_{z_j}(x^0) = z_j^0 (1 \leqq j \leqq q)$, $f_{w_k}(x^0) = w_k^0 (1 \leqq k \leqq r-q)$. On the other hand, since $p_0 \in S$, we have a solution s_g, $g \in V_e$, of $(A)_l$ such that $p_0 = j_{x_0}^l(s_g)$. Since $z_j(s_g(x^0)) = z_j^0 (1 \leqq j \leqq q)$, $w_k(s_g(x^0)) = w_k^0(1 \leqq k \leqq r-q)$ and $\varphi_j(x_1, \cdots, x_n, z_1(s_g(x)), \cdots, z_q(s_g(x)), w_1(s_g(x)), \cdots, w_{r-q}(s_g(x))) = c_j(1 \leqq j \leqq r)$, we have $f_{z_j} = s_g^j (= z_j(s_g))$. Thus from any fundamental system of solutions of D^S at p_0, we can obtain a solution of $(A)_l$, that is, we can integrate $(A)_l$ at p_0.

Corollary 2.2. *Let $(A)_l$ be a system at p_0 satisfying $[\alpha_i]$ $(i=1, 2, 3)$ and $[\beta]$. We set $S^0 = \rho^l(S) \subset N \times Q$ and assume that S^0 is a submanifold of $N \times Q$ defined by $z_i = \varphi_1(x_i, \cdots, x_m, z_1, \cdots, z_t)(t+1 \leqq i \leqq q)$. Then we can induce from $(A)_l$ a Lie-Vessiot system D at $q_0 = \Delta^{-1}(p_0)$ on the trivial principal fiber bundle $(N \times G)(N, G, \pi)$ such that we can integrate $(A)_l$ at p_0 by seeking for an arbitrary fundamental system of solutions of D at q_0.*

Proof. By the same argument, we can obtain $f_{z_j}(x_1, \cdots, x_n)$ $(1 \leqq j \leqq t)$ from a fundamental system of solutions of D at q_0. For $t+1 \leqq i \leqq q$ we set $f_{z_i} = \varphi_i(x_1, \cdots, x_n, f_{z_1}, \cdots, f_{z_t})$. Therefore we can obtain a solution of $(A)_l$ from a fundamental system of solutions of D at q_0.

Let $(A)_l$ be a G-automorphic system satisfying $[\alpha_i]$ $(i=1, 2, 3)$. Let $y_1, \cdots, y_r$ be linearly independent functions on an open subset U of S where $\dim G = r$. Let $\{a_h\}_{h=1}^{r-1}$ be any family of real numbers. As before we set $y = (y_1, \cdots, y_r)$, $U_y^j = \{p \in U_y^{j-1} \mid y_j(p) = a_j\}$ and $y_j^{j-1} = y_j \mid U_y^{j-1}$ where $y_j^0 = y_j$, $U_j^0 = U$. We set $g \cdot y_j^{j-1}(p) = y_j^{j-1}(g \cdot p)$ and $V_y^j = \{g \in V_y^{j-1} \mid g \cdot y_j^{j-1} = y_j^{j-1}\}$, $V_y^0 = V_e$. Clearly each $g \in V_y^j$ operates on U_y^j if g operates on U.

Definition 2.6. Let f: $S \to T$ be a map of a manifold S to a manifold T. We set $\operatorname{graph}(f) = \{(p, f(p)) \mid p \in S\} \subset S \times T$. $\operatorname{graph}(f)$ is called the graph of f.

Definition 2.7. Let D be an $m(=\dim S)$ dimensional distribution on

$S\times \boldsymbol{R}$ defined on a neighbourhood of $(p, t)\in S\times \boldsymbol{R}$. If there exists a local coordinate system $\{\alpha_1, \cdots, \alpha_m, x\}$, where $\{\alpha_1, \cdots, \alpha_m\}$ (resp. $\{x\}$) is a local coordinate system of S at p (resp. of $\boldsymbol{R}$ at t), such that a local basis $\{X_1, \cdots, X_m\}$ of D at (p, t) is expressed as $X_j=\frac{\partial}{\partial\alpha_j}+\phi_j(\alpha_1, \cdots, \alpha_m)\frac{\partial}{\partial x}$, then D is said to be of quadrature type at (p, t).

Definition 2.8. A system $(A)_l$ of partial differential equations at $p_0\in J^l(N, Q)$ satisfying $[\alpha_i]$ $(i=1, 2, 3)$ and $[\beta]$ is said to be solvable if there exists a family of linearly independent functions $\{y_j\}_{j=1}^r$, $r=\dim G$, on a neighbourhood U of p_0 in S which satisfies, for a family of real numbers $\{a_h\}_{h=1}^{r-1}$, the following conditions;

[1] We can induce from $(A)_l$ an $m_j(=\dim U_y^j)$ dimensional distribution D^j on $U_y^j\times \boldsymbol{R}$ such that D^j is of quadrature type at any point in $U_y^j\times \boldsymbol{R}$ and for any $g\in V_y^j$ the graph of $g\cdot y_{j+1}^j$ is an integral manifold of D^j $(j=0, \cdots, r-1)$.

[2] There exists a solution ω of $(A)_l$ defined on $U_{x_0}^\omega\subset U_{x_0}$ such that, if we set $S_\omega=\{j_x^l(\omega)\,|\,x\in U_{x_0}^\omega\}$, the function $y_j\,|\,S_\omega$, the restriction of y_j to S_ω, is constant for each $j(j=1,\cdots, r-1)$ and $y_j\,|\,S_\omega=a_j$.

We shall call $\{y_j\}_{j=1}^r$ satisfying [1], [2], a fundamental family of functions of $(A)_l$.

Theorem 2.2. *Let $(A)_l$ be a system of partial differential equations at $p_0\in J^l(N, Q)$ satisfying $[\alpha_i]$ $(i=1, 2, 3)$ and $[\beta]$. Then the following two statements are equivalent;*

[i] *G is solvable.* [ii] *$(A)_l$ is solvable.*

Proof. First of all, we shall prove [i]$\Rightarrow$[ii]. By Theorem 2.1, we can find a Lie-Vessiot system D at $q_0=\Delta^{-1}(q_0)$ on $(N\times G)(N, G, \pi)$. Since we have a basis $\{X_h\}_{h=1}^m$ of D at p_0 such that $[\mathfrak{g}^*, X_h]=0$ $(1\leqq h\leqq m)$, for any ideal $\mathfrak{h}$ of $\mathfrak{g}$, $E=[\mathfrak{h}^*\cup D]$ (the distribution on $U_{x_0}\times V_e$ generated by $\mathfrak{h}^*$ and D) is involutive. Since $\mathfrak{g}$ is solvable, we have a sequence of subalgebras $\mathfrak{g}=\mathfrak{g}_0\supset\mathfrak{g}_1\supset\cdots\supset\mathfrak{g}_{r-1}\supset\mathfrak{g}_r=\{0\}$ such that $\dim\mathfrak{g}_j-\dim\mathfrak{g}_{j+1}=1$ and $\mathfrak{g}_{j+1}$ is an ideal of $\mathfrak{g}_j(0\leqq j\leqq r-1)$. We set $E_j=[\mathfrak{g}_j^*\cup$

$D]$ $(0\leqq j\leqq r)$. Let $\sigma_1, \cdots, \sigma_r$ be any fundamental system of solutions of $D=E_r$. We set $y_j=\sigma_j\circ\Delta^{-1}(1\leqq j\leqq r)$. Then by Theorem 2.1 for any solution ω of $(A)_l$, $y_j|S_\omega$ is constant for each $1\leqq j\leqq r$. Therefore, if we set $y_j|S_\omega=a_j$ $(1\leqq j\leqq r-1)$, $\{y_j\}_{j=1}^r$ satisfies [2]. We shall prove that we can find a fundamental system of solutions $\{\sigma_j\}_{j=1}^r$ of D such that [1] is also satisfied. We choose a basis $V_1, \cdots, V_r$ such that $\{V_k\}_{k=j}^r$ is a basis of $\mathfrak{g}_{j-1}^*$. By Theorem 1.1, there exists a function σ_1 on $U_{x_0}\times V_e$ such that $x_1\cdot\sigma_1=\cdots=X_m\cdot\sigma_1=V_1\cdot\sigma_1\cdots=V_{r-1}\cdot\sigma_1=0$ and $V_r\cdot\sigma_1=1$. By using a local coordinate system $\{\alpha_1, \cdots, \alpha_{n+r}\}$ on $U_{x_0}\times V_e$, the system

$$(*)_0 \qquad \begin{cases} X_h\cdot\sigma_1=V_k\cdot\sigma_1=0 & (1\leqq h\leqq m,\ 2\leqq k\leqq r) \\ V_1\cdot\sigma_1=1 \end{cases}$$

is expressed as

$$\begin{cases} \displaystyle\sum_{j=1}^{n+r}\xi_k^j\frac{\partial\sigma_1}{\partial\alpha_j}=0 & (1\leqq k\leqq n+r-1) \\ \displaystyle\sum_{j=1}^{n+r}\xi_{n+r}^j\frac{\partial\sigma_1}{\partial\alpha_j}=1. \end{cases}$$

We have therefore $\dfrac{\partial\sigma_1}{\partial\alpha_j}=\phi_j(\alpha_1, \cdots, \alpha_{n+r})$ $(1\leqq j\leqq n+r)$. Let $\{\alpha_1, \cdots, \alpha_{n+r}, x\}$ be accordinate system on $U_{x_0}\times V_e\times \boldsymbol{R}$ and we denote by D^0 the $(n+r)$-dimensional distribution on $S\times\boldsymbol{R}$ generated by $\tilde{\Delta}_*\left(\dfrac{\partial}{\partial\alpha_j}+\phi_j\dfrac{\partial}{\partial x}\right)$ $(1\leqq j\leqq n+r)$ where $\tilde{\Delta}$ is the diffeomorphism of $U_{x_0}\times V_e\times\boldsymbol{R}$ onto $S\times\boldsymbol{R}$ defined by $\tilde{\Delta}(x, g, t)=(\Delta(x, g), t)$. Clearly the graph of $\sigma_1\circ\Delta^{-1}$ is a maximal integral manifold of D^0.

We shall prove that, for any $g\in V_e$, the graph of $g\cdot(\sigma_1\circ\Delta^{-1})$ is also an integral manifold of D^0. Since σ_1 is a first integral of the involutive distribution E_1, and since we have $[\mathfrak{g}^*, E_1]\subset E_1$, $g\cdot\sigma_1$ is also a first integral of E_1, $g\in V_e$. On the other hand $\operatorname{codim} E_1=1$. Therefore we have a function H_g of one variable such that $g\cdot\sigma_1=H_g(\sigma_1)$. Since we have $\dim\mathfrak{g}^*/\mathfrak{g}_1^*=1$, the local transformation group on $\boldsymbol{R}$ consisting of H_g, $g\in V_e$, is commutative. Hence we have $H_{e_t}(H_g(\sigma_1))=H_g(H_{e_t}(\sigma_1))$ where $e_t=\exp t\, V_1$. We have then $V_1(g\cdot\sigma_1)=g(V_1\cdot\sigma_1)$. Therefore we have also

$$\begin{cases} X_h(g\circ\mathscr{s}_1)=V_k(g\circ\mathscr{s}_1)=0 & (1\leqq h\leqq m,\ 2\leqq k\leqq r) \\ V_1(g\circ\mathscr{s}_1)=1 \end{cases}$$

and therefore we have

$$\frac{\partial(g\cdot\mathscr{s}_1)}{\partial\alpha_j}=\phi_j(\alpha_1,\cdots,\alpha_{n+r}) \qquad (1\leqq j\leqq n+r).$$

This implies that the graph of $g(\mathscr{s}_1\circ\varDelta^{-1})$ is also an integral manifold of D^0.

Let $\mathscr{s}_1,\cdots,\mathscr{s}_r$ be a fundamental system of solutions of D such that we have

$$(*)_j \qquad \begin{cases} X^j_h\circ\mathscr{s}^j_{j+1}=V^j_k\circ\mathscr{s}^j_{j+1}=0 & (1\leqq h\leqq m,\ j+2\leqq k\leqq r) \\ V^j_{j+1}\circ\mathscr{s}^j_{j+1}=1 \end{cases}$$

on $(U_{x_0}\times V_e)^j_{\mathscr{s}}$ $(0\leqq j\leqq r-1)$. We choose a coordinate system $\{\alpha^j_1,\cdots,\alpha^j_{n+r-j}\}$ on $S^j_{\mathscr{s}}$. Then we have from $(*)_j$

$$\frac{\partial\mathscr{s}^j_{j+1}}{\partial\alpha^j_k}=\phi^j_k(\alpha^j_1,\cdots,\alpha^j_{n+r-j}) \qquad (1\leqq k\leqq n+r-j)$$

for each $0\leqq j\leqq r-1$. Therefore by similar method we get an $(n+r-j)$-dimensional distribution D^j on $S^j_{\mathscr{s}}\times\boldsymbol{R}$ of quadrature type such that, if we set $V^j_e=V_e\cap\exp\mathfrak{g}_j$, the graph of $g\cdot\mathscr{s}^j_{j+1}$ is an integral manifold of D_j for any $g\in V^j_e$. Note that we have $V^j_e=V^j_{\mathscr{s}}$. This implies $\{\mathscr{s}_j\circ\varDelta^{-1}\}^r_{j=1}$ is a fundamental family of functions of $(A)_l$. This completes the proof of [i]$\Rightarrow$[ii].

Conversely we shall prove [ii]$\Rightarrow$[i]. We denote also by $\mathfrak{g}^*$ the Lie algebra induced from the Lie algebra $\mathfrak{g}$ of G by the action of G on $J^l(N, Q)$. We set $\mathfrak{g}^*_{(0)}=\mathfrak{g}^*$ and inductively we set $\mathfrak{g}^*_{(j)}=\{X\in\mathfrak{g}^*_{(j-1)}\mid X\cdot y^{j-1}_j=0\}\mid U^j_y$ $(j=0,\cdots,r-1)$ where $\{y_j\}^r_{j=1}$ is a fundamental family of functions of $(A)_l$. Note that, since V_e acts freely on S, the restriction map $\mid U^j_y$: $\{X\in\mathfrak{g}^*_{(j-1)}\mid X\cdot y^{j-1}_j=0\}\to\mathfrak{g}^*_{(j)}$ is an isomorphism. By pulling back $\mathfrak{g}^*_{(j)}$ to a Lie subalgebra $\mathfrak{g}^*_j$ of $\mathfrak{g}^*$ through these restriction maps, we get a sequence of Lie subalgebras $\mathfrak{g}^*\supset\mathfrak{g}^*_1\supset\cdots\supset\mathfrak{g}^*_k\supset\cdots$ such that $\mathfrak{g}^*_k$ is isomorphic to $\mathfrak{g}^*_{(k)}$. Therefore we get also a chain of Lie subalgebras $\mathfrak{g}\supset\mathfrak{g}_1\supset\cdots\supset\mathfrak{g}_k\supset\cdots$ such that $\mathfrak{g}_k$ is isomorphic to $\mathfrak{g}^*_k$. We have $V^j_y=V_e\cap\exp\mathfrak{g}_j$ $(j=1, 2, \cdots)$. First of all we prove that $g\cdot y^j_{j+1}=y^j_{j+1}+c^j_g$, $g\in V^j_y$, where c^j_g is constant. By the assumption [1], the graph of $g\cdot y^j_{j+1}$ is an integral manifold of D^j. Since D^j is of quadrature type, there exists a local basis $\{X^j_1, \cdots, X^j_{m_j}\}$ such that, for a coordinate system $\{\alpha^j_1, \cdots, \alpha^j_{m_j}\}$ on U^j_y, X^j_k is expressed as $X^j_k=\dfrac{\partial}{\partial\alpha^j_k}+\phi^j_k(\alpha^j_1, \cdots, \alpha^j_{m_j})\dfrac{\partial}{\partial x}$ $(1\leqq k\leqq m_j)$. Therefore $g\cdot y^j_{j+1}$, $g\in V^j_y$, satisfies $\dfrac{\partial(g\cdot y^j_{j+1})}{\partial\alpha^j_k}=\phi^j_k\ (\alpha^j_1, \cdots, \alpha^j_{m_j})$ for each k. Fix an integer k, $1\leqq k\leqq m_j$. Considering α^j_h $(h\neq k)$ as parameters of an ordinary differential equation $\dfrac{d(g\cdot y^j_{j+1})}{d\alpha^j_k}=\phi^j_k\ (\alpha^j_1, \cdots, \alpha^j_{m_j})$, we get $g\cdot y^j_{j+1}=y^j_{j+1}+c^j_g(\cdots, \alpha^j_h, \cdots)$, $h\neq k$. Since k runs over the set of integers $\{1, 2, \cdots, m_j\}$, c^j_g must be constant. We shall next show that $\mathfrak{g}^*_{j+1}$ is an ideal of $\mathfrak{g}^*_j$. Let σ (resp. g) be any element of V^j_y (resp. V^{j+1}_y). Then we have $(\sigma^{-1}\cdot g\cdot\sigma)\cdot y^j_{j+1}=(\sigma)^{-1}\cdot g\cdot\sigma\cdot y^j_{j+1}=(\sigma)^{-1}\cdot g(y^j_{j+1}+c_\sigma)=(\sigma)^{-1}(y^j_{j+1}+c_\sigma)=y^j_{j+1}$. This implies that $\mathfrak{g}^*_{j+1}$ is an ideal of $\mathfrak{g}^*_j$. By [2] we have a solution ω of $(A)_l$ such that $y_j|S_\omega=a_j$ $(1\leqq j\leqq r-1)$. Therefore we have $S_\omega\cap U\subset U^j_y$ $(0\leqq j\leqq r-1)$. Since any $g\in V^j_y$ leaves $y^j_1, \cdots, y^j_j$ invariant, that is $g\cdot y^j_k=y^j_k$ $(1\leqq k\leqq j)$, we have $S^g_\omega\cap U\subset U^j_y$, $g\in V^j_y$, where $S^g_\omega=g\cdot S_\omega$. Therefore we get $U^j_y\supset\bigcup_{g\in V^j_y}S^g_\omega\cap U$. On the other hand if $g\in V^{j-1}_y$, $\notin V^j_y$, then we have $g\cdot y^{j-1}_j\neq y^{j-1}_j$ and therefore $c^{j-1}_g\neq 0$. This implies that $y^{j-1}_j|S^g_\omega\cap U\neq y^{j-1}_j|S_\omega\cap U$. Therefore $S^g_\omega\cap U\not\subset U^j_y$. We get, therefore, $S^g_\omega\cap U\subset U^j_y$ if and only if $g\in V^j_y$. We shall show that U^j_y is a union of some $S^g_\omega\cap U$, $g\in V_e$. Since we have $g\cdot y_1=y_1+c^1_g$ for any $g\in V_e$, y_1 is constant on each $S^g_\omega\cap U$, $g\in V_e$. On the other hand, since $(A)_l$ is G-automorphic, we have $U=\bigcup_{g\in V_e}S^g_\omega\cap U$ (disjoint union). This implies that U^1_y is a (disjoint) union of some $S^g_\omega\cap U$, $g\in V_e$. Therefore we get $U^1_y=\bigcup_{g\in V^1_y}S^g_\omega\cap U$. Since $g\cdot y^1_2=y^1_2+C^2_g$, $g\in V^1_y$, y^1_2 is constant on each $S^g_\omega\cap U\subset U^1_y$. This implies U^2_y is also a (disjoint) union of some $S^g_\omega\cap U$, $g\in V_e$ which implies $U^2_y=\bigcup_{g\in V^2_y}S^g_\omega\cap U$. Similarly U^j_y is a union of some $S^g_\omega\cap U$, $g\in V_e$ $(0\leqq j\leqq r-1)$. This implies that

$U_y^j = \bigcup_{g \in V_y^j} S_\omega^g \cap U$. Now we have $\dim U_y^j - \dim U_y^{j+1} = 1$ and since $(A)_l$ is G-automorphic, we have $\dim \bigcup_{g \in V_y^j} S_\omega^g \cap U = \dim S_\omega + \dim \mathfrak{g}_j^*$. Therefore we get $\dim \mathfrak{g}_j^* - \dim \mathfrak{g}_{j+1}^* = 1$. This proves that $\mathfrak{g}$ is solvable. This complete the proof of Theorem 2.2.

Corollary 2.3. *Let $(A)_l$ be a system at $p_0 \in J^l(N, Q)$ satisfying $[\alpha_i]$ $(i=1, 2, 3)$ and $[\beta]$. Then $(A)_l$ is solvable if and only if there exists a family of linearly independent functions $\{y_j\}_{j=1}^r$, $r=\dim G$, on a neighbourhood U of p_0 in S which satisfies, for a family of real numbers $\{a_h\}_{h=1}^{r-1}$, the following conditions;*

[1] *We have $g \cdot y_{j+1}^j = H_g^j(y_{j+1}^j)$, $g \in V_y^j$, for a function H_g^j of one variable depending on g and $j\,(j=0, \cdots, r-1)$.*

[2] *There exists a solution ω of $(A)_l$ such that $y_j \mid S_\omega = a_j$ $(j=1, \cdots, r-1)$ and such that if, $g \in V_y^j$, $g \cdot y_{j+1}^j \neq y_{j+1}^j$, then we have $y_{j+1}^j \mid S_\omega \cap U \neq y_{j+1}^j \mid S_\omega^g \cap U$ $(j=0, \cdots, r-1)$.*

Proof. We already showed in the proof of Theorem 2.2. that if $(A)_l$ is solvable, then any fundamental family of functions $\{y_j\}_{j=1}^r$ of $(A)_l$ satisfied [1], [2]. Conversely let $\{y_j\}_{j=1}^r$ be a family of linearly independent functions on U satisfying [1] and [2]. We have only to show that G is solvable. First of all we shall prove that U_y^j is a disjoint union of some $S_\omega^g \cap U$, $g \in V_e$. Since we have $g \cdot y_1 = H_g^1(y_1)$ for any $g \in V_e$, y_1 is constant on each $S_\omega^g \cap U$, $g \in V_e$. Since $(A)_l$ is G-automorphic, we have $U = \bigcup_{g \in V_e} S_\omega^g \cap U$ (disjoint union). Therefore U_y^1 is a disjoint union of some $S_\omega^g \cap U$, $g \in V_e$. We have clearly $U_y^1 \supset \bigcup_{g \in V_y^1} S_\omega^g \cap U$. If $g \in V_e$ and $\notin V_y^1$, then $g \cdot y_1 \neq y_1$. Therefore by the assumption [2], $y_1 \mid S_\omega^g \cap U \neq y_1 \mid S_\omega \cap U$. Hence we get $S_\omega^g \cap U \subset U_y^1$ if and only if $g \in V_y^1$. This implies $U_y^1 = \bigcup_{g \in V_y^1} S_\omega^g \cap U$. By similar considerations we have $U_y^j = \bigcup_{g \in V_y^j} S_\omega^g \cap U$ for $0 \leqq j \leqq r-1$. We have $\dim U_y^j - \dim U_y^{j+1} = 1$ and since $(A)_l$ is G-automorphic, we have $\dim \bigcup_{g \in V_y^j} S_\omega^g \cap U = \dim S_\omega + \dim \mathfrak{g}_j^*$. Therefore we get $\dim \mathfrak{g}_j^* - \dim \mathfrak{g}_{j+1}^* = 1$. Let σ (resp. g) be any element of V_y^j (resp. V_y^{j+1}). Then $(\sigma^{-1} \cdot g \cdot \sigma) \cdot y_{j+1}^j = (\sigma)^{-1} \cdot g(H_\sigma^{j+1}(y_{j+1}^j)) = (\sigma)^{-1} H_\sigma^{j+1}(y_{j+1}^j)$

$=y^j_{j+1}$. This implies that $\mathfrak{g}^*_{j+1}$ is an ideal of $\mathfrak{g}^*_j$. Therefore $\mathfrak{g}$ is solvable. By Theorem 2.2, $(A)_l$ is solvable. This completes the proof of Corollary 2.3.

Corollary 2.4. *Let $(A)_l$ be a system at $p_0 \in J^l(N, Q)$ satisfying $[\alpha_i]$ $(i=1, 2, 3)$ and $[\beta]$. Let D be the Lie-Vessiot system at $q_0=\Delta^{-1}(p_0)$ on $(N\times G)$ (N, G, π) induced from $(A)_l$ by Theorem 2.1. If $(A)_l$ is solvable, then there exists a fundamental family of functions of $(A)_l$ on a neighbourhood U of p_0 in S which satisfies the following conditions*:

[1] *There exists a basis $\{V_k\}^r_{k=1}$ of $\mathfrak{g}^*$ such that we have*

$$\begin{cases} V_{k+1}\cdot(y_{k+1}\circ\Delta)=1 \\ V_{k+j}\cdot(y_{k+1}\circ\Delta)=0 \qquad (2\leqq j\leqq r-k) \end{cases}$$

for $0\leqq k\leqq r-1$.

[2] *$\{y_k\circ\Delta\}^r_{k=1}$ is a fundamental system of solutions of D at q_0.*

Proof. By Theorem 2.2, G is solvable. Then the Lie-Vessiot system D at q_0 on $(N\times G)$ (N, G, π) is solvable. By Theorem 1.1, there exist a fundamental system of solutions $\{\sigma_k\}^r_{k=1}$ of D at q_0 and a basis $\{V_k\}^r_{k=1}$ of $\mathfrak{g}^*$ such that we have

$$\begin{cases} V_{k+1}\cdot\sigma_{k+1}=1 \\ V_{k+j}\cdot\sigma_{k+1}=0 \qquad (2\leqq j\leqq r-k) \end{cases}$$

for $0\leqq k\leqq r-1$. We put $y_j=\sigma_j\circ\Delta^{-1}$ $(j=1, 2,\cdots, r)$. Since $\mathfrak{g}^*$ is solvable, we have a sequence of subalgebras $\mathfrak{g}^*=\mathfrak{g}^*_0\supset\mathfrak{g}^*_1\supset\cdots\supset\mathfrak{g}^*_{r-1}\supset\mathfrak{g}^*_r=\{0\}$ such that $\dim\mathfrak{g}^*_j-\dim\mathfrak{g}^*_{j+1}=1$ and $\mathfrak{g}^*_{j+1}$ is an ideal of $\mathfrak{g}^*_j$. Note that, in Theorem 1.1, we chose a basis $\{V_k\}^r_{k=1}$ of $\mathfrak{g}^*$ such that $\{V_k\}^r_{k=j}$ is a basis of $\mathfrak{g}^*_{j-1}$, from which, as is proved in Theorem 2.2, it follows that $\{y_j\}^r_{j=1}$ is a fundamental family of functions of $(A)_l$.

Corollary 2.5. *Let $(A)_l$ be a system satisfying $[\alpha_i]$ $(i=1, 2, 3)$ and $[\beta]$. Moreover we assume that S is of maximal rank. If $(A)_l$ is solvable, then we can induce a fundamental family of functions $\{y_j\}^r_{j=1}$ of $(A)_l$ by quadratures such that we can obtain a general solution of $(A)_l$ by applying to $\{y_j\}^r_{j=1}$ the implicit function theorem.*

Proof. By Corollary 2.4, there exists a fundamental famliy of functions $\{y_j\}_{j=1}^r$ such that $\{y_j\circ\varDelta\}_{j=1}^r$ satisfies [1] and [2] in Corollary 2.4, which implies, by Corollary 1.1, that $\{y_j\circ\varDelta\}_{j=1}^r$ is obtained by quadrature. By Corollary 2.1, we can obtain a general solution of $(A)_l$ by applying to $\{y_j\}_{j=1}^r$ the implicit function theorem.

Corollary 2.6. *Let $(A)_l$ be a system satisfying $[\alpha_i]$ $(i=1, 2, 3)$ and $[\beta]$. We set $S^0=\rho^l(S)$ and assume that S^0 is a submanifold of $N\times G$ defined by $z_i=\varphi_i(x_1,\cdots, x_n, z,\cdots, z_t)$ $(t+1\leqq i\leqq q)$. If $(A)_l$ is solvable, then we can induce a fundamental family of functions $\{y_j\}_{j=1}^r$ of $(A)_l$ by quadratures such that we can obtain a general solution of $(A)_l$ by applying to $\{y_j\}_{j=1}^r$ the implicit function theorem.*

Proof. Using Corollary 2.2 in place of Corollary 2.1, Corollary 2.6 follows immediately.

§3. Examples

Example 1. We shall consider Riccati's differential equation $\frac{dz}{dx}=\eta_1(x)z^2+\eta_2(x)z+\eta_3(x)$. We set $X_1=z^2\frac{d}{dz}$, $X_2=z\frac{d}{dz}$ and $X_3=\frac{d}{dz}$. Then $\mathfrak{g}=\{\sum_{i=1}^3 c_iX_i\,|\,c_i\in\boldsymbol{R}\}$ is a Lie algebra. Note that $\mathfrak{g}$ is the Lie algebra of the projective transformatiton group on the 1-dimensional projective space and therefore simple. We set $X_*=\frac{\partial}{\partial x}+\eta_1\cdot X_1+\eta_2\cdot X_2+\eta_3\cdot X_3$ and denote by G a connected Lie group with the Lie algebra $\mathfrak{g}$. Then, by considering X_i $(i=1, 2, 3)$ as a right invariant vector field on G, X_* is a vector field on $\boldsymbol{R}\times G$. We can make G a Lie transformation group on G by $g*\alpha=\alpha\cdot g^{-1}$ (resp $g\circ\alpha=g\cdot\alpha$) $\alpha\in G$, $g\in G$ which we denote by G^* (resp. G_*). Let π be the projection of $\boldsymbol{R}\times G$ onto $\boldsymbol{R}$. Then $(\boldsymbol{R}\times G)$ $(\boldsymbol{R}, G^*, \pi)$ (resp. $(\boldsymbol{R}\times G)$ $(\boldsymbol{R}, G_*, \pi)$) is a principal fiber bundle. We denote by $\mathfrak{g}^*$ (resp. $\mathfrak{g}_*$) the Lie algebra induced from the action of G^* (resp. G_*) on $\boldsymbol{R}\times G$. Then any element of G^* commutes with all elements of G_* as a transformation on $\boldsymbol{R}\times G$. Therefore we get $[\mathfrak{g}^*, \mathfrak{g}_*]=0$. This implies that we have $[X_*, \mathfrak{g}^*]=0$, for any right invariant vector field on G is naturally considered as an element of $\mathfrak{g}^*$.

We denote by D the distribution on $\boldsymbol{R}\times G$ generated by X_*. Then D is a Lie-Vessiot system at any point p on $(\boldsymbol{R}\times G)(\boldsymbol{R}, G^*, \pi)$.

Let $p_0=(t_0, g_0)$ be any point $\in \boldsymbol{R}\times G$ and let

$$\mathfrak{N}(A)_1: \quad \begin{aligned} \frac{dz_1}{dx}&=F_1(x, z_1, z_2, z_3) \\ \frac{dz_2}{dx}&=F_2(x, z_1, z_2, z_3) \\ \frac{dz_3}{dx}&=F_3(x, z_1, z_2, z_3) \end{aligned}$$

be the system of ordinary differential equations at $\tilde{p}_0\in J^1(\boldsymbol{R}, G)$, $\tilde{p}_0=\rho_0^1(p_0)$, such that we have $(dz_i-F_i\cdot dx)\,(X_*)=0$ at any point of a neighbourhood of p_0, where $\{x\}$ (resp. $\{z_1, z_2, z_3\}$) is a local coordinate system at t_0 (resp. g_0). We shall show that $\mathfrak{N}(A)_1$ satisfies $[\alpha_i]$ $(i=1, 2, 3)$; Let (z_1^0, z_2^0, z_3^0) be a solution of $\mathfrak{N}(A)_1$ defined on U_{t_0}. We set $s=(z_1^0, z_2^0, z_3^0)$. Then s is a map of $U_{t_0}\to G$. We set $\bar{s}(x, g)=g*s(x)$. Since D is a Lie-Vessiot system at p_0 on $(\boldsymbol{R}\times G)(\boldsymbol{R}, G^*, \pi)$, both $s(U_{t_0})$ and $g*s(U_{t_0})$ are integral manifolds of D, that is, $g*s$ is also a solution of $\mathfrak{N}(A)_1$ if g is in a neighbourhood V_e of the unit of G. Since any integral manifold of D contained in a neighbourhood of p_0 is uniquely expressed as $g*s(U_{t_0})$, $g\in V_e$, $\mathfrak{N}(A)_1$ is G^*-automorphic. Since $1\times s$ is a local cross-section of $(\boldsymbol{R}\times G)(\boldsymbol{R}, G^*, \pi)$ where 1 is the identity of $\boldsymbol{R}$, $j^1(\bar{s})$: $U_{t_0}\times V_e\to J^1(\boldsymbol{R}, G)$ is an embedding. Moreover $S=j^1(\bar{s})(U_{t_0}\times V_e)$ is of maximal rank.

Example 2. Let G be a Lie transformation group acting effectively on a manifold M. Then G acts on the space of l-jets $J^l(M, M)$ naturally. Then, for a sufficiently large l, G acts freely on $J^l(M, M)$, or more precisely, there exists a neighbourhood V_e of the unit element of G such that each element of V_e acts freely on $J^l(M, M)$. We choose and fix an element $g_0\in G$. We set $\bar{s}(x, g)=g\cdot g_0(x)$. Then $\bar{s}$ is a map of $M\times G$ to M. Since V_e acts freely on $J^l(M, M)$, $j^l(\bar{s})$ is an embedding of $M\times$

V_e into $J^l(M, M)$. If G acts transitively on M, $S=j^l(\bar{s})(M\times V_e)$ is of maximal rank. Let $(A)_l$ be the system of defining equations at $p_0\in J^l(M, M)$ of the Lie transformation group G acting on $J^l(M, M)$. Then $(A)_l$ is G-automorphic.

Example 3. Let $(P_i, M_i, \{e\})$ be an $\{e\}$-structure on M_i and ω_i be the basic form of $(P_i, M_i, \{e\})$ $(i=1, 2)$. We assume that $(P_1, M_1, \{e\})$ is locally isomorphic to $(P_2, M_2, \{e\})$ at any point $(x_1, x_2)\in M_1\times M_2$. We denote by Γ (resp. $\tilde{\Gamma}$) the set of all local isomorphisms of $(P_1, M_1, \{e\})$ to $(P_2, M_2, \{e\})$ (resp. the set of all local diffeomorphisms ψ of P_1 to P_2 with $\psi^*\omega_2=\omega_1$). Then it is well-known that the natural lifting of an element of Γ gives a 1-1 correspondence between Γ and $\tilde{\Gamma}$ (cf. Singer, I. M. and S. Sternberg, The infinite groups of Lie and Cartan, *J. Analyse Math.* **15** (1965), 1-114).

Let G be the automorphism group of $(P_2, M_2, \{e\})$ and assume that any local automorphism of $(P_2, M_2, \{e\})$ is a restriction of an element of G. We denote by $(A)_1$ the system of partial differential equations at $p_0\in J^1(P_1, P_2)$ given by $\psi^*\omega_2=\omega_1$. Let ϕ be any local diffeomorphism of a neighbourhood U_{p_1} of $p_1\in P_1$ to a neighbourhood U_{p_2} of $p_2\in P_2$ with $\phi^*\omega_2=\omega_1$ and $\phi(p_1)=p_2$. We set $\bar{s}(p, g)=g\cdot\phi(p)$, $(p, g)\in U_{p_1}\times V_e$. Then $(A)_1$ is G-automorphic and $j^1(\bar{s})$ is an embedding of $U_{p_1}\times V_e$ into $J^1(P_1, P_2)$. Moreover if G acts transitively on P_2, $S=j^1(\bar{s})(U_{p_1}\times V_e)$ is of maximal rank.

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