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A Uniqueness Theorem for Initial-value Problems

By

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Introduction

We shall consider in this note the following linear partial differential operator,

$$P(x, \partial/\partial x) = \sum_{i+j\leqq m} a_{ij}(x)\partial^{i+j}/\partial x_1^i \partial x_2^j, \tag{1}$$

where $a_{ij}(x)$ are analytic functions defined in an open set $\Omega \subset R^2$, and of real-valued if $i+j=m$.

Let $\mathscr{S}$ be an analytic curve defined by $\varphi(x)=\varphi(x^0)$, $x^0 \in \Omega$, where $\varphi(x)$ is a real-valued analytic function defined in Ω. From now on we assume that $\varphi_{x_1}(x^0) \neq 0$.

Now let us assume that $\mathscr{S}$ is a double characteristic curve of (1), that is,

$$\begin{aligned} &P_m(x, \varphi_x)|_{\mathscr{S}}=0, \qquad P_m^{(i)}(x, \varphi_x)|_{\mathscr{S}}=0 \quad \text{for } i=1, 2, \\ &P_m^{(i,j)}(x, \varphi_x)|_{\mathscr{S}} \neq 0 \qquad \text{for some } i, j=1, 2, \end{aligned} \tag{2}$$

where $P_m(x, \xi) = \sum_{i+j=m} a_{ij}(x)\xi_1^i\xi_2^j$, $P_m^{(i)}(x, \xi) = \partial P_m(x, \xi)/\partial \xi_i$ and $P_m^{(i,j)}(x, \xi) = \partial^2 P_m(x, \xi)/\partial \xi_i \partial \xi_j$.

And also we assume that $P_m(x, \varphi_x)$ vanishes at the first order on $\mathscr{S}$, that is,

$$<\varphi_x, \partial/\partial x> P_m(x, \varphi_x)|_{\mathscr{S}} \neq 0, \qquad <\varphi_x, \partial/\partial x> = \sum_{i=1}^{2} \varphi_{x_i} \frac{\partial}{\partial x_i}. \tag{3}$$

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Recently, Y. Hasegawa [3], [4] has proved the existence theorem of the initial-value problem with data on a characteristic surface, and the system case was treated by the author [8]. In those papers, the initial-value problems were classified in many cases using the lower order terms of the differential operator $P(x, \partial/\partial x)$. One of the purposes of this note is to investigate a geometrical meaning of the double characteristic curve under the assumption (3) which did not appear in J. Vaillant [10] but appeared in Y. Hasegawa [3] and the author [8] (see Proposition in § 1).

On the other hand, L. Hörmander [5], [6], F. Trèves [9] and E. C. Zachmanoglou [11], [12] proved uniqueness theorems of the initialvalue problems when the initial surface has simple characteristic points under a convexity condition or modified conditions. And also, J. M. Bony [2] and L. Hörmander [7] proved uniqueness theorems which are extensions of Holmgren's theorem. Another purpose of this note is to show a uniqueness theorem of the distribution solution of $P(x, \partial/\partial x)u=0$ when the initial curve has double characteristic points (see Theorem in § 1).

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§ 1. Statement of Theorem

At first we note that we have $P_m^{(2,2)}(x^0, \varphi_x(x^0)) \neq 0$ from the condition (2) and $\varphi_{x_1}(x^0) \neq 0$. And also we have

$$\operatorname{sgn}[P_m^{(2,2)}(x^0, \varphi_x(x^0))] = \operatorname{sgn}[\sum_{i,j} P_m^{(i,j)}(x^0, \varphi_x(x^0)) \psi_{x_i}(x^0) \psi_{x_j}(x^0)],$$

where $\psi(x)$ is a real-valued analytic function defined in Ω such that $\partial(\varphi, \psi)/\partial(x_1, x_2)|_{x=x^0} \neq 0$.

Without loss of generality, we may assume that $\operatorname{sgn}[<\varphi_x, \frac{\partial}{\partial x}> P_m(x, \varphi_x)|_{x=x^0}]$ and $\operatorname{sgn}[P_m^{(2,2)}(x^0, \varphi_x(x^0))]$ are different. In fact, if they are the same, we then consider $-\varphi(x)$ instead of $\varphi(x)$. Then we have the following

Proposition. *Under the assumptions* (2), (3) *and the above, there exist two and only two analytic characteristic curves through each point in*

$\varphi(x)>\varphi(x^0)$ and sufficiently near $\mathscr{S}$ such that they are tangent at the second order to $\mathscr{S}$. And the other characteristic curves are transversal to $\mathscr{S}$.

From this proposition we may consider that $\mathscr{S}$ is an envelope of characteristic curves of (1). We remark that if we assume that $<\varphi_x, \frac{\partial}{\partial x}>P_m(x, \varphi_x)|_{\mathscr{S}}=0$ instead of the assumption (3), we can not obtain a similar result as in the Proposition.

Let $\mathscr{C}$ be a regular curve defined by $f(x)=f(x^0)$, tangent at x^0 to the double characteristic curve $\mathscr{S}$, where $f(x)$ is a real-valued function of $C^1(\Omega)$, and we may assume

$$f_x(x^0)=\varphi_x(x^0).$$

Now we impose on $\mathscr{C}$ the following conditions:

(4) If $x_2>x_2^0$ (or $x_2<x_2^0$), $f(x)<f(x^0)$ on $\mathscr{S}$,

(5) for $x_2>x_2^0$ (or $x_2<x_2^0$), it holds

$$\operatorname{sgn}[P_m^{(2,2)}(x^0, \varphi_x(x^0))]\cdot P_m(x, f_x)|_{\mathscr{C}}>0.$$

Then we have the following uniqueness theorem.

Theorem. *Under the assumptions* (4) *and* (5), *if $u\in\mathscr{D}'(\Omega)$ is a solution of $P(x, \partial/\partial x)u=0$ and vanishes in $f(x)<f(x^0)$, then $u=0$ in a neighborhood of x^0.*

Remark 1.1. The assumptions (4) and (5) mean the following: the double characteristic curve $\mathscr{S}$ and the characteristic curve in $\varphi(x)>\varphi(x^0)$ which is tangent at x^0 to $\mathscr{S}$ guaranteed its existence from the Proposition, lie in the domain where we require that $u=0$, when $x_2>x_2^0$ (or $x_2<x_2^0$) (see §3).

For the proof of the theorem we use the following lemma due to L. Hörmander [5, p. 125].

Lemma. *Let $P(x, \partial/\partial x)=\sum_{|\alpha|\leq m} a_\alpha(x)(\partial/\partial x)^\alpha$ be a differential operator of order m defined in an open set $\Omega\subset R^n$ with analytic coefficients, and* *assume that the coefficient of* $(\partial/\partial x_n)^m$ *does not vanish in* Ω. *If* $u\in\mathscr{D}'(\Omega)$ *is a solution of* $P(x, \partial/\partial x)u=0$ *in* $\Omega_c=\{x;\ x\in\Omega,\ x_n<c\}$ *for some* c, *then* $u=0$ *in* Ω_c *provided* $\Omega_c\cap\operatorname{supp}[u]$ *is relatively compact in* Ω.

Remark 1.2. We also obtain a uniqueness theorem if we assume that $f(x)>f(x^0)$ on $\mathscr{S}$, $x_2>x_2^0$ (or $x_2<x_2^0$) and $u=0$ in $f(x)>f(x^0)$. We omit the proof, since it is easier than that of Theorem.

§2. Proof of Proposition

Before the proof of our Proposition, we shall prove the following lemma.

Lemma 2.1. *Let us consider the following algebraic equation in* z,

$$x_1z^m+x_1a_1(x)z^{m-1}+a_2(x)z^{m-2}+\cdots+a_m(x)=0, \tag{6}$$

where $a_i(x)$ *are real-valued analytic functions defined in a neighborhood of the origin* $\Omega\subset R^2$. *Assume that* $a_2(x)<-\delta$ *in* Ω *for some positive constant* δ, *and let* $z=z_i(x)$, $i=1,\cdots, m$ *be roots of* (6) *at* $x_1\neq 0$. *Then there exist two and only two real-valued analytic roots in* $\{z_i(x)\}_{i=1}^m$, *say* $z_i(x)$, $i=1, 2$, *in* $\Omega_{+,\varepsilon}=\{x;\ x\in\Omega,\ 0<x_1<\varepsilon\}$ *such that*

$$z_i(x)=c_i(x)\cdot x_1^{-1/2} \quad \text{as} \quad x_1\downarrow 0,$$

where $\varepsilon_1<|c_i(x)|<M$ *for some positive constants* ε_1 *and* M, $c_1\cdot c_2<0$ *and* ε *is a sufficiently small positive constant. And the other roots satisfy* $|z_i(x)|=O(1)$ *uniformly in* x_2 *as* $x_1\downarrow 0$, $i=3,\cdots, m$ *in* $\Omega_+=\{x;\ x\in\Omega,\ x_1>0\}$. *In* $\Omega_-=\{x;\ x\in\Omega,\ x_1<0\}$, *all the real roots satisfy* $z_i(x)=O(1)$ *uniformly in* x_2 *as* $x_1\uparrow 0$.

We prove the lemma by using the following classical result concerning algebraic equation.

Lemma. (Laguerre) *Let us consider the following algebraic equation with real coefficients*,

$$f(z)=a_0z^m+a_1z^{m-1}+\cdots+a_m=0, \qquad a_0\neq 0.$$

And now let N be the number of roots which are greater than a given positive number α, *and M be the number of variation of signs of*

$$f_k(\alpha)=a_0\alpha^k+a_1\alpha^{k-1}+\cdots+a_k, \qquad k=0,\cdots, m.$$

Then it follows that $N=M-2l$, *where* $l\in\{0, 1,\cdots, \left[\frac{m}{2}\right]\}$.

Proof of Lemma 2.1. Let N be a sufficiently large positive constant satisfying

$$N\geqq\max_{i\neq 2}\{\sup_{\Omega}|a_i(x)|\}, \qquad \delta N^{2(k-2)}-\sum_{i=3}^{k}N^{2(k-i)+1}>\delta_1,$$

for $k=3,\cdots, m$, where δ_1 is a positive constant less than δ. Then we see that there exists one and only one real root $z=z_1(x)$ of (6) greater than N^2 in $\Omega_{+,\varepsilon}$, where $\varepsilon=\delta_1/(N^{2m}+N^{2m-1})$ in view of the above lemma. In fact, let

$$f_k(x, z)=x_1z^k+x_1a_1(x)z^{k-1}+a_2(x)z^{k-2}+\cdots+a_k(x),$$

for $k=0, 1,\cdots, m$, then we have that $f_k(x, N^2)>0$, $k=0, 1$ and $f_k(x, N^2)<0$, $k=2,\cdots, m$ in $\Omega_{+,\varepsilon}$. Since the number of variation of signs is one, there exists one and only one real root of (6) greater than N^2. Interchanging z by $-z$ in (6), it also follows that there exists one and only one real root $z=z_2(x)$ of (6) smaller than $-N^2$ in $\Omega_{+,\varepsilon}$. In order to see that $z_i(x)=c_i(x)\cdot x_1^{-1/2}$ as $x_1\downarrow 0$, where $c_1>0$ and $c_2<0$, it sufficies to see that $\varepsilon_1/\sqrt{x_1}<z_1(x)<M/\sqrt{x_1}$ and $-M/\sqrt{x_1}<z_2(x)<-\varepsilon_1/\sqrt{x_1}$ if $x_1>0$ and x_1 is sufficiently small, where ε_1 and M are constants satisfying $\varepsilon_1^m+a_2(x)\varepsilon_1^{m-2}<0$ and $M^m+a_2(x)M^{m-2}>0$ in Ω. The analyticities of them follow immediately from the implicit function theorem. And also it follows that the other roots in $\Omega_{+,\varepsilon}$ are smaller than N^2 in absolute value from Rouché's theorem. In $\Omega_-=\left\{x\,;\,0>x_1>-\frac{\delta_1}{N^{2m}+N^{2m-1}}\right\}$, all the real roots of (6) are smaller than N^2 in absolute value, in view of the above lemma. This completes the proof.

Proof of the Proposition. Let us transform the coordinates as follows,

$$(7)\qquad y_1=\varphi(x)-\varphi(x^0),\qquad y_2=\psi(x)-\psi(x^0)$$

in a neighborhood of x^0, where $\psi(x)$ is a real-valued analytic function defined in Ω satisfying $\partial(\varphi, \psi)/\partial(x_1, x_2)|_{x=x^0}\neq 0$. Let $\tilde{P}(y, \partial/\partial y)$ be a differential operator transformed by (7) defined in a neighborhood of the origin $\tilde{\Omega}\subset R^2$, then the coefficients of $\partial^m/\partial y_1^{m-k}\partial y_2^k$, $k=0, 1, 2$ of $\tilde{P}(y, \partial/\partial y)$ are

$$P_m(x, \varphi_x),\ \sum_{i=1}^{2} P_m^{(i)}(x, \varphi_x)\psi_{x_i}\quad\text{and}\quad \frac{1}{2}\sum_{i,j} P_m^{(i,j)}(x, \varphi_x)\psi_{x_i}\psi_{x_j}$$

respectively. We note at first that we can represent $P_m(x(y), \varphi_x(x(y)))=y_1\tilde{a}_0(y)$ since $P_m(x(y), \varphi_x(x(y)))=0$ when $y_1=0$, and we have easily

$$\tilde{a}_0(0, y_2)=\left[\frac{1}{|\varphi_x|^2}<\varphi_x, \frac{\partial}{\partial x}>P_m(x, \varphi_x)\right]\Big|_{y_1=0}.$$

And also we can represent $\sum_i P_m^{(i)}(x(y), \varphi_x(x(y)))\psi_{x_i}(x(y))=y_1\tilde{a}_1(y)$, since $P_m^{(i)}(x(y), \varphi_x(x(y)))=0$ when $y_1=0$. Thus the characteristic polynomial $\tilde{P}_m(y, \eta)$ of $\tilde{P}(y, \partial/\partial y)$ is represented as follows,

$$\begin{aligned}\tilde{P}_m(y, \eta)=y_1\tilde{a}_0(y)\eta_1^m+y_1\tilde{a}_1(y)\eta_1^{m-1}\eta_2&+\\ +\tilde{a}_2(y)\eta_1^{m-2}\eta_2^2&+\sum_{k=3}^{m}\tilde{a}_k(y)\eta_1^{m-k}\eta_2^k,\end{aligned}$$

where $\tilde{a}_2(y)=\frac{1}{2}\sum_{i,j} P_m^{(i,j)}(x, \varphi_x)\psi_{x_i}\psi_{x_j}$.

Now we remark that if a curve defined by $g(y)=c$, $(g_{y_2}\neq 0)$ is a characteristic curve of $\tilde{P}(y, \partial/\partial y)$, it must satisfy $\tilde{P}_m(y, g_y)=0$. And also it follows that $g(y)=c$ is a solution of the following ordinary differential equation,

$$(8)\qquad \begin{aligned}F(y, dy_2/dy_1)\equiv y_1\tilde{a}_0(y)\Big(-\frac{dy_2}{dy_1}\Big)^m+y_1\tilde{a}_1(y)\Big(-\frac{dy_2}{dy_1}\Big)^{m-1}&+\\ +\tilde{a}_2(y)\Big(-\frac{dy_2}{dy_1}\Big)^{m-2}+\sum_{k=3}^{m}\tilde{a}_k(y)\Big(-\frac{dy_2}{dy_1}\Big)^{m-k}&=0\end{aligned}$$

in view of $-dy_2/dy_1=g_{y_1}/g_{y_2}$ on $g(y)=c$. Conversely it is obvious that the solutions of (8) are also characteristic curves of $\tilde{P}(y, \partial/\partial y)$. Let us consider the ordinary differential equation (8) instead of considering the partial differential equation $\tilde{P}_m(y, g_y)=0$.

In view of the assumptions that $\tilde{a}_0(0, y_2)\neq 0$, and $\mathrm{sgn}[\tilde{a}_0(0)]$ and $\mathrm{sgn}[\tilde{a}_2(0)]$ are different, we may assume

$$\tilde{a}_2(y)/\tilde{a}_0(y)<-\delta$$

in $\tilde{\Omega}$ for some positive constant δ. From Lemma 2.1, there exist two and only two real-valued analytic roots of (8) with respect to dy_2/dy_1 in $\tilde{\Omega}_{+,\varepsilon}$ for a sufficiently small ε, say $dy_2/dy_1=f_i(y)$, $i=1, 2$, such that $f_i=c_i(y)\cdot y_1^{-1/2}$ as $y_1\downarrow 0$, where $\varepsilon_1<|c_i(y)|<M$ for some positive constants ε_1 and M, and $c_1\cdot c_2<0$. Let us consider the following ordinary differential equations in $\tilde{\Omega}^a_{+,\varepsilon}$,

$$\frac{dy_1}{dy_2}=f_i^{-1}(y), \qquad i=1, 2, \tag{9}$$

where $\tilde{\Omega}^a_{+,\varepsilon}$ is a closure of $\tilde{\Omega}_{+,\varepsilon}$. Obviously the solutions of (9) are also characteristic curves of $\tilde{P}(y, \partial/\partial y)$. From the theory of ordinary differential equation, it follows that every solution of (9) is tangent to $y_1=0$ in view of $f_i^{-1}(y)=c_i^{-1}(y)\cdot y_1^{1/2}$ as $y_1\downarrow 0$. In order to see that they are tangent at the second order to $y_1=0$, it suffices to show $\lim_{y_1\downarrow 0} d^2y_1/dy_2^2\neq 0$. Now we use the following relation,

$$d^2y_1/dy_2^2=-[(f_i)_{y_1}/f_i^3+(f_i)_{y_2}/f_i^2].$$

Since $f_i(y)$ satisfies $F(y, f_i)\equiv 0$ in $\tilde{\Omega}_{+,\varepsilon}$, we have

$$(f_i)_{y_1}=-F_{y_1}(y, f_i)/F_z(y, f_i),$$

$$(f_i)_{y_2}=-F_{y_2}(y, f_i)/F_z(y, f_i),$$

where $F_{y_k}(y, z)=\partial F(y, z)/\partial y_k$, $k=1, 2$ and $F_z(y, z)=\partial F(y, z)/\partial z$. Now we shall consider $F_z(y, f_i)$ and $F_{y_k}(y, f_i)$, $k=1, 2$;

$$-F_z(y, f_i)=m y_1\tilde{a}_0(-f_i)^{m-1}+(m-1)y_1\tilde{a}_1(-f_i)^{m-2}$$
$$+(m-2)\tilde{a}_2(-f_i)^{m-3}+\sum_{k=3}^{m-1}(m-k)\tilde{a}_k(-f_i)^{m-k-1},$$
$$F_{y_1}(y, f_i)=\tilde{a}_0(-f_i)^m+y_1(\tilde{a}_0)_{y_1}(-f_i)^m+(y_1\cdot\tilde{a}_1)_{y_1}(-f_i)^{m-1}$$
$$+\sum_{k=2}^{m}(\tilde{a}_k)_{y_1}(-f_i)^{m-k},$$
$$F_{y_2}(y, f_i)=y_1(\tilde{a}_0)_{y_2}(-f_i)^m+y_1(\tilde{a}_1)_{y_2}(-f_i)^{m-1}$$
$$+(\tilde{a}_2)_{y_2}(-f_i)^{m-2}+\sum_{k=3}^{m}(\tilde{a}_k)_{y_2}(-f_i)^{m-k}.$$

Considering $f_i(y)=c_i(y)y_1^{-1/2}$ as $y_1\downarrow 0$, we have

$$F_z(y, f_i)=d_i(y_2)y_1^{-(m-3)/2}+o(y_1^{-(m-3)/2}) \quad \text{as } y_1\downarrow 0,$$

where $d_i(y_2)\neq 0$. And also we have

$$F_{y_1}(y, f_i)=e_i(y_2)y_1^{-m/2}+o(y_1^{-m/2}) \quad \text{as } y_1\downarrow 0,$$
$$F_{y_2}(y, f_i)=O(y_1^{-(m-2)/2}) \quad \text{as } y_1\downarrow 0,$$

where $e_i(y_2)\neq 0$. Thus we have

$$\lim_{y_1\downarrow 0}(f_i)_{y_1}/f_i^3\neq 0,$$
$$(f_i)_{y_2}/f_i^2=O(y_1^{1/2}) \quad \text{as } y_1\downarrow 0.$$

Therefore the solutions of (9) are tangent at the second order to $y_1=0$. Finally, it is obvious that the other characteristic curves of $\tilde{P}(y, \partial/\partial y)$ are transversal to $y_1=0$, because the other real roots of the algebraic equation $F(y, z)=0$ in z at $y_1\neq 0$ are $O(1)$ as $y_1\downarrow 0$ or $y_1\uparrow 0$ (Lemma 2.1). This completes the proof.

§ 3. Proof of Theorem

It suffices to prove our Theorem for the partial differential operator $\tilde{P}(y, \partial/\partial y)$ in $\tilde{\Omega}$ considered in the previous section. Let $\tilde{P}(y, \partial/\partial y)$ be

$$\tilde{P}(y, \partial/\partial y)=\tilde{P}_m(y, \partial/\partial y)+Q(y, \partial/\partial y), \tag{10}$$

where $\tilde{P}_m(y, \partial/\partial y)=y_1\tilde{a}_0(y)\frac{\partial^m}{\partial y_1^m}+y_1\tilde{a}_1(y)\frac{\partial^m}{\partial y_1^{m-1}\partial y_2}+\tilde{a}_2(y)\frac{\partial^m}{\partial y_1^{m-2}\partial y_2^2}+$ $+\sum_{k\geqq 3}\tilde{a}_k(y)\frac{\partial^m}{\partial y_1^{m-k}\partial y_2^k}$, $\tilde{a}_0(y)\cdot\tilde{a}_2(y)<0$ in $\tilde{\Omega}$, and $Q(y, \partial/\partial y)$ is a differential operator of order $m-1$.

Let $\check{\mathscr{C}}$ be an image of $\mathscr{C}$ by the transformation (7), and set

$$\check{\mathscr{C}}: \tilde{f}(y)=\tilde{f}(0),\ ((\tilde{f}_{y_1}(0), \tilde{f}_{y_2}(0))=(1, 0),\ \tilde{f}(y)\in C^1((\tilde{\Omega})).$$

Then the conditions of Theorem become as follows,

$$\begin{aligned}&\tilde{f}(0, y_2)<\tilde{f}(0),\quad y_2>0 \quad (\text{or } y_2<0),\\ &\operatorname{sgn}[\tilde{a}_2(0)]\cdot\tilde{P}_m(y, \tilde{f}_y)|_{\check{\mathscr{C}}}>0,\quad y_2>0 \quad (\text{or } y_2<0).\end{aligned} \tag{11}$$

At first we note the meaning of these conditions. From the Proposition, there exist two and only two characteristic curves through each point in $\tilde{\Omega}_{+,\varepsilon}$ which are tangent at the second order to $y_1=0$, and they are represented by the solutions of the ordinary differential equations (9). Without loss of generality, we may assume that $f_1^{-1}>0$ and $f_2^{-1}<0$ in $\tilde{\Omega}_{+,\varepsilon}$. Then it follows immediately from (11) that

$$\begin{aligned}&-\tilde{f}_{y_2}/\tilde{f}_{y_1}>f_1^{-1} \quad \text{on } \check{\mathscr{C}},\ y_2>0,\\ &(\text{or } -\tilde{f}_{y_2}/\tilde{f}_{y_1}<f_2^{-1} \quad \text{on } \check{\mathscr{C}},\ y_2<0).\end{aligned} \tag{12}$$

These have a geometrical meaning as stated in Remark 1.1 in §1.

We shall prove the theorem only in the case where $-\tilde{f}_{y_2}/\tilde{f}_{y_1}>f_1^{-1}$ on $\check{\mathscr{C}}$, $y_2>0$, since the proof is the same in the other case. We may assume that $f_1^{-1}(y)$ is monotonically increasing with respect to y_1 in $\tilde{\Omega}_{+,\varepsilon}$, because of $f_1^{-1}(y)=c_1^{-1}(y)\cdot y_1^{1/2}$ as $y_1\downarrow 0$. Now let us consider the following ordinary differential equation,

$$\frac{dy_1}{dy_2}=f_1^{-1}(y_1+\alpha, y_2) \quad \text{in } \tilde{\Omega}_\alpha, \tag{13}$$

where α is a sufficiently small positive constant ($\alpha<\varepsilon$) and $\tilde{\Omega}_\alpha$ is defined by $\tilde{\Omega}_\alpha=\{y;(y_1+\alpha, y_2)\in\tilde{\Omega}_{+,\varepsilon}, y_1>-\alpha/2\}$. Since the right hand of (13) is analytic in the domain $\tilde{\Omega}_\alpha$, (13) has not any singular solution, and the solutions of (13) are not characteristic curves of $\tilde{P}(y, \partial/\partial y)$ in $\tilde{\Omega}_\alpha$, because of the assumptions that $f_1^{-1}(y)$ is monotonically increasing with respect to y_1 and the other characteristic curves are transversal to $y_1=0$. And now let

$$\tilde{\varphi}_\alpha(y)=c \quad \text{in } \tilde{\Omega}_\alpha$$

be solutions of (13), then in view of (12) it follows that the domain enclosed by $\check{\mathscr{C}}$ and $\tilde{\varphi}_\alpha(y)=\tilde{\varphi}_\alpha(0)$ is relatively compact in $\tilde{\Omega}_\alpha$ if we choose α sufficiently small. In fact, $\tilde{\varphi}_\alpha(y)=\tilde{\varphi}_\alpha(0)$ is sufficiently near to the solution of $dy_1/dy_2=f_1^{-1}(y)$ which is tangent at the origin to $y_1=0$ if α is small. Thus we see that

$$\{y;\ \tilde{\varphi}_\alpha(y)=\tilde{\varphi}_\alpha(0, y_2), y_2>-\delta\}\cap \text{supp}[u]$$

is relatively compact in $\tilde{\Omega}_\alpha$ if we choose a sufficiently small positive constant δ, since $u=0$ in $\tilde{\Omega}_\alpha\cap\{y; \tilde{f}(y)<\tilde{f}(0)\}$.

Now let us transform the coordinates in $\tilde{\Omega}_\alpha$ as follows,

$$z_1=\tilde{\varphi}_\alpha(y), \qquad z_2=y_2.$$

Then it is obvious that $\partial(z_1, z_2)/\partial(y_1, y_2)\neq 0$ in $\tilde{\Omega}_\alpha$. Let $\tilde{\tilde{P}}(z, \partial/\partial z)$ be a partial differential operator transformed by the above transformation. Then the coefficient of $(\partial/\partial z_1)^m$ of $\tilde{\tilde{P}}(z, \partial/\partial z)$ does not vanish in the domain considered, since $z_1=$ const. is not characteristic. And obviously the assumption of Lemma in §1 is satisfied for the equation $\tilde{\tilde{P}}(z, \partial/\partial z)u=0$, therefore $u=0$ in $\{z; z_1=\tilde{\varphi}_\alpha(0, y_2), y_2>-\delta\}$, that is, $u=0$ in $\{y\in\tilde{\Omega}_\alpha; \tilde{\varphi}_\alpha(y)=\tilde{\varphi}_\alpha(0, y_2), y_2>-\delta\}$. This completes the proof.

References

[1] Bony, J. M., Sur la propagation des maximums et l'unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés du second ordre, *C.R. Acad. Sci. Paris*, **266** (1968), 763-765.

[2] ———, Une extension du théorème de Holmgren sur l'unicité de problème de Cauchy, *C.R. Acad. Sci. Paris*, **268** (1969), 1103-1106.

[3] Hasegawa, Y., On the initial-value problem with data on a double characteristic, *J. Math. Kyoto Univ.* **11** (1971), 357-372.

[4] ———, On the initial-value problem with data on a characteristic plane, to appear.

[5] Hörmander, L., *Linear Partial Differential Operators*, Springer.

[6] ———, Uniqueness theorems and wave front sets for solutions of linear differential equations with analytic coefficients, *Comm. Pure Appl. Math.* **24** (1971), 671-704.

[7] ———, A remark on Holmgren's uniqueness theorem, *J. Diff. Geom.* **6** (1971), 129-134.

[8] Miyake, M., On the initial-value problems with data on a characteristic surface for linear systems of first order equations, *Publ. R.I.M.S. Kyoto Univ.* **8** (1972/73), 231-264.

[9] Trèves, F., *Linear Partial Differential Equations with Constant Coefficients*, Gordon and Breach, New York, 1966.

[10] Vaillant, J., Données de Cauchy portées par une caractéristique double..., *J. Math. Pures Appl.* **47** (1968), 1-40.

[11] Zachmanoglou, E.C., Uniqueness of the Cauchy problem for linear partial differential equations, *Trans. Amer. Math. Soc.* **136** (1969), 517-526.

[12] ———, Propagation of zeros and uniqueness in the Cauchy problem for first order partial differential equations, *Arch. Rat. Mech. Anal.* **38** (1970), 178-188.