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# Some Remarks on the Modified Korteweg-de Vries Equations

By

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**Abstract**

In Section 1 we associate certain linear differential operators to modifications of the KdV equation. An interpretation is given to the non-linear transformation of Miura [4] which converts a solution of one of modified KdV equations into that of the KdV equation.

In Section 2 we construct a family of special solutions of another modification of the KdV equation.

**1.** In this paper we study the modified Korteweg-de Vries (KdV) equations

(1) $$\dot{v} \pm 6v^2v' + v''' = 0$$

where $\dot{v}$ and $v'$ are $t$ and $x$ derivatives of real-valued smooth function $v = v(x, t)$ $(-\infty < x, t < \infty)$ respectively. We shall refer to them as equations $(1+)$ and $(1-)$ according to their signs. These equations appear in Zabusky [6] as generalizations of the KdV equation

(2) $$\dot{u} - 6uu' + u''' = 0$$

and in Miura [5] where the relation between the solutions of (1) and (2) is discussed. The existence theorem for the initial-value problem of (1) has been proved in Kametaka [2].

Lax [4] has rewritten the KdV equation into an evolution equation for a linear operator: For a complex-valued smooth function $u(x)$, let $L_u$ be the one dimensional Schrödinger operator

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$$L_u=-D^2+u$$

and put

$$B_u=-4D^3+3uD+3Du$$

where $D$ stands for the $x$ differentiation. Then the operator

$$[B_u, L_u]=B_uL_u-L_uB_u$$

is the multiplication by the function $6uu'-u'''$. So the operator equation for real-valued function $u(t)=u(t, x)$

$$\dot{L}_{u(t)}=[B_{u(t)}, L_{u(t)}]$$

is equivalent to the KdV equation.

For the modified KdV equations we can give a similar operator interpretation. For a complex-valued smooth function $v(x)$, introduce the first order differential operator

$$\boldsymbol{L}_v=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} D+\begin{bmatrix} 0 & v \\ v & 0 \end{bmatrix}$$

and put

$$\boldsymbol{B}_v=\begin{bmatrix} B_{v'+v^2} & 0 \\ 0 & B_{-v'+v^2} \end{bmatrix}.$$

Then the operator $[\boldsymbol{B}_v, \boldsymbol{L}_v]$ is the multiplication by the matrix valued function

$$\begin{bmatrix} 0 & 6v^2v'-v''' \\ 6v^2v'-v''' & 0 \end{bmatrix}.$$

So for real-valued function $v(t)=v(x, t)$, the operator evolution equations

$$\dot{\boldsymbol{L}}_{v(t)}=[\boldsymbol{B}_{v(t)}, \boldsymbol{L}_{v(t)}]$$

and

$$\dot{\boldsymbol{L}}_{iv(t)}=[\boldsymbol{B}_{iv(t)}, \boldsymbol{L}_{iv(t)}]$$

are equivalent to $(1-)$ and $(1+)$ respectively.

Note that we have an operator identity

$$\boldsymbol{L}_v^2=\begin{bmatrix} L_{v'+v^2} & 0 \\ 0 & L_{-v'+v^2} \end{bmatrix}. \tag{3}$$

Putting a solution $v=v(t, x)$ of the equation $(1-)$ into (3), we differentiate (3) with respect to $t$. Then we have

$$\begin{aligned}\begin{bmatrix} \dot{L}_{v'+v^2} & 0 \\ 0 & \dot{L}_{-v'+v^2} \end{bmatrix} &= \boldsymbol{L}_v\dot{\boldsymbol{L}}_v+\dot{\boldsymbol{L}}_v\boldsymbol{L}_v \\ &= \boldsymbol{L}_v[\boldsymbol{B}_v, \boldsymbol{L}_v]+[\boldsymbol{B}_v, \boldsymbol{L}_v]\boldsymbol{L}_v \\ &= [\boldsymbol{B}_v, \boldsymbol{L}_v^2]\end{aligned}$$

and finally

$$\dot{L}_{\pm v'+v^2}=[B_{\pm v'+v^2}, L_{\pm v'+v^2}].$$

So $\pm v'+v^2$ satisfy the KdV equation. This fact has been discoverd by Miura [5] by a different consideration.

**2.** In this section we construct a family of special solutions of the modified KdV equation $(1+)$. They are analogous to the $N$-tuple wave solutions of the KdV equation, which have been constructed in Gardner, Greene, Kruskal and Miura [1] based on the inverse scattering theory for the Schrödinger equation.

Consider the eigenvalue problem for the operator $\boldsymbol{L}_{iv}$:

$$y_2'+ivy_2=\zeta y_1$$
$$-y_1'+ivy_1=\zeta y_2.$$

Putting $z_1=y_1-iy_2$ and $z_2=y_1+iy_2$, we have

$$iz_1'+vz_2=\zeta z_1$$

$$-iz_2'-vz_1=\zeta z_2.$$

This is a special case of the system of first order differential equations

(4)
$$iz_1'-iqz_2=\zeta z_1$$

$$-iz_2'-iq{*}z_1=\zeta z_2,$$

where $q$ is a complex-valued function and $q*$ denotes its complex conjugate.

The inverse scattering theory for (4), namely the problem of the construction of the potential $q$ from the scattering data, has been discussed by Zakhalov and Shabat [7] and applied to the exact solution of a certain non-linear equation. In what follows we restrict our attention to the case where the fundamental equation of the inverse scattering theory reduces to the system of linear algebraic equations.

Let $\zeta_1, \cdots, \zeta_N$ be complex numbers different from each other in the upper half-plane and $c_1, \cdots, c_N$ be any complex numbers. Put

$$\lambda_j=c_j^{1/2}\exp(i\zeta_j x)$$

and consider a system of linear equations for $\psi_{1j}, \psi_{2j}^*(j=1, \cdots, N)$:

(5a)
$$\psi_{1j}+\sum_k\lambda_j\lambda_k^*(\zeta_j-\zeta_k^*)^{-1}\psi_{2k}^*=0$$

(5b)
$$-\sum_k\lambda_k\lambda_j^*(\zeta_j^*-\zeta_k)^{-1}\psi_{1k}+\psi_{2j}^*=\lambda_j^*$$

(The sums are taken from 1 to $N$ throughout the present paper). Then this system of equations has a non-singular coefficient matrix. Put

$$q(x)=-2i\sum_k\lambda_k^*\psi_{2k}^*.$$

Then for each $j$, the pair $(\psi_{1j}, \psi_{2j})$ satisfies the differential equations

(6)
$$i\psi_{1j}'-iq\psi_{2j}=\zeta_j\psi_{1j}$$

$$-i\psi_{2j}'-iq^*\psi_{1j}=\zeta_j\psi_{2j}.$$

We give a proof of these facts in Appendix.

Multiply $\psi_{2j}^{*}$ on (5b) and take the summation over $j$. Then we have another expression for $q(x)$:

$$q(x)=2i\sum_j(\psi_{1j}^2-\psi_{2j}^{*2}).$$

We pose further restriction on the system (5): Let $M$ be a non-negative integer such that $2M\leqq N$. Let $\sigma$ be the permutation among integers between 1 and $N$ defined by

$$\begin{aligned}\sigma(j)&=j+1 && j \text{ odd}\leqq 2M\\ &=j-1 && j\text{ even}\leqq 2M\\ &=j && j>2M.\end{aligned}$$

We assume that $\zeta_{\sigma(j)}=-\zeta_j^{*}$ and $c_{\sigma(j)}=c_j^{*}(1\leqq j\leqq N)$.

Now let $c_j$ depend on $t$ as

$$c_j(t)=c_j(0)\exp(8i\zeta_j^3 t)$$

and put

$$\lambda_j=\lambda_j(x,t)=c_j(t)^{1/2}\exp(i\zeta_j x).$$

**Theorem.** *Let $\psi_{1j}(x,t)$ and $\psi_{2j}(x,t)$ be the solution of the system (5) for $\lambda_j=\lambda_j(x,t)$ defined above and put*

$$q(x,t)=-2i\sum_j\lambda_j^{*}(x,t)\psi_{2j}^{*}(x,t).$$

*Then $v(x,t)=-iq(x,t)$ is real-valued and satisfies the modified KdV equation* (1+).

*Proof.* Put $\phi_{1j}=i\lambda_j\psi_{1j}$ and $\phi_{2j}=\lambda_j\psi_{2j}$. Then the system (5) is rewritten as

(7a) $$\lambda_j^{-2}\phi_{1j}+i\sum_k(\zeta_j-\zeta_k^{*})^{-1}\phi_{2k}^{*}=0$$

(7b) $$i\sum_k(\zeta_j^{*}-\zeta_k)^{-1}\phi_{1k}+\lambda_j^{*^{-2}}\phi_{2j}^{*}=1.$$

It is easy to verify that $\phi^*_{1\sigma(j)}$ and $\phi^*_{2\sigma(j)}$ satisfy the same equation as $\phi_{1j}$ and $\phi_{2j}$. By the uniqueness of solution we have $\phi^*_{1\sigma(j)}=\phi_{1j}$ and $\phi^*_{2\sigma(j)}=\phi_{2j}$. The function $v(x, t)$ is thus real-valued.

Eliminating $\phi_{1j}$ from (7), we have a system of linear equations for $\phi_{2j}$:

$$\sum_l \alpha_{jl}\phi^*_{2l}=1 \tag{8}$$

where

$$\alpha_{jl}=\alpha_{jl}(x, t)=\sum_k \lambda_k^2(\zeta_j^*-\zeta_k)^{-1}(\zeta_k-\zeta_l^*)^{-1}+\lambda_j^{*^{-2}}\delta_{jl}$$

($\delta_{jl}$ is Kronecker's delta). Now we differentiate (8) with respect to $t$ and obtain a system of linear equations for $\dot{\phi}_{2j}$:

$$\sum_l \alpha_{jl}\dot{\phi}^*_{2l}=\gamma_j$$

where

$$\gamma_j=-8i\sum_{k,l}\zeta_k^3\lambda_k^2(\zeta_j^*-\zeta_k)^{-1}(\zeta_k-\zeta_l^*)^{-1}\phi^*_{2l}-8i\zeta_j^{*^3}\lambda_j^{*^{-2}}\phi^*_{2j}.$$

Let $\beta_{jk}=\beta_{jk}(x, t)$ be the element of the inverse matrix of the matrix $(\alpha_{jk})$. Then we have

$$\phi^*_{2j}=\sum_k \beta_{jk}=\sum_k \beta_{kj} \qquad \dot{\phi}^*_{2j}=\sum_k \beta_{jk}\gamma_k.$$

Using these relations and (7a), we have a formula for the $t$-derivative of $v$:

$$\dot{v}=16i\sum_j(-\zeta_j^3\psi_{1j}^2+\zeta_j^{*3}\psi_{2j}^{*2}).$$

We differentiate

$$v=2\sum_j(\psi_{1j}^2-\psi_{2j}^{*2})$$

successively with respect to $x$ and obtain the formulas for $x$-derivatives of $v$:

$$v'=4i\sum_j(-\zeta_j\psi_{1j}^2+\zeta_j^*\psi_{2j}^{*2})$$

$$v'' = -2v^3 + 8\sum_j(-\zeta_j^2\psi_{1j}^2 + \zeta_j^{*2}\psi_{2j}^{*2})$$

$$v''' = -6v^2v' + 16i\sum_j(\zeta_j^3\psi_{1j}^2 - \zeta_j^{*3}\psi_{2j}^{*2}).$$

Beside the relation (6), we have used the relations

$$\mathrm{Re}(\sum_j\psi_{1j}\psi_{2j}) = 0$$

$$\sum_j\zeta_j\psi_{1j}\psi_{2j} = -8^{-1}v$$

$$\mathrm{Re}(\sum_j\zeta_j^2\psi_{1j}\psi_{2j}) = 0$$

to derive these formulas. Q.E.D.

If $N=1$, then $\zeta_1 = i\eta$ $(\eta>0)$ and $c = c_1(0)$ is real. We have thus solutions

$$v(x, t) = (\mathrm{sgn}\ c)s(x - 4\eta^2 t - \delta, \eta)$$

where

$$s(x, \eta) = -2\eta\,\mathrm{sech}(2\eta x)$$

and

$$\delta = \delta(c, \eta) = (2\eta)^{-1}\log(|c|/2\eta).$$

These solutions coincide with the soliton solutions known to exist for the generalized KdV equations (see Zabusky [6]).

Now let $N=2$ and $M=0$. Then $\zeta_j = i\eta_j$, $0<\eta_1<\eta_2$ and $c_j = c_j(0)$ are real. The solutions decompose into two solitons as $t\to\pm\infty$:

$$v(x, t) - \sum_{j=1}^2(\mathrm{sgn}\,c_j)s(x - 4\eta_j^2 t - \delta_j^{\pm}, \eta_j) \to 0$$

where

$$\delta_1^+ = \delta(c_1, \eta_1) + \eta_1^{-1}\log(\eta_2-\eta_1)(\eta_2+\eta_1)^{-1}$$

$$\delta_2^+ = \delta(c_2, \eta_2) \qquad \delta_1^- = \delta(c_1, \eta_1)$$

$$\delta_2^- = \delta(c_2, \eta_2) + \eta_2^{-1}\log(\eta_2-\eta_1)(\eta_2+\eta_1)^{-1}.$$

More generally in the case $M=0$ (i.e. all of $\zeta_j$ are purely imaginary) the corresponding solutions seem to decompose into solitons as $t\to\pm\infty$.

**Appendix.** The following arguments are quite similar to that of Kay and Moses [3] where the construction of reflectionless potential for Schrödinger equation has been discussed.

The $N\times N$ matrix $A=(i(\zeta_j-\zeta_k^*)^{-1})$ is positive definite because of the identity

$$i(\zeta_j-\zeta_k^*)^{-1}=\int_0^\infty \exp(i\zeta_j t)\exp(i\zeta_k t)^* dt.$$

Eliminating $\psi_{1j}$ from (5), we have a system of $N$ linear equations for $\psi_{j2}^*$:

$$\sum_l b_{jl}\psi_{2l}^*+\psi_{2j}^*=\lambda_j^*,$$

where

$$b_{jl}=\sum_l \lambda_j^*\lambda_k^2\lambda_l^*(\zeta_j^*-\zeta_k)^{-1}(\zeta_k-\zeta_l^*)^{-1}.$$

Putting $B=(b_{jk})$, we have

$$\det B=|\lambda_1\lambda_2\cdots\lambda_N|^4|\det A|^2,$$

so $\det B$ is positive. Any principal minor of $B$ is also positive because it is expressed as the sum of the determinant of the matrices having the same form as $B$. Now the characteristic polynomial of $B$ is

$$\det(B+\lambda I)=\lambda^N+a_1\lambda^{N-1}+\cdots+a_N,$$

where $a_j$ is positive, being the sum of the principal minors of $B$ of order $j$. If we set $\lambda=1$, we see that the matrix $B+I$ is invertible and so is the coefficient matrix of (5).

Differentiating the equations (5) with respect to $x$, we see that $2N$ functions

$$\psi_{1j}'+i\zeta_j\psi_{1j}-q\psi_{2j} \qquad \psi_{2j}'-i\zeta_j\psi_{2j}+q^*\psi_{1j}$$

satisfy the homogeneous system of equations associated with (5) and therefore vanish.

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