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A Remark on an Infinite Tensor Product of von Neumann Algebras

By

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Abstract

Let $H_{\mathfrak{c}}$ be the incomplete infinite tensor product of Hilbert spaces H_ι containing a product vector $\otimes x_\iota$, where $\mathfrak{c}$ denotes the equivalence class of the $\mathfrak{C}_0$-sequence $\{x_\iota\}$. Let $E_{\mathfrak{c}}$ be the projection on $H_{\mathfrak{c}}$ in the complete infinite tensor product H of H_ι. Let $\mathfrak{R}$ be the von Neumann algebra on H generated by von Neumann algebra $\mathfrak{R}_\iota$ on H_ι and $E(\mathfrak{c})$ be the central support of $E_{\mathfrak{c}}$ in $\mathfrak{R}'$. Two $\mathfrak{C}_0$-sequences $\{x_\iota\}$ and $\{y_\iota\}$, and their equivalence classes $\mathfrak{c}$ and $\mathfrak{c}'$, are defined to be p-equivalent if there exist partial isometries $p_\iota \in \mathfrak{R}_\iota'$ such that $\{x_\iota\}$ and $\{p_\iota y_\iota\}$ are equivalent and $p_\iota^* p_\iota y_\iota = y_\iota$. They are defined to be u-equivalent if p_ι can be chosen unitary. We prove that $E(\mathfrak{c})$ is the sum of $E_{\mathfrak{c}'}$ with $\mathfrak{c}'$, p-equivalent to $\mathfrak{c}$. If the index set is countable, p-equivalence and u-equivalence coincide.

§ 1. Introduction

According to von Neumann [8], the complete infinite tensor product $H = \otimes H_\iota$ of Hilbert spaces H_ι, $\iota \in I$, is the (linear topological) span of all product vectors $\otimes x_\iota$ (multilinear in x_ι) such that $x_\iota \neq 0$ and

$$\sum |1 - \|x_\iota\|| < \infty. \tag{1.1}$$

(We have substituted "tensor" into von Neumann's "direct".) Let S denote the set of all $\{x_\iota\}$ satisfying (1.1) and S_0 denote the set of all $\{x_\iota\} \in S$ such that $x_\iota \neq 0$. $\{x_\iota\}$ and $\{y_\iota\}$ are called (strongly) equivalent if

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$$\sum |1-(x_\iota, y_\iota)| < \infty. \tag{1.2}$$

Notation: $\{x_\iota\} \sim \{y_\iota\}$. It defines equivalence relations in S and S_0. Let $\mathfrak{C}$ and $\mathfrak{C}_0$ denote the set of equivalence classes $\mathfrak{c}(\{x_\iota\})$ of $\{x_\iota\}$ in S and S_0, respectively. The subspace of H spanned by $\otimes x_\iota$ with a fixed $\mathfrak{c}(\{x_\iota\}) = \mathfrak{c} \in \mathfrak{C}_0$ is called the incomplete infinite tensor product and is denoted by $H_\mathfrak{c} = \otimes^\mathfrak{c} H_\iota$. Let $E_\mathfrak{c}$ denote the projection on $H_\mathfrak{c}$ in H.

Let $\mathfrak{R}_\iota$ be a von Neumann algebra on H_ι, π be its natural representation on H (namely $\pi(Q)(\otimes x_\iota) = \otimes x'_\iota$, with $x'_\iota = x_\iota$ for $\iota \neq \iota_0$ and $x'_\iota = Qx_\iota$ for $\iota = \iota_0$, if $Q \in \mathfrak{R}_{\iota_0}$). Let $\mathfrak{R} \equiv \otimes \mathfrak{R}_\iota$ be the von Neumann algebra generated by the union of all $\pi(\mathfrak{R}_\iota)$. Since $H_\mathfrak{c}$ is invariant under each $\pi(\mathfrak{R}_\iota)$, E_c is in $\mathfrak{R}'$. Let $E(\mathfrak{c})$ be the central support of $E_\mathfrak{c}$ in $\mathfrak{R}'$.

Definition. *Let* $\{x_\iota\}, \{y_\iota\} \in S$ *and* $\mathfrak{c} = \mathfrak{c}(\{x_\iota\})$, $\mathfrak{c}' = \mathfrak{c}(\{y_\iota\})$.

(1) $\{x_\iota\}$ *and* $\mathfrak{c}$ *are u-equivalent to* $\{y_\iota\}$ *and* $\mathfrak{c}'$, *respectively if* $\{x_\iota\} \sim \{u_\iota y_\iota\}$ *for some unitary* $u_\iota \in \mathfrak{R}'_\iota$. *Notation*: $\{x_\iota\} \underset{u}{\sim} \{y_\iota\}$, $\mathfrak{c} \underset{u}{\sim} \mathfrak{c}'$.

(2) $\{x_\iota\}$ *and* $\mathfrak{c}$ *are p-equivalent to* $\{y_\iota\}$ *and* $\mathfrak{c}'$, *respectively, if* $\{x_\iota\} \sim \{p_\iota y_\iota\}$ *for some partial isometry* $p_\iota \in \mathfrak{R}'_\iota$ *such that* $p_\iota^* p_\iota y_\iota = y_\iota$. *Notation*: $\{x_\iota\} \underset{p}{\sim} \{y_\iota\}$, $\mathfrak{c} \underset{p}{\sim} \mathfrak{c}'$.

(3) $\{x_\iota\}$ *and* $\mathfrak{c}$ *are v-equivalent to* $\{y_\iota\}$ *and* $\mathfrak{c}'$, *respectively, if* $\{x_\iota\} \sim \{v_\iota y_\iota\}$ *for some* $v_\iota \in \mathfrak{R}'_\iota$ *such that* $\|v_\iota\| \leqq 1$. *Notation*: $\{x_\iota\} \underset{v}{\sim} \{y\}$, $\mathfrak{c} \underset{v}{\sim} \mathfrak{c}'$.

Our main result is the following:

Theorem. (1) $E(\mathfrak{c})$ *is the sum of* $E_{\mathfrak{c}'}$ *with* $\mathfrak{c}' \underset{p}{\sim} \mathfrak{c}$.

(2) *If the index set* I *is countable*, $\mathfrak{c}' \underset{p}{\sim} \mathfrak{c}$ *and* $\mathfrak{c}' \underset{u}{\sim} \mathfrak{c}$ *are equivalent.*

Remark. If $\mathfrak{R}_\iota = \mathscr{B}(H_\iota)$, the set of all bounded linear operators on H_ι, then $\underset{u}{\sim}$, $\underset{p}{\sim}$ and $\underset{v}{\sim}$ all coincide with the weak equivalence introduced by von Neumann.

§ 2. Equivalence Relations

The u-equivalence is clearly an equivalence relation. In this section, we shall show that p- and v-equivalence are also equivalence relations and are the same. In the definition of v-equivalence, we have not stated the condition $\{v_\iota y_\iota\}\in S$. This is actually a consequence of $\{x_\iota\}\in S$, $\{y_\iota\}\in S$ and $\{x_\iota\}\sim\{v_\iota y_\iota\}$, as is shown in the next Lemma.

Lemma 1. *If* $\{x_\iota\}\in S$, $\{y_\iota\}\in S$, $\|v_\iota\|\leqq 1$ *and* $\{x_\iota\}\sim\{v_\iota y_\iota\}$, *then* $\{v_\iota y_\iota\}\in S$.

Proof. Since $\{y_\iota\}\in S$ and $\|v_\iota\|\leqq 1$,

$$\sup\|v_\iota y_\iota\|\leqq\sup\|y_\iota\|<\infty.$$

If $\|v_\iota y_\iota\|\geqq 1$, then $0\geqq 1-\|v_\iota y_\iota\|\geqq 1-\|y_\iota\|$ and hence

$$1-\|v_\iota y_\iota\|\geqq|1-\|v_\iota y_\iota\||-2|1-\|y_\iota\||.$$

This inequality obviously holds for $1\geqq\|v_\iota y_\iota\|$. Now assume that $\{v_\iota y_\iota\}\notin S$. Then

$$\begin{aligned}\sum|1-(x_\iota, v_\iota y_\iota)|&\geqq\sum\{1-|(x_\iota, v_\iota y_\iota)|\}\\ &\geqq\sum\{1-\|x_\iota\|\|v_\iota y_\iota\|\}=\sum(1-\|v_\iota y_\iota\|)+\sum\|v_\iota y_\iota\|(1-\|x_\iota\|)\\ &\geqq\sum|1-\|v_\iota y_\iota\||-2\sum|1-\|y_\iota\||-\sup\|v_\iota y_\iota\|\sum|1-\|x_\iota\||\\ &=+\infty\end{aligned}$$

which contradicts with $\{x_\iota\}\sim\{v_\iota y_\iota\}$. Q.E.D.

Lemma 2. $\{x_\iota\}\underset{p}{\sim}\{y_\iota\}$ *and* $\{x_\iota\}\underset{v}{\sim}\{y_\iota\}$ *are equivalent.*

Proof. Obviously $\{x_\iota\}\underset{p}{\sim}\{y_\iota\}$ implies $\{x_\iota\}\underset{v}{\sim}\{y_\iota\}$. To prove the converse, let $\{x_\iota\}\sim\{v_\iota y_\iota\}$ with $\|v_\iota\|\leqq 1$. Let $s'(y_\iota)$ denote the smallest projection $E=s'(y_\iota)\in\mathfrak{R}_\iota'$ such that $Ey_\iota=y_\iota$ (EH_ι is the closure of $R_\iota y_\iota$.) Let $p_\iota q_\iota=v_\iota s'(y_\iota)$ be the polar decomposition with $q_\iota=|v_\iota s'(y_\iota)|$, $p_\iota^*p_\iota=s'(q_\iota)$ (1 minus the spectral projection of q_ι for the eigenvalue 0).

Since $\|q_\iota y_\iota\|=\|v_\iota s'(y_\iota)y_\iota\|=\|v_\iota y_\iota\|$ and $\{v_\iota y_\iota\}\in S$ by Lemma 1, we have $\{q_\iota y_\iota\}\in S$. Since $0\leqq q_\iota\leqq 1$, we have $q_\iota^2\leqq q_\iota$ and hence

$$\sum|1-(p_\iota y_\iota, p_\iota q_\iota y_\iota)|=\sum|1-(y_\iota, q_\iota y_\iota)|$$

$$\leqq \sum |1-\|y_\iota\|^2| + \sum (y_\iota, (1-q_\iota)y_\iota)$$

$$\leqq \sum |1-\|y_\iota\|^2| + \sum (y_\iota, (1-q_\iota^2)y_\iota)$$

$$\leqq 2\sum |1-\|y_\iota\|^2| + \sum |1-\|q_\iota y_\iota\|^2|.$$

Since $\{y_\iota\} \in S$, we have $\sup \|y_\iota\| < \infty$ and hence

$$\sum |1-\|q_\iota y_\iota\|^2| \leqq \sup (1+\|q_\iota y_\iota\|) \sum |1-\|q_\iota y_\iota\|| < \infty.$$

Therefore $\{p_\iota y_\iota\} \sim \{p_\iota q_\iota y_\iota\} = \{v_\iota y_\iota\} \sim \{x_\iota\}$.

Let $s_\iota = s'(y_\iota) - p_\iota^* p_\iota$. If $s_\iota y_\iota = 0$, then we have $\{x_\iota\} \underset{p}{\sim} \{y_\iota\}$. In general, s_ι is a projection in $\mathfrak{R}_\iota'$. Since $\{p_\iota y_\iota\} \in S$ by Lemma 1, we have

$$\sum \|s_\iota y_\iota\|^2 = \sum (\|y_\iota\|^2 - \|p_\iota y_\iota\|^2) \leqq \sum |1-\|y_\iota\|^2| + \sum |1-\|p_\iota y_\iota\|^2|$$

$$\leqq \sup (1+\|y_\iota\|+\|p_\iota y_\iota\|) \sum (|1-\|y_\iota\|| + |1-\|p_\iota y_\iota\||) < \infty.$$

Hence $s_\iota y_\iota = 0$, possibly except for a countable number of $\iota = \iota(l)$, $l=1, 2, \cdots$.

Let F_ι be the central projection in $\mathfrak{R}_\iota'$ such that $F_\iota p_\iota^* p_\iota$ is finite and $(1-F_\iota) p_\iota^* p_\iota$ is properly infinite in $\mathfrak{R}_\iota'$. There exists a partial isometry p_ι' in $\mathfrak{R}_\iota'$ such that $p_\iota'^* p_\iota' = F_\iota (1-p_\iota^* p_\iota)$, $p_\iota' p_\iota'^* = F_\iota (1-p_\iota p_\iota^*)$. There also exist projections $e_{\iota k}$ in $\mathfrak{R}_\iota'$, $k=1, 2, \cdots$ (countably infinite number) such that each $e_{\iota k}$ is equivalent to $(1-F_\iota) p_\iota^* p_\iota$ and $\sum_k e_{\iota k} = (1-F_\iota) p_\iota^* p_\iota$. Since $\sum_k \|e_{\iota k} y_\iota\|^2 \leqq \|y_\iota\|^2$, there exists $k=k(l)$ such that $\|e_{\iota k} y_\iota\|^2 < 2^{-l}$ for $\iota = \iota(l)$. Then there exist a partial isometry $p_{\iota(l)}''$ such that $p_{\iota(l)}''^* p_{\iota(l)}'' = e_{\iota(l)k(l)} + (1-F_{\iota(l)})(1-p_{\iota(l)}^* p_{\iota(l)})$, $p_{\iota(l)}'' p_{\iota(l)}''^* = (1-F_{\iota(l)})(1-p_{\iota(l)}(1-e_{\iota(l)k(l)}) p_{\iota(l)}^*)$. Set $\bar{p}_\iota = p_\iota$ if $\iota \neq \iota(l)$, $l=1, 2, \cdots$, and $\bar{p}_\iota = F_\iota (p_\iota + p_\iota') + (1-F_l) p_l (1-e_{\iota k(l)}) + p_\iota''$ for $\iota = \iota(l)$.

We first see from the construction that $\bar{p}_\iota$ is unitary for $\iota = \iota(l)$ and hence $\bar{p}_\iota^* \bar{p}_\iota y_\iota = y_\iota$ for all ι. For $\iota \neq \iota(l)$, $\bar{p}_\iota y_\iota = p_\iota y_\iota$. For $\iota = \iota(l)$, we have

$$|(\bar{p}_\iota y_\iota, p_\iota y_\iota) - \|p_\iota y_\iota\|^2|$$

$$= |(\bar{p}_\iota (s_\iota + e_{\iota k(l)}) y_\iota, p_\iota e_{\iota k(l)} y_\iota) - \|p_\iota e_{\iota k(l)} y_\iota\|^2|$$

$$\leqq (\|s_\iota y_\iota\| + \|e_{\iota k(l)} y_\iota\|) \|e_{\iota k(l)} y_\iota\| + \|e_{\iota k(l)} y_\iota\|^2$$

which is summable over $l=1, 2, \cdots$. Therefore

$$\begin{aligned}&\sum|1-(\bar{p}_\iota y_\iota, p_\iota y_\iota)|\\ &\leqq \sum|1-\|p_\iota y_\iota\|^2|+\sum_l|(\bar{p}_{\iota(l)}y_{\iota(l)}, p_{\iota(l)}y_{\iota(l)})-\|p_{\iota(l)}y_{\iota(l)}\|^2|\\ &<\infty.\end{aligned}$$

Hence $\{\bar{p}_\iota y_\iota\}\in S$ by Lemma 1 and $\{\bar{p}_\iota y_\iota\}\sim\{p_\iota y_\iota\}\sim\{x_\iota\}$. Q.E.D.

Proof of Theorem (2). In the provious proof $\bar{p}_\iota$ is unitary for $\iota=\iota(l)$. Hence this construction (even if $s_\iota y_\iota=0$ for all ι) gives the equivalence of $\underset{u}{\sim}$ and $\underset{p}{\sim}$ when the index set I is countable. Q.E.D.

Lemma 3. $\underset{p}{\sim}$ *is an equivalence relation.*

Proof. Obviously $\{x_\iota\}\underset{p}{\sim}\{x_\iota\}$ because $\{x_\iota\}\sim\{p_\iota x_\iota\}$ with $p_\iota=1$. Suppose $\{x_\iota\}\sim\{p_\iota y_\iota\}$. Since $(y_\iota, p_\iota^* x_\iota)=(x_\iota, p_\iota y_\iota)^*$, we have $\{y_\iota\}\sim\{p_\iota^* x_\iota\}$ and hence $\{y_\iota\}\underset{v}{\sim}\{x_\iota\}$. By Lemma 2, $\{y_\iota\}\underset{p}{\sim}\{x_\iota\}$.

Finally, suppose $\{x_\iota\}\sim\{p_\iota y_\iota\}$ and $\{y_\iota\}\sim\{p'_\iota z_\iota\}$ with $p_\iota^* p_\iota y_\iota=y_\iota$. Then $(p_\iota y_\iota, p_\iota p'_\iota z_\iota)=(p_\iota^* p_\iota y_\iota, p'_\iota z_\iota)=(y_\iota, p'_\iota z_\iota)$. Hence $\{x_\iota\}\sim\{p_\iota y_\iota\}\sim\{p_\iota p'_\iota z_\iota\}$. Therefore $\{x_\iota\}\underset{v}{\sim}\{z_\iota\}$ and by Lemma 2, $\{x_\iota\}\underset{p}{\sim}\{z_\iota\}$.

Q.E.D.

§3. Central Support $E(\mathfrak{c})$

Lemma 4. *For* $\mathfrak{c}, \mathfrak{c}'\in\mathfrak{C}_0$, *either* $E(\mathfrak{c})E_{\mathfrak{c}'}=E_{\mathfrak{c}'}$ *or* $E(\mathfrak{c})E_{\mathfrak{c}'}=0$.

Proof. Take any $\{y_\iota\}\in S_0$. By Lemma 4.2 of [6],

(3.1) $$E(\mathfrak{c})(\otimes y_\iota)=\lim_{J\subset\subset I}E_J(\otimes y_\iota)$$

where $J\subset\subset I$ indicates that J is a finite subset of I and E_J is the smallest projection in

$$\mathfrak{R}(J^c)=(\bigvee_{\iota\notin J}\pi(\mathfrak{R}_\iota))''$$

scuh that $E_J(\otimes x_\iota)=\otimes x_\iota$ for a fixed $\{x_\iota\}\in\mathfrak{c}$. Let $\mathfrak{c}'=\mathfrak{c}(\{y_\iota\})$. Since $H_{\mathfrak{c}'}$ is invariant under $\mathfrak{R}\supset\mathfrak{R}(J^c)$, each $E_J(\otimes y_\iota)$ as well as its limit $E(\mathfrak{c})(\otimes y_\iota)$ is in $H_{\mathfrak{c}'}$.

By Lemma 3.1 of [2], there exists J for any given $\varepsilon>0$ such that $J\subset\subset I$ and

$$\|E(\mathfrak{c})(\otimes y_\iota)-z_J\otimes y(J^c)\|<\varepsilon \tag{3.2}$$

where $z_J\in\bigotimes_{\iota\in J}H_\iota$ and $y(J^c)=\bigotimes_{\iota\notin J}y_\iota$. For the same ε and J, there exists $K\supset J$, $K\subset\subset I$ such that

$$\|E(\mathfrak{c})(\otimes y_\iota)-E_K(\otimes y_\iota)\|<\varepsilon. \tag{3.3}$$

Since $E_K\in\mathfrak{R}(K^c)\subset\mathfrak{R}(J^c)$, we can write $E_K(\otimes y_\iota)=y(J)\otimes z$ for some $z\in\bigotimes_{\iota\notin J}H_\iota$. From the two inequalities,

$$\|z_J\otimes y(J^c)-y(J)\otimes z\|<2\varepsilon. \tag{3.4}$$

Since $\{y_\iota\}\in S_0$, $a_2=\|y(J^c)\|$ and $b_1=\|y(J)\|$ are bounded away from 0 and ∞ when J runs over all finite subsets of I. Let $a=\|E(\mathfrak{c})(\otimes y_\iota)\|$ and assume that $a\neq 0$. Then we have from (3.2) and (3.3), $|a_1-a/a_2|<\varepsilon/a_2$ and $|b_2-a/b_1|<\varepsilon/b_1$ for $a_1=\|z_J\|$ and $b_2=\|z\|$. Therefore a_1 and b_2 are also bounded away from 0 and ∞ for sufficiently small ε. From (3.4), we also have $|a_1a_2-b_1b_2|<2\varepsilon$.

We set $\Phi_1=z_J/a_1$, $\Phi_2=y(J^c)/a_2$, $\Psi_1=y(J)/b_1$, $\Psi_2=z/b_2$. They are all unit vectors. From (3.4), we obtain, by using $|a_1a_2-b_1b_2|<2\varepsilon$ and separation of a_1a_2, b_1b_2 from 0,

$$\|\Phi_1\otimes\Phi_2-\Psi_1\otimes\Psi_2\|<\varepsilon'(\varepsilon)$$

where $\varepsilon'(\varepsilon)\to 0$ as $\varepsilon\to 0$. Then

$$1-\mathrm{Re}(\Phi_1,\Psi_1)(\Phi_2,\Psi_2)<\varepsilon'(\varepsilon)^2/2.$$

Since $|(\Phi_1,\Psi_1)|\leqq 1$ and $|(\Phi_2,\Psi_2)|\leqq 1$, we have

$$\begin{aligned}\varepsilon'(\varepsilon)^2/2&>1-|(\Phi_1,\Psi_1)|\,|(\Phi_2,\Psi_2)|\\&\geqq\max(1-|(\Phi_1,\Psi_1)|,\,1-|(\Phi_2,\Psi_2)|).\end{aligned}$$

Hence choosing θ and θ' such that $(\Phi_1, e^{i\theta}\Psi_1)$ and $(\Phi_2, e^{i\theta'}\Psi_2)$ are both non-negative, we have

$$||\Phi_1 - e^{i\theta}\Psi_1|| < \varepsilon'(\varepsilon), \ ||\Phi_2 - e^{i\theta'}\Psi_2|| < \varepsilon'(\varepsilon).$$

In particular, we use the first inequality and (3.2) to obtain

$$||E(\mathfrak{c})(\otimes y_\iota) - \lambda_\varepsilon(\otimes y_\iota)|| < \varepsilon + a_1 a_2 \varepsilon'(\varepsilon)$$

where $\lambda_\varepsilon = e^{i\theta} a_1/b_1$ is a complex number depending on ε. We choose a sequence $\varepsilon_n \to 0$ such that $\lambda_{\varepsilon_n} \to \lambda$, which is possible because λ_ε is bounded. Then, by using separation of $a_1 a_2$ from ∞,

$$E(\mathfrak{c})(\otimes y_\iota) = \lambda(\otimes y_\iota). \tag{3.5}$$

In this derivation, we assumed $||E(\mathfrak{c})(\otimes y_\iota)|| \neq 0$. If this is not the case (3.5) holds with $\lambda = 0$. Since $E(\mathfrak{c})^2 = E(\mathfrak{c})$, we have $\lambda^2 = \lambda$ and hence $\lambda = 1$ or 0.

If $c(\{y_\iota\}) = c(\{y'_\iota\})$, then by Lemma 3.1 of [2], there exists $J \subset\subset I$ such that

$$||\otimes y'_\iota - z' \otimes y(J^c)|| < \varepsilon. \tag{3.6}$$

By (3.1), $E(\mathfrak{c})(\otimes y_\iota) = \lambda(\otimes y_\iota)$ with $\lambda = 1$ or 0 implies

$$\lim_K E_K(z' \otimes y(J^c)) = \lambda(z' \otimes y(J^c))$$

and hence by (3.1) and (3.6), we have $E(\mathfrak{c})(\otimes y'_\iota) = \lambda(\otimes y'_\iota)$ with the same λ. Hence $E(\mathfrak{c}) E_{\mathfrak{c}'} = \lambda E_{\mathfrak{c}'}$ with $\lambda = 1$ or 0. Q.E.D.

Let $\pi_{\mathfrak{c}}$ denote the restriction of the representation π to $E_{\mathfrak{c}} H$.

Lemma 5. *Let* $\mathfrak{c}, \mathfrak{c}' \in S_0$. *Either* $E(\mathfrak{c}) = E(\mathfrak{c}')$ *or* $E(\mathfrak{c}) \perp E(\mathfrak{c}')$. *Accordingly*, $\pi_{\mathfrak{c}}$ *and* $\pi_{\mathfrak{c}'}$ *are either quasi-equivalent or disjoint.*

Proof. The first part follows from Lemma 4. It then implies the second part. Q.E.D.

Proof of Theorem (1). First assume that $\mathfrak{c}' \underset{p}{\sim} \mathfrak{c}$. Let x_ι and y_ι in

H_ι be such that $\mathfrak{c}=\mathfrak{c}(\{x_\iota\})$, $\mathfrak{c}'=\mathfrak{c}(\{y_\iota\})$, $\|x_\iota\|=\|y_\iota\|=1$, and $\{x_\iota\}\sim\{p_\iota y_\iota\}$ for partial isometries p_ι with $p_\iota^* p_\iota y_\iota=y_\iota$. Let ω_z generally denote the vector state by z. Then $\omega_{\otimes x_\iota}=\otimes\omega_{x_\iota}$ and $\omega_{\otimes y_\iota}=\otimes\omega_{y_\iota}$.

Let $\otimes p_\iota$ be the mapping from $H_{\mathfrak{c}'}$ to $H_\mathfrak{c}$ defined by

$$(\otimes p_\iota)(y(J^c)\otimes z)=(py)(J^c)\otimes p(J)z \tag{3.7}$$

where J is any finite index set, $y(J^c)=\underset{\iota\notin J}{\otimes}y_\iota$, $(py)(J^c)=\underset{\iota\notin J}{\otimes}p_\iota y_\iota$, $p(J)=\underset{\iota\in J}{\otimes}p_\iota$ and $z\in\underset{\iota\in J}{\otimes}H_\iota$. If $\{p_\iota y_\iota\}\sim\{x_\iota\}$, then $p=\otimes p_\iota$ satisfies $pH_{\mathfrak{c}'}\subset H_\mathfrak{c}$ and $\|p\|\leqq 1$, $\pi_\mathfrak{c}(Q)p=p\pi_{\mathfrak{c}'}(Q)$ for $Q\in\mathfrak{R}_\iota$ and hence for $Q\in\mathfrak{R}$. Furthermore $p(\otimes y_\iota)=\otimes p_\iota y_\iota\neq 0$. Hence $\pi_\mathfrak{c}$ and $\pi_{\mathfrak{c}'}$ have a nonzero intertwining operator p and hence are not disjoint. By Lemma 5, we have $E(\mathfrak{c})E_{\mathfrak{c}'}=E_{\mathfrak{c}'}$.

Conversely, assume $E(\mathfrak{c})E_{\mathfrak{c}'}=E_{\mathfrak{c}'}$. Then $\pi_\mathfrak{c}$ and $\pi_{\mathfrak{c}'}$ are quasi-equivalent by Lemma 5. If x_ι satisfies $\|x_\iota\|=1$, $\mathfrak{c}(\{x_\iota\})=\mathfrak{c}$, then there exist a countable number of vectors ξ_l in $H_{\mathfrak{c}'}$ such that $\omega_{\otimes x_\iota}=\sum_l\omega_{\xi_l}$. Since product vectors are total in $H_{\mathfrak{c}'}$, there exists $y_\iota\in H_\iota$ such that $\|y_\iota\|=1$, $\mathfrak{c}(\{y_\iota\})=\mathfrak{c}'$ and $(\xi_1,\otimes y_\iota)\neq 0$. Then

$$\|\omega_{\otimes x_\iota}-\omega_{\otimes y_\iota}\|\leqq\sum_{l\neq 1}\|\xi_l\|^2+\|\omega_{\xi_1}-\omega_{\otimes y_\iota}\|<2. \tag{3.8}$$

Let $\|\omega_{\otimes x_\iota}-\omega_{\otimes y_\iota}\|_J$ denote the norm of the restricition of $\omega_{\otimes x_\iota}-\omega_{\otimes y_\iota}$ to $\mathfrak{R}(J)=(\bigcup_{\iota\in J}(\mathfrak{R}_\iota))''$. By proposition 1.12 and Corollary 2.6 of [5], we have

$$\begin{aligned}\prod_{\iota\in J}\rho(\omega_{x_\iota},\omega_{y_\iota})&\geqq 2^{-1}(2-\|\omega_{\otimes x_\iota}-\omega_{\otimes y_\iota}\|_J)\\&\geqq 2^{-1}(2-\|\omega_{\otimes x_\iota}-\omega_{\otimes y_\iota}\|)>0\end{aligned} \tag{3.9}$$

where $\rho(\mu,\nu)=2^{-1}(\mu(1)+\nu(1)-d(\mu,\nu)^2)=2^{-1}(2-d(\mu,\nu)^2)$ for states μ and ν. Since each $\rho(\omega_{x_\iota},\omega_{y_\iota})$ is in the interval $[0,1]$, (3.9) for arbitrary J implies the absolute convergence of $\prod\rho(\omega_{x_\iota},\omega_{y_\iota})$ and hence

$$\sum d(\omega_{x_\iota},\omega_{y_\iota})^2=2\sum(1-\rho(\omega_{x_\iota},\omega_{y_\iota}))<\infty. \tag{3.10}$$

By Theorem 4 of [1], there exist x'_ι and y'_ι in H_ι such that

(3.11) $$\omega_{x'_\iota}=\omega_{x_\iota},\ \omega_{y'_\iota}=\omega_{y_\iota},\ \|x'_\iota-y'_\iota\|=d(\omega_{x_\iota},\omega_{y_\iota}),$$

$$(x'_\iota, y'_\iota)>0.$$

Since $\omega_{x'_\iota}=\omega_{x_\iota}$, $p_\iota s'(x_\iota)=p_\iota$ and $p_\iota Qx_\iota=Qx'_\iota$ for all $Q\in\mathfrak{R}_\iota$ defines (by continuity) a partial isometry $p_\iota\in\mathfrak{R}'_\iota$, which satisfies $p^*_\iota p_\iota x_\iota=x_\iota$, $x'_\iota=p_\iota x_\iota$. Similarly there exists a partial isometry $p'_\iota\in\mathfrak{R}'_\iota$ such that $p'^*_\iota p'_\iota y_\iota=y_\iota$ and $y'_\iota=p'_\iota y_\iota$. From (3.10) and (3.11),

$$\sum|1-(x'_\iota, y'_\iota)|=2^{-1}\sum\|x'_\iota-y'_\iota\|^2<\infty$$

and hence $\{x_\iota\}\underset{p}{\sim}\{p_\iota x_\iota\}\sim\{p'_\iota y_\iota\}\underset{p}{\sim}\{y_\iota\}$. Therefore $\{x_\iota\}\underset{p}{\sim}\{y_\iota\}$ by Lemma 3. Q.E.D.

§4. Discussions

If $\{x_\iota\}\underset{p}{\sim}\{y_\iota\}$, then $x_\iota=p_\iota y_\iota$ for a partially isometric $p_\iota\in\mathfrak{R}'_\iota$ for all ι except for a countable number of ι, where p_ι satisfies $p^*_\iota p_\iota y_\iota=y_\iota$. (Note that $\|x_\iota\|=\|y_\iota\|=1$, $(x_\iota, y_\iota)=1$ imply $\|x_\iota-y_\iota\|^2=0$ and hence $x_\iota=y_\iota$.) Then $s'(x_\iota)$ and $s'(y_\iota)$ are equivalent in $\mathfrak{R}'_\iota$. p_ι can be extended to a unitary in $\mathfrak{R}'_\iota$ if and only if $1-s'(x_\iota)$ and $1-s'(y_\iota)$ are equivalent in $\mathfrak{R}'_\iota$.

If $\{x_\iota\}\underset{u}{\sim}\{y_\iota\}$, then $x_\iota=u_\iota y_\iota$ for a unitary $u_\iota\in\mathfrak{R}'_\iota$ for all ι except for a countable number of ι. Therefore both $s'(x_\iota)$ and $1-s'(x_\iota)$ are equivalent to $s'(y_\iota)$ and $1-s'(y_\iota)$ respectively, with a countable exception.

Due to Theorem (2) and its proof, the above argument gives the following:

Theorem (3). $\{x_\iota\}\underset{u}{\sim}\{y_\iota\}$ *if and only if* $\{x_\iota\}\underset{p}{\sim}\{y_\iota\}$ *and* $1-s'(x_\iota)$ *is equivalent to* $1-s'(y_\iota)$ *in* $\mathfrak{R}'_\iota$ *for all* ι *except for a countable many* ι, *where* $s'(x_\iota)$ *is the support projection of* x_ι *in* $\mathfrak{R}'_\iota$. $\mathfrak{c}(\{x_\iota\})\underset{u}{\sim}\mathfrak{c}(\{y_\iota\})$ *if and only if* $\{x_\iota\}\underset{u}{\sim}\{y_\iota\}$.

Proof. The first half is already shown. By definition, if $\{x_\iota\}\underset{u}{\sim}\{y_\iota\}$, then $\mathfrak{c}(\{x_\iota\})\underset{u}{\sim}\mathfrak{c}(\{y_\iota\})$. Therefore it remains to show that $\mathfrak{c}(\{x_\iota\})\underset{u}{\sim}\mathfrak{c}(\{y_\iota\})$

implies $\{x_\iota\}\underset{u}{\sim}\{y_\iota\}$, which is rather trivial consequence of Definition:

By definition, $\mathfrak{c}(\{x_\iota\})\underset{u}{\sim}\mathfrak{c}(\{y_\iota\})$ implies the existence of x'_ι and $y'_\iota \in H_\iota$ such that $\{x_\iota\}\sim\{x'_\iota\}$, $\{y_\iota\}\sim\{y'_\iota\}$ and $\{x'_\iota\}\underset{u}{\sim}\{y'_\iota\}$. Since $\{x_\iota\}\underset{u}{\sim}\{x'_\iota\}$ and $\{y_\iota\}\underset{u}{\sim}\{y'_\iota\}$ trivially, it follows that $\{x_\iota\}\underset{u}{\sim}\{y_\iota\}$. Q.E.D.

Example. Suppose x_ι are cyclic for $\mathfrak{R}_\iota$ and p_ι are isometric operators in $\mathfrak{R}'_\iota$, which are not unitaries. (This can happen for non-finite $\mathfrak{R}'_\iota$.) Then $1-s'(x_\iota)=0$ because $s'(x_\iota)H$ is the closure of $\mathfrak{R}_\iota x_\iota$ and x_ι is cyclic. For $y_\iota=p_\iota x_\iota$, $1-s'(y_\iota)=1-p_\iota p_\iota^*\neq 0$. Hence, if the index set is non-countable, then $\{x_\iota\}\underset{p}{\sim}\{y_\iota\}$ but $\{x_\iota\}$ is not u-equivalent to $\{y_\iota\}$.

In this example, the representation of $\mathfrak{R}$ in $H_\mathfrak{c}$ and $H_{\mathfrak{c}'}$, $\mathfrak{c}=\mathfrak{c}(\{x_\iota\})$, $\mathfrak{c}'=\mathfrak{c}(\{y_\iota\})$, are not unitarily equivalent as is seen by the following argument:

Let $y_{\iota\lambda}$, $\lambda\in\Lambda_\iota$ be an orthonormal basis for H_ι such that $y_{\iota 0}=y_\iota$. Then $\otimes y_{\iota\kappa(\iota)}$, with $\kappa(\iota)=0$ except for a finite number of ι, is an orthonormal basis for $H_{\mathfrak{c}'}$. Any $z\in H_{\mathfrak{c}'}$ has only a countable number of non-zero components on this basis and hence $z=(\underset{\iota\notin A}{\otimes} y_\iota)\otimes z'$, $z'\in\underset{\iota\in A}{\otimes} H_\iota$ for some countable index set A. Since $R=(\bigcup R_\iota)''$ and y_ι is not cyclic for $\mathfrak{R}_\iota$, z can not be cyclic for $\mathfrak{R}$ in $H_{\mathfrak{c}'}$. On the other hand, $\otimes x_\iota\in H_\mathfrak{c}$ is cyclic for $\mathfrak{R}$ in $H_\mathfrak{c}$. Hence $\mathfrak{R}|H_\mathfrak{c}$ and $\mathfrak{R}|H_{\mathfrak{c}'}$ can not be unitarily equivalent.

Theorem (4). *$\pi_\mathfrak{c}$ is quasi-equivalent to $\pi_{\mathfrak{c}'}$ if and only if $\mathfrak{c}\underset{p}{\sim}\mathfrak{c}'$. $\pi_\mathfrak{c}$ is unitarily equivalent to $\pi_{\mathfrak{c}'}$ if $\mathfrak{c}\underset{u}{\sim}\mathfrak{c}'$. If the index set is countable, then $\pi_\mathfrak{c}$ is unitarily equivalent to $\pi_{\mathfrak{c}'}$ if and only if $\pi_\mathfrak{c}$ is quasi-equivalent to $\pi_{\mathfrak{c}'}$.*

Proof. The first part is obvious by Lemma 5 and Theorem (1). To see the second part, assume $\mathfrak{c}=\mathfrak{c}(\{x_\iota\})$, $\mathfrak{c}'=\mathfrak{c}(\{y_\iota\})$ and $\{x_\iota\}\sim\{u_\iota y_\iota\}$, where $u_\iota\in\mathfrak{R}'_\iota$ is unitary. Then $\otimes u_\iota$ defined by the same equation as $\otimes p_\iota$ in the proof of Lemma 5 is obviously isometric and its range contains all $(uy)(J^c)\otimes u(J)H(J)$ where $H(J)=\underset{\iota\in J}{\otimes} H_\iota$. Since $u(J)$ is unitary $u(J)H(J)=H(J)$ and since $\otimes(u_\iota y_\iota)\in H_\mathfrak{c}$, the image of $\otimes u_\iota$ is the whole $H_\mathfrak{c}$.

Hence $\otimes u_\iota$ is a unitary intertwining operator for $\pi_{\mathfrak{c}}$ and $\pi_{\mathfrak{c}'}$, which proves the unitary equivalence of $\pi_{\mathfrak{c}}$ and $\pi_{\mathfrak{c}'}$. The last part follows then from Theorem (2). Q.E.D.

Remark. The unitary equivalence of $\pi_{\mathfrak{c}}$ and $\pi_{\mathfrak{c}'}$ does not necessarily imply $\mathfrak{c}\underset{u}{\sim}\mathfrak{c}'$. For example, consider $\mathfrak{R}_r=\mathscr{B}(H_r')\otimes\mathbf{1}''$ on $H_r'\otimes H_r''=H_r$ with all real $r\neq 0$ as index set and $x_r=\sum_{k=1}^{\infty}2^{-k}e_k\otimes e_k$, $y_r=\sum_{k=1}^{\infty}2^{-k}e_k\otimes e_{k+1}$ for $r>0$ and $x_r=\sum_{k=1}^{\infty}2^{-k}e_k\otimes e_{k+1}$, $y_r=\sum_{k=1}^{\infty}2^{-k}e_k\otimes e_k$ for $r<0$, where all H_r' and H_r'' are identified with a single Hilbert space H and $\{e_k\}$ is its orthonormal basis. Then obviously $\{x_\iota\}\underset{u}{\sim}\{y_\iota\}$ does not hold but $\pi_{\mathfrak{c}}$ is unitarily equivalent to $\pi_{\mathfrak{c}'}$ for $\mathfrak{c}=\mathfrak{c}(\{x_\iota\})$ and $\mathfrak{c}'=\mathfrak{c}(\{y_\iota\})$.

For $Q_\iota\in\mathscr{B}(H_\iota)$ with $\prod\|Q_\iota\|<+\infty$, there exists a unique bounded linear operator $\otimes Q_\iota$ on $\otimes H_\iota=H$ satisfying $(\otimes Q_\iota)(\otimes x_\iota)=\otimes Q_\iota x_\iota$ for all $\{x_\iota\}\in S_0$ by Theorem 3.1 in [6], where $\otimes Q_\iota x_\iota=0$ if $\{Q_\iota x_\iota\}\notin S_0$. If $Q_\iota'\in\mathfrak{R}_\iota'$ and $\prod\|Q_\iota'\|<+\infty$, $\otimes Q_\iota'$ can be defined in exactly the same manner and $\otimes Q_\iota'\in\mathfrak{R}'$ by Theorem 3.2 in [6].

$\otimes p_\iota$ in the proof of Theorem (1) is this $\otimes p_\iota$ with its domain restricted to $H_{\mathfrak{c}}'$.

Theorem (5). *$\mathfrak{R}'$ is generated by the set of all $E_{\mathfrak{c}}$, $\mathfrak{c}\in\mathfrak{C}_0$ and $\otimes p_\iota$ with partial isometries $p_\iota\in\mathfrak{R}_\iota'$. If the index set is countable, p_ι can be restricted to unitaries.*

Proof. Let $\mathfrak{N}$ be the set of all $E_{\mathfrak{c}}$, $\mathfrak{c}\in\mathfrak{C}_0$ and $\otimes p_\iota$ with partial isometries $p_\iota\in\mathfrak{R}_\iota'$. Since $\mathfrak{N}\subset\mathfrak{R}'$, it is enough to prove that $Q\in\mathfrak{R}'$ implies $Q\in\mathfrak{N}$. Let $Q\in\mathfrak{R}'$.

Since isometries $p_\iota\in\mathfrak{R}_\iota'$ generates $\mathfrak{R}_\iota'$,

$$(E_{\mathfrak{c}}\mathfrak{N}E_{\mathfrak{c}})'E_{\mathfrak{c}}=(\otimes\mathfrak{R}_\iota')'E_{\mathfrak{c}}=(\otimes\mathfrak{R}_\iota)E_{\mathfrak{c}}=\mathfrak{R}E_{\mathfrak{c}}$$

by Lemma 6.10 of [3]. Since $QE_{\mathfrak{c}}$ belongs to this set, there exists $Q_1\in\mathfrak{N}$ such that $QE_{\mathfrak{c}}=Q_1E_{\mathfrak{c}}$. Let $\otimes y_\iota\in E(\mathfrak{c})H$. By Theorem (1) and Lemma 3, there exist partial isometries $p_\iota\in\mathfrak{R}_\iota'$ such that $\mathfrak{c}(\{p_\iota y_\iota\})=\mathfrak{c}$ and $p_\iota^*p_\iota y_\iota=y_\iota$. Then $Q\otimes p_\iota y_\iota=Q_1\otimes p_\iota y_\iota$. Hence

$$Q \otimes y_\iota = Q(\otimes p_\iota^*) \otimes p_\iota y_\iota = (\otimes p_\iota^*) Q \otimes p_\iota y_\iota$$

$$= (\otimes p_\iota^*) Q_1 \otimes p_\iota y_\iota = Q_1(\otimes p_\iota^*) \otimes p_\iota y_\iota = Q_1 \otimes y_\iota.$$

This shows that $QE(\mathfrak{c}) = Q_1 E(\mathfrak{c}) \in \mathfrak{R}$. Hence $Q = \sum QE(\mathfrak{c}) \in \mathfrak{R}$ where the sum is over distinct $E(\mathfrak{c})$.

If the index set is countable, then p_ι in the above argument can be taken to be unitaries by Theorem (2) and the latter half of Theorem (5) is obtained. Q.E.D.

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