

The Green Function for $\sum_{n=1}^3 \left(i \frac{\partial}{\partial x_n} - b_n \right)^2 u = \lambda u$ in the Exterior Domain and Its Application to the Eigenfunction Expansion

By

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1. Introduction

Recently exterior boundary value problems for $-\Delta - \lambda$ have been studied by many authors. Ikebe [3] and Mizohata [5], [6] have constructed and investigated the Green function for $-\Delta - \lambda$ for exterior problems in the 3-dimensional Euclidean space R^3 . Ikebe [3] has applied his Green function to obtain the eigenfunction expansion associated with $-\Delta$ and the Schrödinger operator $-\Delta + q$ in the exterior domain, and also Shizuta [9] has obtained the same result for $-\Delta$, employing Mizohata's Green function.

Let Ω be an open unbounded domain $\subset R^3$ exterior to a closed bounded surface $\partial\Omega$ of class C^2 , and ω an open bounded domain interior to $\partial\Omega$. By $C_0^n(\Omega)$ we shall denote the class of functions each of which belongs to $C^n(\Omega)$ and vanishes identically outside a big ball. Let us define

$$\begin{aligned} (1.1) \quad L_\kappa u(x) &= \sum_{n=1}^3 \left(i \frac{\partial}{\partial x_n} - b_n(x) \right)^2 u(x) - \kappa^2 u(x) \\ &= -\Delta u(x) - i \sum_{n=1}^3 \left(\frac{\partial b_n(x) u(x)}{\partial x_n} + b_n(x) \frac{\partial u(x)}{\partial x_n} \right) \\ &\quad + \sum_{n=1}^3 b_n^2(x) u(x) - \kappa^2 u(x) \end{aligned}$$

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where

i) $b_n(x)$, for $n=1, 2, 3$, is real-valued bounded and, belongs to $C_0^1(\Omega)$,

ii) κ is a complex constant with $\text{Im } \kappa \geq 0$.

We consider the following exterior Dirichlet problem for L_κ :

(EDP) Let $f(x)$ be integrable over Ω and Hölder continuous in $\bar{\Omega} = \partial\Omega \cup \Omega$.

Find a function $u(x)$ which satisfies the following conditions:

i) $u(x)$ is continuous in $\bar{\Omega}$ and twice continuously differentiable in Ω ,

ii) $u(x)$ satisfies the equation

$$(1.2) \quad L_\kappa u(x) = f(x) \text{ in } \Omega,$$

iii) $u(x)$ satisfies the boundary condition $u|_{\partial\Omega} = 0$.

In the following section first we shall construct the Green function for L_κ by use of Ikebe and Mizohata's Green function for $-\Delta - \kappa^2$ and show the existence of the solution of the above-mentioned (EDP). Next we shall employ our Green function to obtain the generalized eigenfunctions of L_κ and to expand an arbitrary function of $L^2(\Omega)$ in terms of the generalized eigenfunctions.

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2. The Uniqueness of the Solution of (EDP)

In this section we shall impose an additional condition at infinity on the solution $u(x)$ of (EDP) called the radiation condition, i.e.

$$(2.1) \quad u(x) = O(1/r), \quad \frac{\partial u(x)}{\partial r} - i\kappa u(x) = o(1/r) \quad (r = |x| \rightarrow \infty).$$

Then we get the uniqueness theorem of (EDP).

Lemma 1. *Let $f(x) = 0$ in (EDP), then any solution which satisfies the radiation condition (2.1) vanishes identically in Ω .*

Proof. We shall verify this lemma by the same method as Werner has employed in [10] (Satz 1, Corollar 1 zu Satz 1 and Corollar 2 zu Satz 1). Any solution $u(x)$ of (EDP) possesses the normal derivative on $\partial\Omega$, because the first derivatives of $u(x)$ are continuous in $\bar{\Omega}$ (see [6]). Therefore we can apply Green's formula to $u(x)$ in $\Omega_R = B(R) \cap \bar{\Omega}$ where $B(R)$ denotes the ball of radius R about the origin which contains $\partial\Omega$. We choose R so large that each $b_n(x)$ vanishes in $B(R)^c$, the complement of $B(R)$ in R^3 . Thus we have

$$\begin{aligned}
 (2.2) \quad \int_{\Omega_R} L_n u \cdot \bar{u} dx &= \sum_{n=1}^3 \int_{\Omega_R} \left| i \frac{\partial u}{\partial x_n} - b_n u \right|^2 dx - \kappa^2 \int_{\Omega_R} |u|^2 dx \\
 &\quad - \int_{S(R)} \left\{ \frac{\partial u}{\partial r} + i u \sum_{n=1}^3 b_n \cos(r, x_n) \right\} \bar{u} dS \\
 &\quad - \int_{\partial\Omega} \left\{ \frac{\partial u}{\partial \nu} - i u \sum_{n=1}^3 b_n \cos(\nu, x_n) \right\} \bar{u} dS \\
 &= \sum_{n=1}^3 \int_{\Omega_R} \left| i \frac{\partial u}{\partial x_n} - b_n u \right|^2 dx - \kappa^2 \int_{\Omega_R} |u|^2 dx \\
 &\quad - \int_{S(R)} \frac{\partial u}{\partial r} \bar{u} dS = 0.
 \end{aligned}$$

Here $S(R)$ is the surface of $B(R)$, ν and r denote the outer normal unit vectors to $\partial\Omega$ and $S(R)$, respectively, and $\cos(\nu, x_n)$ and $\cos(r, x_n)$ are the direction cosines.

Let $\kappa = \alpha + i\beta$, where α and β are real. We consider the following three cases.

i) The case $\alpha \neq 0$ and $\beta > 0$. By the radiation condition (2.1), we have

$$(2.3) \quad \int_{S(r)} \frac{\partial u}{\partial r} \bar{u} dS = (\alpha i - \beta) \int_{S(r)} |u|^2 dS + o(1).$$

From (2.2) and (2.3)

$$\begin{aligned}
 (2.4) \quad \sum_{n=1}^3 \int_{\Omega_R} \left| i \frac{\partial u}{\partial x_n} - b_n u \right|^2 dx - (\alpha^2 - \beta^2 + 2\alpha\beta i) \int_{\Omega_R} |u|^2 dx \\
 = (\alpha i - \beta) \int_{S(r)} |u|^2 dS + o(1).
 \end{aligned}$$

Taking the imaginary part of (2.4), we have

$$2\beta \int_{\Omega_R} |u|^2 dx = - \int_{S(r)} |u|^2 dS + o(1) \leq o(1).$$

Thus we get

$$\int_{\Omega} |u|^2 dx = 0.$$

Consequently we have $u(x) = 0$ for all $x \in \bar{\Omega}$.

ii) The case $\alpha \neq 0$ and $\beta = 0$. In $B(R)^C$ L_κ becomes $-\Delta - \alpha^2$, so that $u(x) = 0$ for all $x \in B(R)^C$ by Rellich's uniqueness theorem (see [8]). Thus we have $u(x) = 0$ for all $x \in \bar{\Omega}$ by use of the unique continuation theorem for the solution of an elliptic partial differential equation of second order (see [6]).

iii) The case $\alpha = 0$ and $\beta \geq 0$. For $\alpha = \beta = 0$, from (2.1)

$$(2.5) \quad u(x) = O(1/r), \quad \frac{\partial u}{\partial r} = o(1/r).$$

For $\alpha = 0$ and $\beta > 0$, we get $i\kappa = -\beta < 0$, and the representation

$$(2.6) \quad u(x) = \frac{1}{4\pi} \int_{S(R)} \left\{ u(y) \frac{\partial}{\partial \nu_y} \frac{e^{-\beta|x-y|}}{|x-y|} - \frac{e^{-\beta|x-y|}}{|x-y|} \frac{\partial}{\partial \nu} u(y) \right\} dS_y$$

can easily be shown for $u(x)$ in $B(R)^C$, applying Green's formula to $u(x)$ and $\frac{e^{-\beta|x-y|}}{|x-y|}$. From (2.6)

$$(2.7) \quad u(x) = o(1/r), \quad \frac{\partial u}{\partial r} = o(1/r).$$

In both cases from (2.5) and from (2.7), we get

$$(2.8) \quad \int_{S(r)} \frac{\partial u}{\partial r} \bar{u} dS = o(1).$$

Combining (2.2) and (2.8), and then taking the real part, we have

$$0 = \sum_{n=1}^3 \int_{\Omega_r} \left| i \frac{\partial u}{\partial x_n} - b_n u \right|^2 dx + \beta^2 \int_{\Omega_r} |u|^2 dx + o(1)$$

$$\geq \sum_{n=1}^3 \int_{\Omega_r} \left| i \frac{\partial u}{\partial x_n} - b_n u \right|^2 dx + o(1).$$

Hence we have for $n=1, 2, 3$,

$$i \frac{\partial u}{\partial x_n} - b_n u = 0 \quad \text{for all } x \in \Omega.$$

Therefore from (2.1),

$$u(x) = 0 \quad \text{for all } x \in B(R)^c.$$

Thus by the unique continuation theorem we have $u(x) = 0$ for all $x \in \bar{\Omega}$. Now the proof of our lemma is complete.

From Lemma 1 immediately follows the

Theorem 1. *If a solution of (EDP) which satisfies the radiation condition (2.1) exists, the solution is unique.*

3. The Integral Equation Connected with (EDP)

In this section we shall show the existence of solutions of the integral equation connected with (EDP) and the resolvent kernel of the integral equation.

Let $G(x, y, \kappa)$, for x and $y \in \bar{\Omega}$, $\kappa \in \bar{C}^+$ where $C^+ = \{\kappa \mid \text{Im } \kappa > 0\}$, and $\bar{\quad}$ denotes closure, be the Green function of $-\Delta - \kappa^2$ obtained by Ikebe and Mizohata. The properties of $G(x, y, \kappa)$ are as follows.

Theorem 2. i) *Let $\kappa \in C^+$. Then*

$$(3.1) \quad \|G(\cdot, y, \kappa)\|_{L^1(\Omega)} < C, \quad \|G(\cdot, y, \kappa)\|_{L^2(\Omega)} < C, \quad \left\| \frac{\partial G(x, \cdot, \kappa)}{\partial x_n} \right\|_{L^1(\Omega)} < C$$

for $n=1, 2, 3$, where C depends only on κ .

ii) *Let $\kappa \in \bar{C}^+$. Then*

$$(3.2) \quad |G(x, y, \kappa)| < \frac{C}{|x-y|}, \quad \left| \frac{\partial G(x, y, \kappa)}{\partial x_n} \right| < \frac{C}{|x-y|^2}$$

for $n=1, 2, 3$,

where in the case $\text{Im } \kappa > 0$ the constant C depends only on κ and in the case $\text{Im } \kappa = 0$ C depends only on κ where (x, y) lies in a compact set of $\Omega_x \times \Omega_y$.

iii) $G(\cdot, y, \kappa)$ satisfies the radiation condition.

iv) $G(\cdot, y, \kappa)$ satisfies the null boundary condition, i.e. $G(\cdot, y, \kappa)|_{\partial\Omega} = 0$.

v) Let $\kappa \in \bar{C}^+$. Then $\frac{\partial G(x, y, \kappa)}{\partial \nu_x}$ exists and is continuous in $x \in \partial\Omega$, $y \in \Omega$ and $\kappa \in \bar{C}^+$, where ν_x denotes the outer normal to $\partial\Omega$ at x .

vi) Let $x \neq y$ and $\kappa \in \bar{C}^+$. Then $G(x, y, \kappa)$ is continuous in $\Omega_x \times \Omega_y \times \bar{C}^+$.

vii) $G(x, y, \kappa) = G(y, x, \kappa)$, $\overline{G(x, y, -\bar{\kappa})} = G(x, y, \kappa)$.

viii) We put $u(x) = \int_{\Omega} G(x, y, \kappa) f(y) dy$.

(a) Let $\text{Im } \kappa > 0$. If $f(y)$ is Hölder continuous in $\bar{\Omega}$, and is bounded or integrable over $\bar{\Omega}$. Then $u(x)$ satisfies

$$(-\Delta - \kappa^2)u(x) = f(x) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0,$$

where Δ is taken in the sense of usual differentiation.

(b) Let $\text{Im } \kappa = 0$. If $f(y)$ is Hölder continuous in $\bar{\Omega}$ and vanishes identically outside a big ball, then in the sense of usual differentiation $u(x)$ satisfies

$$(-\Delta - \kappa^2)u(x) = f(x) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0.$$

(c) Let $\text{Im } \kappa > 0$. If $f(y) \in L^2(\Omega)$, then $u(x)$ is bounded and continuous in Ω and $u(x) \in H_2(\Omega)$, and satisfies

$$(-\Delta - \kappa^2)u = f, \quad u|_{\partial\Omega} = 0.$$

Here $H_2(\Omega)$ denotes the class of $L^2(\Omega)$ -functions with distribution derivatives in $L^2(\Omega)$ up to the second order inclusive, and Δ is taken in the distribution sense.

The above theorem is proved in Ikebe [3], and Mizohata [5] and [6], or can easily be shown from their results.

We define the following operator:

$$(3.3) \quad l \cdot f(x) = i \sum_{n=1}^3 \left(\frac{\partial b_n f}{\partial x_n} + b_n \frac{\partial f}{\partial x_n} \right) - \sum_{n=1}^3 b_n^2 f,$$

where $b_n(x)$ are the coefficients of L_κ . We put

$$(3.4) \quad K(x, y, \kappa) = l \cdot G(x, y, \kappa),$$

where l operates on the variable x . Since $b_n(x) \in C_0^1(x)$ for $n=1, 2, 3$, $K(x, y, \kappa)$ vanishes identically for $x \in (R)^c$ with sufficiently large R and from (3.2) we have

$$(3.5) \quad |K(x, y, \kappa)| < \frac{C}{|x-y|^2},$$

where the constant C satisfies the same condition as C in (3.2). We put

$$(3.6) \quad u(x) = \int_{\Omega} G(x, y, \kappa) f(y) dy.$$

As $G(x, y, \kappa)$ is the Green function of $-\Delta - \kappa^2$, if $f(y)$ satisfies suitable conditions, then we obtain

$$L_\kappa u(x) = (-\Delta - \kappa^2)u(x) - l \cdot u(x) = f(x) - \int_{\Omega} K(x, y, \kappa) f(y) dy.$$

Here we shall consider the following integral equation

$$(3.7) \quad g(x) = f(x) - \int_{\Omega} K(x, y, \kappa) f(y) dy, \quad (\kappa \in \bar{C}^+).$$

To the integral kernel $K(x, y, \kappa)$ which is defined by (3.4), we may apply the theory of Fredholm (see [4]), so that we may solve equation (3.7) for $\kappa \in \bar{C}^+$.

Lemma 2. *If $g(x) \in C_0(\Omega)$, then equation (3.7) has a unique solution.*

Proof. As $K(x, y, \kappa)$ and $g(x)$ vanish identically in $B(R)^c$ for sufficiently large R , the integration domain is essentially a bounded domain

$B(R)$. Therefore we can use the theory of Fredholm for continuous functions defined on $B(R)$. Let us introduce the homogeneous equation for (3.7)

$$(3.8) \quad h(x) = \int_{\Omega} K(x, y, \kappa) h(y) dy.$$

If we show that the solution of (3.8) vanishes identically, then we can assert the lemma according to the Fredholm theory. Now we assume that the solution of (3.8) exists. Then since $K(x, y, \kappa) = 0$ for $x \in B(R)^c$, the solution $h(x)$ also vanishes identically in $B(R)^c$. We put

$$v(x) = \int_{\Omega} G(x, y, \kappa) h(y) dy = \int_{B(R)} G(x, y, \kappa) h(y) dy.$$

By theorem 2 we get $(-\Delta - \kappa^2)v = h$, $v|_{\partial\Omega} = 0$ and $v(x)$ satisfies the radiation condition, and we have

$$L_{\kappa}v = (-\Delta - \kappa^2 - l)v = h(x) - \int_{\Omega} K(x, y, \kappa) h(y) dy = 0.$$

Consequently by lemma 1 we get $v(x) = 0$ everywhere in $\bar{\Omega}$. Hence we get

$$\begin{aligned} 0 &= l_x \cdot \int_{\Omega} (x, y, \kappa) h(y) dy = \int_{\Omega} l_x \cdot G(x, y, \kappa) h(y) dy = \int_{\Omega} K(x, y, \kappa) h(y) dy \\ &= h(x). \end{aligned}$$

Therefore $h(x) = 0$ everywhere in Ω . Thus the proof of the lemma is complete.

Let $R(x, y, \kappa)$ be the resolvent kernel of the integral equation (3.7). Then according to lemma 2 $R(x, y, \kappa)$ exists for an arbitrary $\kappa \in \bar{C}^+$, and can be expressed as

$$(3.9) \quad R(x, y, \kappa) = K(x, y, \kappa) + K^{(1)}(x, y, \kappa) + M(x, y, \kappa)/N(\kappa),$$

where $K^{(1)}(x, y, \kappa)$ is the iterated kernel of $K(x, y, \kappa)$, and $M(x, y, \kappa)$ and $N(\kappa)$ are analytic functions of $\kappa \in C^+$.

The properties of $R(x, y, \kappa)$ are summarized as follows.

Lemma 3. i) $R(x, y, \kappa)$ is a continuous function of (x, y, κ) in $\Omega_x \times \Omega_y \times \bar{C}^+$ unless $x = y$ and an analytic function of κ in C^+ .

ii)
$$|R(x, y, \kappa)| \leq \frac{C}{|x - y|^2},$$

where the constant C satisfies the same condition as C in (3.2).

iii) $R(x, y, \kappa)$ satisfies the following resolvent equation:

$$K(x, y, \kappa) = R(x, y, \kappa) - \int_{\Omega} K(x, z, \kappa) R(z, y, \kappa) dz, \tag{3.10}$$

$$K(x, y, \kappa) = R(x, y, \kappa) - \int_{\Omega} K(z, y, \kappa) R(x, z, \kappa) dz.$$

iv) $R(\cdot, y, \kappa)$ vanishes identically outside a big ball which contains $\partial\Omega$.

v) Let $\kappa \in C^+$. Then $\|R(x, \cdot, \kappa)\|_{L^1(\Omega)} < C$, where C depends not on x but only on κ .

Proof. i) and ii) are proved from theorem 2, (3.5) and (3.9). iii) is the fundamental properties of the resolvent kernel of the Fredholm integral equation. iv) is easily shown from the above equation (3.10). From (3.1) and (3.4) we have

$$\|K(x, \cdot, \kappa)\|_{L^1(\Omega)} < C,$$

where C depends not on x but only on κ . So in consideration of the boundedness of $B(R)$ we have

$$\int_{B(R)} dz \int_{\Omega} |K(z, y, \kappa)| |R(x, z, \kappa)| dy \leq C_1 \int_{B(R)} |R(x, z, \kappa)| dz \leq C_2,$$

where C_1 and C_2 depend only on κ . Hence from Fubini's theorem and (3.10) we have v) as follows:

$$\begin{aligned} \int_{\Omega} |R(x, y, \kappa)| dy &\leq \int_{\Omega} |K(x, y, \kappa)| dy \\ &+ \int_{\Omega} dy \int_{B(R)} |K(z, y, \kappa)| |R(x, z, \kappa)| dz \end{aligned}$$

$$\begin{aligned}
&= \int_{\Omega} |K(x, y, \kappa)| dy \\
&\quad + \int_{B(R)} |R(x, z, \kappa)| dz \int_{\Omega} |K(z, y, \kappa)| dy < C.
\end{aligned}$$

Now the proof of our lemma is complete.

4. The Green Function for L_{κ} and Existence of Solutions for (EDP)

In this section we shall define the Green function for L_{κ} for (EDP) by use of $G(x, y, \kappa)$ and $R(x, y, \kappa)$, and construct solutions of (EDP).

In what follows we shall state estimates for some integral kernels as lemma 4 of which a proof is found in Chapter II of [4].

Lemma 4. *Let D be an open bounded domain $\subset R^3$ and \bar{D} its closure.*

i) We put $f(x) = \int_D K(x, y)g(y)dy$.

Here $K(x, y)$ is continuously differentiable on $(x, y) \in \bar{D} \times \bar{D}$ unless $x = y$.

If $g \in L^1$, and $|K(x, y)| \leq \frac{C}{|x-y|}$ and $\left| \frac{\partial K(x, y)}{\partial x_n} \right| \leq \frac{C}{|x-y|^2}$ for $n = 1, 2, 3$, then

$$f(x) \in C^1(\bar{D}) \text{ and } \frac{\partial f(x)}{\partial x_n} = \int_D \frac{\partial K(x, y)}{\partial x_n} g(y) dy \quad \text{for } n = 1, 2, 3.$$

ii) We put $K_3(x, y) = \int_D K_1(x, z)K_2(z, y)dz$.

Here $K_1(x, y)$ and $K_2(x, y)$ are continuous on (x, y) in $\bar{D} \times \bar{D}$ unless $x = y$, and satisfy

$$|K_1(x, y)| \leq C|x-y|^{-m}, \quad |K_2(x, y)| \leq C|x-y|^{-n}, \quad \text{for } m, n = 1, 2.$$

Then $K_3(x, y)$ is continuous on (x, y) in $\bar{D} \times \bar{D}$ unless $x = y$, and

(4.1) if $m + n > 3$, $|K_3(x, y)| \leq C|x - y|^{3-m-n}$,

(4.2) if $m + n = 3$, $|K_3(x, y)| \leq C \log|x - y|$,

(4.3) if $m + n < 3$, $K_3(x, y)$ is continuous in $\bar{D} \times \bar{D}$.

Here we shall define the following function which plays the rôle of the Green function for L_κ as desired:

(4.4)
$$H(x, y, \kappa) = G(x, y, \kappa) + \int_{\Omega} G(x, z, \kappa)R(z, y, \kappa)dz.$$

From (3.2), ii) of lemma 3 and lemma 4 $H(x, y, \kappa)$ is well-defined for $(x, y) \in \Omega \times \Omega$ and $\kappa \in \bar{C}^+$. We put

(4.5)
$$U(x, y, \kappa) = \int_{\Omega} G(x, z, \kappa) R(z, y, \kappa) dz$$

$$= \int_{B(R)} G(x, z, \kappa)R(z, y, \kappa)dz,$$

where R is sufficiently large. Then we get

$$H(x, y, \kappa) = G(x, y, \kappa) + U(x, y, \kappa).$$

We shall exhibit the properties of $H(x, y, \kappa)$ as follows.

Lemma 5. i) If $x \neq y$, then

$$L_\kappa H(x, y, \kappa) = 0,$$

where L_κ operates on the variable x .

ii) $H(\cdot, y, \kappa)|_{\partial\Omega} = 0.$

iii) If $x \neq y$, then $H(x, y, \kappa)$ is continuous in $\Omega_x \times \Omega_y \times \bar{C}^+$.

iv) $|H(x, y, \kappa)| \leq \frac{C}{|x - y|}$, $\left| \frac{\partial H(x, y, \kappa)}{\partial x_n} \right| \leq \frac{C}{|x - y|^2}$

for $n = 1, 2, 3$,

where C satisfies the same condition as C in (3.2).

v) $\frac{\partial H(x, y, \kappa)}{\partial \nu_x}$ exists and is continuous in $x \in \partial\Omega$ and in $y \in \Omega$.

- vi) $H(\cdot, x, y)$ satisfies the radiation condition (2.1).
 vii) Let $\kappa \in C^+$. Then

$$\|H(\cdot, y, \kappa)\|_{L^1(\mathcal{D})} \leq C, \quad \|H(\cdot, y, \kappa)\|_{L^2(\mathcal{D})} \leq C,$$

where C depends not on y but only on κ .

viii) $H(x, y, \kappa) = \overline{H(y, x, -\bar{\kappa})}$,

and moreover

$$(4.6) \quad \int_{\mathcal{D}} G(x, z, \kappa) R(z, y, \kappa) dz = \int_{\mathcal{D}} \overline{G(y, z, -\bar{\kappa}) R(z, x, -\bar{\kappa})} dz \\ = \int_{\mathcal{D}} G(z, y, \kappa) \overline{R(z, x, -\bar{\kappa})} dz.$$

Proof of i). From i) of lemma 4

$$l_x \int_{\mathcal{D}} G(x, z, \kappa) R(z, y, \kappa) dz = \int_{\mathcal{D}} l_x G(x, z, \kappa) R(z, y, \kappa) dz \\ = \int_{\mathcal{D}} K(x, z, \kappa) R(z, y, \kappa) dz.$$

And since $G(x, y, \kappa)$ is the Green function for (EDP) for $-\Delta - \kappa^2$,

$$(-\Delta - \kappa^2)_x G(x, y, \kappa) = 0, \text{ and } (-\Delta - \kappa^2)_x \int_{\mathcal{D}} G(x, z, \kappa) R(z, y, \kappa) dz \\ = R(x, y, \kappa).$$

Therefore we get

$$L_\kappa H(x, y, \kappa) = (-\Delta - \kappa^2)_x G(x, y, \kappa)_x \\ + (-\Delta - \kappa^2)_x \int_{\mathcal{D}} G(x, z, \kappa) R(z, y, \kappa) dz - l_x G(x, y, \kappa) \\ - l_x \int_{\mathcal{D}} G(x, z, \kappa) R(z, y, \kappa) dz \\ = R(x, y, \kappa) - K(x, y, \kappa) - \int_{\mathcal{D}} K(x, z, \kappa) R(z, y, \kappa) dz.$$

Thus $L_\kappa H(x, y, \kappa) = 0$ follows from (3.10).

Proof of ii). The assertion is immediate from the definition of $H(x, y, \kappa)$.

Proof of iii) and iv). In view of theorem 2 it is enough to consider $U(x, y, \kappa)$. From theorem 2 and lemma 3 $G(x, y, \kappa)$ and $R(x, y, \kappa)$ are continuous in $\Omega_x \times \Omega_y \times \bar{C}^+$ unless $x = y$, and

$$|G(x, y, \kappa)| \leq \frac{C}{|x - y|}, \quad |R(x, y, \kappa)| \leq \frac{C}{|x - y|^2}.$$

By means of (4.2) we get the desired results in consideration of the boundedness of $B(R)$.

Proof of v). Clear from theorem 2 and i) of lemma 4.

Proof of vi). That $U(\cdot, y, \kappa)$ satisfies the radiation condition is clear from that $G(\cdot, z, \kappa)$ satisfies the same condition and $B(R)$ is bounded. Hence $H(\cdot, y, \kappa)$ satisfies the radiation condition.

Proof of vii). First we shall show $U(x, y, \kappa)$ is integrable over Ω on x . By means of Fubini's theorem we have

$$\begin{aligned} \int_{\Omega} |U(x, y, \kappa)| dx &\leq \int_{\Omega} dx \int_{B(R)} |G(x, z, \kappa)| |R(z, y, \kappa)| dz \\ &= \int_{B(R)} |R(z, y, \kappa)| dz \int_{\Omega} |G(x, z, \kappa)| dx \\ &\leq C_1 \|G(\cdot, z, \kappa)\|_{L^1(\Omega)} \int_{B(R)} \frac{dz}{|z - y|^2} \leq C_2, \end{aligned}$$

where C_1 and C_2 depend only on κ . Thus we have $H(\cdot, y, \kappa) \in L^1(\Omega)$.

Next we shall show $H(x, y, \kappa)$ is square integrable over Ω on x . By means of Schwarz inequality we have

$$\begin{aligned} (4.7) \quad \int_{\Omega} |U(x, y, \kappa)|^2 dx &\leq \int_{\Omega} dx \left\{ \int_{B(R)} |G(x, z, \kappa)|^2 |R(z, y, \kappa)| dz \right. \\ &\quad \left. \times \int_{B(R)} |R(z, y, \kappa)| dz \right\}. \end{aligned}$$

From (3.1) and ii) of lemma 3 we can apply Fubini's theorem to (4.7), and we have

$$\begin{aligned} \int_{\Omega} |U(x, y, \kappa)|^2 dx &\leq \int_{B(R)} |R(z, y, \kappa)| dz \left\{ \int_{B(R)} |R(z, y, \kappa)| dz \right. \\ &\quad \left. \times \int_{\Omega} |G(x, z, \kappa)|^2 dx \right\} \\ &\leq C_3 \left\{ \int_{B(R)} |R(z, y, \kappa)| dz \right\}^2 \leq C_4, \end{aligned}$$

where C_3 and C_4 depend only on κ . Thus we have obtained the result as desired.

Proof of viii). From the already established i) we have

$$(4.8) \quad H(z, y, -\bar{\kappa})L_{\kappa}H(z, x, \kappa) - H(z, x, \kappa)L_{-\bar{\kappa}}H(z, y, -\bar{\kappa}) = 0,$$

where L_{κ} and $L_{-\bar{\kappa}}$ operate on the variable z . We integrate (4.8) with respect to z and apply Green's formula by virtue of the already established result v). The domain of integration is at first the intersection of $\bar{\Omega}$ and a large ball $B(r)$ containing $\partial\Omega$ excluding small balls of radius ε about x and y where $x, y \in \Omega$. Thus we have

$$\begin{aligned} 0 &= \int_{S(r) \cup S(x, \varepsilon) \cup S(y, \varepsilon)} \left\{ \overline{H(z, y, -\bar{\kappa})} \frac{\partial H(z, x, \kappa)}{\partial \nu} \right. \\ &\quad \left. - H(z, x, \kappa) \frac{\partial \overline{H(z, y, -\bar{\kappa})}}{\partial \nu} \right\} dS \\ &\quad + i \sum_{n=1}^3 \int_{S(x, \varepsilon) \cup S(y, \varepsilon)} \{H(z, x, \kappa) + \overline{H(z, y, -\bar{\kappa})}\} b_n \cos(\nu, x_n) dS, \end{aligned}$$

where $S(r)$, $S(x, \varepsilon)$ and $S(y, \varepsilon)$ are the surface of $B(r) \cap \bar{\Omega}$ and the balls about x and y respectively, and $\frac{\partial}{\partial \nu}$ denotes differentiation with respect to the outer normal ν for $S(r)$, $S(x, \varepsilon)$ and $S(y, \varepsilon)$. Then

$$\lim_{\varepsilon \rightarrow 0} i \sum_{n=1}^3 \int_{S(x, \varepsilon) \cup S(y, \varepsilon)} \{H(z, x, \kappa) + \overline{H(z, y, -\bar{\kappa})}\} b_n \cos(\nu, x_n) ds \rightarrow 0,$$

and by use of the radiation condition

$$\lim_{r \rightarrow +\infty} \int_{S(r)} \left\{ \overline{H(z, y, -\bar{\kappa})} \frac{\partial H(z, x, \kappa)}{\partial \nu} - H(z, x, \kappa) \frac{\partial H(z, y, -\bar{\kappa})}{\partial \nu} \right\} dS \rightarrow 0,$$

and by use of the already established result iv)

$$\begin{aligned} 0 &= \lim_{\varepsilon \rightarrow 0} \frac{1}{4\pi} \int_{S(x, \varepsilon) \cup S(y, \varepsilon)} \left\{ \overline{H(z, y, -\bar{\kappa})} \frac{\partial H(z, x, \kappa)}{\partial \nu} \right. \\ &\quad \left. - H(z, x, \kappa) \frac{\partial \overline{H(z, y, -\bar{\kappa})}}{\partial \nu} \right\} dS \\ &= \overline{H(x, y, -\bar{\kappa})} - H(y, x, \kappa). \end{aligned}$$

Hence we have $H(x, y, \kappa) = \overline{H(y, x, -\bar{\kappa})}$.

Thus the proof of our lemma is complete.

$H(x, y, \kappa)$ defined in (4.4) is the Green function for L_κ for (EDP) in the sense that the function

$$(4.9) \quad u(x) = \int_{\Omega} H(x, y, \kappa) f(y) dy$$

furnishes us with the solution of (EDP) with the inhomogeneous term $f(x)$.

Theorem 3. i) *Let $\text{Im } \kappa > 0$, and $f(x)$ be Hölder continuous in $\bar{\Omega}$ and integrable over Ω . Then $u(x)$ of (4.9) is the unique solution of (EDP) with the null boundary condition and the radiation condition.*

ii)*) *Let $\text{Im } \kappa > 0$, and $f(x)$ be bounded and Hölder continuous in $\bar{\Omega}$. Then $u(x)$ of (4.9) is a solution of (EDP) with the null boundary condition on $\partial\Omega$.*

iii) *Let $\text{Im } \kappa = 0$, and $f(x)$ be Hölder continuous in $\bar{\Omega}$ and vanish outside a ball. Then $u(x)$ of (4.9) is the unique solution of (EDP) with the null boundary condition.*

*) In general $u(x)$ is not the unique solution of (EDP), for $u(x)$ does not always satisfy the radiation condition.

Proof. It is clear from the definition of $u(x)$ in every case that $u(x)$ satisfies the null boundary condition. Next we shall show $L_\kappa u = f$. From theorem 2 we have for $\text{Im } \kappa > 0$

$$(4.10) \quad \|G(x, \cdot, \kappa)\|_{L^1(\Omega)} = \|G(\cdot, x, \kappa)\|_{L^1(\Omega)} \leq C.$$

The case i). From $|R(z, y, \kappa)| < \frac{C}{|z-y|^2}$ and $f \in L^1(\Omega) \cap C(\Omega)$ we have

$$(4.11) \quad \int_{\Omega} |R(z, y, \kappa)| |f(y)| dy \leq C_1 \int_{\Omega - B(z, r)} \frac{f(y)}{|z-y|^2} dy \\ + C_2 \int_{B(z, r)} \frac{|f(y)|}{|z-y|^2} dy \\ \leq C_3 \left\{ \int_{\Omega} |f(y)| dy + \int_{B(z, r)} \frac{1}{|z-y|^2} dy \right\} \leq C_4,$$

where $B(z, r)$ is a small ball of radius r about z and $C_1, C_2, C_3,$ and C_4 depend only on κ . The case ii). From $R(z, \cdot, \kappa) \in L^1(\Omega)$ and from that f is bounded over $\bar{\Omega}$ we have

$$(4.12) \quad \int_{\Omega} |R(z, y, \kappa)| |f(y)| dy \leq C_5 \int_{\Omega} |R(z, y, \kappa)| dy \leq C_6,$$

where C_5 and C_6 depend only on κ . The case iii). From $f(x) \in C_0(\bar{\Omega})$ we have

$$(4.13) \quad \int_{\Omega} |R(z, y, \kappa)| |f(y)| dy \leq C_7 \int_{B(R)} \frac{dy}{|z-y|^2} \leq C_8,$$

where C_7 and C_8 depend only on κ .

From (4.10), (4.11), (4.12), and (4.13) using Fubini's theorem we have

$$(4.14) \quad \int_{\Omega} f(y) dy \int_{B(R)} G(x, z, \kappa) R(z, y, \kappa) dz \\ = \int_{B(R)} G(x, z, \kappa) dz \int_{\Omega} R(z, y, \kappa) f(y) dy.$$

By the same processes as (4.14) above we have

$$(4.15) \quad \int_{B(R)} K(x, z, \kappa) dz \int_{\mathcal{Q}} R(z, y, \kappa) f(y) dy \\ = \int_{\mathcal{Q}} f(y) dy \int_{B(R)} K(x, z, \kappa) R(z, y, \kappa) dz.$$

Here we shall briefly describe that $g(z) = \int_{\mathcal{Q}} R(z, y, \kappa) f(y) dy$ satisfies a Hölder condition at any point $z \in \bar{\mathcal{Q}}$. Let $B(R)$ be such that the support of $R(\cdot, y, \kappa) \subset B(R)$. First we set

$$g_1(z) = \int_{B(R)^c} R(z, y, \kappa) f(y) dy.$$

Then obviously

$$g_1(z) \in C_0^1(\bar{\mathcal{Q}}).$$

Next we set

$$g_2(z) = \int_{B(R)} R(z, y, \kappa) f(y) dy.$$

By the mean-value theorem we have

$$(4.16) \quad \left| \frac{1}{|z_1 - y|^2} - \frac{1}{|z_2 - y|^2} \right| \leq C |z_1 - z_2| \left\{ \frac{1}{|z_1 - y|^3} + \frac{1}{|z_2 - y|^3} \right\}.$$

Let h be a small number and $B(z_1, 3h)$ the small ball of radius $3h$ centered at z_1 . Furthermore let z_2 be such that $|z_1 - z_2| = h$. Then

$$|g_2(z_1) - g_2(z_2)| \leq \int_{B(R)} |f(y)| |R(z_1, y, \kappa) - R(z_2, y, \kappa)| dy \\ \leq C_1 \int_{y \in B(R), |z_1 - y| \leq 2h} \left| \frac{1}{|z_1 - y|^2} - \frac{1}{|z_2 - y|^2} \right| dy \\ + C_2 \int_{y \in B(R), |z_1 - y| \geq 2h} \left| \frac{1}{|z_1 - y|^2} - \frac{1}{|z_2 - y|^2} \right| dy.$$

From (4.16) it follows

$$|g_2(z_1) - g_2(z_2)| \leq C_3 \int_{|z_1 - y| \leq 2h} \frac{dy}{|z_1 - y|^2} + C_4 \int_{|z_2 - y| \leq 3h} \frac{dy}{|z_2 - y|^2} \\ + C_5 |z_1 - z_2| \left\{ \int_{2R \geq |z_1 - y| \geq 2h} \frac{dy}{|z_1 - y|^3} \right.$$

$$\begin{aligned}
& + \int_{4R \leq |z_2 - y| \leq h} \frac{dy}{|z_2 - y|^3} \Big\} \\
& \leq C_6 h + C_7 h \log \frac{R}{h}.
\end{aligned}$$

Hence $g_2(z)$ satisfies a Hölder condition at z_1 , and thus we can show that $g(z) = g_1(z) + g_2(z)$ satisfies a Hölder condition at any point $z \in \bar{Q}$.

Since $G(x, y, \kappa)$ is the Green function for $-\Delta - \kappa^2$ for (EDP), we have

$$\begin{aligned}
(4.17) \quad L_\kappa u(x) &= (-\Delta - \kappa^2 - I) \left\{ \int_Q G(x, y, \kappa) f(y) dy \right. \\
& \quad \left. + \int_{B(R)} G(x, z, \kappa) dz \int_Q R(z, y, \kappa) f(y) dy \right\} \\
&= f(x) - \int_Q K(x, y, \kappa) f(y) dy + \int_Q R(x, y, \kappa) f(y) dy \\
& \quad - \int_{B(R)} K(x, z, \kappa) dz \int_Q R(z, y, \kappa) f(y) dy \\
&= f(x) - \int_Q f(y) \left\{ K(x, y, \kappa) - R(x, y, \kappa) \right. \\
& \quad \left. + \int_Q K(x, z, \kappa) R(z, y, \kappa) dz \right\} dy.
\end{aligned}$$

Here we have freely interchanged the integration order, which is guaranteed by (4.14) and (4.15), and we have used viii) of theorem 2 in consideration of the Hölder continuity of $f(\cdot)$ and $\int_Q R(\cdot, y, \kappa) f(y) dy$. By the resolvent equation (3.10) the expression in $\{ \}$ of the last term of (4.17) vanishes, and thus we have $L_\kappa u = f$.

Finally we shall show the uniqueness of $u(x)$ in the cases i) and iii). From the fact that $H(\cdot, y, \kappa)$ satisfies the radiation condition and the assumption of $f(x)$ in each case, we obtain that $u(x)$ satisfies the radiation condition also. Consequently we get the uniqueness of the solution by theorem 1. Now the proof of our theorem is complete.

The following theorem shows that $H(x, y, \kappa)$ is the Green kernel of

the Green operator in the Hilbert space for L_κ for the exterior problem.

Theorem 4. *Let $\kappa \in C^+$, and $f(x)$ be square integrable over Ω . Then*

$$u(x) = \int_{\Omega} H(x, y, \kappa) f(y) dy$$

satisfies the boundary condition $u|_{\partial\Omega} = 0$ and $u \in H_2(\Omega)$,*) and satisfies the equation $L_\kappa u = f$, where L_κ is taken in the distribution sense.

Proof. $u|_{\partial\Omega} = 0$ and $L_\kappa u = f$ in the distribution sense can easily be shown. By the same procedure as the proof of vii) of lemma 5 we obtain that $\int_{\Omega} |G(x, y, \kappa)| |R(z, y, \kappa)| dz$ is square integrable in y over Ω . Therefore we have

$$\begin{aligned} u(x) &= \int_{\Omega} H(x, y, \kappa) f(y) dy = \int_{\Omega} G(x, y, \kappa) f(y) dy \\ &\quad + \int_{\Omega} f(y) dy \int_{\Omega} G(x, z, \kappa) R(z, y, \kappa) dz \\ &= \int_{\Omega} G(x, y, \kappa) f(y) dy + \int_{\Omega} G(x, z, \kappa) dz \int_{\Omega} R(z, y, \kappa) f(y) dy \\ &= \int_{\Omega} G(x, y, \kappa) f(y) dy + \int_{\Omega} G(x, z, \kappa) g(z) dz. \end{aligned}$$

Here we put $g(z) = \int_{\Omega} R(z, y, \kappa) f(y) dy$. Now we have

$$\begin{aligned} (4.18) \quad \int |g(z)|^2 dz &\leq \int_{\Omega} dz \left(\int_{\Omega} |R(z, y, \kappa)| dy \int_{\Omega} |R(z, y, \kappa)| |f(y)|^2 dy \right) \\ &\leq C_1 \int_{\Omega} dz \int_{\Omega} |R(z, y, \kappa)| |f(y)|^2 dy \\ &\leq C_2 \int_{\Omega} |f(y)|^2 dy \leq C_3. \end{aligned}$$

Thus from viii) of theorem 2 and (4.18) we get $u \in H_2(\Omega)$.

*) See viii) of theorem 2, where $H_2(\Omega)$ was defined.

5. The Eigenfunctions and the Eigenfunction Expansion Associated with \mathbf{H} for (EDP)

We consider the differential operator (1.1) for $\kappa=0$, i.e.

$$L_0 u(x) = \sum_{n=1}^3 \left(i \frac{\partial}{\partial x_n} - b_n(x) \right)^2 u(x).$$

Let the operator L_0 , defined on the set of functions satisfying the null boundary condition on $\partial\Omega$ and belonging to $C_0^2(\bar{\Omega})$, be denoted by \tilde{L}_0 . Then we can see that \tilde{L}_0 can be extended uniquely to a selfadjoint operator in the Hilbert space $L^2(\Omega)$. We denote its unique selfadjoint extension by \mathbf{H} .

Let $\kappa \in C^+$. Define the operator

$$\mathbf{H}(\kappa)f(x) = \int_{\Omega} H(x, y, \kappa)f(y)dy.$$

As is seen from theorem 4 this operator is well-defined for $f \in L^2(\Omega)$. Let

$$(5.1) \quad \mathbf{D} = \{u : u \in H_2(\Omega), u \text{ is continuous in } \bar{\Omega}, u|_{\partial\Omega} = 0\}.$$

Then $\mathbf{H}(\kappa)$ is a bounded linear operator on $L^2(\Omega)$ for any $\kappa \in C^+$, and, moreover, if κ is purely imaginary so that $\kappa^2 < 0$, $\mathbf{H}(\kappa)$ is selfadjoint. The range of $\mathbf{H}(\kappa)$ is \mathbf{D} and is independent of κ . Using theorem 4, we can see that $\mathbf{H}(\kappa)$ is equal to the resolvent operator $R(\kappa^2) = (\mathbf{H} - \kappa^2)^{-1}$ of \mathbf{H} . The above-mentioned results are obtained by the same method as Ikebe's [3] (p. 44, Theorem 3.1). Thus we get the following theorem.

Theorem 5. *The symmetric operator \tilde{L}_0 is uniquely extended to the nonnegative definite selfadjoint operator \mathbf{H} acting in $L^2(\Omega)$, and for every non real λ the resolvent $R(\lambda)$ of \mathbf{H} is a bounded integral operator*

$$R(\lambda)f(x) = \int_{\Omega} H(x, y, \sqrt{\lambda})f(y)dy,$$

where by $\sqrt{\lambda}$ is meant the square root of λ with nonnegative imaginary part, and hence the domain of \mathbf{H} is \mathbf{D} .

Now let us proceed to the construction of the generalized eigenfunctions of \mathbf{H} .

Let $\kappa \in C^+$ and $k \in R^3$. Define the function

$$(5.2) \quad \varphi(x, k, \kappa) = (|k|^2 - \kappa^2) \int_{\mathcal{Q}} H(x, y, \kappa) e^{ik \cdot y} dy,$$

where $k \cdot y$ denotes the scalar product of k and y .

Lemma 6. $\varphi(x, k, \kappa)$ is the unique (EDP) solution associated with $L_{\kappa}\varphi = (|k|^2 - \kappa^2)e^{ik \cdot x}$ and the boundary condition $\varphi|_{\partial\mathcal{Q}} = 0$ such that $\varphi - e^{ik \cdot x}$ satisfies the radiation condition. Moreover $\varphi(x, k, \kappa)$ is continuous in $\bar{\mathcal{Q}} \times R^3 \times C^+$ and can be extended continuously to $\bar{\mathcal{Q}} \times R^3 \times \bar{C}^+$.

Proof. $\varphi|_{\partial\mathcal{Q}} = 0$ is obvious, and it is a direct consequence of ii) of theorem 3 that $\varphi(x, k, \kappa)$ satisfies $L_{\kappa}\varphi = (|k|^2 - \kappa^2)e^{ik \cdot x}$. Next we shall show that $\varphi - e^{ik \cdot x}$ satisfies the radiation condition.

$$\begin{aligned} (|k|^2 - \kappa^2) \int_{\mathcal{Q}} H(x, y, \kappa) e^{ik \cdot y} dy &= (|k|^2 - \kappa^2) \int_{\mathcal{Q}} e^{ik \cdot y} \left\{ G(x, y, \kappa) \right. \\ &\quad \left. + \int_{\mathcal{Q}} G(x, z, \kappa) R(z, y, \kappa) dz \right\} dy \\ &= (|k|^2 - \kappa^2) \int_{\mathcal{Q}} G(x, y, \kappa) e^{ik \cdot y} dy + (|k|^2 - \kappa^2) \int_{\mathcal{Q}} e^{ik \cdot y} dy \\ &\quad \times \int_{\mathcal{Q}} G(x, z, \kappa) R(z, y, \kappa) dz \\ &= (|k|^2 - \kappa^2) \int_{\mathcal{Q}} \tilde{H}(x, y, \kappa) e^{ik \cdot y} dy + (|k|^2 - \kappa^2) \int_{\mathcal{Q}} F(x, y, \kappa) e^{ik \cdot y} dy \\ &\quad + (|k|^2 - \kappa^2) \int_{\mathcal{Q}} e^{ik \cdot y} dy \int_{\mathcal{Q}} G(x, z, \kappa) R(z, y, \kappa) dz \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where $F(x, y, \kappa) = \frac{e^{i\kappa|x-y|}}{4\pi|x-y|}$, and $\tilde{H}(x, y, \kappa)$ is just equal to $H(x, y, \kappa)$ introduced by Ikebe [3] (p. 39, (2.5)) as the compensating part of the Green function for (EDP) for $-\Delta - \kappa^2$. Here we estimate I_1, I_2 and I_3 . I_1 satisfies the radiation condition according to the result of Ikebe [3] (p. 45, Lemma 3.3).

$$\begin{aligned}
 I_2 &= (|k|^2 - \kappa^2) \int_{\mathcal{Q}} F(x, y, \kappa) e^{ik \cdot y} dy = (|k|^2 - \kappa^2) \int_{R^3} F(x, y, \kappa) e^{ik \cdot y} dy \\
 &\quad - (|k|^2 - \kappa^2) \int_{\omega} F(x, y, \kappa) e^{ik \cdot y} dy, \\
 &\quad (\omega = R^3 - \bar{\mathcal{Q}}).
 \end{aligned}$$

Since $F(x, y, \kappa)$ has the conjugate Fourier transform equal to $(2\pi)^{-\frac{3}{2}} (|k|^2 - \kappa^2) e^{ik \cdot x}$, we have

$$I_2 = e^{ik \cdot x} - (|k|^2 - \kappa^2) \int_{\omega} F(x, y, \kappa) e^{ik \cdot y} dy.$$

As $\omega = R^3 - \bar{\mathcal{Q}}$ is a bounded domain, it is easily seen that $(|k|^2 - \kappa^2) \int_{\omega} F(x, y, \kappa) e^{ik \cdot y} dy$ satisfies the radiation condition. Since $R(z, \cdot, \kappa) \in L^1(\mathcal{Q})$, we can interchange the order of integration in I_3 . Thus we have

$$I_3 = (|k|^2 - \kappa^2) \int_{B(R)} G(x, z, \kappa) dz \int_{\mathcal{Q}} e^{ik \cdot y} R(z, y, \kappa) dy.$$

As $\int_{\mathcal{Q}, y} e^{ik \cdot y} R(z, y, \kappa) dy$ is continuous on z and $B(R)$ is a bounded domain, I_3 satisfies the radiation condition. Thus we have shown that

$$\begin{aligned}
 \varphi(x, k, \kappa) - e^{ik \cdot x} &= (|k|^2 - \kappa^2) \left\{ \int_{\mathcal{Q}} \tilde{H}(x, y, \kappa) e^{ik \cdot y} dy - \int_{\omega} F(x, y, \kappa) e^{ik \cdot y} dy \right. \\
 &\quad \left. + \int_{B(R)} G(x, z, \kappa) dz \int_{\mathcal{Q}} e^{ik \cdot y} R(z, y, \kappa) dy \right\}
 \end{aligned}$$

satisfies the radiation condition. Hence by theorem 1 $\varphi(x, k, \kappa) - e^{ik \cdot x}$ is the unique solution of (EDP) with $L_{\kappa} u = l \cdot e^{ik \cdot x}$ in \mathcal{Q} which satisfies the boundary condition $u|_{\partial \mathcal{Q}} = -e^{ik \cdot x}$, and the radiation condition. Consequently by theorem 1 $\varphi(x, k, \kappa)$ is the unique solution.

Now we shall prove the remainder of this lemma. We put

$$h(x, k, \kappa) = (|k|^2 - \kappa^2) \int_{\mathcal{Q}} \tilde{H}(x, y, \kappa) e^{ik \cdot y} dy - \int_{\omega} F(x, y, \kappa) e^{ik \cdot y} dy,$$

where $h(x, k, \kappa)$ is the same function as defined in [3] (p. 45, (3.12)). Ikebe has proved that

$$(5.3) \quad h(x, k, \kappa) = (|k|^2 - \kappa^2) \int_{\Omega} G(x, y, \kappa) e^{ik \cdot y} dy - e^{ik \cdot x} (= I_1 + I_2 - e^{ik \cdot x}),$$

and $h(x, k, \kappa)$ is continuous in $\bar{\Omega} \times R^3 \times C^+$ and can be extended continuously to $\bar{\Omega} \times R^3 \times \bar{C}^+$. We shall consider $I_3 = (|k|^2 - \kappa^2) \int_{\Omega} e^{ik \cdot y} dy \int_{B(R)} G(x, z, \kappa) R(z, y, \kappa) dz$. From (4.6), i.e. $\int_{B(R)} G(x, z, \kappa) R(z, y, \kappa) dz = \int_{B(R)} G(z, y, \kappa) \overline{R(z, x, -\bar{\kappa})} dz$, we have

$$\begin{aligned} I_3 &= (|k|^2 - \kappa^2) \int_{\Omega} e^{ik \cdot y} dy \int_{B(R)} G(x, z, \kappa) R(z, y, \kappa) dz \\ &= (|k|^2 - \kappa^2) \int_{B(R)} \overline{R(z, x, -\bar{\kappa})} dz \int_{\Omega} e^{ik \cdot y} G(z, y, \kappa) dy. \end{aligned}$$

From (5.3)

$$(|k|^2 - \kappa^2) \int_{\Omega} G(z, y, \kappa) e^{ik \cdot y} dy = h(z, k, \kappa) + e^{ik \cdot z}.$$

Thus we have

$$\begin{aligned} I_3 &= (|k|^2 - \kappa^2) \int_{B(R)} \overline{R(z, x, -\bar{\kappa})} e^{ik \cdot z} dz \\ &\quad + (|k|^2 - \kappa^2) \int_{B(R)} \overline{R(z, x, -\bar{\kappa})} h(z, k, \kappa) dz. \end{aligned}$$

Using the above-mentioned properties of $h(x, k, \kappa)$ obtained by Ikebe and the property of $R(x, y, \kappa)$, I_3 is proved continuous in $\bar{\Omega} \times R^3 \times C^+$ and can be extended continuously to $\bar{\Omega} \times R^3 \times \bar{C}^+$. As $\varphi(x, k, \kappa)$ is expressed as $\varphi(x, k, \kappa) = I_1 + I_2 + I_3 = h(x, k, \kappa) + e^{ik \cdot x} + I_3$, the continuity on κ of $\varphi(x, k, \kappa)$ was proved. Now the proof of the lemma is complete.

Since $\varphi(x, k, \cdot)$ is continuous in C^+ and can be extended continuously to \bar{C}^+ , we put

$$(5.4) \quad \varphi(x, k) = \lim_{\kappa \rightarrow |k|} \varphi(x, k, \kappa) = \varphi(x, k, |k|).$$

So from the above-mentioned lemma the following theorem holds.

Theorem 6. *There exists a family of generalized eigenfunctions*

$\varphi(x, k) (\in L^2(\Omega))$ of H satisfying the following conditions:

- i) $\varphi(x, k)$ is continuous in $(x, k) \in \bar{\Omega} \times \mathbb{R}^3$,
- ii) $\varphi(x, k)$, $k \in \mathbb{R}^3$ fixed, is twice continuously differentiable in Ω , and satisfies the equation

$$L_\kappa \varphi(x, k) = 0 \quad \text{in } \Omega,$$

and the boundary condition

$$\varphi(x, k)|_{\partial\Omega} = 0,$$

and the function $\psi(x, k) = \varphi(x, k) - e^{ik \cdot x}$ satisfies the radiation condition

$$\psi(x, k) = O(1/r), \quad \left(\frac{\partial}{\partial r} - i|k| \right) \psi(x, k) = o(1/r) \quad (r = |x| \rightarrow \infty).$$

Now we can get the expansion of an arbitrary function $f \in L^2(\Omega)$ in terms of the generalized eigenfunctions $\varphi(x, k)$. The method of proof employed here is the same as that of Ikebe [1]. Therefore we shall simply sketch the proof.

Let $f \in C_0^\infty(\Omega)$. We introduce the function

$$\Phi(k, \kappa) = (2\pi)^{-\frac{3}{2}} \int_{\Omega} \overline{\varphi(x, k, \kappa)} f(x) dx.$$

Since $H(x, y, \kappa)$ ($\text{Im } \kappa > 0$) belongs to $L^1(\Omega) \cap L^2(\Omega)$, its usual conjugate Fourier transform exists. And Parseval's equality combined with (5.2) leads to

$$\begin{aligned} ([R(\bar{\kappa}^2) - R(\kappa^2)]\bar{f}, \bar{f})_{L^2(\Omega)} &= (\kappa^2 - \bar{\kappa}^2)(R(\kappa^2)\bar{f}, R(\bar{\kappa}^2)\bar{f})_{L^2(\Omega)} \\ &= (\kappa^2 - \bar{\kappa}^2) \int_{\Omega} dx \int_{\Omega} H(x, y, \kappa) \bar{f}(y) dy \int_{\Omega} \overline{H(x, z, \kappa) f(z)} dz \\ &= \iint_{\Omega \times \Omega} \overline{f(y)} f(z) dy dz \int_{\mathbb{R}^3} \varphi(y, k, \kappa) \overline{\varphi(z, k, \kappa)} dk \\ &= \int_{\mathbb{R}^3} \frac{2i\varepsilon}{(|k|^2 - \mu)^2 + \varepsilon^2} |\Phi(k, \sqrt{\mu + i\varepsilon})|^2 dk \end{aligned}$$

where $\kappa^2 = \mu + i\varepsilon$. Next we use the well-known formula

$$\begin{aligned} & \frac{1}{2} [(E(\beta)f + E(\beta - 0)f, f)_{L^2(\Omega)} - (E(\alpha)f + E(\alpha - 0)f, f)_{L^2(\Omega)}] \\ &= \frac{1}{2\pi i} \lim_{\varepsilon \downarrow 0} \int_{\alpha}^{\beta} ([R(\mu + i\varepsilon) - R(\mu - i\varepsilon)]f, f)_{L^2(\Omega)} d\mu, \end{aligned}$$

where $E(\lambda)$ is the resolution of the identity corresponding to the selfadjoint operator H . Thus we get

$$([E(\beta) - E(\alpha)]\bar{f}, \bar{f})_{L^2(\Omega)} = \int_{\sqrt{\alpha} \leq |k| \leq \sqrt{\beta}} \|\hat{f}\|^2 dk,$$

where we put

$$\hat{f}(k) = \lim_{\varepsilon \downarrow 0} \mathcal{O}(k, \sqrt{|k|^2 + i\varepsilon}) = (2\pi)^{-\frac{3}{2}} \int_{\Omega} \overline{\varphi(x, k)} f(x) dx.$$

Letting $\alpha \rightarrow 0$ and $\beta \rightarrow +\infty$, we have

$$(5.5) \quad (\bar{f}, \bar{f})_{L^2(\Omega)} = (f, f)_{L^2(\Omega)} = \|f\|_{L^2(\Omega)}^2 = \int_{0 \leq |k| \leq +\infty} |\hat{f}(k)|^2 dk.$$

This formula (5.5) can be extended by continuity to the case $f \in L^2(\Omega)$. Thus we obtained the generalized Parseval-Plancherel's equality. Here we shall introduce the following mapping Z

$$(Zf)(k) = (2\pi)^{-\frac{3}{2}} \text{l.i.m} \int_{\Omega} \overline{\varphi(x, k)} f(x) dx = \hat{f}(k).$$

From (5.5) Z that takes $f \in L^2(\Omega)$ into $\hat{f}(k) \in L^2(R^3)$ is isometric, and, moreover, we can show that Z maps $L^2(\Omega)$ onto $L^2(R^3)$ by the same method as that of Ikebe [2]. Now it is easy to get the following theorem.

Theorem 7. *The spectrum of H coincides with the positive real line and is absolutely continuous. The mapping Z is unitary from $L^2(\Omega)$ to $L^2(R^3)$ and hence the inverse Z^{-1} from $L^2(R^3)$ onto $L^2(\Omega)$ exists. Moreover, we have the following expansion formulas:*

Let f and $g \in L^2(\Omega)$. Then

$$\begin{aligned} \text{i) } \hat{f}(k) &= (2\pi)^{-\frac{3}{2}} \text{l.i.m} \int_{\Omega} \overline{\varphi(x, k)} f(x) dx, \\ & \text{(the generalized Fourier transform),} \end{aligned}$$

$$f(x) = (2\pi)^{-\frac{3}{2}} \text{l.i.m} \int_{R^3} \varphi(x, k) \hat{f}(k) dk,$$

(the generalized conjugate Fourier transform),

$$\text{ii) } \int_{\Omega} f(x) g(x) dx = \int_{R^3} \hat{f}(k) \hat{g}(k) dk,$$

(the generalized Parseval-Plancherel's equality),

iii) if $f \in \mathbf{D}$ (the domain of H), we have the following representation of H

$$(Hf)(x) = (2\pi)^{-\frac{3}{2}} \text{l.i.m} \int_{R^3} |k|^2 \varphi(x, k) \hat{f}(k) dk.$$

References

- [1] Ikebe, T., Eigenfunction expansions associated with the Schrödinger operators and their applications to scattering theory, *Arch. Rational Mech. Anal.* **5** (1960), 1-34.
- [2] ———, Orthogonality of the eigenfunctions for the exterior problem connected with $-\Delta$, *Arch. Rational Mech. Anal.* **19** (1965), 71-73.
- [3] ———, On the eigenfunction expansion connected with the exterior problem for the Schrödinger equation, *Japan. J. Math.* **36** (1967), 33-55.
- [4] Miranda, C., Partial Differential Equations of Elliptic Type, *Ergebn. Math.*, Springer, 1970.
- [5] Mizohata, S., Sur l'analyticité de la fonction spectrale de l'opérateur Δ relatif au problème extérieur, *Proc. Japan Acad.* **39** (1963), 352-357.
- [6] ———, *Theory of Partial Differential Equations*, Iwanami, 1965 (in Japanese).
- [7] Nirenberg, L., Remarks on strongly elliptic partial differential equations, *Comm. Pure Appl. Math.* **13** (1955), 648-674.
- [8] Rellich, F., Über das asymptotische Verhalten des Lösungen von $\Delta u + \lambda u = 0$ in unendlichen Gebieten, *Jber. Deutsch. Math. Verein.* **53** (1943), 57-65.
- [9] Shizuta, Y., Eigenfunction expansion associated with the operator $-\Delta$ in the exterior domain, *Proc. Japan Acad.* **39** (1963), 656-660.
- [10] Werner, P., Randwertprobleme der mathematischen Akustik, *Arch. Rational Mech. Anal.* **10** (1962), 29-66.