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L^2-well-posedness for Hyperbolic Mixed Problems

By

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Introduction

Strongly hyperbolic differential equations become L^2-well-posed mixed problems under suitable boundary conditions. A concept of uniform Lopatinski's condition, given by S. Agmon [1], gives a sufficient condition for L^2-well-posedness. Moreover, it is known that some types of mixed problems become L^2-well-posed, which do not satisfy uniform Lopatinski's condition (cf. [2], [3], [4]). On the other hand, in the case of constant coefficients and half-space, a necessary and sufficient condition for L^2-well-posedness is given by R. Agemi & T. Shirota [5] by the words of uniform L^2-well-posedness for boundary value problems of ordinary differential equations with parameters. But it is not so concrete to clear the role of uniform Lopatinski's condition. This paper is a trial of more concrete characterizations of L^2-well-posedness for strongly hyperbolic mixed problems.

We consider the problem

$$(P)\begin{cases} A(D_t, D_y, D_x)u = \sum_{i+|\nu|+k\leqq m} a_{i\nu k}\, D_t^i D_y^\nu D_x^k u = f \\ \qquad\qquad \text{in } t>0,\ y\in R^{n-1},\ x>0, \\ B_j(D_t, D_y, D_x)u = \sum_{i+|\nu|+k\leqq r_j} b_{ji\nu k}\, D_t^i D_y^\nu D_x^k u = 0 \\ \qquad\qquad \text{on } t>0,\ y\in R^{n-1},\ x=0 \\ \qquad\qquad (j=1, 2, \cdots, \mu,\ 0\leqq r_j\leqq m-1), \\ D_t^j u = 0 \quad \text{on } t=0,\ y\in R^{n-1},\ x>0 \qquad (j=0, 1, \cdots, m-1). \end{cases}$$

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The problem (P) is said to be L^2-well-posed if there exists $C_T>0$ for any $T>0$ such that $C_T\to+0$ as $T\to+0$ and

$$\sum_{i+|\nu|+k\leqq m-1}\int_0^T dt\int_{R_+^n}|D_t^i D_y^\nu D_x^k u(t,\ y,\ x)|^2 dydx$$

$$\leqq C_T\int_0^T dt\int_{R_+^n}|f(t,\ y,\ x)|^2 dydx.$$

It is obvious that L^2-well-posedness is characterized only by the principal parts of $\{A,\ B_j\}$. Therefore, hereafter, we consider the case when $\{A,\ B_j\}$ are homogeneous. Assumptions are as follows:

i) A is strongly hyperbolic with respect to t-axis,

ii) $x=0$ is non-characteristic of A,

iii) $\{A,\ B_j\}$ satisfy Lopatinski's condition, that is,

$$R(\tau,\ \eta)=\det\left(\frac{1}{2\pi i}\int\frac{B_j(\tau,\ \eta;\ \xi)\xi^{k-1}}{A_+(\tau,\ \eta;\ \xi)}d\xi\right)_{\substack{j=1,\dots,\mu\\ k=1,\dots,\mu}}\neq 0$$

$$\text{for } \operatorname{Im}\tau<0,\ \eta\in R^{n-1},$$

where

$$A(\tau,\ \eta;\ \xi)=c\prod_{j=1}^{\mu}(\xi-\xi_j^+(\tau,\ \eta))\prod_{j=\mu+1}^{m}(\xi-\xi_j^-(\tau,\ \eta))$$

$$(\operatorname{Im}\xi_j^{\pm}(\tau,\ \eta)\gtrless 0 \quad \text{for } \operatorname{Im}\tau<0,\quad \eta\in R^{n-1}),$$

$$A_+(\tau,\ \eta;\ \xi)=\prod_{j=1}^{\mu}(\xi-\xi_j^+(\tau,\ \eta)).$$

Here we remark that iii) is a necessary condition for L^2-well-posedness for (P) under the assumptions i) and ii).

By Laplace-Fourier transform with respect to $(t,\ y)$, the problem (P) becomes to

$$(\hat{\mathrm{P}})\begin{cases} A(\tau,\ \eta;\ D_x)\hat{u}(x)=\hat{f}(x) & \text{for } x>0,\\ B_j(\tau,\ \eta;\ D_x)\hat{u}(x)|_{x=0}=0 & (j=1,\ 2,\ \cdots,\ \mu).\end{cases}$$

Let $G(\tau,\ \eta;\ x,\ y)$ be Green's function of $(\hat{\mathrm{P}})$, that is, the solution $\hat{u}\in L^2$

of (P) for $f\in L^2$ is represented by

$$\hat{u}(x)=\int_0^\infty G(\tau, \eta; x, y)\hat{f}(y)dy.$$

Then we have from the results of R. Agemi and T. Shirota.

Lemma. *In order that* (P) *is* L^2*-well-posed, it is necessary and sufficient that*

$$\sum_{k=0}^{m-1}\left\|\left(\frac{\partial}{\partial x}\right)^k G(\tau, \eta; x, y)\right\|_{\mathscr{L}(L^2, L^2)}\leq\frac{C}{|\operatorname{Im}\tau|}$$

for $\tau\in C^1$, $\operatorname{Im}\tau<0$, $\eta\in R^{n-1}$, $|\tau|^2+|\eta|^2=1$, *where* C *is independent of* (τ, η).

Remark. Let

$$G_0(\tau, \eta; x-y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i(x-y)\xi}}{A(\tau, \eta; \xi)}d\xi,$$

$$G(\tau, \eta; x, y)=G_0(\tau, \eta; x-y)-G_c(\tau, \eta; x, y),$$

then G in Lemma may be replaced by G_c.

§1. Necessary Conditions

1.1. Preliminary. Let (σ_0, η_0) be a real fixed point and let $\{\xi_i=\xi_i(\sigma_0, \eta_0)\}_{i=1,\dots,N}(N=N(\sigma_0, \eta_0))$ be real distinct roots of $A(\sigma_0, \eta_0, \xi)=0$ with multiplicities $\{m_i=m_i(\sigma_0, \eta_0)\}_{i=1,\dots,N}$. Then there exists a complex neighbourhood U of (σ_0, η_0) such that

$$A(\tau, \eta; \xi)=\prod_{i=1}^{N}H_i(\tau,\eta; \xi)E(\tau, \eta; \xi)=H(\tau, \eta; \xi)E(\tau, \eta; \xi),$$

$$H_i(\tau, \eta; \xi)=(\xi-\xi_i)^{m_i}+a_{i1}(\tau, \eta)(\xi-\xi_i)^{m_i-1}+\cdots+a_{im_i}(\tau, \eta)$$

in U, where $a_{ij}(\sigma_0, \eta_0)=0$ and $a_{ij}(\tau, \eta)$ are holomorphic. Moreover from the assumption i), $a_{ij}(\sigma, \eta)((\sigma, \eta)\in R^n\cap U)$ are real valued and $\frac{\partial}{\partial\tau}a_{im_i}(\tau, \eta)\neq 0$.

Now we denote

$$\alpha_i=\frac{\partial a_{im_i}}{\partial\tau}(\sigma_0,\ \eta_0),\qquad \beta_i=\frac{\partial a_{im_i}}{\partial\eta}(\sigma_0,\ \eta_0),$$

and we denote

$$\begin{cases} \Delta_{i\delta}=\{(\tau,\ \eta)\in V_\delta;\ |\alpha_i(\tau-\sigma_0)+\beta_i\cdot(\eta-\eta_0)| \\ \qquad\qquad \geqq \cos\theta_0(|\alpha_i|^2+|\beta_i|^2)^{\frac{1}{2}}d(\tau,\ \eta) \qquad \text{if } m_i\geqq 2, \\ \Delta_{i\delta}=V_\delta \qquad \text{if } m_i=1, \end{cases}$$

and

$$\Delta_\delta=\bigcap_i \Delta_{i\delta},$$

where

$$d=d(\tau,\ \eta)=\text{dis}\,\{(\tau,\eta),\ (\sigma_0,\ \eta_0)\},$$

$$V_\delta=\{\tau,\ \eta);\ \text{Im}\,\tau<0,\ \eta\in R^{n-1},\ d<\delta\},$$

and $\theta_0(0\leqq\theta_0<\pi)$ is an arbitrarily fixed number. Let $\{\xi_{ij}^{\pm}(\tau,\ \eta)\}_{j=1,\dots,m_i^{\pm}}$ ($\text{Im}\,\xi_{ij}^{\pm}\gtrless 0$) be roots of $H_i(\tau,\ \eta;\ \xi)=0$, and $\{\xi_{ij}^{\pm 0}(\tau,\ \eta)\}_{j=1,\dots,m_i^{\pm}}(\text{Im}\,\xi_{ij}^{\pm 0}\gtrless 0)$ be roots of

$$(\xi-\xi_i)^{m_i}+\alpha_i(\tau-\sigma_0)+\beta_i(\eta-\eta_0)=0.$$

Then we have

Lemma 1.1. *There exists* $\delta>0$ *such that*

$$\xi_{ij}^{\pm}(\tau,\ \eta)=\xi_{ij}^{\pm 0}(\tau,\ \eta)+0(d^{\frac{2}{m_i}}) \qquad in\ \Delta_{i\delta},$$

$$\frac{\partial\xi_{ij}^{\pm}}{\partial\tau}(\tau,\ \eta)=\frac{\partial\xi_{ij}^{\pm 0}}{\sigma\tau}(\tau,\ \eta)+0(d^{-1+\frac{2}{m_i}}) \qquad in\ \Delta_{i\delta}.$$

Corollary 1.

$$c_1 d^{\frac{1}{m_i}}\leqq|\xi_{ij}^{\pm}(\tau,\ \eta)-\xi_i|\leqq c_2 d^{\frac{1}{m_i}} \qquad in\ \Delta_{i\delta},$$

$$c_1 d^{\frac{1}{m_i}}\leqq|\xi_{ij}^{\pm}(\tau,\ \eta)-\xi_{ik}^{\pm}(\tau,\ \eta)|\leqq c_2 d^{\frac{1}{m_i}}\,(j\neq k) \qquad in\ \Delta_{i\delta}.$$

Next we consider of $\{\operatorname{Im} \xi^{\pm}_{ij}(\tau, \eta)\}$. Let us denote $i \in I$ if m_i is even, $i \in J$ if m_i is odd, and moreover $i \in J_{\pm}$ if $i \in J$ and $\alpha_i \gtrless 0$. Let

$$\Delta^{\pm}_{i\delta} = \Delta_{i\delta} \cap \{\alpha_i(\operatorname{Re} \tau - \sigma_0) + \beta_i \cdot (\eta - \eta_0) \gtreqless 0\},$$

and let

$$* = (*_1, *_2, \cdots, *_N) \ (*_i = \pm), \qquad \Delta^*_\delta = \bigcap_{i=1}^{N} \Delta^{*_i}_{i\delta},$$

then

$$\Delta_\delta = \bigcup_{*} \Delta^*_\delta.$$

Let $(\tau, \eta) \in \Delta^*_\delta$ and $\operatorname{Im} \tau \to 0$, we have the followings:

i) if $i \in I$, $*_i = +$, then none of $\{\xi^{\pm}_{ij}(\tau, \eta)\}$ has a real limit,

ii) if $i \in I$, $*_i = -$, then only one of $\{\xi^{+}_{ij}(\tau, \eta)\}$ and only one of $\{\xi^{-}_{ij}(\tau, \eta)\}$ have real limits, which we denote especially by $\xi^{\pm}_{i*}(\tau, \eta)$,

iii) if $i \in J_+$, then only one of $\{\xi^{+}_{ij}(\tau, \eta)\}$ has a real limit, which we denote by $\xi^{+}_{i*}(\tau, \eta)$,

iv) if $i \in J_-$, then only one of $\{\xi^{-}_{ij}(\tau, \eta)\}$ has a real limit, which we denote by $\xi^{-}_{i*}(\tau, \eta)$.

Here we denote

$$\delta^*_{ij\pm}(\tau, \eta) = \begin{cases} \left(\dfrac{\gamma}{d}\right)^{m_i} d & \text{if } \xi^{\pm}_{ij} = \xi^{\pm}_{i*}, \\ d & \text{otherwise}, \end{cases}$$

for $i = 1, \cdots, N$, $j = 1, \cdots, m^{\pm}_i$, where $\gamma = -\operatorname{Im} \tau$, then we have

Corollary 2.

$$c_1(\delta^*_{ij\pm})^{\frac{1}{m_i}} \leqq |\operatorname{Im} \xi^{\pm}_{ij}(\tau, \eta)| \leqq c_2(\delta^*_{ij\pm})^{\frac{1}{m_i}} \qquad \text{in } \Delta^*_\delta.$$

Proof.

$$\xi^{\pm}_{i*}(\tau, \eta) = \xi^{\pm}_{i*}(\sigma, \eta) - i\gamma \frac{\partial \xi^{\pm}_{i*}}{\partial \tau}(\sigma - i\theta\gamma, \eta) \qquad (0 < \theta < 1)$$

$$= \xi^{\pm}_{i*}(\sigma, \eta) - i\gamma \left\{ \frac{\partial \xi^{\pm 0}_{i*}}{\partial \tau}(\sigma - i\theta\gamma, \eta) + 0\left(d^{-1+\frac{2}{m_i}}\right) \right\},$$

$$\frac{\partial \xi_{i*}^{\pm 0}}{\partial \tau}(\sigma - i\theta\gamma, \eta) = -\frac{\alpha_i}{m_i(\xi_{i*}^{\pm 0}(\sigma - i\theta\gamma, \eta) - \xi_i)^{m_i - 1}}$$

$$= -\frac{\alpha_i}{m_i(\xi_{i*}^{\pm 0}(\sigma, \eta) - \xi_i + 0(\gamma^{\frac{1}{m_i}}))^{m_i - 1}}$$

$$= -\frac{\alpha_i}{m_i(\xi_{i*}^{\pm 0}(\sigma, \eta) - \xi_i)^{m_i - 1}} \cdot \left\{1 + 0\left(\left(\frac{\gamma}{d}\right)^{\frac{1}{m_i}}\right)\right\}.$$

Therefore we have

$$c_1 \frac{\gamma}{d^{1-\frac{1}{m_i}}} \leqq |\operatorname{Im} \xi_{i1}^{\pm}(\tau, \eta)| \leqq c_2 \frac{\gamma}{d^{1-\frac{1}{m_i}}} \qquad \text{in } \Delta_\delta^*$$

for $\gamma \ll d$. In other cases, the required results follow from

$$\xi_{ij}^{\pm}(\tau, \eta) = \xi_{ij}^{\pm 0}(\tau, \eta) + 0(d^{\frac{2}{m_i}}). \qquad \text{Q.E.D.}$$

Let

$$E(\tau, \eta; \xi) = E_+(\tau, \eta; \xi) E_-(\tau, \eta; \xi),$$

where roots of $E_\pm(\tau, \eta; \xi) = 0$ are on the upper (resp. lower) half plane and M is the degree of $E_\pm$.

Corollary 3.

$$k \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \sum_{i=1}^{N} \sum_{j=1}^{m_i^\pm} C_{ij\pm}^k(\tau, \eta) d^{-1+\frac{1}{m_i}} \begin{pmatrix} B_1(\tau, \eta; \xi_{ij}^{\pm}(\tau, \eta)) \\ \vdots \\ B_\mu(\tau, \eta; \xi_{ij}^{\pm}(\tau, \eta)) \end{pmatrix}$$

$$+ \sum_{j=1}^{M} C_{0j\pm}^k(\tau, \eta) \begin{pmatrix} \frac{1}{2\pi i} \oint \frac{B_1(\tau, \eta; \xi)\xi^{j-1}}{E_\pm(\tau, \eta; \xi)} d\xi \\ \vdots \\ \frac{1}{2\pi i} \oint \frac{B_\mu(\tau, \eta; \xi)\xi^{j-1}}{E_\pm(\tau, \eta; \xi)} d\xi \end{pmatrix},$$

where $\{C_{ij\pm}^k(\tau, \eta), C_{0j\pm}^k(\tau, \eta)\}$ *are bounded in* Δ_δ.

Proof. Let us denote

$$f_{(0)}(x)=f(x),$$

$$f_{(k)}(x_1, x_2, \cdots, x_{k+1})=\frac{f_{(k-1)}(x_1, x_2, \cdots, x_k)-f_{(k-1)}(x_2, x_3, \cdots, x_{k+1})}{x_1-x_{k+1}} \qquad (k=1, 2, 3, \cdots).$$

Then

$$\begin{pmatrix} B_{1(0)}(\tau, \eta; \xi_{11}^+(\tau, \eta))B_{1(1)}(\tau, \eta; \xi_{11}^+(\tau, \eta), \xi_{12}^+(\tau, \eta)) \cdots \\ \vdots \qquad \vdots \\ B_{\mu(0)}(\tau, \eta; \xi_{11}^+(\tau, \eta))B_{\mu(1)}(\tau, \eta; \xi_{11}^+(\tau, \eta), \xi_{12}^+(\tau, \eta)) \cdots \end{pmatrix.$$

$$B_{1(m_1-1)}(\tau, \eta; \xi_{11}^+(\tau, \eta), \cdots, \xi_{1m_1^+}^+(\tau, \eta), \xi_{11}^-(\tau, \eta), \cdots, \xi_{1m_1^-}^-(\tau, \eta))$$
$$\vdots$$
$$B_{\mu(m_1-1)}(\tau, \eta; \xi_{11}^+(\tau, \eta), \cdots, \xi_{1m_1^+}^+(\tau, \eta), \xi_{11}^-(\tau, \eta), \cdots, \xi_{1m_1^-}^-(\tau, \eta))$$

$$B_{1(0)}(\tau, \eta; \xi_{21}^+(\tau, \eta))\cdots\frac{1}{2\pi i}\oint\frac{B_1(\tau, \eta; \xi)}{E_+(\tau, \eta; \xi)}d\xi\cdots\frac{1}{2\pi i}\oint\frac{B_1(\tau, \eta; \xi)\xi^{M-1}}{E_+(\tau, \eta; \xi)}d\xi$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$B_{\mu(0)}(\tau, \eta; \xi_{21}^+(\tau, \eta))\cdots\frac{1}{2\pi i}\oint\frac{B_\mu(\tau, \eta; \xi)}{E_+(\tau, \eta; \xi)}d\xi\cdots\frac{1}{2\pi i}\oint\frac{B_\mu(\tau, \eta; \xi)\xi^{M-1}}{E_+(\tau, \eta; \xi)}d\xi$$

$$\begin{matrix} \frac{1}{2\pi i}\oint\frac{B_1(\tau, \eta; \xi)}{E_-(\tau, \eta; \xi)}d\xi\cdots\frac{1}{2\pi i}\oint\frac{B_1(\tau, \eta; \xi)\xi^{M-1}}{E_-(\tau, \eta; \xi)}d\xi \\ \vdots \qquad \vdots \\ \frac{1}{2\pi i}\oint\frac{B_\mu(\tau, \eta; \xi)}{E_-(\tau, \eta; \xi)}d\xi\cdots\frac{1}{2\pi i}\oint\frac{B_\mu(\tau, \eta; \xi)\xi^{M-1}}{E_-(\tau, \eta; \xi)}d\xi \end{matrix}\Bigg)$$

$$\xrightarrow[d\to 0]{}\begin{pmatrix} B_1(\sigma_0, \eta_0; \xi_1)B'_{1\xi}(\sigma_0, \eta_0; \xi_1)\cdots(m_1-1)!\ B_{1\xi}^{(m_1-1)}(\sigma_0, \eta_0; \xi_1)B_1(\sigma_0, \eta_0; \xi_2) \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ B_\mu(\sigma_0, \eta_0; \xi_1)B'_{\mu\xi}(\sigma_0, \eta_0; \xi_1)\cdots(m_1-1)!\ B_{\mu\xi}^{(m_1-1)}(\sigma_0, \eta_0; \xi_1)B_\mu(\sigma_0, \eta_0; \xi_2) \end{pmatrix.$$

$$\begin{matrix} \cdots\frac{1}{2\pi i}\oint\frac{B_1(\sigma_0, \eta_0; \xi)}{E_+(\sigma_0, \eta_0; \xi)}d\xi\cdots\frac{1}{2\pi i}\oint\frac{B_1(\sigma_0, \eta_0; \xi)\xi^{M-1}}{E_-(\sigma_0, \eta_0; \xi)}d\xi \\ \vdots \qquad \vdots \end{matrix}\Bigg).$$

Since

$$\text{rank}\begin{pmatrix} B_1(\sigma_0, \eta_0; \xi_1) B'_{1\xi}(\sigma_0, \eta_0; \xi_1) \cdots (m_1-1)!\ B_{0\xi}^{(m_1-1)}(\sigma_0, \eta_1; \xi_1) \cdots \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ B_\mu(\sigma_0, \eta_0; \xi_1) B'_{\mu\xi}(\sigma_0, \eta_0; \xi_1) \cdots (m_1-1)!\ B_{\mu\xi}^{(m_1-1)}(\sigma_0, \eta_0; \xi_1) \cdots \end{pmatrix} = \mu,$$

we have

$$k\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} B_{1(0)}(\tau, \eta; \xi_{11}^+(\tau, \eta)) \cdots B_{1(m_1-1)}(\tau, \eta; \xi_{11}^+(\tau, \eta), \cdots, \xi_{1m_1^-}^-(\tau, \eta)) \cdots \\ \vdots \qquad\qquad \vdots \\ B_{\mu(0)}(\tau, \eta; \xi_{11}^+(\tau, \eta)) \cdots B_{\mu(m_1-1)}(\tau, \eta; \xi_{11}^+(\tau, \eta), \cdots, \xi_{1m_1^-}^-(\tau, \eta)) \cdots \end{pmatrix} \begin{pmatrix} \alpha_{11}^k(\tau, \eta) \\ \vdots \\ \alpha_{1m_1-1}^k(\tau, \eta) \\ \vdots \\ \ \end{pmatrix},$$

where $\{\alpha_{ij}^k(\tau, \eta)\}$ are bounded.

Q.E.D.

1.2. Representation of Green's function in Δ_δ. Let us denote

$$\tilde{\boldsymbol{E}}_\pm(\tau, \eta; \xi) = {}^t\Big(\frac{1}{\xi - \xi_{11}^\pm(\tau, \eta)}, \cdots, \frac{1}{\xi - \xi_{1m_1^\pm}^\pm(\tau, \eta)}, \cdots, \frac{1}{E_\pm(\tau, \eta; \xi)}, \cdots, \frac{\xi^{M-1}}{E_\pm(\tau, \eta; \xi)}\Big),$$

$$\boldsymbol{E}_+(\tau, \eta; x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{ix\xi} \tilde{\boldsymbol{E}}_+(\tau, \eta; \xi) d\xi,$$

$$\boldsymbol{E}_-(\tau, \eta; x) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-ix\xi} \tilde{\boldsymbol{E}}_-(\tau, \eta; \xi) d\xi,$$

$$B_\pm(\tau, \eta) = \frac{1}{2\pi i}\oint \begin{pmatrix} B_1(\tau, \eta; \xi) \\ \vdots \\ B_\mu(\tau, \eta; \xi) \end{pmatrix} {}^t\tilde{\boldsymbol{E}}_\pm(\tau, \eta; \xi) d\xi$$

$$=\left|\begin{matrix} B_1(\tau,\eta;\xi_{11}^{\pm}(\tau,\eta))\cdots B_1(\tau,\eta;\xi_{1m_1^{\pm}}^{\pm}(\tau,\eta))\cdots\dfrac{1}{2\pi i}\displaystyle\oint\dfrac{B_1(\tau,\eta;\xi)}{E_{\pm}(\tau,\eta;\xi)}d\xi\cdots & \dfrac{1}{2\pi i}\displaystyle\oint\dfrac{B_1(\tau,\eta;\xi)\xi^{M-1}}{E_{\pm}(\tau,\eta;\xi)}d\xi \\ \vdots & \vdots \\ B_\mu(\tau,\eta;\xi_{11}^{\pm}(\tau,\eta))\cdots B_\mu(\tau,\eta;\xi_{1m_1^{\pm}}^{\pm}(\tau,\eta))\cdots\dfrac{1}{2\pi i}\displaystyle\oint\dfrac{B_\mu(\tau,\eta;\xi)}{E_{\pm}(\tau,\eta;\xi)}d\xi\cdots & \dfrac{1}{2\pi i}\displaystyle\oint\dfrac{B_\mu(\tau,\eta;\xi)\xi^{M-1}}{E_{\pm}(\tau,\eta;\xi)}d\xi \end{matrix}\right|.$$

Let Poisson's kernels $\{P_k(\tau,\eta;x)\}_{k=1,\dots,\mu}$ be the L^2-solutions of

$$\begin{cases} A(\tau,\eta;D_x)P_k(\tau,\eta;x)=0 & \text{for } x>0, \\ B_j(\tau,\eta;D_x)P_k(\tau,\eta;x)|_{x=0}=\delta_{jk} & (j=1,\dots,\mu), \end{cases}$$

that is,

$$(P_1(\tau,\eta;x),\dots,P_\mu(\tau,\eta;x))={}^t\boldsymbol{E}_+(\tau,\eta;x)\{B_+(\tau,\eta)\}^{-1}.$$

Then we have

$$G_c(\tau,\eta;x,y)=(P_1(\tau,\eta;x),\dots,P_\mu(\tau,\eta;x))\left\{\begin{pmatrix} B_1(\tau,\eta;D_x) \\ \vdots \\ B_\mu(\tau,\eta;D_x)\end{pmatrix} G_0(\tau,\eta;x-y)\right\}_{x=0},$$

where

$$\begin{pmatrix} B_1(\tau,\eta;D_x) \\ \vdots \\ B_\mu(\tau,\eta;D_x)\end{pmatrix} G_0(\tau,\eta;x-y)\Big|_{x=0}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\begin{pmatrix} B_1(\tau,\eta;\xi) \\ \vdots \\ B_\mu(\tau,\eta;\xi)\end{pmatrix}\frac{e^{-iy\xi}}{A(\tau,\eta;\xi)}d\xi.$$

By the way, we have

$$\frac{1}{iA(\tau,\eta;\xi)}\begin{pmatrix} B_1(\tau,\eta;\xi) \\ B_\mu(\tau,\eta;\xi)\end{pmatrix}=P_+(\tau,\eta)\tilde{\boldsymbol{E}}_+(\tau,\eta;\xi)+P_-(\tau,\eta)\tilde{\boldsymbol{E}}_-(\tau,\eta;\xi),$$

where

$$B_{\pm}(\tau,\eta)=P_{\pm}(\tau,\eta)\cdot\frac{1}{2\pi}\oint\tilde{\boldsymbol{E}}_{\pm}(\tau,\eta;\xi)^{t}\tilde{\boldsymbol{E}}\,(\tau,\eta;\xi)A(\tau,\eta;\xi)d\xi$$

$$=P_{\pm}(\tau,\eta)\cdot Q_{\pm}(\tau,\eta),$$

therefore we have

$$\begin{pmatrix} B_1(\tau,\eta;D_x) \\ \vdots \\ B_\mu(\tau,\eta;D_x)\end{pmatrix} G_0(\tau,\eta;x-y)|_{x=0}=P_{-}(\tau,\eta)\boldsymbol{E}_{-}(\tau,\eta;y)$$

$$=B_{-}(\tau,\eta)Q_{-}(\tau,\eta)\boldsymbol{E}_{-}(\tau,\eta;y).$$

Here we remark

$$Q_{-}(\tau,\eta)=\begin{pmatrix} \frac{1}{iA'_{\xi}(\tau,\eta;\xi^{-}_{11}(\tau,\eta))} & & & \\ & \frac{1}{iA'_{\xi}(\tau,\eta;\xi^{-}_{12}(\tau,\eta))} & & \\ & & \ddots & \\ & & & \begin{pmatrix} \frac{1}{2\pi}\int\frac{E_{+}(\tau,\eta;\xi)H(\tau,\eta;\xi)}{E_{-}(\tau,\eta;\xi)}d\xi \cdots & \frac{1}{2\pi}\int\frac{E_{+}(\tau,\eta;\xi)H(\tau,\eta;\xi)\xi^{M-1}}{E_{-}(\tau,\eta;\xi)}d\xi \\ \vdots & \\ \frac{1}{2\pi}\int\frac{E_{+}(\tau,\eta;\xi)H(\tau,\eta;\xi)\xi^{M-1}}{E_{-}(\tau,\eta;\xi)}d\xi\cdots & \frac{1}{2\pi}\int\frac{E_{+}(\tau,\eta;\xi)H(\tau,\eta;\xi)\xi^{2M-2}}{E_{-}(\tau,\eta;\xi)}d\xi \end{pmatrix}^{-1}\end{pmatrix}.$$

Let us denote

$$B(\tau,\eta)=(B_{+}(\tau,\eta))^{-1}B_{-}(\tau,\eta)Q_{-}(\tau,\eta)=\mathscr{B}(\tau,\eta)Q_{-}(\tau,\eta),$$

then we have

Lemma 1.2.

$$G_c(\tau,\eta;x,y)={}^{t}\boldsymbol{E}_{+}(\tau,\eta;x)B(\tau,\eta)\boldsymbol{E}_{-}(\tau,\eta;y)$$

$$= {}^t\boldsymbol{E}_+(\tau, \eta; x)\mathscr{B}(\tau, \eta)Q_-(\tau, \eta)\boldsymbol{E}_-(\tau, \eta; y) \qquad \textit{in } \Delta_\delta.$$

1.3. Estimates of Green's function in Δ_δ^*. It is obvious that it holds

$$\|G_c\|_{\mathscr{L}(L^2\times L^2, C^1)} \leqq \|G_c\|_{\mathscr{L}(L^2, L^2)} \leqq \|G_c\|_{L^2\times L^2}$$

in general. On the other hand, we show that it holds

$$\|G_c\|_{\mathscr{L}(L^2\times L^2, C^1)} \geqq c_1\|G_c\|_{\mathscr{L}(L^2, L^2)} \geqq c_2\|G_c\|_{L^2\times L^2}$$

in Δ_δ, where c_1, c_2 are positive constants independent of (τ, η).

Now let

$$N_\pm(\tau, \eta) = \begin{vmatrix} |\mathrm{Im}\, \xi_{11}^\pm(\tau, \eta)|^{-\frac{1}{2}} & & & & & \\ & \ddots & & & & \\ & & |\mathrm{Im}\, \xi_{1m_1^\pm}^\pm(\tau, \eta)|^{-\frac{1}{2}} & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{vmatrix}$$

and

$$\boldsymbol{F}_\pm(\tau, \eta; x) = N_\pm(\tau, \eta)^{-1}\boldsymbol{E}_\pm(\tau, \eta; x),$$

then L^2-norms of $\boldsymbol{F}_\pm(\tau, \eta; x)$ are bounded in Δ_δ, therefore we have

$$\left|\int \overline{\boldsymbol{F}}_+(\tau, \eta; x) G_c(\tau, \eta; x, y)^t\overline{\boldsymbol{F}}_-(\tau, \eta; y)\,dx\,dy\right|$$

$$\leqq C\|G_c\|_{\mathscr{L}(L^2\times L^2, C^1)} \qquad \text{in } \Delta_\delta,$$

where C is independent of (τ, η). Let

$$S_\pm(\tau, \eta) = \int_0^\infty \boldsymbol{F}_\pm(\tau, \eta; x)^t\overline{\boldsymbol{F}}_\pm(\tau, \eta; x)\,dx$$

$$= {}^t\!\left(\int_0^\infty \overline{\boldsymbol{F}}_\pm(\tau, \eta; x)^t\boldsymbol{F}_\pm(\tau, \eta; x)\,dx\right),$$

then we have

$$\int \bar{\boldsymbol{F}}_{\pm}(\tau, \eta; x) G_c(\tau, \eta; x, y)^t \bar{\boldsymbol{F}}_{-}(\tau, \eta; y) dx dy$$

$$= {}^t S_{+}(\tau, \eta) N_{+}(\tau, \eta) B_{+}(\tau, \eta) N_{-}(\tau, \eta) S_{-}(\tau, \eta).$$

Lemma 1.3. *$S_{\pm}(\tau, \eta)$ are positive hermitian matrices in Δ_δ and*

$$c_1 I < S_{\pm}(\tau, \eta) < c_2 I \qquad (I\text{: identity matrix}),$$

where c_1, c_2 are positive constants independent of (τ, η).

Proof. Let

$$S_{i\pm}(\tau, \eta) = \begin{pmatrix} \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \frac{|\mathrm{Im}\, \xi_{i1}^{\pm}(\tau, \eta)|}{|\xi - \xi_{i1}^{\pm}(\tau, \eta)|^2} d\xi & \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \frac{|\mathrm{Im}\, \xi_{i1}^{\pm}(\tau, \eta)|^{\frac{1}{2}} |\mathrm{Im}\, \xi_{i2}^{\pm}(\tau, \eta)|^{\frac{1}{2}}}{(\xi - \xi_{i1}^{\pm}(\tau, \eta))\overline{(\xi - \xi_{i2}^{\pm}(\tau, \eta))}} d\xi \cdots & \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \frac{|\mathrm{Im}\, \xi_{i1}^{\pm}(\tau, \eta)|^{\frac{1}{2}} |\mathrm{Im}\, \xi_{im_i^{\pm}}^{\pm}(\tau, \eta)|^{\frac{1}{2}}}{(\xi - \xi_{i1}^{\pm}(\tau, \eta))\overline{(\xi - \xi_{im_i^{\pm}}^{\pm}(\tau, \eta))}} d\xi \\ \vdots & \vdots & \vdots \\ \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \frac{|\mathrm{Im}\, \xi_{im_i^{\pm}}^{\pm}(\tau, \eta)|^{\frac{1}{2}} |\mathrm{Im}\, \xi_{i1}^{\pm}(\tau, \eta)|^{\frac{1}{2}}}{(\xi - \xi_{im_i^{\pm}}^{\pm}(\tau, \eta))\overline{(\xi - \xi_{i1}^{\pm}(\tau, \eta))}} d\xi & & \end{pmatrix}$$

$$= \pm i \begin{pmatrix} \dfrac{|\mathrm{Im}\, \xi_{i1}^{\pm}(\tau, \eta)|}{\xi_{i1}^{\pm}(\tau, \eta) - \overline{\xi_{i1}^{\pm}(\tau, \eta)}} & \dfrac{|\mathrm{Im}\, \xi_{i1}^{\pm}(\tau, \eta)|^{\frac{1}{2}} |\mathrm{Im}\, \xi_{i2}^{\pm}(\tau, \eta)|^{\frac{1}{2}}}{\xi_{i1}^{\pm}(\tau, \eta) - \overline{\xi_{i2}^{\pm}(\tau, \eta)}} \cdots & \dfrac{|\mathrm{Im}\, \xi_{i1}^{\pm}(\tau, \eta)|^{\frac{1}{2}} |\mathrm{Im}\, \xi_{im_i^{\pm}}^{\pm}(\tau, \eta)|^{\frac{1}{2}}}{\xi_{i1}^{\pm}(\tau, \eta) - \overline{\xi_{im_i^{\pm}}^{\pm}(\tau, \eta)}} \\ \vdots & \vdots & \vdots \\ \dfrac{|\mathrm{Im}\, \xi_{im_i^{\pm}}(\tau, \eta)|^{\frac{1}{2}} |\mathrm{Im}\, \xi_{i1}^{\pm}(\tau, \eta)|^{\frac{1}{2}}}{\xi_{im_i^{\pm}}^{\pm}(\tau, \eta) - \overline{\xi_{i1}^{\pm}(\tau, \eta)}} & & \end{pmatrix}$$

$$S_{0\pm}(\tau,\eta)=\begin{pmatrix} \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{|E_{\pm}(\tau,\eta;\xi)|^2}d\xi & \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\xi}{|E_{\pm}(\tau,\eta;\xi)|^2}d\xi\cdots & \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\xi^{M-1}}{|E_{\pm}(\tau,\eta;\xi)|^2}d\xi \\ \vdots & \vdots & \vdots \\ \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\xi^{M-1}}{|E_{\pm}(\tau,\eta;\xi)|^2}d\xi & & \end{pmatrix},$$

then we have

$$\det S_{\pm}(\tau,\eta)=\prod_{i=1}^{N}\det S_{i\pm}(\tau,\eta)\cdot\det S_{0\pm}(\tau,\eta)+0(1)$$

as $d\to 0$. From Lemma 1.1, we have

$$|\det S_{i\pm}(\tau,\eta)|=\frac{1}{2^{m_i^{\pm}}}\left|\prod_{j<k}\frac{\xi_{ij}^{\pm}(\tau,\eta)-\xi_{ik}^{\pm}(\tau,\eta)}{\xi_{ij}^{\pm}(\tau,\eta)-\overline{\xi_{ik}^{\pm}(\tau,\eta)}}\right|>c>0$$

in $\varDelta_\delta$, where c is independent of (τ,η). Obviously since

$$|\det S_{0\pm}(\tau,\eta)|>c>0,$$

we have

$$|\det S_{\pm}(\tau,\eta)|>c>0 \qquad \text{in } \varDelta_\delta.$$

On the other hand, we have easily that $S_{\pm}(\tau,\eta)$ are positive, hermitian and bounded in $\varDelta_\delta$. Therefore we have $S_{\pm}(\tau,\eta)>c\cdot I$. Q.E.D.

It follows from Lemma 1.3

$$\begin{aligned} c_1|N_+(\tau,\eta)B(\tau,\eta)N_-(\tau,\eta)| &\\ \leqq\left|\int\overline{F}_+(\tau,\eta;x)G_c(\tau,\eta;x,y)^t\overline{F}_-(\tau,\eta;y)dxdy\right| &\\ \leqq c_2|N_+(\tau,\eta)B(\tau,\eta)N_-(\tau,\eta)| \qquad \text{in } \varDelta_\delta. \end{aligned}$$

On the other hand, we have

$$\|G_c\|_{L^2\times L^2}=\|{}^t\boldsymbol{E}_+(\tau,\eta;x)B(\tau,\eta)\boldsymbol{E}_-(\tau,\eta;y)\|_{L^2\times L^2}$$
$$\leqq C|N_+(\tau,\eta)B(\tau,\eta)N_-(\tau,\eta)|$$

in Δ_δ, and moreover

$$\left\|\left(\frac{\partial}{\partial x}\right)^k G_c\right\|_{L^2\times L^2}\leqq C_k|N_+(\tau,\eta)B(\tau,\eta)N_-(\tau,\eta)|$$

in Δ_δ. Hence we have

Lemma 1.4.

$$\sum_{k=0}^{m-1}\left\|\left(\frac{\partial}{\partial x}\right)^k G_c\right\|_{L^2\times L^2}\leqq C|N_+(\tau,\eta)B(\tau,\eta)N_-(\tau,\eta)|$$
$$\leqq C'\|G_c\|_{\mathscr{L}(L^2\times L^2,C^1)} \qquad \textit{in } \Delta_\delta.$$

Let us denote

$$D^*_\pm(\tau,\eta)=\begin{pmatrix}\{\delta^*_{11\pm}(\tau,\eta)\}^{-\frac{1}{2m_1}} & & & & \\ & \ddots & & & \\ & & \{\delta^*_{1m_1^\mp\pm}(\tau,\eta)\}^{-\frac{1}{2m_1}} & & \\ & & & \ddots & \\ & & & & 1 \\ & & & & & \ddots \\ & & & & & & 1\end{pmatrix}$$

in Δ^*_δ, then we have

Proposition 1.

$$c_1|D^*_+(\tau,\eta)B(\tau,\eta)D^*_-(\tau,\eta)|\leqq\|G_c\|_{\mathscr{L}(L^2,H^{m-1})}$$
$$\leqq c_2|D^*_+(\tau,\eta)B(\tau,\eta)D^*_-(\tau,\eta)| \qquad \textit{in } \Delta^*_\delta.$$

Let us denote

$$\mathscr{D}_-(\tau,\eta)=\begin{pmatrix}\overbrace{\{d(\tau,\eta)\}^{-1+\frac{1}{m_1}} \quad \ddots \quad \{d(\tau,\eta)\}^{-1+\frac{1}{m_1}}}^{m_1^-} & & & \\ & \ddots & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1\end{pmatrix}$$

then we have

Proposition 1′.

$$c_1 | D_+^*(\tau, \eta)\mathscr{B}(\tau, \eta)\mathscr{D}_-(\tau, \eta)D_-^*(\tau, \eta)| \leqq \|G_c\|_{\mathscr{L}(L^2, H^{m-1})}$$
$$\leqq c_2 | D_+^*(\tau, \eta)\mathscr{B}(\tau, \eta)\mathscr{D}_-(\tau, \eta)D_-^*(\tau, \eta)| \qquad in\ \Delta_\delta^*.$$

Here we have

Theorem I. *Let* (P) *be* L^2*-well-posed. Then, at any real point* (σ_0, η_0), *there exists* $\delta>0$ *such that*

$$| D_+^*(\tau, \eta)\mathscr{B}(\tau, \eta)\mathscr{D}_-(\tau, \eta)D_-^*(\tau, \eta)| < \frac{C}{\gamma}$$

in Δ_δ^*, *where* C *is a positive constant independent of* (τ, η).

Let us denote

$$\mathscr{B}(\tau, \eta)=\begin{pmatrix} \beta_{11,11}(\tau, \eta) & \cdots & \beta_{11,1m_1^-}(\tau, \eta) & \cdots\cdots & & \\ \vdots & & \vdots & & & \\ \beta_{1m_1^+,11}(\tau, \eta) & \cdots & \beta_{1m_1^+,1m_1^-}(\tau, \eta) & & \vdots & \\ \vdots & & & & & \\ & & & \beta_{01,01}(\tau, \eta) & \cdots & \beta_{01,0M}(\tau, \eta) \\ & & \cdots\cdots & \vdots & & \vdots \\ & & & \beta_{0M,01}(\tau, \eta) & \cdots & \beta_{0M,0M}(\tau, \eta) \end{pmatrix},$$

then Theorem I says that a necessary condition for L^2-well-posedness is

$$\begin{cases} |\beta_{ij,lh}(\tau, \eta)|\,\delta_{ij+}^{*\,-\frac{1}{2m_i}}\, d^{-1+\frac{1}{m_l}}\,\delta_{lh-}^{*\,-\frac{1}{2m_l}} < \dfrac{C}{\gamma} & \begin{pmatrix} i=1, \cdots, N, & l=1, \cdots, N \\ j=1, \cdots, m_i^+, & h=1, \cdots, m_l^- \end{pmatrix}, \\ |\beta_{0j,lh}(\tau, \eta)|\, d^{-1+\frac{1}{m_l}}\,\delta_{lh-}^{*\,-\frac{1}{2m_l}} < \dfrac{C}{\gamma} & \begin{pmatrix} j=1, \cdots, M, & l=1, \cdots, N \\ & h=1, \cdots, m_l^- \end{pmatrix}, \\ |\beta_{ij,0h}(\tau, \eta)|\,\delta_{ij+}^{*\,-\frac{1}{2m_i}} < \dfrac{C}{\gamma} & \begin{pmatrix} i=1, \cdots, N, & h=1, \cdots, M \\ j=1, \cdots, m_i^+, & \end{pmatrix}, \\ |\beta_{0j,0h}(\tau, \eta)| < \dfrac{C}{\gamma} & (j=1, \cdots, M,\ h=1, \cdots, M) \end{cases}$$

in Δ_δ^*.

Since $\delta^*_{ij\pm} \leqq d$, we have

Corollary 1. *If* (P) *is L^2-well-posed, then there exists $\delta > 0$ at every real point (σ_0, η_0) such that*

$$\begin{cases} |\beta_{ij,lh}(\tau, \eta)|\, d^{-\frac{1}{2m_i}-(1-\frac{1}{2m_l})} < \dfrac{C}{\gamma} & (i \neq 0,\ l \neq 0), \\ |\beta_{0j,lh}(\tau, \eta)|\, d^{-(1-\frac{1}{2m_l})} < \dfrac{C}{\gamma} & (l \neq 0), \\ |\beta_{ij,0h}(\tau, \eta)|\, d^{-\frac{1}{2m_i}} < \dfrac{C}{\gamma} & (i \neq 0), \\ |\beta_{0j,0h}(\tau, \eta)| < \dfrac{C}{\gamma} \end{cases}$$

in $\varDelta_\delta$.

Let us denote

$$\{B_+(\tau, \eta)\}^{-1} = \begin{pmatrix} r_{11,1}(\tau, \eta) & \cdots & r_{11,\mu}(\tau, \eta) \\ \vdots & & \vdots \\ r_{1m_1^+,1}(\tau, \eta) & \cdots & r_{1m_1^+,\mu}(\tau, \eta) \\ \vdots & & \vdots \\ r_{01,1}(\tau, \eta) & \cdots & \\ \vdots & & \\ r_{0M,1}(\tau, \eta) & \cdots & \end{pmatrix},$$

then we have from the Corollary 3 of Lemma 1.1 and the Corollary 1 of Theorem I

Corollary 2. *A necessary condition for L^2-well-posedness is*

$$\begin{cases} |r_{ij,k}(\tau, \eta)|\, d^{-\frac{1}{2m_i}} < \dfrac{C}{\gamma} & (i \neq 0), \\ |r_{0j,k}(\tau, \eta)| < \dfrac{C}{\gamma} \end{cases}$$

in $\varDelta_\delta$.

Corollary 3. *Let* (P) *be L^2-well-posed, and let $A(\sigma_0, \eta_0; \xi)$ have not real multiple roots. Let us assume that* $m_i^+ = 1 (i = 1, \cdots, N_0)$, $m_i^- = 1 (i =$

$N_0+1, \cdots, N)$, *and*

$$\operatorname{rank}\begin{pmatrix} \frac{1}{2\pi i}\int\frac{B_1(\sigma_0, \eta_0; \xi)}{E_+(\sigma_0, \eta_0; \xi)}d\xi\cdots & \frac{1}{2\pi i}\int\frac{B_1(\sigma_0, \eta_0; \xi)\xi^{M-1}}{E_+(\sigma_0, \eta_0; \xi)}d\xi \\ \vdots & \vdots \\ \frac{1}{2\pi i}\int\frac{B_\mu(\sigma_0, \eta_0; \xi)}{E_+(\sigma_0, \eta_0; \xi)}d\xi\cdots & \frac{1}{2\pi i}\int\frac{B_\mu(\sigma_0, \eta_0; \xi)\xi^{M-1}}{E_+(\sigma_0, \eta_0; \xi)}d\xi \end{pmatrix}=M_0,$$

then we have

i) $$\left\{\begin{pmatrix} B_1(\sigma_0, \eta_0; \xi_j) \\ \vdots \\ B_\mu(\sigma_0, \eta_0; \xi_j)\end{pmatrix}\right\}_{j=1,\dots,N_0}$$

are linearly independent modulo the space spanned by

$$\left\{\begin{pmatrix} \frac{1}{2\pi i}\int\frac{B_1(\sigma_0, \eta_0; \xi)\xi^{j-1}}{E_+(\sigma_0, \eta_0; \xi)}d\xi \\ \vdots \\ \frac{1}{2\pi i}\int\frac{B_\mu(\sigma_0, \eta_0; \xi)\xi^{j-1}}{E_+(\sigma_0, \eta_0; \xi)}d\xi \end{pmatrix}\right\}_{j=1,\dots,M}$$

ii) $$\left\{\begin{pmatrix} B_1(\sigma_0, \eta_0; \xi_j) \\ \vdots \\ B_\mu(\rho_0, \eta_0; \xi_j)\end{pmatrix}\right\}_{j=N_0+1,\dots,N}$$

belong the space spanned by

$$\left\{\begin{pmatrix} B_1(\sigma_0, \eta_0; \xi_j) \\ \vdots \\ B_\mu(\sigma_0, \eta_0; \xi_j)\end{pmatrix}\right\}_{j=1,\dots,N_0} \quad \text{and} \quad \left\{\begin{pmatrix} \frac{1}{2\pi i}\int\frac{B_1(\sigma_0, \eta_0; \xi)\xi^{j-1}}{E_+(\sigma_0, \eta_0; \xi)}d\xi \\ \vdots \\ \frac{1}{2\pi i}\int\frac{B_\mu(\sigma_0, \eta_0; \xi)\xi^{j-1}}{B_\mu(\sigma_0, \eta_0; \xi)}d\xi \end{pmatrix}\right\}_{j=1,\dots,M}$$

iii) $R(\sigma_0, \eta_0)=R'_\tau(\sigma_0, \eta_0)=\cdots=R_\tau^{(M_1-1)}(\sigma_0, \eta_0)=0, \quad R_\tau^{(M_1)}(\sigma_0, \eta_0)\neq 0,$

where $M_1=M-M_0$.

§2. Sufficient Conditions

2.1. Preliminary. Let us say that a real point (σ_0, η_0) is a regular point, when m_i-ple real root $\xi=\xi_i$ of $A(\sigma_0, \eta_0; \xi)=0$ may be m_i-ple or simple in a neighbourhood of (σ_0, η_0). Let (σ_0, η_0) be a regular point, then m_i $(\geqq 2)$-ple real roots are just all over a real analytic surface S_i: $\sigma=\varphi_i(\eta)$.

Now let (σ_0, η_0) be a regular point. Already we have had a decomposition of A in U with center (σ_0, η_0):

$$A(\tau, \eta; \xi)=\prod_{i=1} H_i(\tau, \eta; \xi)E(\tau, \eta; \xi),$$

where

$$H_i(\tau, \eta; \xi)=(\xi-\xi_i)^{m_i}+a_{i1}(\tau, \eta)(\xi-\xi_i)^{m_i-1}+\cdots+a_{im_i}(\tau, \eta),$$

$$a_{ij}(\sigma_0, \eta_0)=0.$$

Let $(\tilde{\sigma}_0, \tilde{\eta}_0)\in S_i\cap U$ and let $\tilde{\xi}_i$ be the m_i-ple root of $H_i(\tilde{\sigma}_0, \tilde{\eta}_0; \xi)=0$, then we have

$$H_i(\tau, \eta; \xi)=(\xi-\tilde{\xi}_i)^{m_i}+\tilde{a}_{i1}(\tau, \eta)(\xi-\tilde{\xi}_i)^{m_i-1}+\cdots+\tilde{a}_{im_i}(\tau, \eta),$$

$$\tilde{a}_{ij}(\tilde{\sigma}_0, \tilde{\eta}_0)=0.$$

Since

$$\tilde{a}_{im_i-k}(\tau, \eta)=\frac{1}{k!}\frac{\partial^k H_i}{\partial\xi^k}(\tau, \eta; \tilde{\xi}_i),$$

and $\tilde{\xi}_i$ is a continuous function of $(\tilde{\sigma}_0, \tilde{\eta}_0)$, we have a neighbourhood $U'\subset U$ such that

$$|\tilde{a}_{im_i-k}(\tau, \eta)|_{\mathscr{B}^2(U')}<C,$$

$$\left|\frac{\partial\tilde{a}_{im_i}}{\partial\tau}(\tau, \eta)\right|_{\mathscr{B}^0(U')}>c>0,$$

where C and c are independent of $(\tilde{\sigma}_0, \tilde{\eta}_0)$.

Let us denote

$$\tilde{\alpha}_i = \frac{\partial \tilde{a}_{im_i}}{\partial \tau}(\tilde{\sigma}_0, \tilde{\eta}_0), \qquad \tilde{\beta}_i = \frac{\partial \tilde{a}_{im_i}}{\partial \eta}(\tilde{\sigma}_0, \tilde{\eta}_0),$$

$$\begin{cases} \tilde{\Delta}_{i\delta} = \{(\tau, \eta) \in V_\delta;\ |\tilde{\alpha}_i(\tau - \tilde{\sigma}_0) + \tilde{\beta}_i \cdot (\eta - \tilde{\eta})| \\ \qquad\qquad \geqq \cos\theta_0 (|\tilde{\alpha}_i|^2 + |\tilde{\beta}_i|^2)^{\frac{1}{2}} \cdot \tilde{d}\} \qquad \text{if } m_i \geqq 2, \\ \tilde{\Delta}_{i\delta} = V_\delta \qquad \text{if } m_i = 1, \end{cases}$$

where

$$\tilde{d} = \operatorname{dis}\{(\tau, \eta), (\tilde{\sigma}_0, \tilde{\eta}_0)\}.$$

Moreover let $\{\tilde{\xi}_{ij}^{\pm 0}(\tau, \eta)\}$ be roots of

$$(\xi - \tilde{\xi}_i)^{m_i} + \tilde{\alpha}_i(\tau - \tilde{\sigma}_0) + \tilde{\beta}_i \cdot (\eta - \tilde{\eta}_0) = 0,$$

then we have

Lemma 2.1. *Let (σ_0, η_0) be a regular point, then there exist a neighbourhood U of (σ_0, η_0) and $\delta > 0$ $(V_\delta \subset U)$ such that for any point $(\tilde{\sigma}_0, \tilde{\eta}_0) \in S_i \cap U$, we have*

$$|\xi_{ij}^{\pm}(\tau, \eta) - \tilde{\xi}_{ij}^{\pm 0}(\tau, \eta)| < C\tilde{d}^{\frac{2}{m_i}} \qquad \text{in } \tilde{\Delta}_{i\delta},$$

$$\left| \frac{\partial \xi_{ij}^{\pm}}{\partial \tau}(\tau, \eta) - \frac{\partial \tilde{\xi}_{ij}^{\pm 0}}{\partial \tau}(\tau, \eta) \right| < C\tilde{d}^{-1+\frac{2}{m_i}} \qquad \text{in } \tilde{\Delta}_{i\delta},$$

where C is independent not only of (τ, η) but also of $(\tilde{\sigma}_0, \tilde{\eta}_0)$.

Corollary 1. *Let (σ_0, η_0) be a regular point, then for any $(\tilde{\sigma}_0, \tilde{\eta}_0) \in S_i \cap U$*

$$c_1 \tilde{d}^{\frac{1}{m_i}} \leqq |\xi_{ij}^{\pm}(\tau, \eta) - \tilde{\xi}_i| \leqq c_2 \tilde{d}^{\frac{1}{m_i}} \qquad \text{in } \tilde{\Delta}_{i\delta},$$

$$c_1 \tilde{d}^{\frac{1}{m_i}} \leqq |\xi_{ij}^{\pm}(\tau, \eta) - \xi_{ik}^{\pm}(\tau, \eta)| \leqq c_2 \tilde{d}^{\frac{1}{m_i}} \qquad (j \neq k) \text{ in } \tilde{\Delta}_{i\delta},$$

where c_1 and c_2 are independent of (τ, η) and $(\tilde{\sigma}_0, \tilde{\eta}_0)$.

We define $\tilde{\delta}_{ij\pm}^*$ in the same way as $\delta_{ij\pm}^*$, only replacing d by $\tilde{d}$, and $\tilde{\Delta}_\delta^* = \cap \tilde{\Delta}_{i\delta}^{*i}$, then

Corollary 2. *Let* (σ_0, η_0) *be a regular point, then for any* $(\tilde{\sigma}_0, \tilde{\eta}_0) \in S_i \cap U$

$$c_1 \tilde{\delta}^{*}_{ij\pm}{}^{\frac{1}{m_i}} \leqq |\operatorname{Im} \xi^{\pm}_{ij}(\tau, \eta)| \leqq c_2 \tilde{\delta}^{*}_{ij\pm}{}^{\frac{1}{m_i}} \qquad in\ \tilde{\Delta}_{i\delta}$$

where c_1 *and* c_2 *are independent of* (τ, η) *and* $(\tilde{\sigma}_0, \tilde{\eta}_0)$.

Now let us denote

$$V^{\pm}_{i\delta} = \bigcup_{(\tilde{\sigma}_0, \tilde{\eta}_0)} \tilde{\Delta}^{\pm}_{i\delta}, \quad V^{*}_{\delta} = \bigcap_{i=1}^{N} V^{*_i}_{i\delta},$$

then we have

$$V_\delta = \bigcup_{*} V^{*}_{\delta}.$$

Moreover we denote for $m_i \geqq 2$

$$d_i(\tau, \eta) = \operatorname{dis}((\tau, \eta), S_i),$$

$$\bar{\delta}^{*}_{ij\pm}(\tau, \eta) = \begin{cases} \left(\dfrac{\gamma}{d_i}\right)^{m_i} d_i & \text{if } \xi^{\pm}_{ij} = \xi^{\pm}_{i*}, \\ d_i & \text{otherwise}, \end{cases}$$

and $\bar{\delta}^{*}_{ij\pm}(\tau, \eta) = \gamma$ for $m_i = 1$, then we have

Corollary 1′. *Let* (σ_0, η_0) *be a regular point, then we have*

$$c_1 d_i^{\frac{1}{m_i}} \leqq |\xi^{\pm}_{ij}(\tau, \eta) - \xi^{\pm}_{ik}(\tau, \eta)| \leqq c_2 d_i^{\frac{1}{m_i}} \qquad (j \neq k)$$

in V_δ.

Corollary 2′. *Let* (σ_0, η_0) *be a regular point, then we have*

$$c_1 \bar{\delta}^{*}_{ij\pm}{}^{\frac{1}{m_i}} \leqq |\operatorname{Im} \xi^{\pm}_{ij}(\tau, \eta)| \leqq c_2 \bar{\delta}^{*}_{ij\pm}{}^{\frac{1}{m_i}}$$

in V^{*}_{δ}.

Corollary 3. *Let* (σ_0, η_0) *be regular, then we have*

$$k\begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix}=\sum_{i=1}^{N}\sum_{j=1}^{m_i^\pm} C_{ij\pm}^k(\tau,\eta)d_i^{-1+\frac{1}{m_i}}\begin{pmatrix}B_1(\tau,\eta;\xi_{ij}^\pm(\tau,\eta))\\ \vdots\\ \vdots\\ B_\mu(\tau,\eta;\xi_{ij}^\pm(\tau,\eta))\end{pmatrix}$$

$$+\sum_{j=1}^{M} C_{0j\pm}^k(\tau,\eta)\begin{pmatrix}\dfrac{1}{2\pi i}\displaystyle\oint\frac{B_1(\tau,\eta;\xi)\xi^{j-1}}{E_\pm(\tau,\eta;\xi)}d\xi\\ \vdots\\ \dfrac{1}{2\pi i}\displaystyle\oint\frac{B_\mu(\tau,\eta;\xi)\xi^{j-1}}{E_\pm(\tau,\eta;\xi)}d\xi\end{pmatrix}$$

in V_δ, *where* $\{C_{ij\pm}^k(\tau,\eta),\ C_{0j\pm}^k(\tau,\eta)\}$ *are bounded in* V_δ.

2.2. Estimates of Green's function in V_δ^*. Let (σ_0, η_0) be a regular point, then it is shown that

$$c_1 I < S_\pm(\tau,\eta) < c_2 I \qquad \text{in } V_\delta,$$

in the same way as the proof of Lemma 1.3, making use of Lemma 2.1. Hence we have

Lemma 2.2. *Let* (σ_0, η_0) *be regular, then*

$$\sum_{k=0}^{m-1}\left\|\left(\frac{\partial}{\partial k}\right)^k G_c(\tau,\eta;x,y)\right\|_{L^2\times L^2}\leqq C|N_+(\tau,\eta)B(\tau,\eta)N_-(\tau,\eta)|$$

$$\leqq C'\|G_c(\tau,\eta;x,y)\|_{\mathscr{L}(L^2\times L^2,C^1)} \qquad \textit{in } V_\delta.$$

Let us denote

$$\bar{D}_\pm^*(\tau,\eta)=\begin{pmatrix}\overbrace{(\bar\delta_{11\pm}^*)^{-\frac{1}{2m_1}} \ddots (\bar\delta_{1m_1^\pm\pm}^*)^{-\frac{1}{2m_1}}}^{m_1^\pm} & & & \\ & \ddots & & \\ & & 1 & \\ & & & \ddots\ 1\end{pmatrix} \qquad \text{in } V_\delta^*$$

and

$$\bar{\mathscr{D}}_-(\tau,\eta)=\begin{pmatrix} \overbrace{d_1^{-1+\frac{1}{m_1}} \ddots \quad d_1^{-1+\frac{1}{m_1}}}^{m_1^-} & & & \\ & \ddots & & \\ & & 1 & \\ & & & \ddots \; 1 \end{pmatrix} \quad \text{in } V_\delta,$$

then we have

Proposition 2. *Let* (σ_0, η_0) *be regular, then there exist positive constants* δ, c_1 *and* c_2 *such that*

$$c_1|\bar{D}_+^*(\tau,\eta)\mathscr{B}(\tau,\eta)\bar{\mathscr{D}}_-(\tau,\eta)\bar{D}_-^*(\tau,\eta)|\leqq\|G_c(\tau,\eta;x,y)\|_{\mathscr{L}(L^2,H^{m-1})}$$
$$\leqq c_2|\bar{D}_+^*(\tau,\eta)\mathscr{B}(\tau,\eta)\bar{\mathscr{D}}_-(\tau,\eta)\bar{D}_-^*(\tau,\eta)| \qquad \text{in } V_\delta^*.$$

Theorem II. *Let every real point be regular. In order that* (P) *is* L^2*-well-posed, it is necessary and sufficient that there exist positive constants* δ *and* C *for each real point such that it holds*

$$|\bar{D}_+^*(\tau,\eta)\mathscr{B}(\tau,\eta)\bar{\mathscr{D}}_-(\eta,\eta)\bar{D}_-^*(\tau,\eta)|<\frac{C}{\gamma} \qquad \text{in } V_\delta^*,$$

that is,

$$\begin{cases} |\beta_{ij,lh}(\tau,\eta)|\,\bar{\delta}_{ij+}^{*\;-\frac{1}{2m_i}}\,d_l^{-1+\frac{1}{m_l}}\,\bar{\delta}_{lh-}^{*\;-\frac{1}{2m_l}}<\dfrac{C}{\gamma} & (i\neq 0,\ l\neq 0),\\ |\beta_{0j,lh}(\tau,\eta)|\,d_l^{-1+\frac{1}{m_l}}\,\bar{\delta}_{lh-}^{*\;-\frac{1}{2m_l}}<\dfrac{C}{\gamma} & (l\neq 0),\\ |\beta_{ij,0h}(\tau,\eta)|\,\bar{\delta}_{ij+}^{*\;-\frac{1}{2m_i}}<\dfrac{C}{\gamma} & (i\neq 0),\\ |\beta_{0j,0h}(\tau,\eta)|<\dfrac{C}{\gamma} \end{cases}$$

in V_δ^*.

Since $\bar{\delta}_{ij\pm}^*\leqq d_i$, we have

Corollary 1. *Let every real point be regular. In order that* (P) *is*

L^2-well-posed, it is necessary that

$$\begin{cases} |\beta_{ij,lh}(\tau,\eta)|\, d_i^{-\frac{1}{2m_i}}\, d_l^{-1+\frac{1}{2m_l}} < \dfrac{C}{\gamma} & (i \neq 0,\ l \neq 0), \\ |\beta_{0j,lh}(\tau,\eta)|\, d_l^{-1+\frac{1}{2m_l}} < \dfrac{C}{\gamma} & (l \neq 0), \\ |\beta_{ij,0h}(\tau,\eta)|\, d_i^{-\frac{1}{2m_i}} < \dfrac{C}{\gamma} & (i \neq 0), \\ |\beta_{0j,0h}(\tau,\eta)| < \dfrac{C}{\gamma} \end{cases}$$

in V_δ.

Making use of Corollary 3 of Lemma 2.1, it follows from Corollary 1 of Theorem II:

Corollary 2. *Let every real point be regular. In order that* (P) *is L^2-well-posed, it is necessary that*

$$\begin{cases} |r_{ij,k}(\tau,\eta)|\, d_i^{-\frac{1}{2m_i}} < \dfrac{C}{\gamma} & (i \neq 0), \\ |r_{0j,k}(\tau,\eta)| < \dfrac{C}{\gamma} \end{cases}$$

in V_δ.

On the other hand, since $\bar{\delta}^*_{ij\pm} \geqq \left(\dfrac{\gamma}{d_i}\right)^{m_i} d_i$, we have

Corollary 3. *Let every real point be regular. In order that* (P) *is L^2-well-posed, it is sufficient that*

$$\begin{cases} |\beta_{ij,lh}(\tau,\eta)|\, d_l^{\frac{1}{2}-\frac{1}{2m_i}}\, d_l^{\left(\frac{1}{2}-\frac{1}{2m_l}\right)} < C & (i \neq 0,\ l \neq 0), \\ |\beta_{0j,lh}(\tau,\eta)|\, d_l^{-\left(\frac{1}{2}-\frac{1}{2m_l}\right)} < \dfrac{C}{\gamma^{\frac{1}{2}}} & (l \neq 0), \\ |B_{ij,0h}(\tau,\eta)|\, d_i^{\frac{1}{2}-\frac{1}{2m_i}} < \dfrac{C}{\gamma^{\frac{1}{2}}} & (i \neq 0), \\ |\beta_{0j,0h}(\tau,\eta)| < \dfrac{C}{\gamma} \end{cases}$$

in V_δ.

2.3. Semi-uniform Lopatinski's condition. Let us denote in V_δ

$$\tilde{R}(\tau,\eta)=\frac{1}{\prod_{i=1}^{N}\prod_{j<k}(\xi_{ij}^{+}(\tau,\eta)-\xi_{ik}^{+}(\tau,\eta))}$$

$$\times\det\begin{pmatrix} B_1(\tau,\eta;\xi_{11}^{+}(\tau,\eta))\cdots & \frac{1}{2\pi i}\oint\frac{B_1(\tau,\eta;\xi)\xi^{M-1}}{E_+(\tau,\eta;\xi)}d\xi \\ \vdots & \vdots \\ B_\mu(\tau,\eta;\xi_{11}^{+}(\tau,\eta))\cdots & \frac{1}{2\pi i}\oint\frac{B_\mu(\tau,\eta;\xi)\xi^{M-1}}{E_+(\tau,\eta;\xi)}d\xi \end{pmatrix},$$

$$\tilde{R}_{i_0j_0,l_0h_0}(\tau,\eta)$$

$$=\frac{1}{\prod_{i\neq i_0}\prod_{j<k}(\xi_{ij}^{+}(\tau,\eta)-\xi_{ik}^{+}(\tau,\eta))\prod_{\substack{j,k\neq j_0\\ j<k}}(\xi_{i_0j}^{+}(\tau,\eta)-\xi_{i_0k}^{+}(\tau,\eta))}$$

$$\times\frac{1}{\prod_{j}(\xi_{l_0j}^{+}(\tau,\eta)-\xi_{l_0h_0}^{-}(\tau,\eta))}$$

$$\times\det\begin{pmatrix} B_1(\tau,\eta;\xi_{11}^{+}(\tau,\eta))\cdots B_1(\tau,\eta;\xi_{i_0j_0-1}^{+}(\tau,\eta)) & B_1(\tau,\eta;\xi_{l_0h_0}^{-}(\tau,\eta))B_1(\tau,\eta;\xi_{i_0j_0+1}^{+}(\tau,\eta))\cdots \\ \vdots\qquad\qquad\vdots & \vdots\qquad\qquad\vdots \\ B_\mu(\tau,\eta;\xi_{11}^{-}(\tau,\eta))\cdots B_\mu(\tau,\eta;\xi_{i_0j_0-1}^{+}(\tau,\eta)) & B_\mu(\tau,\eta;\xi_{l_0h_0}^{-}(\tau,\eta))B_\mu(\tau,\eta;\xi_{i_0j_0+1}^{+}(\tau,\eta))\cdots \end{pmatrix}$$

$$(i_0\neq 0,\ l_0\neq 0),$$

$$\tilde{R}_{0j_0,l_0h_0}(\tau,\eta)=\frac{1}{\prod_{i}\prod_{j<k}(\xi_{ij}^{+}(\tau,\eta)-\xi_{ik}^{+}(\tau,\eta))\prod_{j}(\xi_{l_0j}^{+}(\tau,\eta)-\xi_{l_0h_0}^{-}(\tau,\eta))}$$

$$\times\det\begin{pmatrix} \cdots\frac{1}{2\pi i}\oint\frac{B_1(\tau,\eta;\xi)\xi^{j_0-2}}{E_+(\tau,\eta;\xi)}d\xi & B_1(\tau,\eta;\xi_{l_0h_0}^{-}(\tau,\eta)) & \frac{1}{2\pi i}\oint\frac{B_1(\tau,\eta;\xi)\xi^{j_0}}{E_+(\tau,\eta;\xi)}d\xi\cdots \\ \vdots & \vdots & \vdots \end{pmatrix}\quad(l_0\neq 0),$$

$$\tilde{R}_{i_0j_0,0h_0}(\tau,\eta)=\frac{1}{\prod_{i\neq i_0}\prod_{j<k}(\xi^+_{ij}(\tau,\eta)-\xi^+_{ik}(\tau,\eta))\prod_{\substack{j,k\neq j_0\\ j<k}}(\xi^+_{i_0j}(\tau,\eta)-\xi^+_{i_0k}(\tau,\eta))}$$

$$\times\det\begin{pmatrix} \cdots B_1(\tau,\eta;\xi^+_{i_0j_0-1}(\tau,\eta)) & \frac{1}{2\pi i}\oint\frac{B_1(\tau,\eta;\xi)\xi^{h_0-1}}{E_-(\tau,\eta;\xi)}d\xi & B_1(\tau,\eta;\xi^+_{i_0j_0+1}(\tau,\eta))\cdots \\ \vdots & \vdots & \vdots \end{pmatrix}\quad (i_0\neq 0),$$

$$\tilde{R}_{0j_0,0h_0}(\tau,\eta)=\frac{1}{\prod_{i}\prod_{j<k}(\xi^+_{ij}(\tau,\eta)-\xi^+_{ik}(\tau,\eta))}$$

$$\times\det\begin{pmatrix} \cdots\frac{1}{2\pi i}\oint\frac{B_1(\tau,\eta;\xi)\xi^{j_0-2}}{E_+(\tau,\eta;\xi)}d\xi & \frac{1}{2\pi i}\oint\frac{B_1(\tau,\eta;\xi)\xi^{h_0-1}}{E_-(\tau,\eta;\xi)}d\xi & \frac{1}{2\pi i}\oint\frac{B_1(\tau,\eta;\xi)\xi^{j_0}}{E_+(\tau,\eta;\xi)}d\xi\cdots \\ \vdots & \vdots & \vdots \end{pmatrix},$$

then these are all bounded in V_δ.

Lemma 2.3. *Let (σ_0, η_0) be regular, then there exist positive constants δ, c_1 and c_2 such that*

$$\begin{cases} c_1|\beta_{ij,lh}(\tau,\eta)|\,d_i^{\frac{m_i^+-1}{m_i}}\,d_l^{-\frac{m_l^+}{m_l}}\leqq\left|\dfrac{\tilde{R}_{ij,lh}(\tau,\eta)}{\tilde{R}(\tau,\eta)}\right| \\ \qquad\leqq c_2|\beta_{ij,lh}(\tau,\eta)|\,d_i^{\frac{m_i^+-1}{m_i}}\,d_l^{-\frac{m_l^+}{m_l}} & (i\neq 0,\ l\neq 0), \\ c_1|\beta_{0j,lh}(\tau,\eta)|\,d_l^{-\frac{m_l^+}{m_l}}\leqq\left|\dfrac{\tilde{R}_{0j,lh}(\tau,\eta)}{\tilde{R}(\tau,\eta)}\right|\leqq c_2|\beta_{0j,lh}(\tau,\eta)|\,d_l^{-\frac{m_l^+}{m_l}} & (l\neq 0), \\ c_1|\beta_{ij,0h}(\tau,\eta)|\,d_i^{\frac{m_i^+-1}{m_i}}\leqq\left|\dfrac{\tilde{R}_{ij,0h}(\tau,\eta)}{\tilde{R}(\tau,\eta)}\right|\leqq c_2|\beta_{ij,0h}(\tau,\eta)|\,d_i^{\frac{m_i^+-1}{m_i}} & (i\neq 0), \\ c_1|\beta_{0j,0h}(\tau,\eta)|\leqq\left|\dfrac{\tilde{R}_{0j,0h}(\tau,\eta)}{\tilde{R}(\tau,\eta)}\right|\leqq c_2|\beta_{0j,0h}(\tau,\eta)| \end{cases}$$

in V_δ.

From Lemma 2.3, we have

Theorem II'. *Let every real point be regular. In order that* (P) *is* L^2*-well-posed, it is necessary and sufficient that*

$$\left|\frac{\tilde{R}_{ij,lh}(\tau,\eta)}{\tilde{R}(\tau,\eta)}\right| d_i^{-\frac{m_i^+-1}{m_i}} \bar{\delta}_{ij+}^{*\;-\frac{1}{2m_i}} d_l^{-\frac{m_l^--1}{m_l}} \bar{\delta}_{lh-}^{*\;-\frac{1}{2m_l}} < \frac{C}{\gamma} \qquad (i \neq 0,\ l \neq 0),$$

$$\left|\frac{\tilde{R}_{0j,lh}(\tau,\eta)}{\tilde{R}(\tau,\eta)}\right| d_l^{-\frac{m_l^--1}{m_l}} \bar{\delta}_{lh-}^{*\;-\frac{1}{2m_l}} < \frac{C}{\gamma} \qquad (l \neq 0),$$

$$\left|\frac{\tilde{R}_{ij,0h}(\tau,\eta)}{\tilde{R}(\tau,\eta)}\right| d_i^{-\frac{m_i^+-1}{m_i}} \bar{\delta}_{ij+}^{*\;-\frac{1}{2m_i}} < \frac{C}{\gamma} \qquad (i \neq 0),$$

$$\left|\frac{\tilde{R}_{0j,0h}(\tau,\eta)}{\tilde{R}(\tau,\eta)}\right| < \frac{C}{\gamma}$$

in V_δ^*.

Remark. The conditions stated in Theorem II' are satisfied if uniform Lopatinski's condition is satisfied, remarking that $\dfrac{m_i^\pm - \frac{1}{2}}{m_i} \leqq \dfrac{1}{2}$.

Now we consider a sufficient condition for L^2-well-posedness, which is stated only by the word of Lopatinski's determinant $R(\tau, \eta)$. Let us denote

$$\Omega = \{(\sigma, \eta) \in R^1 \times R^{n-1};\ \sigma^2 + |\eta|^2 = 1,\ A(\sigma, \eta; \xi) \neq 0 \quad \text{for any } \xi \in R^1\}.$$

We say that semi-uniform Lopatinski's condition is satisfied for (P), when the following conditions are satisfied:

i) let $(\sigma_0, \eta_0) \in (\bar{\Omega})^C$, then $R(\sigma_0, \eta_0) \neq 0$,

ii) let $(\sigma_0, \eta_0) \in \Omega$, then $R(\sigma_0, \eta_0) \neq 0$ or $R'_\tau(\sigma_0, \eta_0) \neq 0$,

iii) let $(\sigma_0, \eta_0) \in \partial\Omega$, then there exists V_δ such that

$$|R(\tau, \eta)| \geqq c \frac{\gamma}{d_0^{1-\frac{1}{\bar{m}}}} \qquad \text{if } (\mathrm{Re}\,\tau, \eta) \in \Omega \cap \bar{V}_\delta,$$

$$|R(\tau, \eta)| \geqq c\, d_0^{\frac{1}{\bar{m}}} \qquad \text{if } (\mathrm{Re}\,\tau, \eta) \in (\bar{\Omega})^C \cap \bar{V}_\delta,$$

where

$$\bar{m}=\max_i \{m_i(\sigma_0, \eta_0)\}, \qquad d_0=\mathrm{dis}((\tau, \eta), \partial\Omega).$$

Theorem III. *Let every point of $\partial\Omega$ be regular. Then semiuniform Lopatinski's condition is a sufficient condition for L^2-well-posedness for* (P).

Proof. Let $(\sigma_0, \eta_0)\in\partial\Omega(\neq\phi)$, then all the indexes $\{m_i(\sigma_0, \eta_0)\}$ are even, that is, $m_i^+(\sigma_0, \eta_0)=m_i^-(\sigma_0, \eta_0)$. Since

$$\begin{cases} d_0\leqq d_i & \text{if } (\mathrm{Re}\,\tau, \eta)\in\Omega, \\ d_0\geqq d_i & \text{if } (\mathrm{Re}\,\tau, \eta)\in(\bar{\Omega})^C, \end{cases}$$

and

$$\begin{cases} \bar{\delta}^*_{ij\pm}=d_i(\geqq d_0) & \text{if } (\mathrm{Re}\,\tau, \eta)\in\Omega, \\ \bar{\delta}^*_{ij\pm}\geqq\left(\dfrac{\gamma}{d_i}\right)^{m_i} d_i & \text{if } (\mathrm{Re}\,\tau, \eta)\in(\bar{\Omega})^C, \end{cases}$$

we have

$$\begin{cases} d_i^{-\frac{1}{2}+\frac{1}{m_i}}\bar{\delta}^{*\ -\frac{1}{2m_i}}_{ij\pm}=d_i^{-\frac{1}{2}+\frac{1}{2m_i}}\leqq d_i^{-\frac{1}{2}+\frac{1}{2\bar{m}}} & \text{if } (\mathrm{Re}\,\tau, \eta)\in\Omega, \\ d_i^{-\frac{1}{2}+\frac{1}{m_i}}\bar{\delta}^{*\ -\frac{1}{2m_i}}_{ij\pm}\leqq\dfrac{d_i^{\frac{1}{2m_i}}}{\gamma^{\frac{1}{2}}}\leqq\gamma^{-\frac{1}{2}}d_0^{\frac{1}{2\bar{m}}} & \text{if } (\mathrm{Re}\,\tau, \eta)\in(\bar{\Omega})^C, \end{cases}$$

therefore we have

$$\left|\frac{\tilde{R}_{ij,lh}(\tau, \eta)}{\tilde{R}(\tau, \eta)}\right| d_i^{-\frac{1}{2}+\frac{1}{m_i}}\bar{\delta}^{*\ -\frac{1}{2m_i}}_{ij+}\, d_l^{-\frac{1}{2}+\frac{1}{m_l}}\bar{\delta}^{*\ -\frac{1}{2m_l}}_{lh-}$$

$$\leqq\frac{Cd_0^{-1+\frac{1}{\bar{m}}}}{|R(\tau, \eta)|}\leqq\frac{C'}{\gamma} \qquad \text{if } (\mathrm{Re}\,\tau, \eta)\in\Omega,$$

$$\left|\frac{\tilde{R}_{ij,lh}(\tau, \eta)}{\tilde{R}(\tau, \eta)}\right| d_i^{-\frac{1}{2}+\frac{1}{m_i}}\bar{\delta}^{*\ -\frac{1}{2m_i}}_{ij+}\, d_l^{-\frac{1}{2}+\frac{1}{m_l}}\bar{\delta}^{*\ -\frac{1}{2m_l}}_{lh-}$$

$$\leqq\frac{C\gamma^{-1}d_0^{\frac{1}{\bar{m}}}}{|R(\tau, \eta)|}\leqq\frac{C'}{\gamma} \qquad \text{if } (\mathrm{Re}\,\tau, \eta)\in(\bar{\Omega})^C,$$

and so on. Q.E.D.

Example. Let (P) be defined by

$$\begin{cases} A=(\xi^2+|\eta|^2-\alpha^2\tau^2)(\xi^2+|\eta|^2-\beta^2\tau^2) & (\alpha>\beta>0), \\ B_1=\xi^2+|\eta|^2-\alpha^2\tau^2, \\ B_2=\xi-i(a\tau+b\eta) & (a,\ b:\ \text{real}). \end{cases}$$

Then uniform Lopatinski's condition is never satisfied, but Lopatinski's condition is satisfied if and only if

$$|b|^2-\frac{a^2}{\alpha^2}\leqq 1.$$

If $b^2-\frac{a^2}{\alpha^2}=1$, then (P) is not L^2-well-posed. In fact, let $\{\xi_1^{\pm}(\tau, \eta)\}$ be roots of $\xi^2+|\eta|^2-\alpha^2\tau^2=0$, let $\{\xi_2^{\pm}(\tau, \eta)\}$ be roots of $\xi^2+|\eta|^2-\beta^2\tau^2=0$, then

$$R(\tau, \eta)=\frac{1}{\xi_2^+(\tau, \eta)-\xi_1^+(\tau, \eta)} \det\begin{pmatrix} B_1(\tau, \eta;\ \xi_1^+(\tau, \eta)) & B_1(\tau, \eta;\ \xi_2^+(\tau, \eta)) \\ B_2(\tau, \eta;\ \xi_1^+(\tau, \eta)) & B_2(\tau, \eta;\ \xi_2^+(\tau, \eta)) \end{pmatrix}.$$

Let $\sigma_0=-\frac{a}{\alpha^2}$, $\eta_0=b$, then $R(\sigma_0, \eta_0)=R'_\tau(\sigma_0, \eta_0)=0$, and

$$\det\begin{pmatrix} B_1(\sigma_0, \eta_0;\ \xi_1^-(\sigma_0, \eta_0)) & B_1(\sigma_0, \eta_0;\ \xi_2^+(\sigma_0, \eta_0)) \\ B_2(\sigma_0, \eta_0;\ \xi_1^-(\sigma_0, \eta_0)) & B_2(\sigma_0, \eta_0;\ \xi_2^+(\sigma_0, \eta_0)) \end{pmatrix}\neq 0$$

therefore

$$\left|\frac{\det\begin{pmatrix} B_1(\sigma_0-i\gamma, \eta_0;\ \xi_1^-(\sigma_0-i\gamma, \eta_0)) & B_1(\sigma_0-i\gamma, \eta_0;\ \xi_2^+(\sigma_0-i\gamma, \eta_0)) \\ B_2(\sigma_0-i\gamma, \eta_0;\ \xi_1^-(\sigma_0-i\gamma, \eta_0)) & B_2(\sigma_0-i\gamma, \eta_0;\ \xi_2^+(\sigma_0-i\gamma, \eta_0)) \end{pmatrix}}{\det\begin{pmatrix} B_1(\sigma_0-i\gamma, \eta_0;\ \xi_1^+(\sigma_0-i\gamma, \eta_0)) & B_1(\sigma_0-i\gamma, \eta_0;\ \xi_2^+(\sigma_0-i\gamma, \eta_0)) \\ B_2(\sigma_0-i\gamma, \eta_0;\ \xi_1^+(\sigma_0-i\gamma, \eta_0)) & B_2(\sigma_0-i\gamma, \eta_0;\ \xi_2^+(\sigma_0-i\gamma, \eta_0)) \end{pmatrix}}\right|$$

$$\geqq\frac{c}{\gamma^2} \qquad (0<\gamma<\gamma_0),$$

which contradict to the necessary condition for L^2-well-posedness. If $b^2-\frac{a^2}{\alpha^2}<1$, then semi-uniform Lopatinski's condition is satisfied, therefore (P) is L^2-well-posed.

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