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Normal Positive Linear Mappings of Norm 1 from a von Neumann Algebra into Its Commutant and Its Application

By

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Abstract

Let M and N be von Neumann algebras such that $N\subset M'$. Let $Z=N\cap M$ and ρ be any normal positive linear functional of $(M\cup N)''$. There exists a unique mapping F_ρ^{NM} from M into N satisfying

$$(1/2)\rho(F_\rho^{NM}(Q_1)Q_2+Q_2F_\rho^{NM}(Q_1))=\rho(Q_1Q_2)$$

for all $Q_1\in M$, $Q_2\in N$ and $s(F_\rho^{NM}(Q_1))\leqq s^N(\rho)$, where s denotes the support and s^N denotes the support in N. The mapping F_ρ^{NM} is Z-linear, positive and transposed-n-positive, of norm 1 and continuous on the unit ball weakly and strongly.

As an application, a generalization of a clustering theorem for an asymptotically abelian case is given.

§ 1. Preliminaries

We consider two von Neumann algebras M and N such that $N\subset M'$ and a normal positive linear functional ρ of $(M\cup N)''$. H_ρ, π_ρ, and Ω_ρ denote a Hilbert space, a representation of $(M\cup N)''$ and a cyclic vector canonically associated with ρ through $\rho=\omega_{\Omega_\rho}$ where ω_Ω denotes the expectation functional by the vector Ω (called a vector state if $\omega_\Omega(1)=1$).

$s(A)$ for an operator A on a Hilbert space denotes the support of A, namely the smallest projection E satisfying $EA=AE=A$. $s(A)$ is in the von Neumann algebra generated by A and A^* and hence the notation $s(A)$ is also used for an element of von Neumann algebra. $s^N(\rho)$ denotes

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the support of ρ relative to N, namely the smallest projection E in N such that $\rho(E)=\rho(1)$. $s^N(\Omega)$ denotes $s^N(\omega_\Omega)$.

Our tool is the following version of the Radon-Nikodym theorem by Sakai [6].

Lemma 1. *Let μ and ν be normal positive linear functionals of a von Neumann algebra N such that $\mu \geqq \nu$. There exists a unique $h_0 \in N$ satisfying*

(1) $\nu(Q)=(1/2)\mu(h_0Q+Qh_0), \qquad Q\in N,$

(2) $s(h_0)\leqq s^N(\mu),$

(3) $0\leqq h_0\leqq 1.$

Proof. The existence of h_0 satisfying (1) and (3) is in [6]. Since $0\leqq \nu(1-s^N(\mu))\leqq \mu(1-s^N(\mu))=0$, we have $s^N(\nu)\leqq s^N(\mu)$. Setting $Q=s^N(\mu)h_0(1-s^N(\mu))$, we obtain from (1)

$$0=\nu(Q)=(1/2)\mu(Qh_0)=(1/2)\mu(QQ^*).$$

Since $s^N(\mu)QQ^*s^N(\mu)=QQ^*$, we obtain $QQ^*=0$, i.e. $Q=Q^*=0$. Hence

$$h_0=h_0'+h_0''$$

where $h_0'=s^N(\mu)h_0s^N(\mu)$ and $h_0''=(1-s^N(\mu))h_0(1-s^N(\mu))$. Since

$$\mu(h_0Q+Qh_0)=\mu(h_0'Q+Qh_0')$$

$h_0'\in N$ satisfies (1), (2) and (3).

The uniqueness holds in the following slightly more general form.

Q.E.D.

Lemma 2. *Let μ and ν be normal linear functionals of N and μ be positive. An operator $h_0\in N$ satisfying* (1) *and* (2) *of Lemma* 1 *is unique, if it exists.*

Proof. Suppose h_0 and h_0' satisfy (1) and (2). Then $h=h_0-h_0'$ satisfy $\mu(hQ+Qh)=0$ for all $Q\in N$. Substituting $Q=h^*$, we have

$$0 \leqq \mu(h^*h) \leqq \mu(hh^* + h^*h) = 0$$

and hence $s^N(\mu)h^*hs^N(\mu)=0$. Since $s(h) \leqq s^N(\mu)$, we have $h^*h=0$ and hence $h_0 - h_0' = h = 0$. Q.E.D.

We use Lemma 1 in the following complex form.

Lemma 3. *Let μ and ν be normal linear functionals of N,*

$$\nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4),$$

μ, ν_1, ν_2, ν_3, and ν_4 be positive and $\nu_k \leqq \lambda\mu$, $k=1, 2, 3, 4$, $\lambda > 0$. There exists a unique $h_0 \in N$ satisfying the conditions (1) *and* (2) *of Lemma* 1.

Proof. Immediate from Lemmas 1 and 2. Q.E.D.

A linear mapping F from a von Neumann algebra M into N is called n-positive if the mapping $F \otimes 1$ from $M \otimes \mathscr{B}(C^n)$ to $N \otimes \mathscr{B}(C^n)$ is positive, where C^n is an n-dimensional Hilbert space, $\mathscr{B}(C^n)$ is the set of all linear operators on C^n and $(F \otimes 1)(Q \otimes Q') = F(Q) \otimes Q'$ for $Q \in M$, $Q' \in \mathscr{B}(C^n)$. If F is n-positive for all positive integers n, F is called completely positive.

F is called transposed-n-positive if $F \otimes t$ from $M \otimes \mathscr{B}(C^n)$ to $N \otimes \mathscr{B}(C^n)$ is positive where t is any transposition of matrices relative to any fixed orthonormal basis. The positivity of $F \otimes t$ does not depend on t because two transpositions t and t' relative to different orthonormal bases are always related by $t'(Q) = ut(Q)u^*$ for some unitary $u \in \mathscr{B}(C^n)$.

If F is n-positive or transposed-n-positive, then $Q \geqq 0$ implies $Q \otimes 1 \geqq 0$ and hence $F(Q) \otimes 1 \geqq 0$ and hence $F(Q) \geqq 0$. (More generally it is n'-positive or transposed-n'-positive for $n' \leqq n$.) Considering $F((z+Q)^*(z+Q)) \geqq 0$ for $z=1$ and i, we then have the selfadjointness $F(Q)^* = F(Q^*)$.

Lemma 4. *If a linear map F from M into N is 2-positive and satisfies $F(1)F(Q) = F(Q)$, $Q \in M$, then*

$$F(Q^*Q) \geqq F(Q)^*F(Q), \qquad Q \in M. \tag{1.1}$$

If a linear map F from M into N is transposed-2-positive and satisfies $F(1)F(Q)=F(Q)$, $Q\in M$, then

$$F(Q^*Q)\geqq F(Q)F(Q)^*, \qquad Q\in M. \tag{1.2}$$

Proof. Consider

$$\hat{Q}=\begin{pmatrix}1 & Q\\ Q^* & Q^*Q\end{pmatrix}\in M\otimes\mathscr{B}(C^2)$$

for $Q\in M$ relative to a fixed orthonormal basis e_1 and e_2 in C^2. Let x_1 and x_2 be vectors in defining Hilbert space of M and N and $x=x_1\otimes e_1+x_2\otimes e_2$. Then

$$(x, \hat{Q}x)=||x_1+Qx_2||^2\geqq 0$$

and hence $\hat{Q}\geqq 0$.

If F is 2-positive then

$$0\leqq(x, (F\otimes 1)(\hat{Q})x)=(x_1, F(1)x_1)+2\mathrm{Re}(x_1, F(Q)x_2)+(x_2, F(Q^*Q)x_2)$$

where we have used $F(Q)^*=F(Q^*)$. Setting $x_1=-F(Q)x_2$, we have

$$0\leqq(x_2, F(Q^*Q)x_2)-(x_2, F(Q)^*F(Q)x_2)$$

for any x_2 where we have used $F(1)F(Q)=F(Q)$. Hence we have (1.1).

If F is transposed-2-positive, we have

$$0\leqq(x, (F\otimes t)(\hat{Q})x)=(x_1, F(1)x_1)+2\mathrm{Re}(x_1, F(Q)^*x_2)+(x_2, F(Q^*Q)x_2).$$

Hence, by setting $x_1=-F(Q)^*x_2$, we obtain (1.2). Q.E.D.

For a cyclic and separating vector Ω for M, the polar decomposition

$$\bar{S}=J_\Omega\Delta_\Omega^{1/2}$$

of the closure $\bar{S}$ of the operator S defined on $M\Omega$ by

$$SQ\Omega=Q^*\Omega, \qquad Q\in M$$

defines the modular operator Δ_Ω, which is a strictly positive selfadjoint operator satisfying $\Delta_\Omega \Omega = \Omega$ and $J_\Omega \Delta_\Omega = \Delta_\Omega^{-1} J_\Omega$, and the modular conjugation J_Ω which is an antiunitary involution satisfying $J_\Omega \Omega = \Omega$.

If Ω is not a cyclic and separating vector, we consider the restrictions of M and M' to $s^M(\Omega) s^{M'}(\Omega) H$, and define J_Ω and Δ_Ω on $s^M(\Omega) s^{M'}(\Omega) H$ as above and 0 on $(1 - s^M(\Omega) s^{M'}(\Omega)) H$. The mapping

$$\tau_\Omega(t) Q \equiv \Delta_\Omega^{it} Q \Delta_\Omega^{-it} s^M(\Omega) s^{M'}(\Omega)$$

maps M onto $s^{M'}(\Omega) s^M(\Omega) M s^M(\Omega)$ and M' onto $s^M(\Omega) s^{M'}(\Omega) M' s^{M'}(\Omega)$. It is an automorphism of $s^{M'}(\Omega) s^M(\Omega) M s^M(\Omega)$ and $s^M(\Omega) s^{M'}(\Omega) M' s^{M'}(\Omega)$.

We denote

$$j_\Omega(Q) = J_\Omega Q J_\Omega.$$

It brings M onto $s^{M'}(\Omega) M' s^{M'}(\Omega) s^M(\Omega)$ and M' onto $s^M(\Omega) M s^M(\Omega) s^{M'}(\Omega)$.

For a normal positive linear functional ρ on M, we denote J_Ω, Δ_Ω, $\tau_\Omega(t)$, j_Ω for $\pi_\rho(M)$ and $\Omega = \Omega_\rho$ by J_ρ, Δ_ρ, $\tau_\rho(t)$ and j_ρ. We sometimes denote the expectation functional of $B(H_\rho)$ by the vector Ω_ρ again by ρ.

We need the following.

Lemma 5. *Let ρ be a normal positive linear functional of M and Z_ρ be the set of $x \in M$ such that $\rho(xQ) = \rho(Qx)$ for all $Q \in M$. Then for every $z \in Z_\rho$, $[s^M(\rho), z] = 0$, $[\Delta_\rho, \pi_\rho(z)] = 0$ and*

$$\tau_\rho(t) \pi_\rho(z) = \pi_\rho(z\, s^M(\rho)).$$

If $z \in M \cap M'$, then

$$j_\rho(\pi_\rho(z)) = \pi_\rho(z^* s^M(\rho)).$$

Proof. Substituting $Qs^M(\rho)^\perp$ into Q of $\rho(xQ) = \rho(Qx)$, we obtain $\rho(Qs^M(\rho)^\perp x) = 0$ where $s^M(\rho)^\perp = 1 - s^M(\rho)$. Hence $\pi_\rho(s^M(\rho)^\perp x) \Omega_\rho = 0$. Multiplying $\pi_\rho(M)'$, we obtain $0 = \pi_\rho(s^M(\rho)^\perp x) s^{\pi_\rho(M)}(\Omega_\rho) = \pi_\rho(s^M(\rho)^\perp x s^M(\rho))$. Substituting $s^M(\rho)^\perp Q$ into Q of $\rho(xQ) = \rho(Qx)$, we also obtain $\pi_\rho(s^M(\rho) x s^M(\rho)^\perp) = \pi_\rho(s^M(\rho)^\perp x^* s^M(\rho))^* = 0$. Hence $\pi_\rho([x, s^M(\rho)]) = 0$. Hence $s_c(\rho)[x, s^M(\rho)] = 0$ where $s_c(\rho)$ is the central support of ρ. Since $[1 - s_c(\rho)] s^M(\rho) = 0$, we have $[x, s^M(\rho)] = 0$.

Since Ω_ρ is cyclic for $R\equiv\pi_\rho(M)$, $s^{R'}(\Omega_\rho)=1$. Since $\tau_\rho(t)\pi_\rho(z)=\tau_\rho(t)\pi_\rho(zs^M(\rho))$ and $j_\rho(\pi_\rho(z))=j_\rho(\pi_\rho(z)s^M(\rho))$ by definitions of τ_ρ and j_ρ, it is enough to prove

$$\tau_\rho(t)\pi_\rho(z)=\pi_\rho(z)$$

$$j_\rho(\pi_\rho(z))=\pi_\rho(z)^*$$

for $z\in Z_\rho s^M(\rho)$ on $\pi_\rho(s^M(\rho))H_\rho\equiv H'_\rho$. Since Ω_ρ is cyclic and separating for $R_\rho\equiv\pi_\rho(s^M(\rho)Ms^M(\rho))$ on H'_ρ, the first equation is known. [8] It implies $[\Delta_\rho, \pi_\rho(z)]=0$. From $j_\rho(\bar{z})\Omega_\rho=\Delta_\rho^{1/2}\bar{z}^*\Omega_\rho=\bar{z}^*\Omega_\rho$ we have $j_\rho(\bar{z})=\bar{z}^*$ for $\bar{z}=\pi_\rho(z)$, $z\in M\cap M's^M(\rho)$. Q.E.D.

§2. Mapping F_ρ^{NM} from a von Neumann Algebra M into M'

Theorem 1. *Let M and N be von Neumann algebras such that $N\subset M'$. Let ρ be a normal positive linear functional of $(M\cup N)''$. There exists a unique mapping F_ρ^{NM} from M into N satisfying*

$$\rho(QQ')=\rho(F_\rho^{NM}(Q)Q'+Q'F_\rho^{NM}(Q))/2 \tag{2.1}$$

for all $Q\in M$, $Q'\in N$, and

$$s(F_\rho^{NM}(Q))\leqq s^N(\rho). \tag{2.2}$$

It has the following properties:

(1) *F_ρ^{NM} is $(M\cap N)$-linear. $F_\rho^{NM}(Q)^*=F_\rho^{NM}(Q^*)$.*

(2) *$F_\rho^{NM}(1)=s^N(\rho)$.*

(3) *F_ρ^{NM} is transposed-n-positive for all positive integers n. (In particular, F_ρ^{NM} is positive and $F_\rho^{NM}(Q)^*=F_\rho^{NM}(Q^*)$.)*

(4) *$\|F_\rho^{NM}\|=1$ for $\rho\neq 0$.*

(5) *F_ρ^{NM} is σ-weakly continuous (i.e. normal). It is continuous on the unit ball relative to the strong topology on M and * strong topology on N.*

(6) *For any automorphism* τ *of* $(M\cup N)''$ *satisfying* $\tau(M)=M$ *and* $\tau(N)=N$,

$$F_{\rho}^{NM}(\tau Q)=\tau F_{\rho*\tau}^{NM}(Q) \tag{2.3}$$

where $\tau^*\rho$ *is defined by* $(\tau^*\rho)(Q)=\rho(\tau Q)$. *In particular, if* $u\in M$ *is unitary,*

$$F_{\rho}^{NM}(uQu^*)=F_{u^*\rho u}^{NM}(Q) \tag{2.4}$$

and if $v\in N$ *is unitary*

$$vF_{\rho}^{NM}(Q)v^*=F_{v\rho v^*}^{NM}(Q) \tag{2.5}$$

where $(t_1\rho t_2)(Q)=\rho(t_2Qt_1)$.

(7) *For any* $A\in M\cap N$, $A\geqq 0$,

$$F_{A\rho}^{NM}(Q)=F_{\rho}^{NM}(Q)\,s(A).$$

(8) *If* $\lim_n \|\rho_n-\rho\|=0$ *and* $\lim_n s^N(\rho_n)=s^N(\rho)$, *then*

$$\lim_n F_{\rho_n}^{NM}(Q)=F_{\rho}^{NM}(Q),\ \lim_n F_{\rho_n}^{NM}(Q)^*=F_{\rho}^{NM}(Q)^*$$

uniformly for a bounded set of Q. (*If* $s^N(\rho_n)\leqq s^N(\rho)$, *then* $\lim_n \|\rho_n-\rho\|=0$ *implies* $\lim s^N(\rho_n)=s^N(\rho)$.)

Proof. Let $Q\in M$ and $Q'\in N$. Consider

$$f_Q(Q')=\rho(QQ'). \tag{2.6}$$

If $Q\geqq 0$, then

$$f_Q(Q')=\rho(Q^{1/2}Q'Q^{1/2})$$

is normal positive linear functional on N. If $Q'\geqq 0$ in addition,

$$f_Q(Q')=\rho(Q'^{1/2}QQ'^{1/2})\leqq \|Q\|\rho(Q'). \tag{2.7}$$

Hence $f_Q\leqq \|Q\|\rho$.

For general Q, we have

$$Q=Q_1-Q_2+i(Q_3-Q_4) \tag{2.8}$$

where Q_1 and Q_2 are positive and negative parts of $(Q+Q^*)/2$, Q_3 and Q_4 are positive and negative parts of $(Q-Q^*)/(2i)$. Then

$$f_Q=f_{Q_1}-f_{Q_2}+i(f_{Q_3}-f_{Q_4})$$

where $f_{Q_k}\leqq\|Q_k\|\rho$.

By Lemma 3, there exists a unique $h_0=F_\rho^{NM}(Q)\in N$ such that

$$f_Q(Q')=\rho(F_\rho^{NM}(Q)Q'+Q'F_\rho^{NM}(Q))/2 \tag{2.9}$$

for all $Q'\in N$ and

$$s(F_\rho^{NM}(Q))\leqq s^N(\rho). \tag{2.10}$$

This shows the existence and uniqueness of F_ρ^{NM}.

(1) Let $z_1, z_2\in M\cap N$ and $Q_1, Q_2\in M$. Note that $M\cap N$ is in the center of $(N\cup M)''$ by $N\subset M'$. We have, for $Q=z_1Q_1+z_2Q_2$,

$$\begin{aligned}\rho(F_\rho^{NM}(Q)Q'+Q'F_\rho^{NM}(Q))/2&=\rho(QQ')\\&=\rho(Q_1z_1Q')+\rho(Q_2z_2Q')\\&=\rho(F_\rho^{NM}(Q_1)z_1Q'+z_1Q'F_\rho^{NM}(Q_1))/2\\&\quad+\rho(F_\rho^{NM}(Q_2)z_2Q'+z_2Q'F_\rho^{NM}(Q_2))/2\\&=\rho(F'Q'+Q'F')/2\end{aligned}$$

where

$$F'=z_1F_\rho^{NM}(Q_1)+z_2F_\rho^{NM}(Q_2).$$

Since $s(F_\rho^{NM}(Q_k))\leqq s^N(\rho)$, $k=1, 2$, we also have $s(F')\leqq s^N(\rho)$. By the uniqueness, we have

$$F'=F_\rho^{NM}(z_1Q_1+z_2Q_2).$$

From $\rho(Q^*Q')=\rho(Q(Q')^*)^*$ and the uniqueness, we obtain $F_\rho^{NM}(Q)^*=F_\rho^{NM}(Q^*)$.

(2) The substitution of $Q=1$ and $F_\rho^{NM}(Q)=s^N(\rho)$ into (2.1) and (2.2) immediately prove this statement.

(3) If $Q\geqq 0$, then $F_\rho^{NM}(Q)\geqq 0$ from Lemma 1. Hence F_ρ^{NM} is positive.

To prove transposed-n-positivity for $n>1$, let $e_1, \cdots, e_n$ be an orthonormal basis of C^n,

$$\Omega=n^{-1/2}\sum_{k=1}^{n} e_k\otimes e_k\in C^n\otimes C^n,$$

J_Ω be the modular conjugation for Ω ($J_\Omega\sum c_{ij}e_i\otimes e_j=\sum \bar{c}_{ij}e_j\otimes e_i$), and the transposition t be chosen to be

$$ {}^tQ=J_\Omega Q^*J_\Omega. \tag{2.11}$$

which maps $Q\in\mathscr{B}(C^n)\otimes 1$ onto $1\otimes\mathscr{B}(C^n)$. Consider (on $H\otimes(C^n\otimes C^n)$)

$$\bar{M}=M\otimes(\mathscr{B}(C^n)\otimes 1),$$

$$\bar{N}=N\otimes(1\otimes\mathscr{B}(C^n)),$$

$$\bar{\rho}=\rho\otimes\omega_\Omega.$$

Then $F_\rho^{NM}\otimes t$ from $\bar{M}$ to $\bar{N}$ coincides with $F_{\bar{\rho}}^{\bar{N}\bar{M}}$ due to the following computation and hence is positive by our earlier result.

Let $Q_1\in M$, $Q_1'\in N$, $Q_2\in\mathscr{B}(C^n)\otimes 1$, $Q_2'\in 1\otimes\mathscr{B}(C^n)$. Then

$$\begin{aligned}\bar{\rho}((Q_1\otimes Q_2)(Q_1'\otimes Q_2'))&=\rho(Q_1Q_1')(\Omega, Q_2Q_2'\Omega)\\&=\rho(F_\rho^{NM}(Q_1)Q_1')(Q_2^*\Omega, Q_2'\Omega)/2\\&\quad+\rho(Q_1'F_\rho^{NM}(Q_1))(\Omega, Q_2'Q_2\Omega)/2\\&=\rho(F_\rho^{NM}(Q_1)Q_1')(j_\Omega(Q_2)\Omega, Q_2'\Omega)/2\\&\quad+\rho(Q_1'F_\rho^{NM}(Q_1))(\Omega, Q_2'j_\Omega(Q_2^*)\Omega)/2\end{aligned}$$

where we have used the fact that the modular operator for a faithful trace vector Ω is 1 and hence $j_\Omega(Q)\Omega = J_\Omega Q\Omega = \Delta_\Omega^{1/2}Q^*\Omega = Q^*\Omega$. Substituting the definition of ${}^tQ$, we have

$$\bar\rho((Q_1\otimes Q_2)Q') = \bar\rho(\{F_\rho^{NM}(Q_1)\otimes {}^tQ_2\}Q')/2$$
$$+\bar\rho(Q'\{F_\rho^{NM}(Q_1)\otimes {}^tQ_2\})/2$$

for $Q'=Q_1'\otimes Q_2'$. Since such Q' linearly span $N\otimes(1\otimes\mathscr{B}(C^n))$, the same equation holds for all Q' in $\bar N$. Since $s^{\bar N}(\bar\rho)=s^N(\rho)\otimes 1$ because Ω is cyclic for $1\otimes\mathscr{B}(C^n)$, we have $s(F_\rho^{NM}(Q_1)\otimes {}^tQ_2)\leqq s(F_\rho^{NM}(Q_1))\otimes 1\leqq s^{\bar N}(\bar\rho)$. Hence

$$F_{\bar\rho}^{\bar N\bar M}(Q_1\otimes Q_2) = (F_\rho^{NM}\otimes t)(Q_1\otimes Q_2). \tag{2.12}$$

(4) From Lemma 1 (3) and (2.7), we have

$$\|F_\rho^{NM}(Q)\|\leqq\|Q\|$$

for $Q\geqq 0$. Due to Lemma 4, we have

$$\|F_\rho^{NM}(Q)\|^2 = \|F_\rho^{NM}(Q)F_\rho^{NM}(Q)^*\|$$
$$\leqq\|F_\rho^{NM}(Q^*Q)\|\leqq\|Q^*Q\|=\|Q\|^2$$

for arbitrary Q. From (2), we obtain $\|F_\rho^{NM}\|=1$ if $\rho\neq 0$.

(5) Assume that a net $Q_\alpha\in M$ has a weak limit Q and $\|Q_\alpha\|\leqq 1$. Then

$$\lim_\alpha \rho(F_\rho^{NM}(Q_\alpha)Q'+Q'F_\rho^{NM}(Q_\alpha)) = \rho(F_\rho^{NM}(Q)Q'+Q'F_\rho^{NM}(Q)). \tag{2.13}$$

Since $\|F_\rho^{NM}(Q_\alpha)\|\leqq\|Q_\alpha\|\leqq 1$, the set of accumulation points

$$\bigcap_\beta\Big(\bigcup_{\alpha>\beta}F_\rho^{NM}(Q_\alpha)\Big)^{-(\text{weak})} \tag{2.14}$$

is non-empty due to the weak compactness. Let $\bar Q$ be in this set. Then from (2.13), we have

$$\rho(F_\rho^{NM}(Q)Q'+Q'F_\rho^{NM}(Q)) = \rho(\bar QQ'+Q'\bar Q).$$

From the uniqueness in Lemma 2, we have

$$\bar{Q}=F_{\rho}^{NM}(Q)$$

and hence the set (2.14) consists of a single point $F_{\rho}^{NM}(Q)$. Thus

$$\text{w-}\lim_{\alpha} F_{\rho}^{NM}(Q_{\alpha})=F_{\rho}^{NM}(\text{w-}\lim_{\alpha} Q_{\alpha}).$$

The weak continuity on bounded sets implies the normality and the σ-weak continuity for a positive linear mapping.

Next, we assume that a net $Q_{\alpha}\in M$ has a strong limit Q and $\|Q_{\alpha}\|\leqq 1$. Then $\|F_{\rho}^{NM}(Q_{\alpha}-Q)\|\leqq\|Q_{\alpha}-Q\|\leqq 2$. Hence

$$\lim_{\alpha} \rho(\{F_{\rho}^{NM}(Q_{\alpha}-Q)\}^{*}(Q_{\alpha}-Q))=0.$$

By using (2.1) with $Q=(Q_{\alpha}-Q)^{*}$, $Q'=F_{\rho}^{NM}(Q_{\alpha}-Q)$, we have

$$0\leqq\rho(\{F_{\rho}^{NM}(Q_{\alpha}-Q)\}^{*}F_{\rho}^{NM}(Q_{\alpha}-Q))+\rho(F_{\rho}^{NM}(Q_{\alpha}-Q)\{F_{\rho}^{NM}(Q_{\alpha}-Q)\}^{*})$$

$$=2\rho(\{F_{\rho}^{NM}(Q_{\alpha}-Q)\}^{*}(Q_{\alpha}-Q))\rightarrow 0$$

and hence

$$\lim \pi_{\rho}\{F_{\rho}^{NM}(Q_{\alpha}-Q)\}\Omega_{\rho}=0,$$

$$\lim \pi_{\rho}\{F_{\rho}^{NM}(Q_{\alpha}-Q)\}^{*}\Omega_{\rho}=0.$$

Multiplying $\hat{Q}\in\pi_{\rho}(N)'$, we have

$$\lim \pi_{\rho}\{F_{\rho}^{NM}(Q_{\alpha}-Q)\}\Psi=0,$$

$$\lim \pi_{\rho}\{F_{\rho}^{NM}(Q_{\alpha}-Q)^{*}\}\Psi=0.$$

for $\Psi=\hat{Q}\Omega_{\rho}$. Since $\|F_{\rho}^{NM}(Q_{\alpha}-Q)\|\leqq 2$, the same hold on the closure of $\pi_{\rho}(N)'\Omega_{\rho}$, which is $\pi_{\rho}(s^{N}(\rho))H_{\rho}$. Hence

$$\lim \pi_{\rho}\{F_{\rho}^{NM}(Q_{\alpha}-Q)s^{N}(\rho)\}=0, \qquad \lim \pi_{\rho}\{F_{\rho}^{NM}(Q_{\alpha}-Q)^{*}s^{N}(\rho)\}=0.$$

Since π_{ρ} is faithful at least on $s^{N}(\rho)Ns^{N}(\rho)$, π_{ρ}^{-1} is continuous on $s^{N}(\rho)Ns^{N}(\rho)$ and

$$0=\lim_{\alpha} s^{N}(\rho)F_{\rho}^{NM}(Q_{\alpha}-Q)s^{N}(\rho)=\lim_{\alpha} \{F_{\rho}^{NM}(Q_{\alpha})-F_{\rho}^{NM}(Q)\},$$

$$0=\lim_{\alpha} s^N(\rho)F_\rho^{NM}(Q_\alpha-Q)^*s^N(\rho)=\lim_{\alpha}\{F_\rho^{NM}(Q_\alpha)^*-F_\rho^{NM}(Q)^*\},$$

due to (2.2) and (1).

(6) For $Q\in M$ and $Q'\in N$, we have

$$\begin{aligned}\rho(\tau(Q)Q')&=\rho(\tau\{Q\tau^{-1}Q'\})\\ &=\tau^*\rho(Q\tau^{-1}Q')\\ &=\tau^*\rho(F_{\tau^*\rho}^{NM}(Q)\tau^{-1}Q')/2+\tau^*\rho(\{\tau^{-1}Q'\}F_{\tau^*\rho}^{NM}(Q))/2\\ &=\rho(\{\tau F_{\tau^*\rho}^{NM}(Q)\}Q')/2+\rho(Q'\tau F_{\tau^*\rho}^{NM}(Q))/2.\end{aligned}$$

We also have

$$\begin{aligned}s(\tau F_{\tau^*\rho}^{NM}(Q))&=\tau\{s(F_{\tau^*\rho}^{NM}(Q))\}\\ &\leqq\tau\{s^N(\tau^*\rho)\}=s^N(\rho).\end{aligned}$$

Hence (2.3) holds by the uniqueness.

(2.4) and (2.5) are special cases of (2.3) where $\tau(A)=uAu^*$ and $\tau(A)=vAv^*$ for $A\in(N\cup M)''$.

(7) Since $N\subset M'$, $M\cap N$ is in the center of $(N\cup M)''$. We have

$$\begin{aligned}(A\rho)(QQ')&=\rho(QQ'A)\\ &=\rho(F_\rho^{NM}(Q)Q'A)/2+\rho(Q'AF_\rho^{NM}(Q))/2\\ &=A\rho(s(A)F_\rho^{NM}(Q)Q'+Q's(A)F_\rho^{NM}(Q))/2.\end{aligned}$$

We also have

$$s\{s(A)F_\rho^{NM}(s)\}=s(A)s(F_\rho^{NM}(s))\leqq s(A)s^N(\rho)=s^N(A\rho).$$

Hence, by uniqueness, we have

$$F_{A\rho}^{NM}(Q)=F_\rho^{NM}(Q)\,s(A).$$

(8) We have for $\delta_n\equiv F_\rho^{NM}(Q)-F_{\rho_n}^{NM}(Q)$ the following estimate

$$|\rho(\delta_nQ'+Q'\delta_n)|\leqq2|\rho(QQ')-\rho_n(QQ')|$$

$$+|\rho_n(F_{\rho_n}^{NM}(Q)Q'+Q'F_{\rho_n}^{NM}(Q))-\rho(F_{\rho_n}^{NM}(Q)Q'+Q'F_{\rho_n}^{NM}(Q))|$$

$$\leqq 4\|Q\|\|Q'\|\|\rho-\rho_n\|.$$

Setting $Q'=\delta_n^*$ and using $\|\delta_n\|\leqq 2\|Q\|$, we have

$$0\leqq\rho(\delta_n^*\delta_n)\leqq\rho(\delta_n\delta_n^*+\delta_n^*\delta_n)\leqq 8\|Q\|^2\|\rho-\rho_n\|$$

$$0\leqq\rho(\delta_n\delta_n^*)\leqq 8\|Q\|^2\|\rho-\rho_n\|.$$

Hence we have

$$\lim_{n\to\infty}\pi_\rho(\delta_n)\Psi=0, \qquad \lim_{n\to\infty}\pi_\rho(\delta_n^*)\Psi=0,$$

for $\Psi=\Omega_\rho$ and hence for $\Psi=Q'\Omega_\rho$, $Q'\in\pi_\rho(N)'$. Since $\|\pi_\rho(\delta_n)\|\leqq 2\|Q\|$ is uniformly bounded, the same holds for $\Psi\in s^N(\Omega_\rho)H_\rho$ and hence

$$\lim_{n\to\infty}\pi_\rho(\delta_n s^N(\rho))=0, \qquad \lim_{n\to\infty}\pi_\rho(\delta_n^* s^N(\rho))=0,$$

uniformly for a bounded set of Q. Since π_ρ^{-1} is continuous on $Ns_c^N(\rho)$, where $s_c^N(\rho)$ is the central support of $s^N(\rho)$, we have

$$\lim_{n\to\infty}\{F_\rho^{NM}(Q)-F_{\rho_n}^{NM}(Q)s^N(\rho)\}=0,$$

$$\lim_{n\to\infty}\{F_\rho^{NM}(Q)^*-F_{\rho_n}^{NM}(Q)^*s^N(\rho)\}=0.$$

If $\lim s^N(\rho_n)=s^N(\rho)$, then as $\|F_{\rho_n}^{NM}(Q)\|\leqq\|Q\|$ we have

$$\lim_{n\to\infty}\{F_{\rho_n}^{NM}(Q)s^N(\rho)-F_{\rho_n}^{NM}(Q)\}$$

$$=\lim_{n\to\infty}F_{\rho_n}^{NM}(Q)(s^N(\rho)-s^N(\rho_n))=0$$

and we obtain

$$\lim F_{\rho_n}^{NM}(Q)=\lim F_\rho^{NM}(Q)$$

uniformly for a bounded set of Q. Similar equation for adjoint also holds.

If $s^N(\rho_n)\leqq s^N(\rho)$, then

$$|\rho(1-s^N(\rho_n))|=|\rho(1-s^N(\rho_n))-\rho_n(1-s^N(\rho_n))|\leqq\|\rho-\rho_n\|$$

and hence

$$\lim \pi_\rho(1-s^N(\rho_n))\Omega_\rho=0.$$

As before, we have

$$\lim (s^N(\rho)-s^N(\rho_n))=0. \qquad \text{Q.E.D.}$$

The proof of (3) implies the following corollaries.

Corollary 1. *If $M'=N$, $\rho=\omega_\Omega$ and Ω is a faithful trace vector for M as well as for N, then*

$$F_\rho^{NM}(Q)=J_\Omega Q^* J_\Omega.$$

Corollary 2. *Let $M=M_1\otimes M_2$, $N=N_1\otimes N_2$, $\rho=\rho_1\otimes\rho_2$. If ρ_1 is a trace on N_1 or if ρ_2 is a trace on N_2, then*

$$F^{NM}(Q_1\otimes Q_2)=F_{\rho_1}^{N_1}(Q_1)\otimes F_{\rho_2}^{N_2}(Q_2)$$

for all $Q_1\in M_1$, $Q_2\in M_2$. (In particular, if either N_1 or N_2 is abelian then this holds for any normal states ρ_1 and ρ_2.)

Remark 1. $F_{\rho_M}^{NM}(Q)=F_\rho^{NN'}(Q)$ for $Q\in M$, where ρ_M is the restriction of ρ (which is a functional on $(N\cup N')''$) to $(M\cup N)''$. In this sense, the case $M=N'$ is most canonical and we shall study it from different viewpoint in the next section.

Remark 2. In order to define $F_\rho^{NM}(Q)$, ρ need not be normal on the whole $(M\cup N)''$, but it is sufficient that ρ is normal on N. The uniqueness and existence together with properties (1), (2), (3), (4), (6), (7) and (8) hold for such non-normal ρ. Note that f_Q defined by (2.6) is normal due to (2.7) if ρ is normal on N.

Remark 3. Theorem 1 holds also for the case where N is a weakly closed * subalgebra of M' even if the unit in N is not the identity operator in M'.

§3. Mapping G_ρ^M from a von Neumann Algebra M into Itself

Theorem 2. *Let ρ be a normal positive linear functional of M. There exists a unique mapping G_ρ^M from M into $s^M(\rho)Ms^M(\rho)$ satisfying*

$$(\Omega_\rho, \pi_\rho(Q)\Delta_\rho^{1/2}\pi_\rho(Q')\Omega_\rho) = \rho(G_\rho^M(Q)Q' + Q'G_\rho^M(Q))/2 \tag{3.1}$$

for all $Q, Q' \in M$.

It has the following properties:

(1) *G_ρ^M is Z_ρ-linear, where Z_ρ is the set of $x \in M$ such that $\rho(xQ) = \rho(Qx)$ for all $Q \in M$, and M is considered as two-sided Z_ρ module. In particular, G_ρ^M is Z-linear for the center $Z = M \cap M'$.*

(2) $G_\rho^M(1) = s^M(\rho)$.

(3) *G_ρ^M is completely positive. (In particular, it is positive and $G^M(Q)^* = G^M(Q^*)$.)*

(4) $\|G_\rho^M\| = 1$ *for* $\rho \neq 0$.

(5) *G_ρ^M is σ-weakly continuous (i.e. normal). It is continuous on the unit ball relative to the strong topology for Q and $*$ strong topology for $G_\rho^M(Q)$.*

(6) *If τ is an automorphism of M and $\tau^*\rho = \rho$, then*

$$G_\rho^M(\tau Q) = \tau G_\rho^M(Q).$$

(7) *If $z \in Z$, $z \geqq 0$, then*

$$G_{z\rho}^M(Q) = G^M(Q)s^M(z).$$

(8) *The kernel of G_ρ^M is*

$$s^M(\rho)M(1 - s^M(\rho)) + (1 - s^M(\rho))M,$$

which implies

$$G_\rho^M(Q) = G_\rho^M(s^M(\rho)Qs^M(\rho)).$$

The image of G_ρ^M is strongly dense in $s^M(\rho)Ms^M(\rho)$.

Proof. Let $R=\pi_\rho(M)$ and $Q, Q'\in R$. From the formula $j_\Omega(Q)\Omega=\Delta_\Omega^{1/2}Q^*\Omega$ and (2.1), we have

$$(\Omega_\rho, Q\Delta_\rho^{1/2}Q'\Omega_\rho)=(\Omega_\rho, Qj_\rho\{Q'^*\}\Omega_\rho) \tag{3.2}$$
$$=(\Omega_\rho, F_\rho^{R'R}(Q)j_\rho(Q'^*)\Omega_\rho)/2$$
$$+(j_\rho(Q')\Omega_\rho, F_\rho^{R'R}(Q)\Omega_\rho)/2$$

where ρ is also used for $\rho(Q)=(\Omega_\rho, Q\Omega_\rho)$, $Q\in(R\cup R')''$, in writing $F_\rho^{R'R}$. Since $(J_\rho x, y)=(J_\rho^2J_\rho x, y)=(J_\rho x, J_\rho^2 y)=\overline{(x, J_\rho y)}=(J_\rho y, x)$ where $J_\rho^2=s^R(\Omega_\rho)$ is hermitian ($s^{R'}(\Omega_\rho)=1$ due to the cyclicity of Ω_ρ), and since $J_\rho\Omega_\rho=\Omega_\rho$, we have

$$(\Omega_\rho, Q\Delta_\rho^{1/2}Q'\Omega_\rho)=(Q'^*\Omega_\rho, j_\rho(F_\rho^{R'R}(Q)^*)\Omega_\rho)/2 \tag{3.3}$$
$$+(j_\rho(F_\rho^{R'R}(Q))\Omega_\rho, Q'\Omega_\rho)/2.$$

Since $s^R(j_\rho(F_\rho^{R'R}(Q)))\leqq s^R(\Omega_\rho)=\pi_\rho(s^M(\rho))$, there exists $G\in s^M(\rho)Ms^M(\rho)$ such that

$$\pi_\rho(G)=j_\rho(F_\rho^{R'R}(Q^*)). \tag{3.4}$$

From (3.2) and (3.3), $G_\rho^M(Q)=G$ satisfies (3.1) for all $Q'\in M$. Hence the existence is proved.

If $G_\rho^M(Q)=G$ and G' both satisfy (3.1), then $G-G'$ also satisfies $\rho((G-G')Q'+Q'(G-G'))=0$ for all $Q'\in M$. In particular, we have $\rho((G-G')^*(G-G'))=0$ for $Q'=(G-G')^*$. Since ρ is faithful on $s(\rho)Ms(\rho)$, we have $G-G'=0$ and hence the uniqueness.

(1) From (3.4) and Theorem 1 (1), G_ρ^M is linear. If $z\in Z_\rho$, then $\bar{z}=\pi_\rho(z)$ commutes with Δ_ρ (Lemma 5) and we have

$$(\Omega_\rho, \bar{Q}\bar{z}\Delta_\rho^{1/2}\bar{Q}'\Omega_\rho)=(\Omega_\rho, \bar{Q}\Delta_\rho^{1/2}\bar{z}\bar{Q}'\Omega_\rho)$$
$$=\rho(G_\rho^M(Q)zQ'+zQ'G_\rho^M(Q))/2$$
$$=\rho(G_\rho^M(Q)zQ'+Q'G_\rho^M(Q)z)/2$$

for $\bar{Q}=\pi_\rho(Q)$ and $\bar{Q}'=\pi_\rho(Q')$, $Q\in M$, $Q'\in M$. Since z commutes with $s(\rho)$ by Lemma 5, $s(G_\rho^M(Q)z)\leqq s(\rho)$ and hence

$$G_\rho^M(Qz)=G_\rho^M(Q)z.$$

Since $[j_\rho(\bar{z}), \varDelta_\rho]=j_\rho([\bar{z}, \varDelta_\rho^{-1}])=0$, we also have

$$\begin{aligned}(\Omega_\rho, \bar{z}\bar{Q}\varDelta_\rho^{1/2}\bar{Q}'\Omega_\rho)&=(\bar{z}^*\Omega_\rho, \bar{Q}\varDelta_\rho^{1/2}\bar{Q}'\Omega_\rho)\\ &=(j_\rho(\bar{z})\Omega_\rho, \bar{Q}\varDelta_\rho^{1/2}\bar{Q}'\Omega_\rho)\\ &=(\Omega_\rho, \bar{Q}\varDelta_\rho^{1/2}\bar{Q}'j_\rho(\bar{z}^*)\Omega_\rho)\\ &=(\Omega_\rho, \bar{Q}\varDelta_\rho^{1/2}\bar{Q}'\bar{z}\Omega_\rho)\\ &=\rho(G_\rho^M(Q)Q'z+Q'zG_\rho^M(Q))/2\\ &=\rho(zG_\rho^M(Q)Q'+Q'zG_\rho^M(Q))/2.\end{aligned}$$

Hence we have

$$G_\rho^M(zQ)=zG_\rho^M(Q).$$

(2), (4) and (5) follow from the corresponding results in Theorem 1 and (3.4).

(3) Let $Q_{ij}\in R$ such that $\sum_{i,j}(x_i, Q_{ij}x_j)\geqq 0$ for any $x_j\in H_\rho$ where the indices i, j run from 1 to n. By Theorem 1 (3),

$$\sum(x_i, F_\rho^{R'R}(Q_{ji})x_j)\geqq 0$$

for any vectors $x_j\in H_\rho$. Hence

$$\begin{aligned}&\sum(x_i, \pi_\rho(G_\rho^M(Q_{ij}))x_j)\\ &\quad=\sum(x_i, J_\rho^2J_\rho F_\rho^{R'R}(Q_{ij})^*J_\rho x_j)\\ &\quad=\sum(J_\rho^2x_i, J_\rho F_\rho^{R'R}(Q_{ij})^*J_\rho x_j)\\ &\quad=\sum(F_\rho^{R'R}(Q_{ij})^*J_\rho x_j, J_\rho x_i)\\ &\quad=\sum(J_\rho x_j, F_\rho^{R'R}(Q_{ij})J_\rho x_i)\geqq 0.\end{aligned}$$

Since π_ρ is faithful on $s^M(\rho)Ms^M(\rho)$, this proves n-positivity of G_ρ^M.

(6) If $\tau^*\rho=\rho$, there exists a unitary operator $U_\rho(\tau)$ on H_ρ such that

$$U_\rho(\tau)\pi_\rho(Q)\Omega_\rho=\pi_\rho(\tau Q)\Omega_\rho.$$

Applying S, we have

$$SU_\rho(\tau)\pi_\rho(Q)\Omega_\rho=\pi_\rho(\tau Q^*)\Omega_\rho$$

$$=U_\rho(\tau)S\pi_\rho(Q)\Omega_\rho.$$

Hence $U_\rho(\tau)$ also commutes with closure $\bar{S}$ and hence with Δ_ρ and J_ρ. We also have $\tau s^M(\rho)=s^M(\rho)$. From Theorem 1 (6), we now have, for $\bar{\tau}Q\equiv U_\rho(\tau)QU_\rho(\tau)^*$,

$$\pi_\rho(G_\rho^M(\tau Q))=j_\rho(F_\rho^{R'R}(\bar{\tau}\bar{Q}^*))$$

$$=j_\rho(\bar{\tau}F_\rho^{R'R}(\bar{Q}^*))$$

$$=\bar{\tau}\pi_\rho(G_\rho^M(Q))$$

$$=\pi_\rho(\tau G_\rho^M(Q)).$$

Since $s^M(\tau G_\rho^M(Q))\leqq s^M(\tau\rho)=s^M(\rho)$, we have (6).

(7) It follows from Theorem 1 (7) and $j_\pi(\pi_\rho(s^M(z)^*))=\pi_\rho(s^M(z))$. The latter equation is due to Lemma 5.

(8) From $G_\rho^M(Q)=0$ and (3.1), we obtain

$$0=(\Omega_\rho,\ \pi_\rho(Q)\Delta_\rho^{1/2}\pi_\rho(Q')\Omega_\rho)$$

$$=(\Omega_\rho,\ \pi_\rho(Q)j_\rho(\pi_\rho(Q')^*)\Omega_\rho)$$

$$=(j_\rho(\pi_\rho(Q'))\Omega_\rho,\ \pi_\rho(Q)\Omega_\rho).$$

Since $j_\rho(\pi_\rho(M))\Omega_\rho=\pi_\rho(M)'\Omega_\rho$ span $\pi_\rho(s^M(\rho))H_\rho$ $(=s^R(\Omega_\rho)H_\rho)$, we have

$$\pi_\rho(s^M(\rho)Q)\Omega_\rho=\pi_\rho(s^M(\rho))\pi_\rho(Q)\Omega_\rho=0.$$

By multiplying $Q'\in\pi_\rho(M)'$, we obtain

$$\pi_\rho(s^M(\rho)Qs^M(\rho))\Psi = \pi_\rho(s^M(\rho)Q)s^M(\Omega_\rho)\Psi = 0$$

for $s^R(\Omega_\rho)\Psi = Q'\Omega_\rho$ and hence for all Ψ. Therefore

$$\pi_\rho(s^M(\rho)Qs^M(\rho)) = 0$$

and hence $s^M(\rho)Qs^M(\rho) = 0$. Thus Q must be in $s^M(\rho)M(1-s^M(\rho)) + (1-s^M(\rho))M$. On the other hand, if Q is in this set, (3.1) vanishes and hence by the uniqueness of $G_\rho^M(Q)$, we have $G_\rho^M(Q) = G_\rho^M(0) = 0$.

To prove that the image of G_ρ^M is strongly dense in $s^M(\rho)Ms^M(\rho)$, it is enough to prove that the image of G_ρ^M is strongly dense in M for faithful ρ because ρ is faithful on $s^M(\rho)Ms^M(\rho)$. Assume that ρ is faithful on M.

Let $\bar{Q} \in \pi_\rho(M)$ and

$$\bar{Q}_\beta \equiv \int \tau_\rho(t)\bar{Q}\exp(-t^2/\beta)\,dt/(\beta\pi)^{1/2}.$$

It satisfies $\|\bar{Q}_\beta\| \leqq \|Q\|$, $\lim_{\beta\to 0} \bar{Q}_\beta = \bar{Q}$. Furthermore,

$$\tau_\rho(t)\bar{Q}_\beta = \int \tau_\rho(s)\bar{Q}\exp(-(t-s)^2/\beta)\,ds/(\beta\pi)^{1/2}$$

is analytic for all t. Hence, for $Q' \in \pi_\rho(M)$, we have

$$(\Omega_\rho, (\bar{Q}_\beta Q' + Q'\bar{Q}_\beta)\Omega_\rho) = (\Omega_\rho, (\bar{Q}_\beta + \tau_\rho(i)\bar{Q}_\beta)Q'\Omega_\rho)$$

$$= (\Omega_\rho, (\tau_\rho(-i/2)\bar{Q}_\beta + \tau_\rho(i/2)\bar{Q}_\beta)\Delta_\rho^{1/2}Q'\Omega_\rho)$$

where the first equality is due to KMS condition. Hence we have for $Q \in M$, $\bar{Q} = \pi_\rho(Q)$, $\bar{Q}_\beta = \pi_\rho(Q_\beta)$

$$G_\rho^M(\{\tau_\rho(-i/2)Q_\beta + \tau_\rho(i/2)Q_\beta\}) = 2Q_\beta. \tag{3.5}$$

Thus the image of G_ρ^M is strongly dense in M for faithful ρ.

Q.E.D.

§4. Projections of a von Neumann Algebra into Its Center

Theorem 3. *Let Z denote the center of M and $N\subset Z$. Then F_{ρ}^{NM} has the following properties besides the properties* (1)–(8) *of Theorem* 1.

(9) *F_{ρ}^{NM} is a projection from M onto $Ns^{N}(\rho)$.*

(10) *Define ρ and ρ' to be N-equivalent if $s^{N}(\rho)=s^{N}(\rho')$ and ρ' is in the norm closure of the set of all $A\rho$, $A\in N$, $A\geqq 0$. It is an equivalence relation and $F_{\rho}^{NM}=F_{\rho'}^{NM}$ if and only if ρ is N-equivalent to ρ'.*

(11) *Let $s^{N'}(\Omega_{\rho})$ be the projection on the closure of $\pi_{\rho}(N)\Omega_{\rho}$. The mapping from $Q\in Ns^{N}(\rho)$ to $s^{N'}(\Omega_{\rho})\pi_{\rho}(Q)\in s^{N'}(\Omega_{\rho})\pi_{\rho}(s^{N}(\rho)M)$ is bijective. Let the inverse mapping be α. Then*

$$F_{\rho}^{NM}(Q)=\alpha s^{N'}(\Omega_{\rho})\pi_{\rho}(Q)s^{N'}(\Omega_{\rho}). \tag{4.1}$$

(12) *If $K\subset N$, then $F_{\rho}^{KN}F_{\rho}^{NM}=F_{\rho}^{KM}$.*

Proof. (9) $F_{\rho}^{NM}(Q)=Qs^{N}(\rho)=Q$ for $Q\in Ns(\rho)$ due to Theorem 1 (1) and (2). Hence F_{ρ}^{NM} is a projection onto $Ns^{N}(\rho)$.

(10) If ρ is N-equivalent to ρ', then ρ' is a norm limit of $A_n\rho$, where we may restrict $s^{N}(\rho)A_n\rho\equiv\rho_n$. Then by Theorem 1 (7) and (8), we have $F_{\rho'}^{NM}(Q)=\lim F_{\rho_n}^{NM}(Q)=F_{\rho}^{NM}(Q)$.

Next assume that $F_{\rho'}^{NM}=F_{\rho}^{NM}$. From Theorem 1 (2), we have

$$s^{N}(\rho)=s^{N}(\rho').$$

By the Radon-Nikodym theorem, there exists a non-negative self-adjoint operator A affiliated with N such that $s(A)=s^{N}(\rho)$ and

$$\rho(QA)=\rho'(Q),\qquad Q\in N.$$

Let E_{λ}^{A} be the spectral projection of A and $A_n=AE_n^{A}\in N$, $\rho_n=A_n\rho$. Let $\bar{\rho}\equiv A\rho=\lim_{n} A_n\rho$ which exists as a state of M, because $0\leqq\rho(A_nQ)-\rho(A_mQ)\leqq\|Q\|\rho(A_n-A_m)\to 0$ for $Q\geqq 0$, $Q\in M$ and $n\geqq m$. Then the restriction of $\bar{\rho}$ to N is the same as the restriction of ρ' to N. By what we have already proved, $F_{\bar{\rho}}^{NM}=F_{\rho}^{NM}=F_{\rho'}^{NM}$. Hence we have

$$\bar{\rho}(QQ')=\rho'(QQ')$$

for all $Q\in M$ and $Q'\in N$. Setting $Q'=1$, we have $\bar{\rho}=\rho'$ as a functional on M. This shows that ρ is N-equivalent to ρ'.

$F_{\rho}^{NM}=F_{\rho'}^{N,M}$ is certainly an equivalence relation for ρ and ρ'.

(11) Since ρ is faithful on $Ns^{N}(\rho)$, $s^{N'}(\Omega_{\rho})\pi_{\rho}(Q)=0$ for $Q\in Ns^{N}(\rho)$ implies $\|s^{N'}(\Omega_{\rho})\pi_{\rho}(Q)\Omega_{\rho}\|^{2}=\rho(Q^{*}Q)=0$ and hence $Q=0$. Thus $Q\rightarrow s^{N'}(\Omega_{\rho})\pi_{\rho}(Q)$ is bijective from $Ns^{N}(\rho)$ to $s^{N'}(\Omega_{\rho})\pi_{\rho}(Ns^{N}(\rho))$.

We have, for $Q\in M$, $Q'\in N$,

$$\rho(QQ')=(\Omega_{\rho},\ \pi_{\rho}(Q)\pi_{\rho}(Q')\Omega_{\rho})$$

$$=(\Omega_{\rho},\ s^{N'}(\Omega_{\rho})\pi_{\rho}(Q)s^{N'}(\Omega_{\rho})\pi_{\rho}(Q')\Omega_{\rho}).$$

If we prove that

$$s^{N'}(\Omega_{\rho})\pi_{\rho}(M)s^{N'}(\Omega_{\rho})=\pi_{\rho}(Ns^{N}(\rho))s^{N'}(\Omega_{\rho}), \tag{4.2}$$

then we have

$$\rho(QQ')=\rho(\{\alpha s^{N'}(\Omega_{\rho})\pi_{\rho}(Q)s^{N'}(\Omega_{\rho}))\}Q').$$

Due to the commutativity of elements of N, we have (4.1).

To prove (4.2), we note that Ω_{ρ} is a cyclic vector for abelian $\pi_{\rho}(N)$ on $s^{N'}(\Omega_{\rho})H_{\rho}$ by definition and hence maximal abelian there. Furthermore, $\pi_{\rho}(1-s^{N}(\rho))Q\Omega_{\rho}=0$ for $Q\in\pi_{\rho}(N)$ by the commutativity and hence $s^{N'}(\Omega_{\rho})\pi_{\rho}(s^{N}(\rho))=s^{N'}(\Omega_{\rho})$. Thus any $Q\in\mathscr{B}(s^{N'}(\Omega_{\rho})H_{\rho})$ satisfying $[Q,Q_{1}]=0$ for all $Q_{1}\in\pi_{\rho}(N)$ belongs to $\pi_{\rho}(Ns^{N}(\rho))s^{N'}(\Omega_{\rho})$.

Since $s^{N'}(\Omega_{\rho})\in\pi_{\rho}(N)'$ and N commutes with M, $Q\in s^{N'}(\Omega_{\rho})\pi_{\rho}(M)s^{N'}(\Omega_{\rho})$ commutes with any $Q_{1}\in\pi_{\rho}(N)$. Hence

$$s^{N'}(\Omega_{\rho})\pi_{\rho}(M)s^{N'}(\Omega_{\rho})\subseteqq\pi_{\rho}(Ns^{N}(\rho))s^{N'}(\Omega_{\rho}).$$

Since $M\supset Ns^{N}(\rho)$, the equality holds.

(12) This is immediate from the defining equations (2.1) and (2.2) and the abelian property of N. Q.E.D.

Corollary. (10), (11) *and* (12) *of Theorem* 3 *holds if* $N\subset M'$ *and* N *is abelian.*

Proof. Let $R=(N\cup M)''$. Then N is in the center of R. Furthermore

$$F_\rho^{NM}(Q)=F_\rho^{NR}(Q), \qquad Q\in M.$$

Hence by applying Theorem 3 (10) and (11) to F_ρ^{NR}, we obtain (10) and (11) for F_ρ^{NM}. Note that $F_\rho^{NR}(Q)$ for $Q\in M$ determines F_ρ^{NR} due to the property (1) of Theorem 1. Q.E.D.

Remark. If N is abelian, $Q\in N$ can be identified with continuous function on its spectrum and any normal linear function on N with a Radon measure on its spectrum. Denoting the measure corresponding to the normal linear functional $\rho(QQ')=f_Q(Q')$ for $Q'\in N$ and $Q\in M$ by μ_Q, F_ρ^{NM} is given by the Radon-Nikodym derivative:

$$F_\rho^{NM}(Q)=d\mu_Q/d\mu_1$$

where we define $d\mu_Q/d\mu_1=0$ outside the support of $s^N(\rho)$.

$F_\rho^{NM}(Q)$ for an abelian N has been introduced through the equation (4.1) by D. Ruelle [5] in his theory of decomposition of state. If μ_ρ denotes the measure on the spectrum Ξ_N of N, corresponding to the restriction of ρ to N, then

$$\rho(Q)=\int_{\Xi_N}\xi(F_\rho^{NM}(Q))\,d\mu_\rho(\xi)$$

is his decomposition.

§5. Asymptotically Abelian System

A net Q_α of elements of a von Neumann algebra M is called weakly central if there exists a weakly total selfadjoint subset M_0 of M such that

$$[x, Q_\alpha]\to 0 \tag{5.1}$$

in the weak topology for every $x\in M_0$. If (5.1) holds with the strong limit, then Q_α is called strongly central.

The following result is an extension of Proposition 4 of [1] to non-factors.

Theorem 4. *If* Q_α *is a uniformly bounded weakly central net in* M, *then*

$$\text{w-}\lim_\alpha (Q_\alpha - F_\rho^{ZM}(Q_\alpha))s^Z(\rho) = 0 \tag{5.2}$$

for any normal positive linear functional ρ *on* M, *where* $Z = M \cap M'$.

For any two normal positive linear functionals ρ *and* ρ',

$$\text{w-}\lim_\alpha (F_\rho^{ZM}(Q_\alpha) - F_{\rho'}^{Z,M}(Q_\alpha))(s^Z(\rho) \wedge s^Z(\rho')) = 0. \tag{5.3}$$

In particular, if $s^Z(\rho) = s^Z(\rho')$,

$$\text{w-}\lim_\alpha (F_\rho^{ZM}(Q_\alpha) - F_{\rho'}^{Z,M}(Q_\alpha)) = 0. \tag{5.4}$$

When $s^Z(\rho') \leqq s^Z(\rho)$, *let* $A^Z(\rho'/\rho)$ *be the Radon-Nikodym derivative of* ρ' *by* ρ *relative to* Z, *namely*,

$$A^Z(\rho'/\rho) = \int \lambda \, dE_\lambda, \qquad E_\lambda \in Z,$$

$$s(A^Z(\rho'/\rho)) = s^Z(\rho'),$$

$$\rho'(z) = \rho(zA^Z(\rho'/\rho)), \qquad z \in Z,$$

where $A^Z(\rho'/\rho)$ *can be unbounded and* $\rho(zA^Z(\rho'/\rho)) \equiv \int \lambda \, d(\rho(zE_\lambda))$.

If $s^Z(\rho') \leqq s^Z(\rho)$, *then*

$$\lim_\alpha \{\rho'(Q_\alpha) - \rho(Q_\alpha A(\rho'/\rho))\} = 0. \tag{5.5}$$

In particular, if $A(\rho'/\rho) = 1$ (*i.e. if* $\rho | Z = \rho' | Z$), *then*

$$\lim_\alpha \{\rho'(Q_\alpha) - \rho(Q_\alpha)\} = 0. \tag{5.6}$$

Proof. Consider H_ρ, π_ρ, Ω_ρ canonically associated with $\rho \neq 0$. Let $\bar{Q}_\alpha = \pi_\rho(Q_\alpha)$. Let R_0 be the linear hull of $\pi_\rho(M_0)$, $R = \pi_\rho(M) = \pi_\rho(M_0)'' = \bar{R}_0$, $\bar{Z} = \pi_\rho(Z)$, $s' = s^{\bar{Z}'}(\Omega z_\rho)$.

Given $\varepsilon>0$ and vectors $\Phi_j\in H$, $j=1,\cdots,n$, $\Phi_j\neq 0$, there exist $Q_1'',\cdots,Q_k''\in R_0$ and $Q_1',\cdots,Q_k'\in R'$ such that

$$P_\varepsilon\equiv\sum_{j=1}^{k}Q_j''Q_j'$$

satisfies

$$\|P_\varepsilon\Omega_\rho-\Omega_\rho\|=\|\{P_\varepsilon-s'\}\Omega_\rho\|$$

$$\leqq\{\sup_\alpha\|Q_\alpha\|\}^{-1}\{\sup\|\Phi_j\|\}^{-1}\varepsilon/4,$$

$$\|\{P_\varepsilon^*-s'\}\Phi_j\|\leqq\{\sup_\alpha\|Q_\alpha\|\}^{-1}\|\Omega_\rho\|^{-1}\varepsilon/4,$$

because $s'\in\bar{Z}'$ and linear hull of R_0R_0' is $*$ strongly dense in $\bar{Z}'$.

For this set of operators, there exists α_ε such that for all $\alpha>\alpha_\varepsilon$,

$$|(\Phi_j,[\bar{Q}_\alpha,P_\varepsilon]\Omega_\rho)|<\varepsilon/2,$$

due to the weakly central property.

Then for $\alpha>\alpha_\varepsilon$, we have

$$|(\Phi_j,[\bar{Q}_\alpha,s']\Omega_\rho)|$$

$$\leqq|(\Phi_j,[\bar{Q}_\alpha,P_\varepsilon]\Omega_\rho)|+\varepsilon/2$$

$$<\varepsilon.$$

Hence

$$\text{w-}\lim_\alpha[\bar{Q}_\alpha,s']\Omega_\rho=0. \tag{5.7}$$

By Theorem 3 (11), we have

$$s'\bar{Q}_\alpha s'=\pi_\rho(F_\rho^{ZM}(Q_\alpha))s'. \tag{5.8}$$

Since $s'\Omega_\rho=\Omega_\rho$, we obtain from (5.7) and (5.8)

$$\text{w-}\lim_\alpha\pi_\rho(Q_\alpha-F_\rho^{ZM}(Q_\alpha))\Omega_\rho=0.$$

Take any $Q''\in R_0$, $Q'\in R_0'$. Since $\pi_\rho(F_\rho^{ZM}(Q_\alpha))\in\bar{Z}$, it commutes with

$Q''Q'$. By weakly central property,

$$\text{w-}\lim_{\alpha} [\pi_\rho(Q_\alpha), Q''Q']\Omega_\rho = 0.$$

Hence

$$\text{w-}\lim_{\alpha} \pi_\rho(Q_\alpha - F_\rho^{ZM}(Q_\alpha))\Psi = 0 \tag{5.9}$$

for $\Psi = Q''Q'\Omega_\rho$. Since Q_α is assumed to uniformly bounded and $\|F_\rho^{ZM}\| = 1$, (5.9) holds for all Ψ in the closure of $\bar{Z}'\Omega_\rho$, which is $s^{\bar{Z}}(\Omega_\rho)H_\rho = \pi_\rho(s^Z(\rho))H_\rho$. Hence

$$\text{w-}\lim_{\alpha} \pi_\rho(\{Q_\alpha - F_\rho^{ZM}(Q_\alpha)\}s^Z(\rho)) = 0.$$

Since π_ρ is faithful on $s^Z(\rho)M$, we have (5.2).

From (5.2) for ρ and ρ', we have (5.3) and in the special case $s^Z(\rho') = s^Z(\rho)$, we obtain (5.4), where we use $F_\rho^{ZM}(Q_\alpha)s^Z(\rho) = F_\rho^{ZM}(Q_\alpha)$.

If $s^Z(\rho') \leqq s^Z(\rho)$, we obtain from (5.2)

$$\lim_{\alpha} \{\rho'(Q_\alpha) - \rho'(F_\rho^{ZM}(Q_\alpha))\} = 0. \tag{5.10}$$

Using the definition of $A^Z(\rho'/\rho)$ and (2.1) with $Q' = 1$, we obtain

$$\begin{aligned}\rho'(F_\rho^{ZM}(Q_\alpha)) &= \int \lambda\, d\rho(F_\rho^{ZM}(Q_\alpha)E_\lambda) \\ &= \int \lambda\, d\rho(Q_\alpha E_\lambda) = \rho(Q_\alpha A^Z(\rho'/\rho)).\end{aligned}$$

This proves (5.5). (5.6) then follows. Q.E.D.

If a subset $\mathfrak{A}$ of a von Neumann algebra M and a net of * automorphisms τ_α of M satisfy the property that $\tau_\alpha Q$ for every $Q \in \mathfrak{A}$ is weakly (or strongly) central, then $\mathfrak{A}$ is called weakly (or strongly) τ_α central in M.

Corollary. *If $\mathfrak{A}$ is weakly τ_α central in M and ρ is a τ_α invariant normal positive linear functional on M, then*

$$\text{w-}\lim_{\alpha} (\tau_\alpha Q - \tau_\alpha F_\rho^{ZM}(Q))s^Z(\rho) = 0 \tag{5.11}$$

for all $Q\in\mathfrak{A}$ *where* Z *is the center of* M.

If ρ' *is another normal positive linear functional on* M *and* $s^Z(\rho')\leqq s^Z(\rho)$, *then*

$$\lim_{\alpha}\{\rho'(\tau_\alpha Q)-\rho(Q\tau_\alpha^{-1}A^Z(\rho'/\rho))\}=0 \tag{5.12}$$

for all $Q\in\mathfrak{A}$ *where* $\tau_\alpha^{-1}A^Z(\rho'/\rho)=\int\lambda d(\tau_\alpha^{-1}E_\lambda)$. *In particular if* $\rho'(z)=\rho(z)$ *for all* $z\in Z$, *then*

$$\lim_{\alpha}\rho'(\tau_\alpha Q)=\rho(Q),\qquad Q\in\mathfrak{A}. \tag{5.13}$$

Proof. Since $||\tau_\alpha Q||=||Q||$, $\tau_\alpha Q$ is uniformly bounded. By (5.2), (2.3) and $\tau_\alpha^*\rho=\rho$, we have (5.11). (5.12) follows from (5.5) and the invariance of ρ. (5.13) is a special case of (5.12) where $s^Z(\rho')=s^Z(\rho)$ and $A^Z(\rho'/\rho)=1$. Q.E.D.

Remark. If Q_α is weakly central and uniformly bounded, then w-lim $[x, Q_\alpha]=0$ for all $x\in M$, because it holds for any x in the linear hull M_1 of M_0, which, being a weakly dense linear subset, is $*$ strongly dense in M, and hence for given $x\in M$, $\varepsilon>0$, Ψ_j, Φ_j, there exist $x'\in M_1$ and α_0 such that $||Q_\alpha||<L$,

$$||\Psi_j||L||(x-x')\Phi_j||<\varepsilon/3,\ ||(x-x')^*\Psi_j||L||\Phi_j||<\varepsilon/3$$

and

$$|(\Psi_j, [x', Q_\alpha]\Phi_j)|<\varepsilon/3\qquad \text{for } \alpha>\alpha_0$$

which imply $|(\Psi_j, [x, Q_\alpha]\Phi_j)|<\varepsilon$, $j=1, \cdots, n$.

Hence, if $\mathfrak{A}$ is weakly τ_α central, then the norm closure $\mathfrak{A}_1$ of the linear hull of $\mathfrak{A}\cup\mathfrak{A}^*$ is obviously weakly τ_α central and (5.2)–(5.6) for $Q_\alpha=\tau_\alpha Q$ and (5.11)–(5.13) hold for any $Q\in\mathfrak{A}_1$.

(5.11)–(5.13) hold for σ-weak colsure of $\mathfrak{A}_1$ if $s^Z(\rho)$ and $s^Z(\rho')\leqq s^Z(\rho)$ are replaced by $s^M(\rho)$ and $s^M(\rho')\leqq s^M(\rho)$, because (5.11) implies

$$\text{w-lim}\,\pi_\rho(\tau_\alpha Q-\tau_\alpha F_\rho^{ZM}(Q))\Psi=0$$

for $\Psi=\Omega_\rho$ and Q in the strong closure of the unit ball of $\mathfrak{A}_1$ and hence for $\Psi\in\overline{R'\Omega_\rho}$ and Q in the σ-weak closure of $\mathfrak{A}_1$.

The next theorem has an application in [2].

Description of situation. A von Neumann algebra M, a net of $*$ automorphisms τ_α, a faithful normal positive linear functional $\rho\neq 0$ on M, invariant under all τ_α and a C^* subalgebra $\mathfrak{A}$ of M are given. Let U_α be the unique unitary operator on H_ρ satisfying $U_\alpha\pi_\rho(Q)\Omega_\rho=\pi_\rho(\tau_\alpha Q)\Omega_\rho$ for all $Q\in M$. Let $\bar{\tau}_\alpha Q\equiv U_\alpha Q U_\alpha^*$ for all $Q\in\mathscr{B}(H_\rho)$. Let J_ρ be the modular conjugation operator for the cyclic and separating Ω_ρ relative to $\pi_\rho(M)$ and $j_\rho(Q)\equiv J_\rho Q J_\rho$, $Q\in\mathscr{B}(H_\rho)$. Let $\hat{\mathfrak{A}}$ be the C^* algebra generated by

$$\pi_\rho(\mathfrak{A})j_\rho\{\pi_\rho(\mathfrak{A})\} \tag{5.14}$$

and $\hat{R}\equiv(\pi_\rho(M)\cup\pi_\rho(M)')''$.

Theorem 5. *Assume that $\mathfrak{A}$ is strongly τ_α central in M. For any normal positive linear functional ρ' on $\mathscr{B}(H_\rho)$, all $Q\in\hat{\mathfrak{A}}$ satisfy*

$$\lim_\alpha\{\rho'(\bar{\tau}_\alpha Q)-(\Omega_\rho, Q\bar{\tau}_\alpha^{-1}A^{\bar{Z}}(\rho'/\bar{\rho})\Omega_\rho)\}=0 \tag{5.15}$$

where $\bar{\rho}=\omega_\Omega$, $\bar{Z}=\pi_\rho(M)\cap\pi_\rho(M)'$ which is the center of $\hat{R}$ and $A^{\bar{Z}}(\rho'/\bar{\rho})$ is as in Theorem 4. In particular, if $\rho(z)=\rho'(\pi_\rho(z))$ for all z in the center of M, then

$$\lim_\alpha \rho'(\bar{\tau}_\alpha Q)=\bar{\rho}(Q),\qquad Q\in\hat{\mathfrak{A}}. \tag{5.16}$$

Proof. Let $S_\rho=J_\rho\Delta_\rho^{1/2}$. We have

$$\begin{aligned}U_\alpha S_\rho Q\Omega_\rho&=U_\alpha Q^*\Omega_\rho=(\bar{\tau}_\alpha Q)^*\Omega_\rho\\&=S_\rho(\bar{\tau}_\alpha Q)\Omega_\rho=S_\rho U_\alpha Q\Omega_\rho.\end{aligned}$$

where $Q\in\pi_\rho(M)$, which implies $\bar{\tau}_\alpha Q\in\pi_\rho(M)$. Thus U_α commutes with S_ρ and hence with $\Delta_\rho=S_\rho^*S_\rho$ and J_ρ.

Let $Q, Q'\in\mathfrak{A}$, $Q_0\in M_0(Q)$, $Q_0'\in M_0(Q')$ where $M_0(Q)$ is a selfadjoint total subset of M such that (5.1) is satisfied in the strong topology for all $x\in M_0(Q)$ and $Q_\alpha=\tau_\alpha(Q)$ and $M_0(Q')$ is the same for Q'.

$$[\pi_\rho(Q_0)j_\rho\{\pi_\rho(Q_0')\},\ \bar{\tau}_\alpha(\pi_\rho(Q)j_\rho\{\pi_\rho(Q')\})] \tag{5.17}$$
$$=\pi_\rho([Q_0,\ \tau_\alpha Q])j_\rho\{\pi_\rho(Q_0'\tau_\alpha Q')\}$$
$$+\pi_\rho(\{\tau_\alpha Q\}Q_0)j_\rho\{\pi_\rho([Q_0',\ \tau_\alpha Q'])\}.$$

Since both $[Q_0, \tau_\alpha Q]$ and $[Q_0', \tau_\alpha Q']$ tends to 0 strongly, and all operators are bounded uniformly in α, (5.17) tends to 0 strongly.

Since $M_0(Q)$ and $M_0(Q')$ are selfadjoint and total,

$$\hat{M}_0=\pi_\rho(M_0(Q))j_\rho\{\pi_\rho(M_0(Q'))\}$$

is also selfadjoint and total in $\hat{R}$. Hence (5.14) is strongly τ_α central in $\hat{R}$.

By (5.12) and (5.13), we obtain (5.15) and (5.16) when Q is in (5.14). Note that $s^Z(\rho)=1$ because ρ is assumed to be faithful. Note also that $\bar{Z}=\pi_\rho(Z)$.

By Remark after Corollary to Theorem 4, Q in (5.15) and (5.16) can be in the norm closure of the linear hull of (5.14), which is $\hat{\mathfrak{A}}$.

Q.E.D.

Remark. Let $\mathfrak{A}_1$ be a C^* algebra, ρ be a state on $\mathfrak{A}_1$ and τ_α be a net of $*$ automorphisms of $\mathfrak{A}_1$. If

$$\pi_\rho([Q_1,\ \tau_\alpha Q_2])$$

tends to 0 weakly (or strongly) for all $Q_1, Q_2\in\mathfrak{A}_1$, then $\mathfrak{A}_1$ is said to be weakly (or strongly) τ_α asymptotically abelian. We can apply Theorem 4 to such a situation by taking $M_0=\pi_\rho(\mathfrak{A}_1)$, $M=\pi_\rho(\mathfrak{A}_1)''$ and $Q_\alpha=\pi_\rho(\tau_\alpha Q)$ for $Q\in\mathfrak{A}_1$. If ρ is τ_α invariant and $\mathfrak{A}_1$ is strongly τ_α asymptotically abelian, then we can apply Theorem 5.

Method of big translation in [3] can be formulated as follows. (See Theorem 6).

Lemma 6. *Let ρ_α be a net of (not necessarily normal) positive linear functionals on $(M\cup N)''$ such that $\lim_\alpha \rho_\alpha=\rho$ (i.e. $\lim_\alpha \rho_\alpha(Q)=\rho(Q)$ for each $Q\in(M\cup N)''$). Assume that the restriction of ρ_α to N is normal and independent of α. Assume also that N is abelian. Then*

$$\text{w-}\lim_\alpha F_{\rho_\alpha}^{NM}(Q)=F_\rho^{NM}(Q), \qquad Q\in M. \tag{5.18}$$

(See Remark 2 of §2.)

Proof. Let the restriction of ρ_α to N be denoted by σ which is independent of α by assumption. Then we obtain, from $\lim_\alpha \rho_\alpha=\rho$,

$$\lim_\alpha \sigma(F_{\rho_\alpha}^{NM}(Q)Q_1'Q_2')=\sigma(F_\rho^{NM}(Q)Q_1'Q_2')$$

for all $Q_1'\in N$, $Q_2'\in N$. Setting $x_\alpha=(F_{\rho_\alpha}^{NM}-F_\rho^{NM})(Q)$, we have

$$\lim_\alpha (\Psi,\ \pi_\sigma(x_\alpha)\Phi)=0$$

for $\Psi=\pi_\sigma(Q_1')^*\Omega_\sigma$ and $\Phi=\pi_\sigma(Q_2')\Omega_\sigma$. Since $\|x_\alpha\|\leqq 2\|Q\|$, we have $\text{w-}\lim_\alpha \pi_\sigma(x_\alpha)=0$. Since $s(x_\alpha)\leqq s^N(\sigma)$, we obtain $\text{w-}\lim_\alpha x_\alpha=0$.

Theorem 6. *Let $\mathfrak{A}$ be a weakly τ_α central C^* subalgebra of a von Neumann algebra M. Assume that the center Z of M is elementwise τ_α invariant and has a faithful normal state ρ. Let $\bar{\mathfrak{A}}$ be the C^* algebra generated by $\mathfrak{A}$ and Z. Then there exists a subnet $\tau_{\alpha(\beta)}$ such that*

$$L(Q)=\text{w-}\lim_\beta \tau_{\alpha(\beta)}Q \tag{5.19}$$

exists for all $Q\in\bar{\mathfrak{A}}$, where L is Z-linear, completely positive projection of norm 1 from $\bar{\mathfrak{A}}$ onto Z and $L(1)=1$. If Z is trivial, then $L(Q)=\omega(Q)1$ for a state ω on $\bar{\mathfrak{A}}$.

Proof. Let $\bar\rho$ be any extension of ρ to a state on M. By weak compactness, there exists a subnet $a(\beta)$ such that

$$\lim \tau_{\alpha(\beta)}^*\bar\rho=\rho_\infty$$

exists.

Since Z is elementwise invariant under τ_α, the restriction of $\tau_\alpha^*\bar{\rho}$ to Z is always ρ and hence the restriction of ρ_∞ to Z is also ρ. Since ρ is faithful on Z, $s^Z(\rho)=1$. By (5.2), (2.3) and Lemma 6,

$$\begin{aligned}\text{w-}\lim_\beta \tau_{\alpha(\beta)}Q &= \text{w-}\lim_\beta F_{\bar{\rho}}^{ZM}(\tau_{\alpha(\beta)}Q)\\ &= \text{w-}\lim_\beta F_{\tau^*_{\alpha(\beta)}\bar{\rho}}^{ZM}(Q)\\ &= F_{\rho_\infty}^{ZM}(Q).\end{aligned}$$

Hence (5.19) holds with $L=F_{\rho_\infty}^{ZM}$. The properties of L follow from Theorem 1 applied for $F_{\rho_\infty}^{ZM}$ (see Remark 2 of § 2) except possibly for the complete positivity.

Since Z is abelian, $J_\rho z^* J_\rho = z$, $z\in\pi_\rho(Z)$ for a faithful state ρ. Hence $Q\to {}^tQ\equiv J_\rho Q^* J_\rho$ is a transposition on $\mathscr{B}(H_\rho)$ leaving Z invariant. Hence if L is transposed-n-positive then

$$L\otimes 1_n=(\pi_\rho^{-1}\otimes 1_n)(t\otimes t_n)(\pi_\rho\otimes 1_n)(L\otimes t_n)$$

is also positive and hence F is n positive. Here 1_n and t_n denote the identity mapping and a transposition of $n\times n$ matrices. Q.E.D.

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