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Bures Distance Function and a Generalization of Sakai's Non-commutative Radon-Nikodym Theorem

By

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Abstract

For normal positive linear functionals μ and ν of a W^* algebra $\mathfrak{R}$, the following extension of a noncommutative Radon-Nikodym theorem by Sakai is given.

There exist decompositions $\mu=\mu_1+\mu_2$, $\nu=\nu_1+\nu_2$ such that ν_2 is the smallest normal positive linear functional on $\mathfrak{R}$ satisfying $\nu\geqq\nu_2$ and $s(\nu_2)\perp s(\mu)$, where $s(\alpha)$ denotes the support projection of α, and μ_2 is the smallest normal positive linear functional on $\mathfrak{R}$ satisfying $\mu\geqq\mu_2$ and $s(\mu_2)\perp s(\nu)$. Further, there exists a non-negative self-adjoint operator $A_1=A_1(\nu/\mu)$ (in general unbounded) such that $A_1=\int\lambda dE_\lambda^1$ with its spectral projections E_λ^1 in $\mathfrak{R}$, $\lim_{\lambda\downarrow 0}E_\lambda^1=1-s_\mu^\nu$ and

$$\nu(s(\mu_1)Qs(\mu_1))=\mu_1(A_1QA_1)\equiv\lim_{\lambda,\lambda'}\mu_1(A_1E_\lambda^1QA_1E_{\lambda'}^1)$$

for all $Q\in\mathfrak{R}$, where $s_\mu^\nu=s(\mu_1)-s(\mu_1)\wedge(1-s(\nu))$. There also exists another non-negative self-adjoint operator $A_2=A_2(\nu/\mu)$ such that its spectral projections E_λ^2 are in $\mathfrak{R}$, $\lim_{\lambda\downarrow 0}E_\lambda^2=1-s_\nu^\mu$ and, for all $Q\in\mathfrak{R}$,

$$\nu_1(s_\nu^\mu Qs_\nu^\mu)=\mu(A_2QA_2).$$

They are related by $A_1(\nu/\mu)A_2(\mu/\nu)=A_2(\mu/\nu)A_1(\nu/\mu)=s_\mu^\nu$.

The Bures distance function $d(\mu,\nu)$ is given by

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$$d(\mu, \nu)^2 = \mu(1) + \nu(1) - 2\mu_1(A_1)$$

$$= \mu(1) + \nu(1) - 2\mu(A_2).$$

In any representation π of $\mathfrak{R}$, if two vectors Ψ and Φ satisfy $\omega_\Psi = \mu$, $\omega_\Phi = \nu$ and $\|\Psi - \Phi\| = d(\mu, \nu)$, where ω_Ψ denotes the vector state by Ψ, then there is a decomposition $\pi = \pi_1 \oplus \pi'$, $\Psi = x_1 \oplus x'$, $\Phi = y_1 \oplus y'$, $\omega_{x'} = \mu_2$, $\omega_{y'} = \nu_2$, x_1 and y_1 are cyclic vectors of π_1, $\pi_1(s(\mu_1)) y_1 = \pi_1(A_1) x_1$, $\pi_1(s_\nu^\mu) y_1 = \pi_1(A_2) x_1$, and such that triplet π_1, x_1 and y_1 are unique up to unitary equivalence for given μ and ν.

§ 1. Introduction

For two normal positive linear functionals μ and ν of a W^*-algebra $\mathfrak{R}$ satisfying $\mu \geqq \nu$, Sakai [5] has shown the existence of a unique $t_0 \in \mathfrak{R}$ such that $0 \leqq t_0 \leqq 1$ and

$$\nu(Q) = \mu(t_0 Q t_0) \tag{1.1}$$

for all $Q \in \mathfrak{R}$. We shall generalize this Radon-Nikodym theorem of Sakai to the case where $\mu \geqq \nu$ does not necessarily hold.

Our investigation originally started from a search for a standard form of vectors Ψ and Φ such that their vector "states" are μ and ν and $\|\Psi - \Phi\|$ is minimal. The minimal value of $\|\Psi - \Phi\|$ is defined to be $d(\mu, \nu)$ by Bures [2]. It is easily shown that, if $\mu \geqq \nu$ holds, then Φ is uniquely given by

$$\Phi = \pi(t_0)\Psi \tag{1.2}$$

and hence

$$d(\mu, \nu)^2 = \mu(1) + \nu(1) - 2\mu(t_0). \tag{1.3}$$

We shall first show the existence of a pair Ψ and Φ giving the mimimal distance for general μ and ν. An analysis of their mutual relation leads to a generalization of t_0. The result reduces to the Radon-Nikodym theorem by Sakai if $\mu \geqq \nu$.

Notations and Conventions: All representations of $\mathfrak{R}$ in the present work will be normal representations. We denote the set of all normal positive linear functionals on $\mathfrak{R}$ by $S(\mathfrak{R})$. The expectation functional on $\mathfrak{R}$ by a vector x in a representation space $\mathfrak{H}$ of $\mathfrak{R}$ is denoted by ω_x. The support $s(\mu)$ of $\mu \in S(\mathfrak{R})$ is the smallest projection operator $E \in \mathfrak{R}$ satisfying $\mu(E)=\mu(1)$. The support $s(x)$ of a vector x in $\mathfrak{H}$ relative to a representation π of $\mathfrak{R}$ on $\mathfrak{H}$ is the smallest projection $E \in \pi(\mathfrak{R})$ satisfying $Ex=x$. $E\mathfrak{H}$ is the closure of $\pi(\mathfrak{R})'x$ and $s(x)=\pi(s(\omega_x))$. The support $s(\pi)$ of a representation π is the smallest central projection $E \in \mathfrak{R}$ satisfying $\pi(E)=1$. π is faithful on $\mathfrak{R}s(\pi)$. The support $s(Q)$ of an operator Q is the smallest projection E such that $EQ=QE=Q$. $s(Q)=s(Q^*Q)\vee s(QQ^*)$ and it belongs to $\mathfrak{R}$ if $Q \in \mathfrak{R}$.

Our main results are following theorems:

Theorem 1. *Let* $\mu, \nu \in S(\mathfrak{R})$.

(1) *There exists a unique decomposition* $\mu=\mu_1+\mu_2$, $\nu=\nu_1+\nu_2$ *such that* ν_2 *is the largest* $\rho \in S(\mathfrak{R})$ *satisfying* $\nu \geqq \rho$ *and* $s(\mu)\perp s(\rho)$, *and* μ_2 *is the largest* $\rho \in S(\mathfrak{R})$ *satisfying* $\mu \geqq \rho$ *and* $s(\nu)\perp s(\rho)$.

(2) *There exists a non-negative self-adjoint operator*

$$A_1=A_1(\nu/\mu)=\int \lambda dE_\lambda^1 \tag{1.4}$$

such that $E_\lambda^1 \in \mathfrak{R}$, $\lim_{\lambda\downarrow 0} E_\lambda^1 = 1-s_\mu^\nu$ *and*

$$\begin{aligned}\nu(s(\mu_1)Qs(\mu_1)) &= \nu(s_\mu^\nu Q s_\mu^\nu) \\ &= \mu_1(A_1QA_1) \equiv \lim_{\lambda,\lambda'\to+\infty} \mu_1(A_1E_\lambda^1 QA_1E_{\lambda'}^1),\end{aligned} \tag{1.5}$$

where

$$s_\mu^\nu = s(\mu_1)-s(\mu_1)\wedge(1-s(\nu)). \tag{1.6}$$

(3) *There exists a non-negative self-adjoint operator*

$$A_2=A_2(\nu/\mu)\equiv\int \lambda dE_\lambda^2 \tag{1.7}$$

such that $E_\lambda^2 \in \mathfrak{R}$, $\lim_{\lambda\downarrow 0} E_\lambda^2 = 1-s_\nu^\mu$, $A_1(\mu/\nu)A_2(\nu/\mu)=s_\nu^\mu$, *and*

(1.8) $$\nu_1(s^\mu_\nu Q s^\mu_\nu)=\mu(A_2 Q A_2).$$

(4) *In a representation* π_1 *of* $\mathfrak{R}$ *with a cyclic vector* x_1 *satisfying* $\omega_{x_1}=\mu_1$, *there exists a unique vector* y_1 *such that* $\omega_{y_1}=\nu_1$ *and*

(1.9) $$s(x_1)y_1=\pi_1(A_1)x_1\equiv\lim_{\lambda\to+\infty}\pi_1(A_1E^1_\lambda)x_1.$$

It satisfies

(1.10) $$\{s(y_1)-s(y_1)\wedge(1-s(x_1))\}y_1=\pi_1(A_2)x_1.$$

Theorem 2. *For any* μ, $\nu\in S(\mathfrak{R})$,

$$d(\mu,\ \nu)^2=\mu(1)+\nu(1)-2\mu_1(A_1)$$
$$=\mu(1)+\nu(1)-2\mu(A_2)$$

where A_1 *and* A_2 *are as in Theorem* 1 *and* $\mu_1(A_1)=\lim_{\lambda\to+\infty}\mu_1(A_1E^1_\lambda)$.

For any vectors Ψ *and* Φ *in a representation* π *of* $\mathfrak{R}$ *satisfying* $\omega_\Psi=\mu$, $\omega_\Phi=\nu$, *and* $d(\mu,\nu)=\|\Psi-\Phi\|$, *there exists a decomposition* $\pi=\pi_1\oplus\pi^1$, $\Psi=x_1\oplus x^1$, $\Phi=y_1\oplus y^1$, *such that* $\omega_{x^1}=\mu_2$, $\omega_{y^1}=\nu_2$, x_1 *and* y_1 *are cyclic for* $\pi_1(\mathfrak{R})$, *the triplet* π_1, x_1 *and* y_1 *are unitarily equivalent to* π_1, x_1 *and* y_1 *of Theorem* 1 (4) *and is unique up to unitary equivalence.*

Takesaki ([8] §15) considers the case $s(\mu)=1$. His h_0 has the same matrix element as our A_2 on the dense domain $\pi_\mu(\mathfrak{R})'\Psi$.

§2. Bures Distance Function

The Bures distance for $\mu,\nu\in S(\mathfrak{R})$ is

(2.1) $$d(\mu,\nu)=\inf\{\|x-y\|;\ \omega_x=\mu,\ \omega_y=\nu\}$$

where x and y can be in an arbitrary representation space of $\mathfrak{R}$. The following lemma shows that the infimum is actually reached.

Lemma 1. *For* $\mu\in S(\mathfrak{R})$, *there exist a representation* π_μ *of* $\mathfrak{R}$ *on* $\mathfrak{H}_\mu$ *and a vector* Ψ *in* $\mathfrak{H}_\mu$ *such that* $\mu=\omega_\Psi$ *and for any* $\nu\in S(\mathfrak{R})$ *there exists* $\Phi\in\mathfrak{H}_\mu$ *satisfying* $\omega_\Phi=\nu$ *and* $\|\Psi-\Phi\|=d(\mu,\ \nu)$.

Proof. By Proposition 1.6 of Bures [2], there exists a representation π_B of $\mathfrak{R}$ on $\mathfrak{H}_B$ and a vector x_B in $\mathfrak{H}_B$ such that $\omega_{x_B}=\mu$ and

$$d(\mu, \nu)=\inf\{\|x_B-y\|;\ y\in\mathfrak{H}_B, \omega_y=\nu\},$$

for any $\nu\in S(\mathfrak{R})$.

Let y_n be such that $y_n\in\mathfrak{H}_B$, $\omega_{y_n}=\nu$ and

$$\lim_n \|x_B-y_n\|=d(\mu, \nu).$$

By weak sequential compactness, there exists a subsequence $n(k)$ and $y\in\mathfrak{H}_B$ such that

$$\text{w-}\lim_k y_{n(k)}=y\,.$$

Then

$$\nu-\omega_y=\lim_k \omega_{(y_{n(k)}-y)}\geqq 0.$$

Therefore $\nu-\omega_y\in S(\mathfrak{R})$ and there exists $y^1\in\mathfrak{H}_B$ satisfying $\nu-\omega_y=\omega_{y^1}$. We also have

$$\begin{aligned}\|x_B-y\|^2&=\|x_B\|^2+\|y\|^2-2\mathrm{Re}\lim(x_B,\ y_{n(k)})\\&=\lim\|x_B-y_{n(k)}\|^2-\lim\|y_{n(k)}-y\|^2\\&=d(\mu, \nu)^2-\omega_{y^1}(1).\end{aligned}$$

Hence $\mathfrak{H}_\mu=\mathfrak{H}_B\oplus\mathfrak{H}_B$, $\pi_\mu=\pi_B\oplus\pi_B$, $\Psi=x_B\oplus 0$ and $\Phi=y\oplus y^1$ satisfy all the requirements. Q.E.D.

The next Lemma is not needed in the proof of the main Theorems and is a special case of Theorem 2. We present it here because it gives a motivation for the proof technique in the following sections.

Lemma 2. *Let $\mu\in S(\mathfrak{R})$, $t_0\in\mathfrak{R}$, $t_0\geqq 0$, and $\nu(Q)=\mu(t_0Qt_0)$ for all $Q\in\mathfrak{R}$. Then*

(2.2) $$d(\mu, \nu)^2=\mu(1)+\nu(1)-2\mu(t_0).$$

Proof. Let $\mathfrak{H}_\mu$, π_μ, Ψ and Φ be as in Lemma 1. Let $\Phi'=\pi_\mu(t_0)\Psi$. Then $\omega_{\Phi'}=\nu$ and

$$d(\mu, \nu)^2 \leqq \|\Psi-\Phi'\|^2=\mu(1)+\nu(1)-2\mu(t_0).$$

Let α be defined on $\pi_\mu(\mathfrak{R})\Phi'$ by

$$\alpha\pi_\mu(Q)\Phi'=\pi_\mu(Q)\Phi, \; Q\in\mathfrak{R}.$$

Then α is isometric on $\pi_\mu(\mathfrak{R})\Phi'$:

$$\|\pi_\mu(Q)\Phi\|^2=\nu(Q^*Q)=\|\pi_\mu(Q)\Phi'\|^2.$$

Hence α is well-defined on $\pi_\mu(\mathfrak{R})\Phi$, linear there and $\|\alpha\|\leqq 1$. Let $s'(\Phi)$ be the projection on the closure of $\pi_\mu(\mathfrak{R})\Phi$. Then $s'(\Phi)\in\pi_\mu(\mathfrak{R})'$ and $\hat{\alpha}\equiv\alpha s'(\Phi)\in\pi_\mu(\mathfrak{R})'$. We have

$$\begin{aligned}|(\Psi, \Phi)| &= |(\Psi, \hat{\alpha}\Phi')| \\ &= |(\pi_\mu(t_0)^{1/2}\Psi, \; \hat{\alpha}\pi_\mu(t_0)^{1/2}\Psi)| \\ &\leqq \|\hat{\alpha}\|\|\pi_\mu(t_0)^{1/2}\Psi\|^2 \leqq \mu(t_0).\end{aligned}$$

Hence

$$\begin{aligned}d(\mu, \nu)^2 &= \mu(1)+\nu(1)-2\,\mathrm{Re}(\Psi, \Phi) \\ &\geqq \mu(1)+\nu(1)-2\mu(t_0).\end{aligned}$$

Q.E.D.

Remark. Lemma 2 gives the uniqueness of t_0 satisfying

(i) $t_0\in\mathfrak{R}$, $t_0\geqq 0$,

(ii) $\mu(t_0Qt_0)=\nu(Q)$, $Q\in\mathfrak{R}$,

(iii) $s(t_0)\leqq s(\mu)$,

for given μ and ν by the following argument.

Consider the representation π_μ of $\mathfrak{R}$ on $\mathfrak{H}_\mu$ with a cyclic vector Ω_μ such that $\omega_{\Omega_\mu}=\mu$. Assume that t_0 and t_0' satisfy (i)-(iii). From the proof of Corollary, which gives the uniqueness of y satisfying $\omega_y=\nu$,

$\omega_x=\mu$, $d(\mu, \nu)=2(1-(x, y))$ (for given μ, ν, x), we obtain $\pi_\mu(t_0)\Omega_\mu =\pi_\mu(t_0')\Omega_\mu$. Hence $\pi_\mu(t_0)Q'\Omega_\mu=\pi_\mu(t_0')Q'\Omega_\mu$ for any $Q'\in\pi_\mu(\mathfrak{R})'$. Therefore $\pi_\mu(t_0-t_0')s(\Omega_\mu)=0$. Since $s(\Omega_\mu)=\pi_\mu(s(\mu))$ and the representation π_μ is faithful at least for $s(\mu)\mathfrak{R}s(\mu)$, we have $s(\mu)(t_0-t_0')s(\mu)=0$. By (iii), we have $t_0=t_0'$.

Corollary. *Let μ, ν, t_0 be as in Lemma 2. Let π be a representation of $\mathfrak{R}$ on $\mathfrak{H}$ and Ψ, $\Phi\in\mathfrak{H}$ satisfy $\omega_\Psi=\mu$, $\omega_\Phi=\nu$ and $d(\mu, \nu)^2=\|\Psi-\Phi\|^2$. Then*

$$\Phi=\pi(t_0)\Psi. \tag{2.3}$$

Proof. From the preceding proof, we have

$$\begin{aligned}\mathrm{Re}(\Psi, \Phi)&=\mathrm{Re}(\pi(t_0)^{1/2}\Psi, \hat{\alpha}\pi(t_0)^{1/2}\Psi)\\ &=\|\pi(t_0)^{1/2}\Psi\|^2.\end{aligned}$$

Hence $|\hat{\alpha}|\leqq 1$ implies

$$\hat{\alpha}\pi(t_0)^{1/2}\Psi=\pi(t_0)^{1/2}\Psi.$$

Since $\hat{\alpha}\in\pi(\mathfrak{R})'$, we have

$$\Phi=\pi(t_0)^{1/2}\hat{\alpha}\pi(t_0)^{1/2}\Psi=\pi(t_0)\Psi.$$

Q.E.D.

Remark. If $\mathfrak{R}$ is a type I factor, $\mu(Q)=\mathrm{tr}(\rho Q)$, $\nu(Q)=\mathrm{tr}(\sigma Q)$ for $Q\in\mathfrak{R}$, $\rho\geqq\sigma$, $\rho>0$, then

$$t_0=(\rho^{-1/2}|\sigma^{1/2}\rho^{1/2}|\rho^{-1/2})^- \tag{2.4}$$

and

$$\mu(t_0)=\mathrm{tr}|\sigma^{1/2}\rho^{1/2}| \tag{2.5}$$

where $|\beta|$ denotes $(\beta^*\beta)^{1/2}$.

§3. Construction of A

The following construction of A_0 is similar to the method of Takesaki [8]. $\mathfrak{H}_0$ is not assumed to be separable.

Lemma 3. *Let x_0 be a cyclic and separating vectors for a von Neuman algebra $\mathfrak{R}_0$ on $\mathfrak{H}_0$ and z be a separating vector for $\mathfrak{R}_0$ satisfying*

$$(x_0, Qz) \geqq 0 \tag{3.1}$$

for all $Q \geqq 0$, $Q \in \mathfrak{R}_0'$. Then there exists a positive self-adjoint operator

$$A_0 = \int_0^\infty \lambda dE_\lambda^0 \tag{3.2}$$

such that $E_\lambda^0 \in \mathfrak{R}_0$, $z = A_0 x_0$, $E_{+0}^0 \equiv \lim_{\lambda \downarrow 0} E_\lambda^0 = 0$.

Proof. Let S be defined on $\mathfrak{D} = \mathfrak{R}_0' x_0$ by

$$SQx_0 = Qz, \quad Q \in \mathfrak{R}_0'. \tag{3.3}$$

Since x_0 is cyclic for $\mathfrak{R}_0$, $Qx_0 = 0$ for $Q \in \mathfrak{R}_0'$ implies $Q = 0$ and hence $Qz = 0$. Therefore S is well-defined, linear operator. Since x_0 is separating for $\mathfrak{R}_0$, the domain $\mathfrak{D} = \mathfrak{R}_0' x_0$ of S is dense.

By assumption (3.1), $(x_0, (c-Q)^*(c-Q)z)$ is real for $Q \in \mathfrak{R}_0'$ and any complex number c. This implies that

$$(x_0, Qz) = (x_0, Q^* z)^* = (z, Qx_0).$$

Therefore for $Q_1, Q_2 \in \mathfrak{R}_0'$

$$\begin{aligned}(Q_2 x_0, SQ_1 x_0) &= (x_0, Q_2^* Q_1 z) \\ &= (z, Q_2^* Q_1 x_0) \\ &= (SQ_2 x, Q_1 x_0).\end{aligned}$$

Hence S is symmetric. S is non-negative on $\mathfrak{D}$ by (3.1).

$\mathfrak{D}$ is obviously invariant under $\mathfrak{R}_0'$. For $Q, Q_1, Q_2 \in \mathfrak{R}_0'$,

$$
\begin{aligned}
(Q_2 x_0, SQQ_1 x_0) &= (Q_2 x_0, QQ_1 z) \\
&= (Q^* Q_2 x_0, Q_1 z) \\
&= (Q^* Q_2 x_0, SQ_1 x_0) \\
&= (Q_2 x_0, QSQ_1 x_0).
\end{aligned}
$$

Hence S commutes with any Q in $\mathfrak{R}_0'$.

We now consider the Friedrichs extension of S. Let

$$(3.4) \qquad (\Psi_1, \Psi_2)_{\mathfrak{K}} = (\Psi_1, S\Psi_2) + (\Psi_1, \Psi_2)$$

for all $\Psi_1, \Psi_2 \in \mathfrak{D}$. Since

$$(Qx_0, Qx_0)_{\mathfrak{K}} = (x_0, Q^* Qz) + \|Qx_0\|^2 > 0$$

for non-zero $Q \in \mathfrak{R}_0'$, (Ψ_1, Ψ_2) is an inner product on $\mathfrak{D}$. Let $\mathfrak{K}$ be its completion, which is a Hilbert space with $(\Psi_1, \Psi_2)_{\mathfrak{K}}$ as an inner product. Let α be the mapping from Qx_0 in $\mathfrak{K}$ to Qx_0 in $\mathfrak{H}_0$. It is densely defined, linear and $|\alpha| \leqq 1$. Let $\bar{\alpha}$ be its closure.

Since $\|Q\| - Q \geqq 0$ for any self-adjoint Q, we have

$$(Q_1 x_0, (\|Q\| - Q) Q_1 z) \geqq 0$$

for any $Q_1 \in \mathfrak{R}_0'$. Replacing Q by Q^*Q, we obtain

$$\|QQ_1 x_0\|_{\mathfrak{K}}^2 \leqq \|Q^*Q\| \|Q_1 x_0\|_{\mathfrak{K}}^2 = \|Q\|^2 \|Q_1 x_0\|_{\mathfrak{K}}^2.$$

Therefore $\alpha^{-1} Q \alpha$ is linear and bounded on $\alpha^{-1}\mathfrak{D}$. Let $\pi_{\mathfrak{K}}(Q)$ be its closure on $\mathfrak{K}$. $\alpha \pi_{\mathfrak{K}}(Q) = Q\alpha$ on $\mathfrak{D}$ implies

$$(3.5) \qquad \bar{\alpha} \pi_{\mathfrak{K}}(Q) = Q \bar{\alpha}.$$

$\pi_{\mathfrak{K}}$ is clearly a $*$ representation of $\mathfrak{R}_0'$. If Q_α is a non-decreasing monotonous net in $\mathfrak{R}_0'$ with $\lim_{\alpha \uparrow} Q_\alpha = Q$, then $\lim_{\alpha \uparrow} \|\pi_{\mathfrak{K}}(Q - Q_\alpha)\Psi\|_{\mathfrak{K}}^2 = 0$ for $\Psi \in \alpha^{-1}\mathfrak{D}$ and hence for $\Psi \in \mathfrak{K}$. Therefore $\pi_{\mathfrak{K}}$ is normal.

From the Schwarz inequality

$$|(\Psi_1, S\Psi_2)|^2 \leqq (\Psi_1, S\Psi_1)(\Psi_2, S\Psi_2)$$

for $\Psi_1, \Psi_2 \in \mathfrak{D}$ and the majorization

$$(\alpha\Psi, S\alpha\Psi) \leqq \|\Psi\|^2_{\mathfrak{K}},$$

we obtain the existence of a bounded non-negative self-adjoint T on $\mathfrak{K}$ such that $1 \geqq T$ and

$$(\bar{\alpha}\Psi_1, S\alpha\Psi_2) = (\Psi_1, T\Psi_2)_{\mathfrak{K}} \tag{3.6}$$

for all $\Psi_2 \in \alpha^{-1}\mathfrak{D}$, $\Psi_1 \in \mathfrak{K}$.

Since S commutes with $Q \in \mathfrak{R}_0'$, we have from (3.5) and (3.6)

$$(\Psi_1, T\pi_{\mathfrak{K}}(Q)\Psi_2)_{\mathfrak{K}} = (\Psi_1, \pi_{\mathfrak{K}}(Q)T\Psi_2)_{\mathfrak{K}}$$

for all $\Psi_1 \in \mathfrak{K}$ and $\Psi_2 \in \alpha^{-1}\mathfrak{D}$. Hence $T \in \pi_{\mathfrak{K}}(\mathfrak{R}_0')'$.

According to Sakai ([6], 1.11.3), there exists a projection $e_\lambda \in \pi_{\mathfrak{K}}(\mathfrak{R}_0')'$ for each real λ, having the following properties:

(1) $e_\lambda \leqq e_{\lambda'}$ if $\lambda \leqq \lambda'$.

(2) $\lim_{\lambda_n \uparrow \lambda} e_{\lambda_n} = e_\lambda$.

(3) $e_{1+\varepsilon} = 1$ for $\varepsilon > 0$ and $e_0 = 0$.

(4) $T = \int_0^\infty \lambda de_\lambda$.

Let the closure of $\bar{\alpha}e_\lambda\mathfrak{K}$ be $\mathfrak{H}_\lambda$ and the projection onto $\mathfrak{H}_{f(\lambda)}$ be E^0_λ where $f(\lambda) = (1+\lambda)^{-1}\lambda$. f is a monotonously increasing function on $[0, \infty)$ with the range $[0, 1)$. From (1), we have

$$E^0_\lambda \leqq E^0_{\lambda'} \quad \text{if } \lambda \leqq \lambda'. \tag{3.7}$$

From (2), we have

$$\lim_{\lambda_n \uparrow \lambda} E^0_{\lambda_n} = E^0_\lambda. \tag{3.8}$$

From (3), we have

$$E^0_0 = 0. \tag{3.9}$$

For $\Psi \in e_\lambda\mathfrak{K}$ and $Q \in \mathfrak{R}_0'$, we have

$$Q\bar{\alpha}\Psi = \bar{\alpha}\pi_{\mathfrak{K}}(Q)\Psi \in \bar{\alpha}e_{\lambda}\mathfrak{K} \subset \mathfrak{H}_{\lambda}$$

due to (3.5) and $e_{\lambda} \in \pi_{\mathfrak{K}}(\mathfrak{R}_0')'$. Hence $Q\Psi \in \mathfrak{H}_{\lambda}$ for any $\Psi \in \mathfrak{H}_{\lambda}$, $Q \in \mathfrak{R}_0'$ and hence

$$E_{\lambda}^0 \in \mathfrak{R}_0. \tag{3.10}$$

From the definition (3.4), we have for $\Psi_2 \in \alpha^{-1}\mathfrak{D}$,

$$(\Psi_1, \Psi_2)_{\mathfrak{K}} = (\bar{\alpha}\Psi_1, S\alpha\Psi_2) + (\bar{\alpha}\Psi_1, \alpha\Psi_2) \tag{3.11}$$

for all $\Psi_1 \in \alpha^{-1}\mathfrak{D}$ and hence for all $\Psi_1 \in \mathfrak{K}$ by continuity. If $\bar{\alpha}\Psi_1 = 0$, then $(\Psi_1, \Psi_2)_{\mathfrak{K}} = 0$ for all Ψ_2 in the dense subset $\alpha^{-1}\mathfrak{D}$ of $\mathfrak{K}$ and hence $\Psi_1 = 0$. Namely the kernel of $\bar{\alpha}$ is 0.

From (3.11) and (3.6), we have

$$\begin{aligned}(\bar{\alpha}\Psi_1, \bar{\alpha}\Psi_2) &= (\Psi_1, \Psi_2)_{\mathfrak{K}} - (\bar{\alpha}\Psi_1, S\bar{\alpha}\Psi_2) \\ &= (\Psi_1, (1-T)\Psi_2)_{\mathfrak{K}}\end{aligned} \tag{3.12}$$

for $\Psi_1, \Psi_2 \in \alpha^{-1}\mathfrak{D}$ and hence for all $\Psi_1, \Psi_2 \in \mathfrak{K}$ by continuity. From this equality, we obtain the following three conclusions.

(i) If $(1-T)\Psi = 0$, then from (3.12) with $\Psi_1 = \Psi_2 = \Psi$, we obtain $\Psi = 0$. Hence $e_1 = 1$ and

$$\lim_{\lambda \to +\infty} E_{\lambda}^0 = 1. \tag{3.13}$$

(ii) Since e_{λ} commutes with T, we have

$$(\bar{\alpha}(1-e_{\lambda})\Psi_1, \bar{\alpha}e_{\lambda}\Psi_2) = 0$$

for all $\Psi_1, \Psi_2 \in \mathfrak{K}$. Hence $\bar{\alpha}(1-e_{\lambda})\Psi \perp \mathfrak{H}_{\lambda}$ and

$$\begin{aligned}E_{\lambda}^0\bar{\alpha}\Psi &= \bar{\alpha}e_{f(\lambda)}\Psi + E_{\lambda}^0\bar{\alpha}(1-e_{f(\lambda)})\Psi \\ &= \bar{\alpha}e_{f(\lambda)}\Psi.\end{aligned} \tag{3.14}$$

(iii) For all $\Psi_1, \Psi_2 \in \mathfrak{K}$, we have

$$\begin{aligned}d(\bar{\alpha}\Psi_1, E_{\lambda}^0\bar{\alpha}\Psi_2) &= d(\Psi_1, (1-T)e_{f(\lambda)}\Psi_2)_{\mathfrak{K}} \\ &= (1+\lambda)^{-1}d(\Psi_1, e_{f(\lambda)}\Psi_2)_{\mathfrak{K}}.\end{aligned} \tag{3.15}$$

This also implies that $\bar{\alpha}^{-1}e_{f(\lambda)}$ is bounded for finite λ and hence

$$\bar{\alpha}e_{\lambda}\mathfrak{K}=\mathfrak{H}_{\lambda},\ \lambda<1. \tag{3.16}$$

From (3.7), (3.8), (3.9) and (3.13), we can define a non-negative self-adjoint operator associated with $\mathfrak{N}_0$ on $\mathfrak{H}_0$ by

$$B=\int_0^{\infty}\lambda^{1/2}dE_{\lambda}^{0}. \tag{3.17}$$

Its domain $D(B)$ is the set of all $\Psi\in\mathfrak{H}$ such that

$$(\|B\Psi\|^2=)\int_0^{\infty}\lambda d(\Psi,\ E_{\lambda}^{0}\Psi)<\infty.$$

By (3.15), we have

$$(\|B\bar{\alpha}\Psi\|^2=)\int_0^{\infty}\lambda d(\bar{\alpha}\Psi,\ E_{\lambda}^{0}\bar{\alpha}\Psi) \tag{3.18}$$

$$=\int_0^{\infty}f(\lambda)d(\Psi,\ e_{f(\lambda)}\Psi)_{\mathfrak{K}}=(\Psi,\ T\Psi)_{\mathfrak{K}}<\infty$$

and hence $\bar{\alpha}\ \mathfrak{K}\subset D(B)$. Further, by (3.11), (3.6) and (3.18),

$$\|\Psi\|_{\mathfrak{K}}^2=\|B\bar{\alpha}\Psi\|^2+\|\bar{\alpha}\Psi\|^2. \tag{3.19}$$

Since the union of (3.16) is dense in $D(B)$ relative to the metric $\{\|B\Psi\|^2+\|\Psi\|^2\}^{1/2}$ and since $\bar{\alpha}\mathfrak{K}$ is complete relative to the same metric due to (3.19), we have

$$D(B)=\bar{\alpha}\mathfrak{K}. \tag{3.20}$$

By polarization, we obtain from (3.18),

$$(B\bar{\alpha}\Psi_1,\ B\bar{\alpha}\Psi_2)=(\Psi_1,\ T\Psi_2)_{\mathfrak{K}}.$$

Combining with (3.6), we obtain $\mathfrak{D}\subset D(B^2)$ and

$$B^2\Psi=S\Psi,\ \Psi\in\mathfrak{D}. \tag{3.21}$$

Hence

$$A_0 = B^2 = \int_0^\infty \lambda dE_\lambda^0$$

satisfies $E_\lambda^0 \in \mathfrak{R}_0$ and $z = A_0 x_0$.

If $\lim_{\lambda \downarrow 0} E_\lambda^0 \Psi = \Psi$, then

$$\begin{aligned}(Qz, \Psi) &= (SQx_0, \Psi) \\ &= (Qx_0, A_0 \Psi) = 0.\end{aligned}$$

Since z is assumed to be separating for $\mathfrak{R}_0$ and hence is cyclic for $\mathfrak{R}_0'$, we have $\Psi = 0$. Therefore

$$\lim_{\lambda \downarrow 0} E_\lambda^0 = 0. \tag{3.22}$$

Q.E.D.

Remark. A_0 satisfying $E_\lambda^0 \in \mathfrak{R}_0$ and $z = A_0 x_0$ can be constructed exactly in the same way even if z is not separating for $\mathfrak{R}_0$, except that $\lim_{\lambda \downarrow 0} E_\lambda^0$ is in general a non-zero projection.

In the present case, $A_0 \geqq 0$ and hence the equality in (3.1) holds only if $Qx_0 = 0$, namely $Q = 0$. Therefore z is separating for $\mathfrak{R}_0'$ and hence is cyclic for $\mathfrak{R}_0$.

§ 4. Proof of Main Theorems

The unique decompositions $\mu = \mu_1 + \mu_2$ and $\nu = \nu_1 + \nu_2$ are essentially given by the following lemma.

Lemma 4. *Let $\mathfrak{R}_2$ be a von Neumann algebra on $\mathfrak{H}$ and let Ψ and Φ be two vectors in $\mathfrak{H}$ such that*

$$(\Psi, Q\Phi) \geqq 0 \tag{4.1}$$

for all non-negative self-adjoint Q in $\mathfrak{R}_2'$. Then there exists the largest projection E in $\mathfrak{R}_2'$ such that

$$(\Psi, E\Phi) = 0. \tag{4.2}$$

It satisfies

$$\omega_\Psi = \omega_{E\Psi} + \omega_{(1-E)\Psi}, \quad \omega_\Phi = \omega_{E\Phi} + \omega_{(1-E)\Phi}, \tag{4.3}$$

$$s(\omega_{E\Psi}) \perp s(\omega_{(1-E)\Phi}), \quad s(\omega_{(1-E)\Psi}) \perp s(\omega_{E\Phi}) \tag{4.4}$$

$$s(\omega_{E\Psi}) \perp s(\omega_{E\Phi}). \tag{4.5}$$

$\omega_{E\Phi}$ is the largest $\rho \in S(\mathfrak{R}_2)$ such that $\omega_\Phi \geqq \rho$ and $s(\rho) \perp s(\omega_\Psi)$. $\omega_{E\Psi}$ is the largest $\rho \in S(\mathfrak{R}_2)$ such that $\omega_\Psi \geqq \rho$ and $s(\rho) \perp s(\omega_\Phi)$.

Proof. Let $(\Psi, Q\Phi) = 0$ for $Q \in \mathfrak{R}_2'$, $Q \geqq 0$. Let $e_\lambda \in \mathfrak{R}_2'$ be the spectral projection of Q (Sakai [6]) and

$$e(n) = e_{1/(n-1)} - e_{1/n}$$

where $e_\infty = 1$ and $n = 1, 2, \cdots$. Since

$$Q \geqq Qe(n) \geqq n^{-1} e(n),$$

we have

$$(\Psi, e(n)\Phi) = 0.$$

Hence $(\Psi, Q\Phi) = 0$ for $Q \in \mathfrak{R}_2'$ implies

$$(\Psi, s(Q)\Phi) = 0, \quad s(Q) = \sum_n e(n). \tag{4.6}$$

For a finite number of projections $E_i \in \mathfrak{R}_2'$, satisfying $(\Psi, E_i\Phi) = 0$, we obtain from (4.6)

$$(\Psi, \bigvee_i E_i \Phi) = (\Psi, s(\sum_i E_i)\Phi) = 0. \tag{4.7}$$

From the normality, the same holds for any number of E_i. Let E be the supremum of $E_\alpha \in \mathfrak{R}_2'$ satisfying $(\Psi, E_\alpha \Phi) = 0$. Then, by (4.7), we have $(\Psi, E\Phi) = 0$, and by construction, E is the largest such projection in $\mathfrak{R}_2'$.

From $E \in \mathfrak{R}_2'$, we have (4.3). From Schwarz inequality for positive linear functional $(\Psi, Q\Phi)$, we have

$$(Q_1\Psi, Q_2 E\Phi) = 0$$

for any $Q_1, Q_2 \in \mathfrak{R}_2'$. Setting $Q_1 = Q_3 E$ or $Q_1 = Q_3(1-E)$, we obtain $s(\omega_{E\Phi}) \perp s(\omega_{E\Psi})$ and $s(\omega_{E\Phi}) \perp s(\omega_{(1-E)\Psi})$. Interchanging the role of Ψ and Φ, we obtain $s(\omega_{E\Psi}) \perp s(\omega_{(1-E)\Phi})$.

Let $\rho \in S(\mathfrak{R}_2)$ be such that

$$\rho \leqq \omega_\Phi, \; s(\rho) \perp s(\omega_\Psi). \tag{4.8}$$

Then there exists $Q \in \mathfrak{R}_2'$, $1 \geqq Q \geqq 0$ satisfying

$$\rho = \omega_{Q\Phi}$$

due to $\rho \leqq \omega_\Phi$. Since $\rho(s(\omega_\Psi)) = 0$, we have $s(\Psi)Q\Phi = 0$. Hence $(\Psi, Q\Phi) = 0$, which implies by (4.6)

$$s(Q) \leqq E$$

and we have $Q\Phi = EQ\Phi = QE\Phi$. Therefore

$$\rho = \omega_{Q\Phi} = \omega_{QE\Phi} \leqq \omega_{E\Phi}.$$

This proves that $\omega_{E\Phi}$ is the largest ρ satisfying (4.8).

The same proof holds for $\omega_{E\Psi}$. Q.E.D.

Proof of Theorem 1 (1). By Lemma 1, there exists a representation π_μ of $\mathfrak{R}$ on $\mathfrak{H}_\mu$ and vectors Ψ and $\Phi \in \mathfrak{H}_\mu$ such that

$$\omega_\Psi = \mu, \; \omega_\Phi = \nu, \; d(\mu, \nu) = \|\Psi - \Phi\|^2.$$

We shall show that for $Q \in \pi(\mathfrak{R})'$, $Q \geqq 0$

$$(\Psi, Q\Phi) \geqq 0. \tag{4.9}$$

This will prove Theorem 1 (1) due to Lemma 4, where $\mathfrak{R}_2 = \pi_\mu(\mathfrak{R})$, $\mathfrak{H} = \mathfrak{H}_\mu$.

Suppose E' is a projection in $\pi(\mathfrak{R})'$ and $(\Psi, E'\Phi)$ is not a non-negative real number. Then there exists real numbers θ_1 and θ_2 such that θ_1 is not an integer multiple of 2π and

$$\alpha \equiv (\Psi, e^{i\theta_1} E'\Phi) \geqq 0, \; \beta \equiv (\Psi, e^{i\theta_2}(1-E')\Phi) \geqq 0.$$

Then

$$\mathrm{Re}(\Psi, \Phi) < \alpha + \beta. \tag{4.10}$$

Now consider the representation $\pi \oplus \pi$ of $\mathfrak{R}$ on $\mathfrak{H} \oplus \mathfrak{H}$ and vectors

$$\Psi' = E'\Psi \oplus (1-E')\Psi,$$

$$\Phi' = e^{i\theta_1} E'\Phi \oplus e^{i\theta_2}(1-E')\Phi.$$

They satisfy $\omega_{\Psi'} = \omega_{\Psi} = \mu$, $\omega_{\Phi'} = \omega_{\Phi} = \nu$ and, by (4.10),

$$\|\Psi' - \Phi'\|^2 = \mu(1) + \nu(1) - 2(\alpha+\beta) < \|\Psi - \Phi\|^2,$$

which is a contradiction with the minimality of $\|\Psi - \Phi\|^2$.

Therefore $(\Psi, E'\Phi) \geqq 0$ for any projection E' in $\pi(\mathfrak{R})'$ and hence (4.9) holds for any $Q \geqq 0$, $Q \in \pi(\mathfrak{R})'$. Q.E.D.

To apply Lemma 3, we need a further reduction:

Lemma 5. *Let $\mathfrak{R}_1$ be a von Neumann algebra on $\mathfrak{H}_1$ and let x_1 and y_1 be vectors in $\mathfrak{H}_1$. Let*

$$P \equiv s(s(x_1)s(y_1)). \tag{4.11}$$

Then

$$P \equiv s(x_1) \vee s(y_1) - s(x_1) \wedge (1-s(y_1)) - s(y_1) \wedge (1-s(x_1)). \tag{4.12}$$

Let

$$x_0 \equiv Px_1 = x_1 - \{s(x_1) \wedge (1-s(y_1))\} x_1, \tag{4.13}$$

$$y_0 \equiv Py_1 = y_1 - \{s(y_1) \wedge (1-s(x_1))\} y_1. \tag{4.14}$$

Then

$$s(x_0) = s(x_1) - s(x_1) \wedge (1-s(y_1)), \tag{4.15}$$

$$s(y_0) = s(y_1) - s(y_1) \wedge (1-s(x_1)), \tag{4.16}$$

$$s(x_0) \vee s(y_0) = P, \tag{4.17}$$

(4.18) $$s(x_0)\wedge(1-s(y_0))=0,\ s(y_0)\wedge(1-s(x_0))=0,$$

(4.19) $$(x_1, Qy_1)=(x_0, Qy_0)$$

for all $Q\in\mathfrak{R}_1'$.

If

(4.20) $$(x_1, Qy_1)>0$$

holds for all $Q\in\mathfrak{R}_1'$, $Q\geqq 0$, $Q\neq 0$, *then both* x_0 *and*

(4.21) $$z=s(x_0)y_0=s(x_1)y_0=s(x_1)y_1$$

are cyclic and separating for the restriction

(4.22) $$s(x_0)\mathfrak{R}_1 s(x_0)\equiv\mathfrak{R}_0$$

of $\mathfrak{R}_1$ *in* $s(x_0)\mathfrak{H}_1\equiv\mathfrak{H}_0$.

Proof. $s(x_1)s(y_1)\Psi=0$ implies

$$s(y_1)\Psi\in(1-s(x_1))\mathfrak{H}_1$$

and hence

$$\Psi=(1-s(y_1))\Psi+s(y_1)\Psi$$
$$\in(1-s(y_1))\mathfrak{H}_1+\{s(y_1)\mathfrak{H}_1\cap(1-s(x_1))\mathfrak{H}_1\}.$$

The converse is also true. Therefore

$$\ker s(x_1)s(y_1)=(1-s(y_1))\mathfrak{H}_1+\{s(y_1)\wedge(1-s(x_1))\}\mathfrak{H}_1.$$

Similar formula holds for $s(y_1)s(x_1)$. Since

$$(1-P)\mathfrak{H}_1=\ker s(x_1)s(y_1)\cap\ker(s(x_1)s(y_1))^*$$

by definition, we obtain (4.12). (4.13) and (4.14) then follow.

From (4.13), the set of Qx_0, $Q\in\mathfrak{R}_1'$ is the same as $s(x_1)-s(x_1)\wedge(1-s(y_1))$ times the set of Qx_1, $Q\in\mathfrak{R}_1'$ and the set of Qx_1, $Q\in\mathfrak{R}_1'$ spans $s(x_1)\mathfrak{H}_1$. Hence we obtain (4.15). Similarly we have (4.16). (4.17) and (4.18) then follow.

Since $Qy_1 \in s(y_1)\mathfrak{H}_1$ for $Q \in \mathfrak{R}_1'$, we have

$$(x_1, Qy_1) = (x_0, Qy_1).$$

Since $Qx_0 \in s(x_0)\mathfrak{H}_1 \subset s(x_1)\mathfrak{H}_1$, we have

$$(x_0, Qy_1) = (Q^* x_0, y_1) = (Q^* x_0, y_0).$$

Therefore (4.19) holds.

If (4.20) holds, then for any $Q \in \mathfrak{R}_1'$, $Q \geqq 0$, $Q \neq 0$, we have, by (4.19),

$$(x_0, Qy_0) = (x_0, Qs(x_0)y_0) > 0$$

and hence $Qx_0 \neq 0$, $Qs(x_0)y_0 \neq 0$. Therefore x_0 and $s(x_0)y_0$ are separating for $\mathfrak{R}_1'$. ($Q \in \mathfrak{R}_1'$ and $Qx_0 = 0$ implies $Q^*Qx_0 = 0$, hence $Q^*Q = 0$.) Therefore both x_0 and $s(x_0)y_0$ are cyclic for $\mathfrak{R}_1$ and hence cyclic for $s(x_0)\mathfrak{R}_1 s(x_0)$ on $s(x_0)\mathfrak{H}_1$. x_0 is obviously cyclic for $s(x_0)\mathfrak{R}_1'$ on $s(x_0)\mathfrak{H}_1$ and hence is separating for $s(x_0)\mathfrak{R}_1 s(x_0)$.

Suppose that $Q \in s(x_0)\mathfrak{R}_1 s(x_0)$ and

$$Qs(x_0)y_0 = 0.$$

Then $s(Q^*Q) \leqq 1 - s(y_0)$ because $Qy_0 = Qs(x_0)y_0 = 0$. Since $s(Q^*Q) \leqq s(x_0)$, we have by (4.18)

$$s(Q^*Q) \leqq s(x_0) \wedge (1 - s(y_0)) = 0.$$

Therefore we have $Q = 0$. Hence $z = s(x_0)y_0$ is separating for $s(x_0)\mathfrak{R}_1 s(x_0)$. Q.E.D.

Proof of Theorem 1 (2). In the proof of Theorem 1 (1), we set

$$\mathfrak{H}_1 = (1-E)\mathfrak{H}_\mu, \ \mathfrak{R}_1 = \pi_\mu(\mathfrak{R})(1-E),$$

$$x_1 = (1-E)\Psi, \quad y_1 = (1-E)\Phi,$$

where E is taken from Lemma 4.

If $Q \in (1-E)\pi_\mu(\mathfrak{R})'(1-E)$, $Q \geqq 0$, $Q \neq 0$, we have

$$(\Psi, Q\Phi) \neq 0$$

due to the maximality of E. Therefore we have (4.20).

We now apply Lemma 3 to $\mathfrak{H}_0, \mathfrak{R}_0, x_0$ and z of Lemma 4, and obtain a positive self-adjoint operator (3.2), where

$$E_\lambda^0 \in \mathfrak{R}_0 = s(x_0)\mathfrak{R}_1 s(x_0) \subset \mathfrak{R}_1.$$

By $\omega_{x_1} = \mu_1$, $\pi_\mu(Q)(1-E)$ is faithful certainly for

$$Q \in s(\mu_1)\mathfrak{R}s(\mu_1)$$

and

$$\pi_\mu(s(\mu_1))(1-E) = s(x_1)(1-E) \tag{4.23}$$

because x_1 is cyclic for $\mathfrak{R}_1$ on $\mathfrak{H}_1$ due to (4.20). Therefore there exists a unique E_λ^1 such that for $\lambda > 0$ $(1-s(\mu_1)) \leqq E_\lambda^1$ and

$$(1-E)\pi_\mu(E_\lambda^1) = E_\lambda^0(1-E).$$

By the faithfulness of $(1-E)\pi_\mu$, we have

(1) $E_\lambda^1 \geqq E_{\lambda'}^1$ for $\lambda \geqq \lambda'$,

(2) $\lim_{\lambda_n \uparrow \lambda} E_{\lambda_n}^1 = E_\lambda^1$,

(3) $E_0^1 = 0$, $\lim_{\lambda \uparrow \infty} E_\lambda^1 = 1$.

We now define $A_1(\nu/\mu)$ by (1.4). We have

$$\mu_1(A_1 Q A_1) = (\pi_\mu(A_1)x_1, \ \pi_\mu(Q)\pi_\mu(A_1)x_1), \ Q \in \mathfrak{R}. \tag{4.24}$$

Since $\pi_\mu(E_\lambda^1)x_1 = E_\lambda^0 x_1$ with $E_\lambda^0 \in s(x_0)\mathfrak{R}_1 s(x_0)$, we have

$$\pi_\mu(A_1)x_1 = A_0 x_0 = s(x_0)y_1 = s(x_1)y_1.$$

By $\omega_{y_1} = \nu_1$ and (4.23), we obtain

$$(s(x_1)y_1, \ \pi_\mu(Q)s(x_1)y_1) = \nu_1(s(\mu_1)Qs(\mu_1)). \tag{4.25}$$

By the same argument as for (4.23), we obtain

$$\pi_\mu(s(\nu_1))(1-E) = s(y_1)(1-E). \tag{4.26}$$

From (4.15), (4.23) and (4.26), we have

$$\pi_\mu(s_\mu^\nu)(1-E)=s(x_0)(1-E). \tag{4.27}$$

Therefore we also have

$$(s(x_0)y_1,\ \pi_\mu(Q)s(x_0)y_1)=\nu_1(s_\mu^\nu Q s_\mu^\nu). \tag{4.28}$$

By (4.24), (4.25), and (4.28), we obtain (1.5).

From (4.27), we have

$$\lim_{\lambda\downarrow 0} E_\lambda^1 = 1-s_\mu^\nu.$$

Proof of Theorem 1 (3). Since the initial assumptions are symmetric in μ and ν, we define

$$A_2(\mu/\nu)\equiv\int_0^\infty \lambda^{-1}dE_\lambda^1 s(x_0) \tag{4.29}$$

and prove the corresponding properties. By definition of E_λ^1,

$$\pi_\mu(A_2(\mu/\nu)A_1(\nu/\mu))^-(1-E)=s(x_0)(1-E),$$

where unbounded operators A_k are always defined as the limit of $A_k E_L^k$. By (4.27), we have

$$(A_2(\mu/\nu)A_1(\nu/\mu))^- s_\mu^\nu.$$

Since $s(\nu_2)\perp s(\mu)\geqq s_\mu^\nu$, we have $\pi_\mu(A_2(\mu/\nu))E\Phi=0$. Hence, by using $\pi_\mu(s_\mu^\nu)y_1=s(x_0)y_1$ and $\pi_\mu(s_\mu^\nu)x_1=x_0$,

$$\begin{aligned}\pi_\mu(A_2(\mu/\nu))\Phi &= \pi_\mu(A_2(\mu/\nu))y_1\\ &= \pi_\mu(A_2(\mu/\nu))s(x_0)y_1\\ &= x_0=\pi_\mu(s_\mu^\nu)x_1.\end{aligned} \tag{4.30}$$

Therefore we have

$$\nu(A_2(\mu/\nu)QA_2(\mu/\nu))=\mu_1(s_\mu^\nu Q s_\mu^\nu).$$

Q.E.D.

Proof of Theorem 1 (4). We have already the existence because the vector y_1 in the proof of Theorem 1 (2) satisfies all requirements. To prove the uniqueness, suppose that $y' \in \mathfrak{H}_1$ satisfies $\omega_{y'} = \nu_1$ and $s(x_1)y' = z$. Then there exists a partial isometry $u \in \mathfrak{R}_1'$ such that $u^* u y_1 = y_1$ and $u y_1 = y'$ due to $\omega_{y'} = \omega_{y_1}$. We have

$$us(x_1)y_1 = s(x_1)uy_1 = s(x_1)y' = s(x_1)y_1$$

and hence $u-1$ is 0 on z. By applying $\pi_\mu(A_2(\mu/\nu))$, we have

$$(u-1)x_0 = 0.$$

Since x_0 is separating for $\mathfrak{R}_1'$, we have $u=1$ and $y'=y$. Hence the uniqueness. Q.E.D.

Proof of Theorem 2. From the construction of A_1 and A_2, we have

$$d(\mu, \nu)^2 = \mu(1) + \nu(1) - 2(x_1, y_1)$$

$$= \mu(1) + \nu(1) - 2\mu_1(A_1)$$

$$= \mu(1) + \nu(1) - 2\mu(A_2).$$

To prove the uniqueness, suppose Ψ_1 and Φ_1 be given, satisfying $\omega_{\Psi_1} = \mu$, $\omega_{\Phi_1} = \nu$ and $d(\mu, \nu) = \|\Psi_1 - \Phi_1\|$. By expanding the representation, we can identify Ψ_1 with Ψ in the proof of Theorem 1 (2), where the representation contains π_μ and Φ_1 is not necessarily the same as Φ.

Since $\omega_\Phi = \omega_{\Phi_1} = \nu$, there exists a partial isometry $u \in \pi(\mathfrak{R})'$, satisfying $u^* u \Phi = \Phi$ and $u\Phi = \Phi_1$. We also have

$$(\Psi, \Phi) = d(\mu, \nu)^{1/2} = (\Psi, \Phi_1).$$

Since

$$s(\Psi)uE\Phi = us(\Psi)E\Phi$$

$$= u\pi(s(\omega_\Psi))E\Phi$$

$$= u\pi_\mu(s(\mu))E\Phi = 0,$$

we have

$$(\Psi, \Phi_1)=(\Psi, u(1-E)\Phi)=(A_0^{1/2}x_1, uA_0^{1/2}x_1).$$

Equality of this expression with $(\Psi, \Phi)=\|A_0^{1/2}x_1\|^2$ implies

$$(u-1)A_0^{1/2}x_1=0.$$

By multiplying $\pi(A_2(\mu/\nu))^{1/2}$, we obtain

$$(u-1)x_0=0.$$

Since x_0 is cyclic in $\mathfrak{H}_1$, we have $u=1$ on $\mathfrak{H}_1$. Therefore

$$(1-E)\Phi_1=y_1=(1-E)\Phi.$$

Setting $E\Phi_1=y'$, we obtain the statement of Theorem 2.

§5. $d_\pi(\mu, \nu)$ and $d(\mu, \nu)$

In [1], we have defined

$$(5.1)\qquad d_\pi(\mu, \nu)=\inf\{\|\Psi-\Phi\|;\ \omega_\Psi=\mu,\ \omega_\Phi=\nu,\ \Psi\in\mathfrak{H}_\pi,\ \Phi\in\mathfrak{H}_\pi\}$$

where π is a fixed representation on $\mathfrak{H}_\pi$. Obviously $d_\pi(\mu, \nu)\geqq d(\mu, \nu)$. We shall now discuss when the equality holds.

We shall start by considering whether there exists Φ giving $d(\mu, \nu)=\|\Psi-\Phi\|$ for the fixed representation π and a fixed vector Ψ. It already gives some cases where $d(\mu, \nu)=d_\pi(\mu, \nu)$.

Theorem 3. *Let $\mu, \nu\in S(\mathfrak{R})$ and π be a fixed representation of $\mathfrak{R}$ on $\mathfrak{H}_\pi$. Let Ψ be a fixed vector in $\mathfrak{H}_\pi$ satisfying $\omega_\Psi=\mu$.*

(1) *Let E_1 be the projection on the closure of*

$$(5.2)\qquad \pi(\mathfrak{R})\pi(A_2(\nu/\mu))\Psi.$$

Then there exists $\Phi\in\mathfrak{H}_\pi$ satisfying

$$(5.3)\qquad \omega_\Phi=\nu,\ \|\Psi-\Phi\|=d(\mu, \nu)$$

if and only if there exists a vector y' *in* $(1-E_1)H$ *such that* $\omega_{y'}=\nu_2$.

(2) *If* $s(\mu)\geqq s(\nu)$, *then there always exists* $\Phi\in\mathfrak{H}_\pi$ *satisfying* (5.3).

(3) *If* Ψ *is separating for* $\pi(\mathfrak{R})$, *there always exists* $\Phi\in\mathfrak{H}_\pi$ *satisfying* (5.3).

Proof. We first extend π to sufficiently large representation $\hat{\pi}$ of $\mathfrak{R}$ on $\hat{\mathfrak{H}}\in\mathfrak{H}_\pi$ such that Φ in Theorem 1 is in $\hat{\mathfrak{H}}$.

By (4.30) with μ and ν exchanged, we have

$$\pi(A_2(\mu/\nu))\Psi=\hat{\pi}(A_2(\mu/\nu))\Psi=y_0\in\mathfrak{H}_\pi.$$

Since y_0 is cyclic for $\hat{\pi}(\mathfrak{R})$ on $\mathfrak{H}_1$, we have

$$E_1\hat{\mathfrak{H}}=\mathfrak{H}_1\subset\mathfrak{H}_\pi.$$

Therefore we have (1).

If $s(\mu)\geqq s(\nu)$, then $\nu_2=0$ and (2) follows from (1).

If Ψ is separating for $\pi(\mathfrak{R})$, then $s(\mu)=1$ and $s(\mu)\geqq s(\nu)$ for any ν. Hence (3) follows from (2). Q.E.D.

Theorem 4. *Let* π *be a fixed representation of* $\mathfrak{R}$ *on* $\mathfrak{H}_\pi$ *and* $x, y\in\mathfrak{H}_\pi$. *Then*

$$d_\pi(\omega_x, \omega_y)=d(\omega_x, \omega_y) \tag{5.4}$$

and there exist Ψ *and* Φ *in* $\mathfrak{H}_\pi$ *such that*

$$\omega_\Psi=\omega_x,\ \omega_\Phi=\omega_y,\ d(\omega_x, \omega_y)=\|\Psi-\Phi\|.$$

Proof. Let $\mu=\omega_x$, $\nu=\omega_y$ and E_1 be the projection on the closure of

$$\pi(\mathfrak{R})\pi(A_2(\mu/\nu))x$$

and E_1' be the projection on the closure of

$$\pi(\mathfrak{R})\pi(A_2(\nu/\mu))y.$$

Since $\pi(\mathfrak{R})$ on $E_1\mathfrak{H}_\pi$ and $E_1'\mathfrak{H}_\pi$ are unitarily equivalent by the uniqueness in Theorem 1 (4), there exists a partial isometry $u\in\pi(\mathfrak{R})'$ such that

(5.5) $$u^*u=E_1,\ uu^*=E_1',$$

(5.6) $$(x,\ u^*y)=\omega_x(A_2(\nu/\mu))$$

There exist a central projection F and partial isometries $u_1,\ u_2\in\pi(\mathfrak{R})'$ such that

(5.7) $$u_1^*u_1=F(1-E_1),\quad u_1u_1^*\leqq F(1-E_1'),$$

(5.8) $$u_2^*u_2\leqq(1-F)(1-E_1),\ u_2u_2^*=(1-F)(1-E_1').$$

We set

$$\Psi=F(u_1+u)x+(1-F)x$$

$$\Phi=Fy+(1-F)(u_2^*+u^*)y.$$

Since F is a central projection, we have

$$\omega_\Psi=\omega_{x_F}+\omega_{(1-F)x},\ x_F\equiv F(u_1+u)x.$$

Since $E_1'u=u$, $(1-E_1')u_1=u_1$, we have $u^*u_1=u_1^*u=0$. Therefore

$$F(u_1+u)^*(u_1+u)=F(u_1^*u_1+u^*u)=F.$$

Since $u_1+u\in\pi(\mathfrak{R})'$, we have

$$\omega_{x_F}=\omega_{Fx},\ \omega_\Psi=\omega_{Fx}+\omega_{(1-F)x}=\omega_x.$$

Similarly, we have

$$\omega_\Phi=\omega_y.$$

Since

$$\omega_{u_1x}=\omega_{F(1-E_1)x}\leqq\omega_{(1-E_1)x}=\omega_x-\omega_{E_1x}=\mu_2(x),$$

we obtain

$$s(u_1x)\leqq\pi(s(\mu_2))\perp s(\nu).$$

Hence

$$(Fu_1x,\ Fy)=(F\pi(s(\mu_2))u_1x,\ Fy)=(Fu_1x,\ F\pi(s(\mu_2))y)$$
$$=0.$$

Similarly

$$((1-F)x,\ (1-F)u_2^*y)=0.$$

Therefore

$$(\Psi,\ \Phi)=(Fux,\ Fy)+((1-F)x,\ (1-F)u^*y)=(x,\ u^*y)$$
$$=\omega_x(A_2(\nu/\mu)). \qquad \text{Q.E.D.}$$

Remark. By [2], $d(\omega, \omega')^2 \leqq \|\omega-\omega'\|$. Using (5.4), we then have

$$d_\pi(\omega_x,\ \omega_y)^2 \leqq \|\omega_x-\omega_y\|.$$

The remark at the end of [1] is thus incorrect. The counterexample mentioned there was a counter-example only to the method of [1].

§6. Discussions

If $\mu \geqq \nu$, then we can obtain $\|A_2\| \leqq 1$ as follows. From $\mu \geqq \nu$, we have

$$0=\mu(1-s(\mu)) \geqq \nu(1-s(\mu))$$

and hence $s(\mu) \geqq s(\nu)$. Thus $\nu_2=0$, $s_\nu^\mu=s(\nu)$ and

$$\nu_1(Q)=\nu(Q)=\mu(A_2QA_2). \tag{6.1}$$

Let $b>1$. Then

$$\nu(1-E_b^2)=\mu((1-E_b^2)A_2^2) \geqq b^2\mu(1-E_b^2).$$
$$\geqq b^2\nu(1-E_b^2).$$

Therefore $\nu(1-E_b^2)=0$ and hence $\mu(1-E_b^2)=0$. This implies $E_b^2 \geqq s(\mu)$. Since $s_\nu^\mu A_2=A_2$ and $s(\mu) \geqq s_\nu^\mu$, we have $(1-E_b^2)A_2=0$ and hence $\|A_2\|=1$. From (6.1), we see that A_2 is the same as Sakai's t_0.

As for A_1, we have for $Q_1, Q_2 \in \pi_1(\mathfrak{R})'$

$$\begin{aligned}(Q_1x_0, \pi_1(A_2)Q_2x_0) &= (Q_1x_0, Q_2\pi_1(A_2)\pi_1(s_\nu^\mu)x_0) \\ &= (Q_1x_0, Q_2y_0) = (Q_1x_0, Q_2s(x_0)y_0) \\ &= (Q_1x_0, Q_2z) = (Q_1x_0, \pi_1(A_1)Q_2x_0).\end{aligned}$$

Therefore we have, from $s(x_0)\pi_1(A_1) = \pi_1(A_1)$,

$$\pi_1(A_1) = s(x_0)\pi_1(A_2)s(x_0) = \pi_1(s_\mu^\nu A_2 s_\mu^\nu)$$

and hence

$$A_1 = s_\mu^\nu A_2 s_\mu^\nu.$$

In particular $||A_1|| \leqq 1$.

If $\mathfrak{R}$ is commutative, $\mu, \nu \in S(\mathfrak{R})$ are measures on the spectrum of $\mathfrak{R}$ and $A_1 = A_2$ is the square root of the Radon-Nikodym derivative. The decomposition $\nu = \nu_1 + \nu_2$ is the decomposition of the measure ν into absolutely continuous and singular parts relative to the measure μ.

From proof of Theorem 1, it is seen that $\pi_\mu(A_1)$ is unique on $(1 - s(x_0))\mathfrak{H}_1 + \mathfrak{D}$. We would obtain the uniqueness of A_1 if $\mathfrak{D}$ is the core of A_0 in Lemma 3.

As for the appearance of two operators A_1 and A_2, we have the following result.

Theorem 5. *A_1 and A_2 coincide if and only if $s(\mu_1)$ commutes with $s(\nu_1)$. If $s(\mu_1)$ and $s(\nu_1)$ commute, then $s_\mu^\nu = s_\nu^\mu = s(\mu_1)s(\nu_1)$.*

Proof. From the construction of A_1 and A_2 in the proof of Theorem 1, it is clear that $A_1 = A_2$ if and only if $s_\mu^\nu = s_\nu^\mu$.

If $s(\mu_1)$ and $s(\nu_1)$ commute, then obviously $s_\mu^\nu = s_\nu^\mu = s(\mu_1)s(\nu_1)$. If $s(\mu_1)$ does not commute with $s(\nu_1)$ then $[s_\mu^\nu, s(\mu_1)] = 0$ while $[s(\mu_1), s_\nu^\mu] = [s(\mu_1), s(\nu_1)] \neq 0$ and hence $s_\mu^\nu \neq s_\nu^\mu$. [Note that $s_\nu^\mu = s(\nu_1) - s(\nu_1) \wedge (1 - s(\mu)) = s(\nu_1) - s(\nu_1) \wedge (1 - s(\mu_1))$.] Q.E.D.

If $\mu \geqq \nu$, then $[s(\mu), s(\nu)] = 0$. The following example gives the case where $\mu \geqq \nu$ and $[s(\mu_1), s(\nu_1)] \neq 0$.

Example. Let $\mathfrak{R}=\mathscr{B}(\mathfrak{H})$ and p and q be mutually orthogonal unit vectors in $\mathfrak{H}$. Let

$$\mu=\omega_p+\omega_{p+q},\ \nu=\frac{1}{2}\omega_q. \tag{6.2}$$

First we prove $\mu \geqq \nu$. For $Q \geqq 0$, $Q \in \mathfrak{R}$,

$$\begin{aligned}\mu(Q)&=(p, Qp)+(p+q,\ Q(p+q))\\ &=2(p,\ Qp)+(q, Qq)+2\mathrm{Re}(p,\ Qq).\end{aligned}$$

Since

$$\begin{aligned}2\,|(p, Qq)| &\leqq 2\,(p, Qp)^{\frac{1}{2}}(q,\ Qq)^{\frac{1}{2}}\\ &\leqq 2\,(p, Qp)+\frac{1}{2}(q, Qq),\end{aligned}$$

we obtain

$$\mu(Q) \geqq \frac{1}{2}(q, Qq)=\nu(Q).$$

Next, we see that any ρ satisfying $\mu \geqq \rho$ and $s(\rho) \perp s(\nu)$ must be proportional to ω_p because $\mu \geqq \rho$ implies $s(\rho) \leqq s(p)+s(q)$ and $(1-s(\nu)) \wedge (s(p)+s(q))=s(p)$ due to $s(\nu)=s(q)$. Since $\mu \geqq \omega_p$ and $\mu(Q)=\omega_p(Q) \neq 0$ if $Q=s(p-q)$, we see that $\mu_2=\omega_p$ and hence $\mu_1=\omega_{p+q}$. Therefore $s(\mu_1)=s(p+q)$ does not commute with $s(\nu_1)=s(\nu)=s(q)$.

In this example $s^{\mu}_{\nu}=s(q)$, $s^{\nu}_{\mu}=s(p+q)$ and $A_1=2^{-3/2}s(p+q)$, $A_2=2^{-1/2}s(q)$.

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References

[1] Araki, H., *Publ. RIMS Kyoto Univ.* **6** (1970/71), 477-487.
[2] Bures, D.J.C., *Trans. Amer. Math. Soc.* **135** (1969), 199-212
[3] Dixmier, J., *Les Algèbres d'Opérateurs dans l'Espace Hilbertien*, 2nd ed., Gauthier Villars, Paris, 1969.
[4] Dye, H.A., *Trans. Amer. Math. Soc.* **70** (1952), 243-280.
[5] Sakai, S., *Bull. Amer. Math. Soc.* **71** (1965), 149-151.
[6] Sakai, S., *C*-Algebras and W*-Algebras*, Springer-Verlag, New York, Heidelberg, Berlin, 1971.
[7] Segal, I.E., *Ann. of Math.* **57** (1953), 401-457.
[8] Takesaki, M., *Tomita's Theory of Modular Hilbert Algebras and Its Applications*, Springer-Verlag, Berlin, Heidelberg, New York, 1970.

Note Added in Proof. Professor J. Dixmier has pointed out that $1-E$ in Lemma 4 is the support of the normal state $(\Psi, Q\Phi)$ and hence its existence is well-known.