Publ. RIMS, Kyoto Univ.
9 (1973), 1–44

Structure of Some von Neumann Algebras with Isolated Discrete Modular Spectrum

By

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Abstract

From a finite von Neumann algebra $\mathfrak{F}$ with a faithful normal trace and its normal injective $*$ endomorphism ϕ satisfying $\phi(\mathfrak{F})=\phi(1)\mathfrak{F}\phi(1)$, we construct another von Neumann algebra $M(\mathfrak{F}, \phi)$ by a method which reduces to group-measure construction when $\mathfrak{F}$ is commutative and ϕ is an automorphism. If ϕ satisfies $\phi(z)=\phi(1)z$ for all central elements z of $\mathfrak{F}$ and $\phi(1)^\natural=e^{-a}$ for a positive number a, then $M(\mathfrak{F}, \phi)$ has the following 3 properties: (1) It has a faithful normal state ρ whose modular operator Δ_ρ has the spectrum $\{0\}\cup\{e^{na}; n=0, \pm1,\dots\}=S_{e^a}$. (2) The set $\mathfrak{M}_0$ of all elements of $M(\mathfrak{F}, \phi)$, commuting with Δ_ρ is isomorphic to $\mathfrak{F}$. (3) The center of $\mathfrak{M}_0$ coincides with the center of $\mathfrak{M}$.

Conversely, any von Neumann algebra with a faithful normal state ρ such that $\log\Delta_\rho$ has exclusively an isolated point spectrum and the center of $\mathfrak{M}_0$ coincides with its center is a direct sum of $M(\mathfrak{F}_j, \phi_j)$, $j=1,\dots$, and possibly a finite von Neumann algebra, where each ϕ_j satisfies $\phi_j(\mathfrak{F})=\phi_j(1)\,\mathfrak{F}\,\phi_j(1)$, $\phi_j(z)=z\phi_j(1)$ for all central element z of $\mathfrak{F}$ and $\phi_j(1)^\natural=e^{-a_j}$.

If ρ is a *KMS* state under time translation of a C^* algebra, which is asymptotically abelian with respect to (either discrete or continuous) space translation and if the spectrum of generator of time translation has exclusively an isolated point spectrum in the representation associated with ρ, then the associated von Neumann algebra has the above structure where the asymptotic ratio set of $\mathfrak{F}_j$ as well as that of a possible finite summand (if non-zero) is $\{1\}$ and $r_\infty(M(\mathfrak{F}_j, \phi_j))=S_{x_j}$ where $x_j=e^{a_j}$. The last result on asymptotic ratio set is limited to the case where the representation space is separable.

A generalization of $M(\mathfrak{F}, \phi)$ for a commutative semigroup of endomorphisms of a finite von Neumann algebra, instead of one ϕ, is given.

§1. Notation and Main Results

Let $\mathfrak{M}$ be a von Neumann algebra, Ω be a cyclic and separating unit

* Received August 4, 1972.

vector, Δ be the modular operator for Ω, $\tau(t)Q=\Delta^{it}Q\Delta^{-it}$, J be the modular conjugation operator for Ω and $j(Q)=JQJ$.

$\mathfrak{H}_\alpha$ denotes the eigenspace of $\log \Delta$ belonging to an eigenvalue α and $\mathfrak{M}_\alpha$ denotes the set of $Q\in\mathfrak{M}$ such that $\tau(t)Q=e^{it\alpha}Q$. It is known that $\mathfrak{M}_0$ is a finite von Neumann algebra containing the center $\mathfrak{Z}$ of $\mathfrak{M}$ and Ω is a cyclic and separating trace vector for $\mathfrak{M}_0$ restricted to $\mathfrak{H}_0$. $\mathfrak{Z}_0$ denotes the center of $\mathfrak{M}_0$. $\mathfrak{Z}_0\supset\mathfrak{Z}$.

We are interested in the structure of $\mathfrak{M}$ and we can analyze it when $\log \Delta$ has exclusively an isolated point spectrum and $\mathfrak{Z}_0=\mathfrak{Z}$.

For any given finite von Neumann algebra $\mathfrak{F}$ with a faithful normal trace vector Ψ such as $\mathfrak{M}_0$ with Ω and its normal injective $*$ endomorphism ϕ satisfying $\phi(\mathfrak{F})=\phi(1)\mathfrak{F}\phi(1)$, we present in section 2 a method of constructing another von Neumann algebra, denoted as $M(\mathfrak{F}, \phi)$, with a cyclic and separating vector $\Omega(\mathfrak{F}, \phi)$. If $(\Psi, \phi(z)\Psi)=e^{-a}(\Psi, z\Psi)$ for all $z\in\mathfrak{F}_c$ (the center of $\mathfrak{F}$), then the spectrum of modular operator $\Delta(\mathfrak{F}, \phi)$ for $\Omega(\mathfrak{F}, \phi)$ is

$$S_{e^a}=\{0\}\cup\{e^{na};\ n=0,\ \pm1,\dots\}.$$

If $\phi(z)=z\phi(1)$ for $z\in\mathfrak{F}_c$, then $\mathfrak{Z}_0=\mathfrak{Z}$.

The following main result proved in section 3 gives a converse.

Theorem 1. *Let $\mathfrak{M}$ be a von Neumann algebra with a cyclic and separating vector Ω such that* $\log \Delta$ *has exclusively an isolated point spectrum*

$$\{0,\ \pm a_1,\ \pm a_2,\ \dots\},\ 0<a_1<a_2\dots,$$

and the center $\mathfrak{Z}_0$ of the set $\mathfrak{M}_0$ of all elements in $\mathfrak{M}$ commuting with Δ coincides with the center $\mathfrak{Z}$ of $\mathfrak{M}$. Then there exists central projections P_n of $\mathfrak{M}$, $n=0, 1, \dots$ such that

(i) $P_n\perp P_m$ *for* $n\neq m$, $\sum P_n=1$ *and* $P_1\neq 0$,

(ii) $P_0\mathfrak{M}$ *is finite if* $P_0\neq 0$,

(iii) $P_n\mathfrak{M}$ *is $*$ isomorphic to* $M(P_n\mathfrak{M}_0, \phi_n)$ *for some normal injective $*$ endomorphism* ϕ_n *of* $P_n\mathfrak{M}_0$ *satisfying*

$$\phi_n(P_n\mathfrak{M}_0)=\phi_n(P_n)P_n\mathfrak{M}_0\phi_n(P_n)$$

$$\phi_n(z)=\phi_n(1)z,\ z\in P_n\mathfrak{Z},$$

$$\phi_n(P_n)^{\natural}=e^{-a_n}P_n$$

if $P_n\neq 0$, *where* $\natural$ *is the canonical* $\natural$ *mapping in* $\mathfrak{M}_0$.

In a special case where $\mathfrak{M}$ is a factor, the spectrum of $\log\Delta$ is necessarily an additive group. (This statement is always true if all subspaces $\mathfrak{H}(I)=(E_{\beta-0}-E_{\alpha+0})\mathfrak{H}$ for $\Delta=\int\lambda\, dE_\lambda$, $I=(\alpha,\ \beta)$, is cyclic for $\mathfrak{M}$ (and hence separating for $\mathfrak{M}$ due to $J\mathfrak{H}(I)=\mathfrak{H}(-I)$) even when Δ has a continuous spectrum.)

Examples where the center of $\mathfrak{M}_0$ does not coincide with the center of $\mathfrak{M}$ are tensor product of ITPFI of the class S_{01} (type III or II) with any finite von Neumann algebra where the vector Ω is product of a defining product vector of the ITPFI with a cyclic and separating trace vector for the finite von Neumann algebra. Another example for $\mathfrak{Z}_0\neq\mathfrak{Z}$ is $R_x\otimes R$ with $\Omega=\Omega_x\otimes\Phi$ where Ω_x is the defining product vector of R_x, R is type I_2 and spectrum of modular operator for $(R,\ \Phi)$ is $\{y^{-1},\ 1,\ y\}$ with $y\notin S_x$.

The condition $\mathfrak{Z}_0=\mathfrak{Z}$ is satisfied for a KMS state of an asymptotically abelian C^* algebra. More precisely, a net of operator Q_α in a von Neumann algebra $\mathfrak{M}$ is called strongly central if there exists a weakly total self-adjoint subset $\mathfrak{W}$ of $\mathfrak{M}$ such that $\lim_{\alpha}[Q_\alpha,\ w]=0$ strongly for every $w\in\mathfrak{W}$. A subset $\mathfrak{A}$ of $\mathfrak{M}$ is called strongly τ_α central relative to a net τ_α of $*$ automorphisms of $\mathfrak{M}$ if $\tau_\alpha Q$ is strongly central in $\mathfrak{M}$ for each $Q\in\mathfrak{A}$. We have

Theorem 2. *Let* $\mathfrak{M}$ *be a von Neumann algebra,* τ_α *be a net of* $*$ *automorphisms of* $\mathfrak{M}$, ρ *be a faithful normal* τ_α *invariant state of* $\mathfrak{M}$ *and* $\mathfrak{A}$ *be a weakly dense* C^* *subalgebra of* $\mathfrak{M}$, *which is invariant under modular automorphism* $\tau_\rho(t)$ *for* ρ *and is strongly* τ_α *central. Assume that modular operator* Δ_ρ *for* ρ *is such that* $\log\Delta_\rho$ *has exclusively an isolated point spectrum.*

Then $\mathfrak{Z}_0=\mathfrak{Z}$ *and there exists central projections* P_n *satisfying* (i)$\sim$(iii) *of Theorem* 1. *If the representation space is separable, in addition, then*

$$r_\infty(P_0\mathfrak{M})=\{1\}\qquad \text{if}\qquad P_0\neq 0,$$

$$r_\infty(P_n\mathfrak{M}_0)=\{1\}\qquad \text{if}\qquad P_n\neq 0,$$

$$r_\infty(P_n\mathfrak{M}) = S_{x_n} \quad \textit{if} \quad P_n \neq 0, \textit{ where } x_n = e^{-a_n},\ n > 0.$$

This theorem is applicable to a situation where ρ is a KMS state for time translation of a C^* algebra A, which is asymptotically abelian for (discrete or continuous) space translation, the generator of time translation has exclusively an isolated point spectrum in the representation associated with ρ and the representation space is separable.

§2. Construction of $M(\mathfrak{F}, \phi)$

Let $\mathfrak{F}$ be a finite von Neumann algebra acting on a Hilbert space $\mathfrak{K}$ with a cyclic and separating unit trace vector Ψ and $\mathfrak{F}_c$ be the center of $\mathfrak{F}$. Let $\natural$ be the canonical $\natural$ mapping on $\mathfrak{F}_c$, J_Ψ be the modular conjugation operator for Ψ and $j_\Psi(Q) = J_\Psi Q J_\Psi$.

Let ϕ be a normal injective $*$ endomorphism of $\mathfrak{F}$, ω_Ψ be the vector state by Ψ, and $\phi^*\omega_\Psi$ be the normal positive linear functional defined by

$$\phi^*\omega_\Psi(Q) = \omega_\Psi(\phi(Q)).$$

Both ω_Ψ and $\phi^*\omega_\Psi$ are faithful and tracial.

By the Radon-Nikodym theorem, there exists a strictly positive selfadjoint operator $A_\phi = \int \lambda\, dE_\lambda^A$ such that $E_\lambda^A \in \mathfrak{F}_c$ and

$$\lim_{L\to\infty} \phi^*\omega_\Psi(A_L z A_L) = \omega_\Psi(z),\ A_L = A_\phi E_L^A,\ z \in \mathfrak{F}_c.$$

In other words, Ψ is in the domain of $\phi(A_\phi) \equiv \lim_L \phi(A_L)$ and

$$(\phi(A_\phi)\Psi,\ \phi(z)\phi(A_\phi)\Psi) = (\Psi,\ z\Psi),\ z \in \mathfrak{F}_c. \tag{2.1}$$

(These equations hold for $z \in \mathfrak{F}$ as will be seen in the following proof.)

Lemma 1. *There exists a unique isometric operator V satisfying*

$$VQ\Psi = \phi(Q)\phi(A_\phi)\Psi. \qquad Q \in \mathfrak{F}. \tag{2.2}$$

It satisfies

$$V^*V = 1, \tag{2.3}$$

$$VQ = \phi(Q)V, \quad V^*\phi(Q) = QV^*, \quad Q \in \mathfrak{F}, \tag{2.4}$$

(2.5) $$[J_\Psi, V]=0.$$

Proof. Because $\phi^*\omega_\Psi$ and ω_Ψ are tracial, we have

$$\begin{aligned}||\phi(Q)\phi(A_L)\Psi||^2 &= \phi^*\omega_\Psi(A_LQ^*QA_L)\\ &= \phi^*\omega_\Psi((A_LQ^*QA_L)^\natural)\\ &= \phi^*\omega_\Psi(A_L(Q^*Q)^\natural A_L)\\ &\to \omega_\Psi((Q^*Q)^\natural)=\omega_\Psi(Q^*Q)=||Q\Psi||^2, \quad Q\in\mathfrak{F}.\end{aligned}$$

Since $\mathfrak{F}\Psi$ is dense in $\mathfrak{K}$, there exists a unique isometric V satisfying (2.2). (2.3) says that V is isometric.

The range of V is the closure of $\phi(\mathfrak{F})\phi(A_\phi)\Psi$, which is invariant under $\phi(\mathfrak{F})$. Hence VV^* commutes with $\phi(Q)$, $Q\in\mathfrak{F}$. From (2.2), we have for $Q\in\mathfrak{F}$ and $Q_1\in\mathfrak{F}$

$$\begin{aligned}V^*\phi(Q)VQ_1\Psi &= V^*\phi(QQ_1)\phi(A_\phi)\Psi\\ &= V^*VQQ_1\Psi=QQ_1\Psi.\end{aligned}$$

Hence

(2.6) $$V^*\phi(Q)V=Q.$$

Hence

$$VQ=VV^*\phi(Q)V=\phi(Q)VV^*V=\phi(Q)V,$$

$$QV^*=V^*\phi(Q)VV^*=V^*VV^*\phi(Q)=V^*\phi(Q).$$

Since $J_\Psi Q\Psi=Q^*\Psi$, we have

$$\begin{aligned}VJ_\Psi Q\Psi &= VQ^*\Psi=\phi(Q^*)\phi(A_\phi)\Psi\\ &= \lim_{L\to\infty}\phi(QA_L)^*\Psi=J_\Psi\phi(Q)\phi(A_\phi)\Psi\\ &= J_\Psi VQ\Psi.\end{aligned}$$

Hence (2.5) holds. Q.E.D.

Lemma 2. *If z is a closed operator affiliated with $\mathfrak{F}$ and Ψ is in the domain of z, then Ψ is in the domain of $\phi(zA_\phi)$ and $Vz\Psi=\phi(zA_\phi)\Psi$.*

Proof. By polar decomposition of z, it is enough to prove the statement for a positive selfadjoint $z=\int\lambda\, dE_\lambda^z$.
Let $z_\lambda = zE_\lambda^z$. Then

$$Vz\Psi = \lim_{\lambda\to\infty} Vz_\lambda\Psi = \lim_{\lambda\to\infty}\lim_{L\to\infty}\phi(z_\lambda A_L)\Psi.$$

Hence Ψ is in the domain of $\phi(zA_\phi)$ and $Vz\Psi=\phi(zA_\phi)\Psi$. (Note that A_ϕ is affiliated with $\mathfrak{F}_c$ and hence E_λ^z and E_L^A commute.) Q.E.D.

Lemma 3. *Define*

$$D_\phi^{(n)} = \phi(D_\phi^{(n-1)}A_\phi),\ n=1, 2, \dots,$$

$$D_\phi^{(0)}=1.$$

Then $V^n\Psi = D_\phi^{(n)}\Psi$, $D_\phi^{(n)}$ is affiliated with $\mathfrak{F}$, commutes with $\phi^n(Q)$, $Q\in\mathfrak{F}$ and its support $s(D_\phi^{(n)})$ is $\phi^n(1)$.

Proof. By repeated use of Lemma 2, we have $V^n\Psi=D_\phi^{(n)}\Psi$. It is affiliated with $\mathfrak{F}$ because it is a product of mutually commuting positive selfadjoint operators

$$D_\phi^{(n)} = \prod_{k=1}^{n}\phi^k(A_\phi). \tag{2.7}$$

Since spectral projections of A_ϕ is in the center of $\mathfrak{F}$, spectral projections of $\phi^k(A_\phi)$ is in the center of $\phi^k(\mathfrak{F})\supset\phi^n(\mathfrak{F})$ $(k\leqq n)$ and hence $D_\phi^{(n)}$ commutes with $\phi^n(Q)$, $Q\in\mathfrak{F}$. Since A_ϕ is strictly positive, $s(A_\phi)=1$. If $z=\int\lambda\, dE_\lambda$ is a positive selfadjoint operator, then $z\geqq\lambda(1-E_\lambda)$ implies $s(\phi(z))\geqq\sup_\lambda\phi(1-E_\lambda)=\phi(s(z))$. Hence $s(\phi(z))=\phi(s(z))$. In particular,

$$s(D_\phi^{(n)})=\phi(s\{D_\phi^{(n-1)}\})=\phi^n(1)$$

by induction. Q.E.D.

Lemma 4. *Assume*

$$\phi(\mathfrak{F})=\phi(1)\mathfrak{F}\phi(1). \tag{2.8}$$

Then $D_{\phi}^{(n)} \eta \phi^n(1)\mathfrak{F}_c$ *and*

$$V^n V^{*n} = \phi^n(1) j_{\Psi}\{\phi^n(1)\}, \tag{2.9}$$

where η *denotes an operator affiliated with a von Neumann algebra.*

Proof. From (2.7), $\phi^k(A_\phi)\, \eta\, \phi^k(\mathfrak{F}_c) = \mathfrak{F}_c \phi^k(1)$ and $\prod_{k=1}^{n} \phi^k(1) = \phi^n(1)$, we have $D_{\phi}^{(n)} \eta \phi^n(1)\mathfrak{F}_c$. The range of V is the closure of

$$\begin{aligned}\phi(\mathfrak{F})\Psi &= \phi(1)\mathfrak{F}\phi(1)\Psi = \phi(1)\mathfrak{F}J_{\Psi}\phi(1)\Psi \\ &= \phi(1) j_{\Psi}\{\phi(1)\}\mathfrak{F}\Psi\end{aligned}$$

and hence $VV^* = \phi(1) j_{\Psi}\{\phi(1)\}$.

By (2.4) and (2.5), we have inductively

$$\begin{aligned}V^n V^{*n} &= V\phi^{n-1}(1) j_{\Psi}\{\phi^{n-1}(1)\} V^* \\ &= \phi^n(1) j_{\Psi}\{\phi^n(1)\} VV^* \\ &= \phi^n(1) j_{\Psi}\{\phi^n(1)\}.\end{aligned}$$

Q.E.D.

We define

$$\phi'(y) = j_{\Psi}\{\phi(j_{\Psi}(y))\}, \quad y \in \mathfrak{F}'. \tag{2.10}$$

Then from Lemma 1,

$$V^*\phi'(y) = yV^*, \quad Vy = \phi'(y)V, \quad y \in \mathfrak{F}'. \tag{2.11}$$

We also note

$$\phi'^n(1) = j_{\Psi}\{\phi^n(1)\}. \tag{2.12}$$

We now construct $M(\mathfrak{F}, \phi)$ on a Hilbert space

$$\mathfrak{H} = \{\bigoplus_{-\infty}^{-1} \phi'^{|n|}(1)\mathfrak{R}\} \oplus \{\bigoplus_{0}^{\infty} \phi^n(1)\mathfrak{R}\},$$

namely we have partially isometric mappings p_n from $\mathfrak{R}$ into $\mathfrak{H}$ such that

$$p_n^* p_n = \begin{cases} \phi^n(1), & n \geqq 0, \\ \phi'^{|n|}(1), & n \leqq 0, \end{cases} \tag{2.13}$$

and subspaces $\mathfrak{H}_n \equiv p_n \mathfrak{K}$ are mutually orthogonal and span the whole space $\mathfrak{H}$.

Let $x \in \mathfrak{F}$, $y \in \mathfrak{F}'$. We define faithful representations of $\mathfrak{F}$ and $\mathfrak{F}'$ by

$$\pi(x) = \sum_n p_n \pi_n(x) p_n^*, \tag{2.14}$$

$$\pi_n(x) = \begin{cases} \phi^n(x), & n \geqq 0, \\ x, & n \leqq 0, \end{cases} \tag{2.15}$$

$$\pi'(y) = \sum_n p_n \pi'_n(y) p_n^*, \tag{2.16}$$

$$\pi'_n(y) = \begin{cases} y, & n \geqq 0 \\ \phi'^{|n|}(y), & n \leqq 0. \end{cases} \tag{2.17}$$

We also define partially isometric operators

$$U = \sum_{n=-\infty}^{\infty} p_{n+1} U_n p_n^*, \tag{2.18}$$

$$U_n = \begin{cases} 1, & n \geqq 0, \\ V^*, & n < 0, \end{cases} \tag{2.19}$$

$$U' = \sum_{n=-\infty}^{\infty} p_{n-1} U'_n p_n^*, \tag{2.20}$$

$$U'_n = \begin{cases} V^* & n > 0, \\ 1 & n \leqq 0. \end{cases} \tag{2.21}$$

The von Neumann algebra $M(\mathfrak{F}, \phi)$ and a candidate for its commutant are defined by

$$M(\mathfrak{F}, \phi) = \{\pi(\mathfrak{F}), U, U^*\}'', \tag{2.22}$$

$$M'(\mathfrak{F}, \phi) = \{\pi'(\mathfrak{F}'), U', U'^*\}''. \tag{2.23}$$

Theorem 3. *Assume $\phi(F) = \phi(1)\mathfrak{F}\phi(1)$. $\Omega(\mathfrak{F}, \phi) \equiv p_0 \Psi$ is a cyclic and separating vector for $M(\mathfrak{F}, \phi)$, with modular operator $\Delta(\mathfrak{F}, \phi)$ and modular conjugation operator $J(\mathfrak{F}, \phi)$ given by*

$$\Delta(\mathfrak{F}, \phi) = \sum_{-\infty}^{-1} p_n j_\Psi \{D_\phi^{(|n|)}\}^{-2} p_n^* + \sum_0^\infty p_n \{D_\phi^{(n)}\}^2 p_n^*, \tag{2.24}$$

$$J(\mathfrak{F}, \phi)=\sum_n p_{-n}J_{\Psi}p_n^*, \tag{2.25}$$

where the inverse in $j_{\Psi}\{D^{(n)}\}^{-2}$ *is to be taken in* $\phi'^n(1)\mathfrak{R}=J_{\Psi}\phi^n(1)\mathfrak{R}$. $M(\mathfrak{F}, \phi)$ *satisfies*

$$M(\mathfrak{F}, \phi)'=M'(\mathfrak{F}, \phi). \tag{2.26}$$

If $A_\phi>1$, *then the set of* $Q\in M(\mathfrak{F}, \phi)$ *commuting with* $\Delta(\mathfrak{F}, \phi)$ *is* $\pi(\mathfrak{F})$.

If $A_\phi=e^{a/2}$ *for a strictly positive number* a, *then the spectrum of* $\Delta(\mathfrak{F}, \phi)$ *is*

$$S_x=\{x^n;\ n=0, \pm 1, \dots\}\cup\{0\},\ x=e^a. \tag{2.27}$$

If $A_\phi>1$ *and* $\phi(z)=z\phi(1)$ *for all* $z\in\mathfrak{F}_c$, *then the center of* $\pi(\mathfrak{F})$ *coincides with the center of* $M(\mathfrak{F}, \phi)$. *Under the assumption* $A_\phi>1$, *the center of* $M(\mathfrak{F}, \phi)$ *consists of all* $z\in\mathfrak{F}_c$ *such that* $\phi(z)=z\phi(1)$,

If $\phi(z)=z\phi(1)$ *for all* $z\in\mathfrak{F}_c$, $A_\phi=\{\phi(1)^\natural\}^{-1/2}$.

Proof. $J(\mathfrak{F}, \phi)$ defined by (2.25) is antiunitary and satisfies

$$J(\mathfrak{F}, \phi)^2=1,\ J(\mathfrak{F}, \phi)\Omega(\mathfrak{F}, \phi)=\Omega(\mathfrak{F}, \phi), \tag{2.28}$$

$$J(\mathfrak{F}, \phi)\pi(x)J(\mathfrak{F}, \phi)=\pi'(j_{\Psi}(x)),\ x\in\mathfrak{F}, \tag{2.29}$$

$$J(\mathfrak{F}, \phi)UJ(\mathfrak{F}, \phi)=U', \tag{2.30}$$

where (2.5) has been used in (2.30).

We obtain

$$M'(\mathfrak{F}, \phi)\subset M(\mathfrak{F}, \phi)' \tag{2.31}$$

from the following calculations.

$$\begin{aligned}[\pi(x), \pi'(y)]&=\sum_n p_n\pi_n(x)(p_n^*p_n)\pi_n'(y)p_n^*\\ &\quad-\sum_n p_n\pi_n'(y)(p_n^*p_n)\pi_n(x)p_n^*\\ &=\Sigma\ p_n[\pi_n(x), \pi_n'(y)]p_n^*=0,\end{aligned}$$

$$\begin{aligned}[U, \pi'(y)]&=\sum_0^\infty p_{n+1}[1,\ y]p_n^*\\ &\quad+\sum_{-\infty}^{-1} p_{n+1}\{V^*\phi'^{|n|}(1)\phi'^{|n|}(y)-\phi'^{|n|-1}(y)\phi'^{|n|-1}(1)V^*\}p_n^*\\ &=0,\end{aligned}$$

$$[U', \pi(x)] = J(\mathfrak{F}, \phi)[U, \pi'(j_\Psi(x))]J(\mathfrak{F}, \phi) = 0,$$

$$\begin{aligned}[U, U'] = &\sum_1^\infty p_n\{p_{n-1}^* p_{n-1} V^* - V^* p_{n+1}^* p_{n+1}\} p_n^* \\ &+ p_0\{V^* p_{-1}^* p_{-1} - V^* p_1^* p_1\} p_0^* \\ &+ \sum_{-\infty}^{-1} p_n\{V^* p_{n-1}^* p_{n-1} - p_{n+1}^* p_{n+1} V^*\} p_n^* \\ = &\, 0,\end{aligned}$$

$$\begin{aligned}[U^*, U'] = &\sum_2^\infty p_{n-2}\{p_{n-1}^* p_{n-1} V^* - V^* p_{n-1}^* p_{n-1}\} p_n^* + p_{-1}\{VV^* - p_0^* p_0\} p_1^* \\ &+ \sum_{-\infty}^0 p_{n-2}\{V p_{n-1}^* p_{n-1} - p_{n-1}^* p_{n-1} V\} p_n^* \\ = &\, 0,\end{aligned}$$

where (2.4), (2.11), (2.8) and $p_n p_k^* p_k = p_n$, $p_k^* p_k p_n^* = p_n^*$ for $|n| \geqq |k|$, $nk \geqq 0$ are used.

From definitions, we have for $n \geqq 0$

$$\pi(\mathfrak{F}) U^{*n} \Omega(\mathfrak{F}, \phi) = p_{-n} \mathfrak{F} D_\phi^{(n)} \Psi.$$

Since $s(D_\phi^{(n)}) = \phi^n(1)$, the closure of $p_{-n}\mathfrak{F} D_\phi^{(n)}\Psi$ contains

$$\begin{aligned}p_{-n}\mathfrak{F}\phi^n(1)\Psi &= p_{-n}\mathfrak{F} J_\Psi \phi^n(1)\Psi = p_{-n} j_\Psi\{\phi^n(1)\}\mathfrak{F}\Psi \\ &= p_{-n}\mathfrak{F}\Psi\end{aligned}$$

which is dense in $\mathfrak{H}_{-n}$. We also have for $n > 0$,

$$U^n \pi(\mathfrak{F}) \Omega(\mathfrak{F}, \phi) = p_n \mathfrak{F}\Psi$$

which is dense in $\mathfrak{H}_n$. Hence $\Omega(\mathfrak{F}, \phi)$ is cyclic for $M(\mathfrak{F}, \phi)$.

(2.29) and (2.30), together with (2.28), imply that $\Omega(\mathfrak{F}, \phi)$ is cyclic for $M'(\mathfrak{F}, \phi)$ and hence separating for $M(\mathfrak{F}, \phi)$ by (2.31).

Setting $S = J(\mathfrak{F}, \phi)\Delta(\mathfrak{F}, \phi)^{1/2}$, we obtain for $n \geqq 0$

$$S\pi(x) U^{*n} \Omega(\mathfrak{F}, \phi) = p_n J_\Psi x D_\phi^{(n)} j_\Psi(D_\phi^{(n)})^{-1} \phi'^n(1)\Psi.$$

Since $j_\Psi(D_\phi^{(n)})^{-1}\phi'^n(1)\Psi = J_\Psi(D_\phi^{(n)})^{-1}\phi^n(1)\Psi = (D_\phi^{(n)})^{-1}\phi^n(1)\Psi$, we obtain

$$S\pi(x)U^{*n}\Omega(\mathfrak{F}, \phi) = p_n J_\Psi x\phi^n(1)\Psi = p_n x^*\Psi$$

$$= U^n\pi(x^*)\Omega(\mathfrak{F}, \phi).$$

From (2.24) and (2.25), $J(\mathfrak{F}, \phi)\Delta(\mathfrak{F}, \phi) = \Delta(\mathfrak{F}, \phi)^{-1}J(\mathfrak{F}, \phi)$. Hence $S^2 = 1$ whenever S is defined. Hence

$$SU^n\pi(x)\Omega(\mathfrak{F}, \phi) = \pi(x^*)U^{*n}\Omega(\mathfrak{F}, \phi).$$

Due to (2.4), we have

$$\begin{aligned}\pi(x)U^* &= \sum_1^\infty p_{n-1}\phi^n(x)p_n^* + \sum_{-\infty}^0 p_{n-1}Vxp_n^* \\ &= \sum_1^\infty p_{n-1}\phi^{n-1}(\phi(x))p_n^* + \sum_{-\infty}^0 p_{n-1}\phi(x)Vp_n^* \\ &= \pi(\phi(x))U^*, \\ \pi(x)U &= \{U^*\pi(x^*)\}^* = \{\pi(\phi(x))^*U^*\}^* \\ &= U\pi(\phi(x)).\end{aligned}$$

Due to (2.9) with $n=1$, we have

$$\begin{aligned}\pi(x)U^* &= \sum_1^\infty p_{n-1}\phi^{n-1}(x)p_n^* + \sum_{-\infty}^0 p_{n-1}xVp_n^* \\ &= \sum_1^\infty p_{n-1}\phi^{n-1}(x\phi(1))p_n^* + \sum_{-\infty}^0 p_{n-1}x\phi(1)Vp_n^* \\ &= \pi(x\phi(1))U^*, \\ U\pi(x) &= (\pi(x^*)U^*)^* = U\pi(\phi(1)x), \\ U\pi(x)U^* &= \pi(\phi^{-1}\{\phi(1)x\phi(1)\})UU^*, \\ UU^*\Omega(\mathfrak{F}, \phi) &= p_0V^*V\Psi \\ &= \Omega(\mathfrak{F}, \phi) \\ U^*U\Omega(\mathfrak{F}, \phi) &= p_0p_1^*p_1\Psi = \pi(\phi(1))\Omega(\mathfrak{F}, \phi).\end{aligned}$$

The last two equations imply

$$UU^* = 1, \qquad U^*U = \pi(\phi(1)). \tag{2.32}$$

Therefore $\pi(F)U^{*n}$ and $U^n\pi(F)$ with varying n together are total in $M(\mathfrak{F}, \pi)$. Hence

$$(J(\mathfrak{F}, \phi)\Delta(\mathfrak{F}, \phi)^{1/2})Q\Omega(\mathfrak{F}, \phi)=Q^*\Omega(\mathfrak{F}, \phi)$$

for all $Q\in M(\mathfrak{F}, \phi)$.

Let $D_\phi^{(n)}=\int L\,dE_L$ and $j_\Psi\{(D_\phi^{(n)})^{-1}\phi'^n(1)\}=\int L\,dj_\Psi(E_L')$.

Since $E_L\mathfrak{F}\Psi$ and

$$\mathfrak{F}E_L'D_\phi^{(n)}\Psi=\mathfrak{F}D_\phi^{(n)}E_L'\Psi=\mathfrak{F}D_\phi^{(n)}j_\Psi(E_L')\Psi=j_\Psi(E_L')\mathfrak{F}D_\phi^{(n)}\Psi$$

are total sets of analytic vectors in $\phi^n(1)\mathfrak{K}$ and $\phi'^n(1)\mathfrak{K}$, respectively, $M(\mathfrak{F}, \phi)\ \Omega(\mathfrak{F}, \phi)$ is a core of $\Delta(\mathfrak{F}, \phi)^{1/2}$.

Hence $\Delta(\mathfrak{F}, \phi)$ and $J(\mathfrak{F}, \phi)$ are modular operator and modular conjugation operator for $\Omega(\mathfrak{F}, \phi)$.

By (2.29) and (2.30), we have

$$M'(\mathfrak{F}, \phi)=J(\mathfrak{F}, \phi)M(\mathfrak{F}, \phi)J(\mathfrak{F}, \phi)=M(\mathfrak{F}, \phi)'.$$

If $A_\phi>1$, then $\phi^n(A_\phi)>\phi^n(1)$ and hence $D_\phi^{(n)}>\phi^n(1)$. Then $\mathfrak{H}_0$ is the eigenspace of $\Delta(\mathfrak{F}, \phi)$ belonging to 1. If $Q\in M(\mathfrak{F}, \phi)$ is invariant under modular automorphisms for $\Omega(\mathfrak{F}, \phi)$, then Q leaves $\mathfrak{H}_0$ invariant and hence the restriction of Q to $\mathfrak{H}_0$ must be in the von Neumann algebra generated by $E_0M(\mathfrak{F}, \phi)E_0$ where E_0 is the projection on $\mathfrak{H}_0$. Since $\pi(F)U^{*n}$ and $U^n\pi(F)$ are total in $M(\mathfrak{F}, \phi)$ and $E_0\pi(F)U^{*n}E_0=E_0U^n\pi(F)E_0=0$ for $n\neq 0$, we have $E_0M(\mathfrak{F}, \phi)E_0=\pi(\mathfrak{F})E_0$. Hence $QE_0=\pi(x)E_0$ for some $x\in\mathfrak{F}$. Since $\Omega(\mathfrak{F}, \phi)$ is separating, $Q=\pi(x)\in\pi(\mathfrak{F})E_0$. Conversely, all elements in $\pi(\mathfrak{F})$ commute with $\Delta(\mathfrak{F}, \phi)$, which is most easily demonstrated by

$$\{\Delta(\mathfrak{F}, \phi)^{it}\pi(x)\Delta(\mathfrak{F}, \phi)^{-it}-\pi(x)\}\Omega(\mathfrak{F}, \phi)=0.$$

If $A_\phi=e^{a/2}$, then $D_\phi^{(n)}=e^{na/2}\phi^n(1)$ and hence $\mathfrak{H}_n$ is the eigenspace of $\Delta(\mathfrak{F}, \phi)$ belonging to an eigenvalue e^{na} and hence the spectrum of $\Delta(\mathfrak{F}, \phi)$ is S_x, $x=e^a$.

If z is in the center of $M(\mathfrak{F}, \phi)$, then by a general result it is in the center of the set of elements in $M(\mathfrak{F}, \phi)$ commuting with $\Delta(\mathfrak{F}, \phi)$. Under the assumption $A_\phi>1$, $z=\pi(\bar{z})$, $\bar{z}\in\mathfrak{F}_c$. It is in the center of $M(\mathfrak{F}, \phi)$ if and only if

$$U\pi(\bar{z})=\pi(\bar{z})U.$$

Since $UU^*=1$ and $U\pi(\bar{z})U^*=\pi(\phi^{-1}\{\phi(1)\bar{z}\phi(1)\})$, this condition is equivalent to $\phi(\bar{z})=\phi(1)\bar{z}\phi(1)=\bar{z}\phi(1)$. Hence the center of $M(\mathfrak{F}, \phi)$ consists of $\pi(z)$ such that $z\in\mathfrak{F}_c$ and $\phi(z)=z\phi(1)$.

If $\phi(z)=z\phi(1)$ for all $z\in\mathfrak{F}_c$, then

$$(\Psi, \phi(z)\Psi)=(\Psi, z\phi(1)\Psi)=(\Psi, z\phi(1)^{\natural}\Psi).$$

Since ϕ and $\natural$ are faithful, $\phi(z)^{\natural}=z\phi(1)^{\natural}\neq 0$ for $z\in\mathfrak{F}_c$, $z\neq 0$. Hence $s(\phi(1)^{\natural})=1$ and $(\phi(1)^{\natural})^{-1}$ exists. Then

$$(\Psi, \phi(AzA)\Psi)=(\Psi, z\Psi)$$

if $A=\{\phi(1)^{\natural}\}^{-1/2}$. Since such A in $\mathfrak{F}_c$ is unique, we have $A_{\phi}=(\phi(1)^{\natural})^{-1/2}$.

Q.E.D.

Remark 1. If $\phi(z)=z\phi(1)$ for all $z\in\mathfrak{F}_c$, then the condition $A_{\phi}>1$ is equivalent to the condition that ϕ is a proper injective endomorphism of $z\mathfrak{F}$, for every central projection $z\neq 0$, as is seen by the following argument.

If $A_{\phi}=1$, then $(\Psi, (1-\phi(1))\Psi)=0$. Since Ψ is separating, $\phi(1)=1$ and ϕ is an automorphism. If p is the projection on the eigenspace of A_{ϕ} belonging to 1, then the same argument shows $p=\phi(p)$ and hence ϕ is an automorphism on $p\mathfrak{F}$. Since $p\geqq p\phi(1)=\phi(p)$ for every central projection, $A_{\phi}\geqq 1$ and hence $A_{\phi}>1$ is equivalent to ϕ not being an automorphism on $z\mathfrak{F}$ for every central projection $z\neq 0$.

Remark 2. If $\mathfrak{F}$ is commutative and ϕ is an automorphism, then $M(\mathfrak{F}, \phi)$ is the von Neumann algebra obtained by group-measure construction with the additive group of integers.

§3. Proof of Theorem 1

We consider the von Neumann algebra $\mathfrak{M}$ on a space $\mathfrak{H}$ in Theorem 1. We start with preliminary analysis.

In [3], we have considered the set $\mathfrak{M}_J$ of operators Q in $\mathfrak{M}$ satisfying $Q\mathfrak{H}(I)\subset\mathfrak{H}(I+J)$ for every finite open interval I where $\mathfrak{H}((\alpha, \beta))=$

$(E_{\beta-0}-E_{\alpha+0})H$, E_λ is the spectral projection of $\log\Delta=\int\lambda\, dE_\lambda$ and J is a finite open interval. We assume that the spectrum of $\log\Delta$ is $\{0, \pm\log x_1, \pm\log x_2,\ldots\}$.

If the spectrum of $\log\Delta$ contained in J is just one point α, then $\mathfrak{H}(J)=\mathfrak{H}_\alpha$. Let $Q\in\mathfrak{M}_J$ for such J. Then

$$\{\tau(t)Q\}\Omega=\Delta^{it}Q\Omega=e^{i\alpha t}Q\Omega.$$

Since Ω is separating, $\tau(t)Q=e^{i\alpha t}Q$ and $Q\in\mathfrak{M}_\alpha$.
Hence $\mathfrak{M}_J=\mathfrak{M}_\alpha$.

By Lemma 7, Lemma 5 and (2.1) of [3], we have

$$\mathfrak{M}_\alpha\mathfrak{M}_\beta\subset\mathfrak{M}_{\alpha+\beta}, \tag{3.1}$$

$$\mathfrak{M}_\alpha^*=\mathfrak{M}_{-\alpha}, \tag{3.2}$$

$$\overline{\mathfrak{M}_\alpha\Omega}=\mathfrak{H}_\alpha. \tag{3.3}$$

Lemma 5. $\bigcup_\alpha\mathfrak{M}_\alpha$ is total in $\mathfrak{M}$.

Proof. Let $J_n=(\log x_{n-1}, \log x_{n+1})$, $n=0, \pm1,\ldots$ where $x_0=1$. Let $\tilde{\phi}_n$, $n=0, \pm1, \ldots$ be non-negative C^∞ functions with a compact support in J_n such that $\Sigma\tilde{\phi}_n=1$. Let $\tilde{f}$ be any C^∞ function with a compact support. Let

$$f(t)=(2\pi)^{-1}\int\tilde{f}(\lambda)e^{-i\lambda t}d\lambda,$$

$$f_n(t)=(2\pi)^{-1}\int\tilde{f}(\lambda)\tilde{\phi}_n(\lambda)e^{-i\lambda t}d\lambda.$$

Then $f=\sum_n f_n$ (finite sum) and $Q(f_n)\in\mathfrak{M}_{\log x_n}$ by Lemma 6 of [3]. Hence $Q(f)\in\bigcup_\alpha\mathfrak{M}_\alpha$.

Let $\tilde{g}$ be a non-negative C^∞ function with a compact support such that $\tilde{g}(0)=1$ and $g(t)=(2\pi)^{-1}\int e^{-i\lambda t}\tilde{g}(\lambda)d\lambda$. Let $g_n(t)=n\,g(nt)$ and $\tilde{g}_n(\lambda)=\tilde{g}(\lambda/n)$. It is then easy to see $\lim_{n\to\infty}Q(g_n)=Q$. Since $Q(g_n)\in\bigcup_\alpha\mathfrak{M}_n$, we have the lemma. Q.E.D.

Lemma 6. $\overline{(\mathfrak{M}_\alpha)_{p.i.}\mathfrak{H}_0}=\mathfrak{H}_\alpha$, *where* $(\mathfrak{M}_\alpha)_{p.i.}$ *is the set of all partial isometries in* $\mathfrak{M}_\alpha$ *and the bar denotes the strong closure.*

Proof. Let $\Psi\in\mathfrak{H}_\alpha$ and $\varepsilon>0$ be given. By (3.3), there exists $Q\in\mathfrak{M}_\alpha$ such that $\|Q\Omega-\Psi\|<\varepsilon/2$.

Let $Q^*Q=\int\lambda\, de_\lambda$. Since $Q^*Q\in\mathfrak{M}_0$ by (3.1) and (3.2) and since $\mathfrak{M}_0$ is a von Neumann algebra, $e_\lambda\varepsilon\mathfrak{M}_0$. Let $\delta>0$ be such that $Q_\delta=Q(1-e_\delta)$ satisfies $\|Q_\delta\Omega-Q\Omega\|<\varepsilon/2$. Let $|Q|_\delta=(Q^*Q)^{1/2}(1-e_\delta)+e_\delta$. Then $|Q|_\delta$ has a bounded inverse $|Q|_\delta^{-1}\in\mathfrak{M}_0$, $U=Q_\delta|Q|_\delta^{-1}\in(\mathfrak{M}_\alpha)_{p.i.}$, $\Phi\equiv|Q|_\delta\Omega\in\mathfrak{H}_0$ and $U\Phi=Q_\delta\Omega$ satisfies $\|U\Phi-\Psi\|<\varepsilon$. Q.E.D.

We need a Lemma on the mapping F_ρ^{ZR} introduced in [2].

Lemma 7. *If Z is a center of a finite von Neumann algebra R and ρ is a trace on R, then*

$$F_\rho^{ZR}(Q)=Q^\natural s^Z(\rho). \tag{3.4}$$

Proof. As proved in [2], F_ρ^{ZR} is Z-linear, positive normal mapping from R onto $Zs^Z(\rho)$, vanishing on $(1-s^Z(\rho))R+R(1-s^Z(\rho))$ and strictly positive on $s^Z(\rho)R$. Since ρ is a trace state, we have

$$\rho(Q_1Q_2z)=\rho(Q_1zQ_2)=\rho(Q_2Q_1z)$$

for $z\in Z$. Hence $F_\rho^{ZR}(Q_1Q_2)=F_\rho^{ZR}(Q_2Q_1)$, which implies $F_\rho^{ZR}(UQU^*)=F_\rho^{ZR}(Q)$ for all unitary U in R. Hence F_ρ^{ZR} is the canonical $\natural$ mapping of $Rs^Z(\rho)$ and we have (3.4). Q.E.D.

Proof of Theorem 1

Step 1. Let be s_α the support of $\mathfrak{H}_\alpha$ in $\mathfrak{M}$, namely the smallest projection in $\mathfrak{M}$ satisfying $(1-s_\alpha)\mathfrak{H}_\alpha=0$. We prove $s_\alpha\in\mathfrak{Z}$.

Since $\mathfrak{H}_\alpha$ is invariant under Δ^{it} (as a set), $[s_\alpha, \Delta^{it}]=0$ for all real t. Hence $s_\alpha\in\mathfrak{M}_0$.

Since $\mathfrak{M}_0\mathfrak{H}_\alpha=\mathfrak{H}_\alpha$, any $Q\in\mathfrak{M}_0$ commutes with s_α. Hence $s_\alpha\in\mathfrak{Z}_0$.

By assumption $\mathfrak{Z}=\mathfrak{Z}_0$, we have $s_\alpha\in\mathfrak{Z}$.

Step 2. $\sup\{UU^*;U\in(\mathfrak{M}_\alpha)_{p.i.}\}=s_\alpha$.

Since $U\Omega\in\mathfrak{H}_\alpha$, $(1-s_\alpha)U\Omega=0$ which implies $(1-s_\alpha)U=0$. Hence $s_\alpha\geqq UU^*$ for all $U\in(\mathfrak{M}_\alpha)_{p.i.}$.

By Lemma 6, $\{U\mathfrak{H}_0\}$ is dense in $\mathfrak{H}_\alpha$. Hence

$$\{UU^*\mathfrak{H}\}=\{U\mathfrak{H}\}=\{(U\mathfrak{M}'\mathfrak{H}_0)^-\}=\{(\mathfrak{M}'U\mathfrak{H}_0)^-\}$$

is dense in $(\mathfrak{M}'\mathfrak{H}_\alpha)^-=s_\alpha\mathfrak{H}$.

Step 3. $\sup\{U^*U;\ U\in(\mathfrak{M}_\alpha)_{p.i.}\}=s_\alpha$.

Since $J\Delta=\Delta^{-1}J$, we have $J\mathfrak{H}_\alpha=\mathfrak{H}_{-\alpha}$. Hence $s_{-\alpha}=j(s'_\alpha)$ where s'_α is the support of $\mathfrak{H}_\alpha$ relative to $\mathfrak{M}'$. Since $s_{-\alpha}\in Z$ by Step 1, $s'_\alpha=j(s_{-\alpha})=s_{-\alpha}\in Z$. We now have $\mathfrak{H}_\alpha\subset s'_\alpha\mathfrak{H}$ and hence $s_\alpha\mathfrak{H}=\overline{\mathfrak{M}'\mathfrak{H}_\alpha}\subset s'_\alpha\mathfrak{H}$, which implies $s_\alpha\leqq s'_\alpha$. At the same time $\mathfrak{H}_\alpha\subset s_\alpha\mathfrak{H}$. Hence $s'_\alpha\mathfrak{H}=\overline{\mathfrak{M}\mathfrak{H}_\alpha}\subset s_\alpha\mathfrak{H}$ which implies $s'_\alpha\leqq s_\alpha$. Hence $s_\alpha=s'_\alpha=s_{-\alpha}$. By (3.2) and Step 2, we obtain Step 3.

Step 4. For $U\in(\mathfrak{M}_\alpha)_{p.i.}$.

$$(UU^*)^\natural=e^\alpha(U^*U)^\natural. \tag{3.5}$$

Proof. For $z\in\mathfrak{Z}_0$, we have

$$\begin{aligned}(\Omega,\ UU^*z\Omega)&=(\Omega,\ UzU^*\Omega) && (\text{by } \mathfrak{Z}=\mathfrak{Z}_0)\\ &=(\Omega,\ UzJ\Delta^{1/2}U\Omega)\\ &=(\Omega,\ Uzj(U)\Omega)e^{\alpha/2} && (\text{by } U\Omega\in\mathfrak{H}_\alpha)\\ &=(j(U^*)\Omega,\ Uz\Omega)e^{\alpha/2}\\ &=(\Omega,\ U^*Uz\Omega)e^\alpha.\end{aligned}$$

Hence

$$F_{\omega_\Omega}^{\mathfrak{Z}_0\mathfrak{M}_0}(UU^*)=e^\alpha F_{\omega_\Omega}^{\mathfrak{Z}_0\mathfrak{M}_0}(U^*U).$$

By Lemma 7, we have (3.5).

Step 5. There exists $U_\alpha\in(\mathfrak{M}_\alpha)_{p.i.}$ such that $U_\alpha U_\alpha^*=s_\alpha$ for $\alpha>0$.

Consider a maximal family of $U_\nu\in(\mathfrak{M}_\alpha)_{p.i.}$ such that $U_\nu U_\nu^*$ are mutually orthogonal and $U_\nu^*U_\nu$ are mutually orthogonal. Then $U_\alpha\equiv\Sigma U_\nu\in(\mathfrak{M}_\alpha)_{p.i.}$. Assume $s_\alpha-U_\alpha U_\alpha^*\neq0$. By Step 2, there exists $U_{(0)}\in(\mathfrak{M}_\alpha)_{p.i.}$.

such that $c(s_\alpha - U_\alpha U_\alpha^*)U_{(0)}U_{(0)}^* \neq 0$ where $c(\cdot)$ denotes the central support. By the comparability theorem, there exists a projection e in $\mathfrak{Z}_0$ and partial isometries u_1 and u_2 in $\mathfrak{M}_0$ such that $u_1^*u_1 = e(s_\alpha - U_\alpha U_\alpha^*)$, $u_1u_1^* \leqq eU_{(0)}U_{(0)}^*$, $u_2^*u_2 = (1-e)(U_{(0)}U_{(0)}^*)$ and $u_2u_2^* \leqq (1-e)(s_\alpha - U_\alpha U_\alpha^*)$. Due to $c(s_\alpha - U_\alpha U_\alpha^*)U_{(0)}U_{(0)}^* \neq 0$, we have either $u_1 \neq 0$ or $u_2 \neq 0$. If $u_1 \neq 0$, we set $U' = eu_1^*U_{(0)}$. Then $U'U'^* = u_1^*u_1 \leqq s_\alpha - U_\alpha U_\alpha^*$, $U' \neq 0$ and $U' \varepsilon (\mathfrak{M}_\alpha)_{p.i.}$. If $u_2 \neq 0$, we set $U' = (1-e)u_2U_{(0)}$. Then $U'U'^* = u_2u_2^* \leqq s_\alpha - U_\alpha U_\alpha^*$, $U' \neq 0$ and $U' \varepsilon (\mathfrak{M}_\alpha)_{p.i.}$.

By Step 4,

$$
\begin{aligned}
(s_\alpha - U_\alpha^* U_\alpha)^\natural &= s_\alpha - e^{-\alpha}(U_\alpha U_\alpha^*)^\natural \\
&\geqq e^{-\alpha}(s_\alpha - U_\alpha U_\alpha^*)^\natural \\
&\geqq e^{-\alpha}(U'U'^*)^\natural = (U'^*U')^\natural.
\end{aligned}
$$

Hence there exists $u \in (\mathfrak{M}_0)_{p.i.}$ such that

$$uu^* = U'^*U', \ u^*u \leqq s_\alpha - U_\alpha^* U_\alpha.$$

Setting $U'' = U'u$, we have $U'' \in (\mathfrak{M}_\alpha)_{p.i.}$ and

$$
\begin{aligned}
U''^*U'' &= u^*u \leqq s_\alpha - U_\alpha^* U_\alpha, \\
U''U''^* &= U'U'^* \leqq s_\alpha - U_\alpha U_\alpha^*.
\end{aligned}
$$

We also have $U''U''^* = U'U'^* \neq 0$ and hence $U'' \neq 0$. This contradicts with the maximality of $\{U_\nu\}$.

Step 6. Fix U_α such that $U_\alpha \in (\mathfrak{M}_\alpha)_{p.i.}$, $U_\alpha U_\alpha^* = s_\alpha$. Then $(U_\alpha^* U_\alpha)^\natural = e^{-\alpha}s_\alpha$. Such U_α exists for $\alpha > 0$ by Step 5. Define an injective endomorphism ϕ_α of $s_\alpha \mathfrak{M}_0$ by $\phi_\alpha(Q) = U_\alpha^* Q U_\alpha$. Then

$$\phi_\alpha(\mathfrak{M}_0) = \phi_\alpha(1)\mathfrak{M}_0\phi_\alpha(1), \tag{3.6}$$

$$\phi_\alpha(z) = z\phi_\alpha(1), \ z \in \mathfrak{Z}_0, \tag{3.7}$$

$$\phi_\alpha(1)^\natural = e^{-\alpha}s_\alpha. \tag{3.8}$$

(3.8) is the same as $(U_\alpha^* U_\alpha)^\natural = e^{-\alpha}s_\alpha$. (3.7) follows from $\mathfrak{Z}_0 = \mathfrak{Z}$, which implies $U_\alpha^* z U_\alpha = z U_\alpha^* U_\alpha = z\phi_\alpha(1)$ if $z \in \mathfrak{Z}_0$. Let e be a projection in $\phi_\alpha(1)\mathfrak{M}_0\phi_\alpha(1)$. Then $e \leqq \phi_\alpha(1) = U_\alpha^* U_\alpha$. Let $e' = U_\alpha e U_\alpha^* \in \mathfrak{M}_0$. Then $e = U_\alpha^* e' U_\alpha = \phi_\alpha(e')$. Hence $\phi_\alpha(1)\mathfrak{M}_0\phi_\alpha(1) \subset \phi_\alpha(\mathfrak{M}_0)$. Conversely $\phi_\alpha(\mathfrak{M}_0) \subset \phi_\alpha(1)\phi_\alpha(\mathfrak{M}_0)\phi_\alpha(1) \subset \phi_\alpha(1)\mathfrak{M}_0\phi_\alpha(1)$. Hence (3.6) holds.

ϕ_α is injective on $\mathfrak{M}_0 s_\alpha$ because $\phi_\alpha(Q) = 0$ implies $0 = U_\alpha \phi_\alpha(Q) U_\alpha^* = s_\alpha Q s_\alpha = Q$ for $Q \in s_\alpha \mathfrak{M}_0$. ϕ_α is a $*$ homomorphism because $\phi_\alpha(Q_1)\phi_\alpha(Q_2) = \phi_\alpha(Q_1 s_\alpha Q_2) = \phi_\alpha(Q_1 Q_2)$ for $Q_1, Q_2 \in s_\alpha \mathfrak{M}_0$ and $\phi_\alpha(Q)^* = \phi_\alpha(Q^*)$.

Step 7. $s_\alpha(\mathfrak{M}_{n\alpha})_{p.i.} \subset U_\alpha^n(\mathfrak{M}_0)_{p.i.}$ for $n \geqq 0$ and $s_\alpha(\mathfrak{M}_{n\alpha})_{p.i.} \subset (\mathfrak{M}_0)_{p.i.} U_\alpha^{*n}$ for $n \leqq 0$, where U_α is from Step 6.

Let $n > 0$ and $V \in s_\alpha(\mathfrak{M}_{n\alpha})_{p.i.}$. By (3.1) and (3.2), $w = U_\alpha^{*n} V \in (\mathfrak{M}_0)$ and $U_\alpha^n w = s_\alpha V = V$. w is a partial isometry because $ww^* = \phi_\alpha^n(VV^*)$ and $w^* w = s_\alpha V^* V$ are both projections. Hence $s_\alpha(\mathfrak{M}_{n\alpha})_{p.i.} \subset U_\alpha^n(\mathfrak{M}_0)_{p.i.}$. Taking adjoint and using (3.2), we have $s_\alpha(\mathfrak{M}_{n\alpha})_{p.i.} \subset (\mathfrak{M}_0)_{p.i.} U_\alpha^{*|n|}$ for $n < 0$. The case $n = 0$ is trivial.

Step 8. Let $\{0, \pm a_1, \pm a_2, \ldots\}$ be spectrum of $\log \Delta$ such that $0 < a_1 < a_2 < \cdots$. Let

$$P_n = s_{a_n} \prod_{k=1}^{n-1} (1 - s_{a_k}), \; n = 1, 2, \ldots,$$

$$P_0 = 1 - \sum_{n=1}^{\infty} P_n.$$

By definition and Step 1, P_n are mutually orthogonal central projections with the sum $\sum_{k=0}^{\infty} P_k = 1$. $P_1 = s_{a_1} \neq 0$. On $P_0 \mathfrak{H}$, Δ has no eigenvalue larger than 1 and hence $\Delta = 1$. (Each $P_n \mathfrak{H}$ is invariant under J and Δ because $P_n \in \mathfrak{Z}$.) Consequently, $P_0 \Omega$ is a cyclic and separating trace vector for $P_0 \mathfrak{M}$.

Let $U_n = P_n U_{a_n}$, $\Omega_n = P_n \Omega$, $\mathfrak{F}_n = P_n \mathfrak{M}_0$. $\mathfrak{F}_n$ is known to be a finite von Neumann algebra with Ω_n as a cyclic (in $P_n \mathfrak{H}$) and separating trace vector. Let $\phi_n = \phi_{a_n} | \mathfrak{F}_n$. It satisfies (3.6)$\sim$(3.8), where ϕ_α is replaced by ϕ_n, $\mathfrak{M}_0$ by $\mathfrak{F}_n$, $\mathfrak{Z}_0$ by the center $\mathfrak{F}_{nc}$ of $\mathfrak{F}_n$, α by a_n, and s_α by P_n.

Step 9. $U_n^m \mathfrak{F}_n \Omega_n$ and $\mathfrak{F}_n (U_n^*)^m \Omega_n$, $m = 0, 1, 2, \ldots$ span the whole space $P_n \mathfrak{H}$.

By Step 7 and (3.3), they span $\sum_{m=-\infty}^{\infty} P_n \mathfrak{H}_{m a_n}$. Assume that $P_n \mathfrak{H}_b \neq 0$, $b \neq m a_n$ for any integer m. Since $P_n \mathfrak{H}_{-b} = P_n J \mathfrak{H}_b = J P_n \mathfrak{H}_b \neq 0$, we may assume $b>0$. There exists a non-negative integer m such that $ma_n<b<(m+1)a_n$.

$$U_n^m(U_n^m)^* = P_n \geqq U_b' U_b'^*$$

for $U_b' = P_n U_b$. We define $W = (U_n^m)^* U_b'$. $W \in P_n \mathfrak{M}_{b-ma_n}$. We also have

$$W^* W = U_b'^* P_n U_b' = U_b'^* U_b' \neq 0.$$

Hence $0 \neq W\Omega_n \in P_n \mathfrak{H}_{b-ma_n}$. Since $0<b-ma_n<a_n$, this contradicts the definition of P_n.

Step 10. $P_n\mathfrak{M}$, Ω_n, $\mathfrak{F}_n$ and U_n are unitarily equivalent to $M(\mathfrak{F}_n, \phi_n)$, $\Omega(\mathfrak{F}_n, \phi_n)$, $\pi(\mathfrak{F}_n)$ and U of Section 2, where Ψ is to be taken equal to Ω_n.

$U_n^m \mathfrak{F}_n \Omega_n$, $m=0, 1, \dots$ and $\mathfrak{F}_n (U_n^*)^{m'} \Omega_n$, $m'=1, 2, \dots$ are orthogonal family of subspaces of $P_n\mathfrak{H}$ because they belong to different eigenvalues of Δ. They span the whole space $P_n\mathfrak{H}$ by Step 9.

We have for x, $y \in \mathfrak{F}_n$ and $m \geqq 0$,

$$\begin{aligned}(U_n^m x\Omega_n, U_n^m y\Omega_n) &= (\Omega_n, x^*\phi_n^m(1) y\Omega_n) \\ &= (U^m\pi(x)\Omega(\mathfrak{F}_n, \phi_n), U^m\pi(y)\Omega(\mathfrak{F}_n\phi_n))\end{aligned}$$

where we have used $(U^*)^m U^m = \pi(\phi_n^m(1))$ which follows from (2.32) and $U^*\pi(x)U = \pi(\phi(x))$. For $m>0$,

$$(x(U_n^*)^m\Omega_n, y(U_n^*)^m\Omega_n) = (\Omega_n, U_n^m x^* y(U_n^*)^m\Omega_n).$$

Since $U_n\phi_n(1) = U_n U_n^* U_n = U_n$, $U_n^m = U_n\phi_n(1)U_n^{m-1} = U_n^m\phi_n^m(1)$.
Hence $U_n^m x^* y(U_n^*)^m = U_n^m\phi_n^m(1)x^* y\phi_n^m(1)(U_n^*)^m$. There exists $Q \in \mathfrak{F}_n$ such that $\phi_n^m(Q) = \phi_n^m(1)x^* y\phi_n^m(1)$. Then $U_n^m\phi_n^m(Q)(U_n^*)^m = U_n^m(U_n^*)^m Q U_n^m(U_n^*)^m = P_nQP_n = Q$. Hence

$$\begin{aligned}(x(U_n^*)^m\Omega_n, y(U_n^*)^m\Omega_n) &= (\Omega_n, \phi_n^{-m}\{\phi_n^m(1)x^* y\phi_n^m(1)\}\Omega_n) \\ &= (\Omega(\mathfrak{F}_n, \phi_n), U^m\pi(x^* y)U^{*m}\Omega(\mathfrak{F}_n, \phi_n)) \\ &= (\pi(x)U^{*m}\Omega(\mathfrak{F}_n, \phi_n), \pi(y)U^{*m}\Omega(\mathfrak{F}_n, \phi_n)).\end{aligned}$$

Therefore there exists a unitary mapping p from $\mathfrak{H}$ of Section 2 to $P_n\mathfrak{H}$ such that for all $x \in \mathfrak{F}_n$,

$$pU^m\pi(x)\Omega(\mathfrak{F}_n, \phi_n) = U_n^m x\Omega_n,\ m=0, 1,\dots,$$

$$p\pi(x)U^{*m}\Omega(\mathfrak{F}_n, \phi_n) = x(U_n^*)^m\Omega_n,\ m=1, 2,\dots.$$

It is then immediately seen that

$$p^* x p = \pi(x),$$

$$p^* U_n p = U.$$

By Lemma 5 and Step 7, $\mathfrak{F}_n$ and U_n generate $\mathfrak{M}P_n$. Hence $\mathfrak{M}P_n = pM(\mathfrak{F}_n, \phi_n)p^*$. Q.E.D.

§4. Isomorphism among $M(\mathfrak{F}, \phi)$

Theorem 4. *Different choices of cyclic and separating trace vector Ψ yield unitarily equivalent $M(\mathfrak{F}, \phi)$.*

Proof. Let ρ' be another faithful tracial state of $\mathfrak{F}$. Then there exists a strictly positive selfadjoint operator α affiliated with $\mathfrak{F}_c$ such that the vector state by $\Psi' = \alpha\Psi$ is ρ'. It is sufficient to show that $P_0\Psi'$, $\pi(\mathfrak{F})$ and U have exactly the same structure as in the construction in section 2 where Ψ is to be replaced by Ψ'.

Let $A'_\phi = \int \lambda\, dE_\lambda^{A'}$, $E_\lambda^{A'} \in \mathfrak{F}_c$ be defined similar to A_ϕ:

$$(\phi(A'_\phi)\Psi', \phi(z)\phi(A'_\phi)\Psi') = (\Psi', z\Psi'), \qquad z \in \mathfrak{F}_c.$$

Let $\alpha = \int \lambda\, dE_\lambda^\alpha$, $\alpha_L = \alpha E_L^\alpha$. Then

$$||z_1\alpha_L\Psi||^2 = ||\phi(z_1\alpha_L A_\phi)\Psi||^2, \qquad z_1 \in \mathfrak{F}_c.$$

Hence $\phi(A_\phi)\Psi$ is in the domain of α and

$$(\Psi', z\Psi') = (\phi(\alpha)\phi(A_\phi)\Psi, \phi(z)\phi(\alpha)\phi(A_\phi)\Psi)$$

for positive z in $\mathfrak{F}_c$ and hence for any z in $\mathfrak{F}_c$. Since αA_ϕ and α^{-1} are strictly positive, $\phi(\alpha A_\phi)\alpha^{-1}$ is strictly positive on $\phi(1)\mathfrak{K}$ and hence there exists a positive selfadjoint operator

$$A' = \int \lambda_1\lambda_2\lambda_3^{-1}\, dE_{\lambda_1}^\alpha\, dE_{\lambda_2}^A\, d\{\phi^{-1}(E_{\lambda_3}^\alpha\phi(1))\}$$

affiliated with $\mathfrak{F}_c$ such that $\phi(A')=\phi(\alpha A_\phi)\alpha^{-1}$ and hence

$$(\Psi', z\Psi')=(\phi(A')\Psi', \phi(z)\phi(A')\Psi').$$

From the uniqueness of $A'_\phi=(\text{Radon-Nikodym derivative})^{1/2}$, we have $A'_\phi=A'$ and

$$\phi(A'_\phi)\alpha=\phi(\alpha A_\phi).$$

Hence

$$\begin{aligned} VQ\Psi' &= \lim_L VQ\alpha_L\Psi \\ &= \lim_L \phi(Q\alpha_L)\phi(A_\phi)\Psi \\ &= \phi(Q)\phi(\alpha A_\phi)\Psi = \phi(Q)\phi(A'_\phi)\Psi'. \end{aligned}$$

This shows that V for Ψ' is the same as V for Ψ. Hence U and $\pi(x)$ constructed relative to Ψ and Ψ' coincide. Q.E.D.

Corollary. *Let ϕ_0 be $*$ automorphism of $\mathfrak{F}$ and $\phi'=\phi_0^{-1}\phi\phi_0$. Then $M(\mathfrak{F}, \phi')$ is unitarily equivalent to $M(\mathfrak{F}, \phi)$.*

Since the triplet F, ϕ, ω_Ψ is isomorphic to $\phi_0^{-1}\mathfrak{F}=\mathfrak{F}$, ϕ', $\phi_0^*\omega_\Psi$, the triplet $\mathfrak{F}$, ϕ, Ψ is unitarily equivalent to $\mathfrak{F}$, ϕ', Ψ' if $\omega_{\Psi'}=\phi_0^*\omega_\Psi$ and Ψ' is cyclic and separating. Since $\Psi'=\alpha\Psi$ for strictly positive selfadjoint α affiliated with $\mathfrak{F}_c$ is cyclic and separating, we have Corollary from Theorem 4. Q.E.D.

It is also obvious that $M(\mathfrak{F}, \phi)$ is unitarily equivalent to $M(\mathfrak{F}, \phi\phi_0)$ for any inner $*$ automorphism ϕ_0. (If $\phi_0(x)=uxu^*$, then consider $\pi(u^*)U$ instead of U.)

Theorem 5. *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be finite von Neumann algebras on $\mathfrak{K}_1$ and $\mathfrak{K}_2$ with cyclic and separating unit trace vectors Ψ_1 and Ψ_2. Let ϕ_1 and ϕ_2 be injective endomorphisms of $\mathfrak{F}_1$ and $\mathfrak{F}_2$ such that $\phi_k(\mathfrak{F}_k)=\phi_k(1)\mathfrak{F}_k\phi_k(1)$, $\phi_k(z)=z\phi_k(1)$ for all z in the center of $\mathfrak{F}_k$ and $\phi_k(1)^\natural=e^{-a}$, $k=1, 2$, $a>0$. The pairs $M(\mathfrak{F}_1, \phi_1)$, $\Omega(\mathfrak{F}_1, \phi_1)$ and $M(\mathfrak{F}_2, \phi_2)$, $\Omega(\mathfrak{F}_2, \phi_2)$ are unitarily equivalent if and only if there exists a unitary mapping w from $\mathfrak{K}_1$ onto $\mathfrak{K}_2$ and a unitary operator v in $\mathfrak{F}_2$ such that $w\mathfrak{F}_1w^*=\mathfrak{F}_2$, $w\Psi_1=\Psi_2$ and*

$\phi_2^{-1}\phi_v\phi_w\phi_1\phi_{w^*}$ *is an inner* $*$ *automorphism of* $\mathfrak{F}_1$ *where* $\phi_u(x)=uxu^*$.

Proof. Let $\bar{w}$ be a unitary mapping such that $\bar{w}\Omega(\mathfrak{F}_1, \phi_1)=\Omega(\mathfrak{F}_2, \phi_2)$ and $\bar{w}M(\mathfrak{F}_1, \phi_1)\bar{w}^*=M(\mathfrak{F}_2, \phi_2)$. Since $J(\mathfrak{F}, \phi)$ and $\Delta(\mathfrak{F}, \phi)$ are defined by a polar decomposition of the closure $\bar{s}$ of S defined by $SQ\Omega(\mathfrak{F}, \phi)=Q^*\Omega(\mathfrak{F}, \phi)$, $Q\in M(\mathfrak{F}, \phi)$, we have $\bar{w}\Delta(\mathfrak{F}_1, \phi_1)\bar{w}^*=\Delta(\mathfrak{F}_2, \phi_2)$. Hence $\bar{w}M(\mathfrak{F}_1, \phi_1)_0\bar{w}^*=M(\mathfrak{F}_2, \phi_2)_0$. Hence $\bar{w}$ restricted to the eigenspace $\mathfrak{H}_0$ of $\Delta(\mathfrak{F}_1, \phi_1)$ belonging to an eigenvalue 1, gives a unitary mapping from the pair $M(\mathfrak{F}_1, \phi_1)_0$, $\Omega(\mathfrak{F}_1, \phi_1)$ $(\sim\mathfrak{F}_1, \Psi_1)$ to $M(\mathfrak{F}_2, \phi_2)_0$, $\Omega(\mathfrak{F}_2, \phi_2)$ $(\sim\mathfrak{F}_2, \Psi_2)$. Let w be the corresponding unitary mapping from $\mathfrak{K}_1$, $\mathfrak{F}_1$, Ψ_1 to $\mathfrak{K}_2$, $\mathfrak{F}_2$, Ψ_2.

Let $\bar{w}\pi(\phi_1(1))\bar{w}^*=e_1$, $\pi(\phi_2(1))=e_2$. By assumption, $e_1^\natural=e_2^\natural=e^{-a}$ where $\natural$ denotes the canonical $\natural$-mapping in $M(\mathfrak{F}_2, \phi_2)_0$. Hence there exists a unitary $\bar{v}\in M(\mathfrak{F}_2, \phi_2)_0$ such that $\bar{v}e_1\bar{v}^*=e_2$. Let $\bar{v}=\pi(v)$, $v\in\mathfrak{F}$.

By construction, there exist isometric operators U_1^* in $M(\mathfrak{F}_1, \phi_1)_{-a}$ and U_2^* in $M(\mathfrak{F}_2, \phi_2)_{-a}$ such that $U_1^*\pi(x)U_1=\pi(\phi_1(x))$ and $U_2^*\pi(x)U_2=\pi(\phi_2(x))$. Then $\bar{u}=U_2\bar{v}\bar{w}U_1^*\bar{w}^*$ is a unitary operator in $M(\mathfrak{F}_2, \phi_2)_0$ and we have $\bar{u}\pi(x)\bar{u}^*=\pi\{\phi_2^{-1}\phi_v\phi_w\phi_1\phi_{w^*}(x)\}$. Hence $\phi_2^{-1}\phi_v\phi_w\phi_1\phi_{w^*}$ is inner.

The converse is immediate. Q.E.D.

Theorem 6. *Let* $\mathfrak{F}$ *be a finite von Neumann algebra with a cyclic and separating trace vector and* ϕ_1, ϕ_2 *be two injective endomorphisms of* $\mathfrak{F}$ *such that* $\phi_k(\mathfrak{F})=\phi_k(1)\mathfrak{F}\phi_k(1)$, $\phi_k(z)=z\phi_k(1)$ *for all* $z\in\mathfrak{F}_c$ *and* $\phi_1(1)^\natural=\phi_2(1)^\natural=e^{-a}, a>0$. *Two triplets* $\mathfrak{F}$, ϕ_1, Ψ_1 *and* $\mathfrak{F}$, ϕ_2, Ψ_2 *for some cyclic and separating trace vectors* Ψ_1 *and* Ψ_2 *satisfy the relation in Theorem* 5 *if and only if there exists a* $*$ *automorphism* ϕ_0 *of* $\mathfrak{F}$ *such that*

$$\phi_2^{-1}\phi_v\phi_0^{-1}\phi_1\phi_0$$

is an inner $*$ *automorphism of* $\mathfrak{F}$ *where* v *is any unitary element in* $\mathfrak{F}$ *satisfying*

$$v[\phi_0^{-1}\phi_1(1)]v^*=\phi_2(1).$$

Proof. The "only if" part follows from the condition stated in Theorem 6 because ϕ_{w^*} is a $*$ isomorphism of $\mathfrak{F}$ if $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are both $*$ isomorphic to $\mathfrak{F}$. The "if" part is also immediate because $x\in\mathfrak{F}$ and its cyclic and separating trace vector Ψ is mapped unitarily to $\phi_0^{-1}(x)\in\mathfrak{F}$ and Ψ' if Ψ' is another trace vector such that $\phi_0^*\omega_\Psi=\omega_{\Psi'}$. Q.E.D.

§5. Asymptotic Abelian System

Proof of Theorem 2

Notation The Hilbert space, representation of $\mathfrak{M}$ and a cyclic and separating unit vector associated with ρ are denoted by $\mathfrak{H}_\rho$, π_ρ and Ω_ρ. Modular operator and modular conjugation operator for Ω_ρ are denoted by $\varDelta_\rho$ and J_ρ. $\bar{\tau}_\rho(t)Q\equiv\varDelta_\rho^{it}Q\varDelta_\rho^{-it}$, $j_\rho(Q)=J_\rho QJ_\rho$. $\mathfrak{H}_a$ denotes the eigenspace of $\log\varDelta_\rho$ belonging to an eigenvalue a and $\bar{\mathfrak{M}}_a$ is the set of $Q\in\bar{\mathfrak{M}}\equiv\pi(\mathfrak{M})$ satisfying $\bar{\tau}_\rho(t)Q=e^{iat}Q$. U_α is a unitary operator satisfying $U_\alpha\pi_\rho(Q)\Omega_\rho=\pi_\rho(\tau_\alpha Q)\Omega_\rho$ for all $Q\in\mathfrak{M}$. $\bar{\tau}_\alpha(Q)=U_\alpha QU_\alpha^*$.

Step 1. U_α commutes with $\varDelta_\rho$.

For $S_\rho=J_\rho\varDelta_\rho^{1/2}$ and $Q\in\mathfrak{M}$, we have

$$S_\rho U_\alpha\pi_\rho(Q)\Omega_\rho=\pi_\rho(\tau_\alpha Q)^*\Omega_\rho=U_\alpha S_\rho\pi_\rho(Q)\Omega_\rho,$$

which implies $[U_\alpha, S_\rho]=0$ and hence $[U_\alpha, \varDelta_\rho]=0$.

Step 2. Let $\bar{s}_a$ be the support of $\mathfrak{H}_a$ in $\bar{\mathfrak{M}}$. Then $\bar{s}_a\in\bar{\mathfrak{Z}}(\subset\bar{\mathfrak{Z}}_0)$ and $\bar{s}_{-a}=\bar{s}_a$.

Since $\mathfrak{H}_a$ is invariant under $\varDelta^{it}$, $\bar{s}_a\in\bar{\mathfrak{M}}_0$. Since U_α commute with $\varDelta_\rho$, $\mathfrak{H}_a$ is invariant under U_α and hence $\bar{s}_a$ commutes with U_α.

Let $s_a=\pi_\rho^{-1}(\bar{s}_a)$.

$$\psi_a(Q)\equiv\rho(s_aQ)=\rho(s_aQs_a),\qquad Q\in\mathfrak{M}$$

It is a normal positive linear functional on $\mathfrak{M}$. By using the mapping $F_\rho^{\mathfrak{Z}\mathfrak{M}}$ of [2], we have

$$\rho(s_aQ)=\rho(F_\rho^{\mathfrak{Z}\mathfrak{M}}(s_a)Q),\qquad Q\in\mathfrak{Z}.$$

By Theorem 1 (6) of [2],

$$\tau_\alpha F_\rho^{\mathfrak{Z}\mathfrak{M}}(s_a)=F_\rho^{\mathfrak{Z}\mathfrak{M}}(\tau_\alpha s_a)=F_\rho^{\mathfrak{Z}\mathfrak{M}}(s_a).$$

Hence for $Q\in\mathfrak{A}$, we have by Lemma 9 of [3]

$$\rho(s_aQ)=\rho((\tau_a^{-1}s_a)Q)=\rho(s_a\tau_\alpha Q)$$
$$=\lim_\alpha \rho(s_a\tau_\alpha Q)=\lim_\alpha \rho(F_\rho^{\mathfrak{Z}\mathfrak{M}}(s_a)\tau_\alpha Q)$$
$$=\lim_\alpha \rho(F_\rho^{\mathfrak{Z}\mathfrak{M}}(s_a)Q)=\rho(F_\rho^{\mathfrak{Z}\mathfrak{M}}(s_a)Q).$$

This implies

$$\pi_\rho\{F_\rho^{\mathfrak{Z}\mathfrak{M}}(s_a)-s_a\}\Omega=0.$$

Since ρ is faithful, we have

$$s_a=F_\rho^{\mathfrak{Z}\mathfrak{M}}(s_a)\in\mathfrak{Z}.$$

By the same argument as Step 3 of the proof of Theorem 1, we have $s_a=s_{-a}$.

Step 3. Let $s_0(Q)$ denotes the support in $\overline{\mathfrak{Z}}_0$ of $Q\in\overline{\mathfrak{M}}$, namely the smallest projection $e\in\overline{\mathfrak{Z}}_0$ such that $eQ=Qe=Q$. Then $s_0(U_1^*U_1)\perp s_0(U_2^*U_2)$ is equivalent to $s_0(U_1U_1^*)\perp s_0(U_2U_2^*)$ for U_1, $U_2\in(\overline{\mathfrak{M}}_a)_{p.i.}$

Assume that $s_0(U_1^*U_1)\perp s_0(U_2^*U_2)$ and $s_0(U_1U_1^*)\wedge s_0(U_2U_2^*)\neq 0$. Then there exists $u\in(\overline{\mathfrak{M}}_0)_{p.i.}$ such that $U_2^*uU_1\neq 0$. However by (3.1) and (3.2), $U_2^*uU_1\in\overline{\mathfrak{M}}_0$ and $s_0(U_2^*U_2)U_2^*uU_1s_0(U_1^*U_1)=U_2^*uU_1\neq 0$, which is a contradiction. Hence $s_0(U_1^*U_1)\perp s_0(U_2^*U_2)$ implies $s_0(U_1U_1^*)\perp s_0(U_2U_2^*)$. Similarly the converse is proved.

Step 4. $s_0(U_1^*U_1)=s_0(U_2^*U_2)$ is equivalent to $s_0(U_1U_1^*)=s_0(U_2U_2^*)$ for U_1, $U_2\in(\overline{\mathfrak{M}}_a)_{p.i.}$.

Assume that a projection $e\in\overline{\mathfrak{Z}}_0$ satisfies

$$s_0(U_1^*U_1)\perp e,\ s_0(U_2^*U_2)\geqq e\neq 0.$$

Then $u_2=U_2e\in(\overline{\mathfrak{M}}_a)_{p.i.}$, $u_2^*u_2=eU_2^*U_2e\neq 0$, $s_0(u_2^*u_2)=e$. By Step 3, $s_0(u_2u_2^*)\perp s_0(U_1U_1^*)$. We also have $s_0(u_2u_2^*)=s_0(U_2eU_2^*)\leqq s_0(U_2U_2^*)$. Similarly, $e\in\overline{\mathfrak{Z}}_0$, $s_0(U_2^*U_2)\perp e$ and $s_0(U_1^*U_1)\geqq e\neq 0$ imply $s_0(U_2U_2^*)\perp s_0(u_1u_1^*)$ and $s_0(U_1U_1^*)\geqq s_0(u_1u_1^*)\neq 0$ for $u_1=U_1e$. Hence $s_0(U_1^*U_1)\neq s_0(U_2^*U_2)$ implies $s_0(U_1U_1^*)\neq s_0(U_2U_2^*)$.

The converse is similarly proved.

Step 5. Let $\phi_a^0(z)=s_0(UU^*)$ whenever $U\in(\bar{\mathfrak{M}}_a)_{p.i.}$ and $z=s_0(U^*U)$. Then ϕ_a^0 is a lattice automorphism of projections in $\bar{s}_a\bar{\mathfrak{Z}}_0$.

By Step 4, ϕ_a^0 is single-valued. By Step 3, $z_1\perp z_2$ and $\phi_a^0(z_1)\perp\phi_a^0(z_2)$ are equivalent,

If $U\in(\bar{\mathfrak{M}}_a)_{p.i.}$, then $U\Omega_\rho=s_aU\Omega_\rho$ and hence $UU^*\leqq\bar{s}_a$. We also have $U^*\in(\bar{\mathfrak{M}}_{-a})_{p.i.}$ and hence $U^*U\leqq\bar{s}_{-a}=\bar{s}_a$.

Let U_ν be a maximal family of elements in $(\bar{\mathfrak{M}}_a)_{p.i.}$ such that $s_0(U_\nu^*U_\nu)$ is mutually orthogonal. Then $s_0(U_\nu U_\nu^*)$ is mutually orthogonal by Step 3 and hence $U=\sum_\nu U_\nu$ is in $(\bar{\mathfrak{M}}_a)_{p.i.}$. If $s_0(U^*U)<\bar{s}_a$, then there exists non-zero $u\in(\bar{\mathfrak{M}}_a)_{p.i.}$ with $s_0(u^*u)\perp s_0(U^*U)$, by Step 3 of Section 3. By Step 3, $s_0(uu^*)\perp s_0(UU^*)$. This contradicts the maximality. Hence $s_0(U^*U)=\bar{s}_a$. Similarly $s_0(UU^*)=\bar{s}_a$ by Step 2 of Section 3, Step 3 and maximality.

For any projection $e\in\bar{\mathfrak{Z}}_0\bar{s}_a$, $Ue\in(\bar{\mathfrak{M}}_a)_{p.i.}$, $eU\in(\bar{\mathfrak{M}}_a)_{p.i.}$, $s_0((Ue)^*(Ue))=es_0(U^*U)e=e$ and $s_0((eU)(eU)^*)=es_0(UU^*)e=e$. Hence the domain and range of ϕ_a^0 is all projections in $\bar{s}_a\bar{\mathfrak{Z}}_0$.

Since $z_1\geqq z_2$ is equivalent to $z_2\perp z$ for all $z\perp z_1$, $z_1\geqq z_2$ and $\phi_a^0(z_1)\geqq\phi_a^0(z_2)$ are equivalent. Hence ϕ_a^0 is a lattice automorphism.

Step 6. If $z=\int\lambda\,de_\lambda+i\int\lambda\,de'_\lambda\in\bar{\mathfrak{Z}}_0\bar{s}_a$, then define $\phi_a^0(z)=\int\lambda\,d\phi_a^0(e_\lambda)+i\int\lambda\,d\phi_a^0(e_\lambda')$. Then ϕ_a^0 is an automorphism of $\bar{\mathfrak{Z}}_0\bar{s}_a$, which follows from Step 5.

Step 7. If $z\in\bar{\mathfrak{Z}}_0\bar{s}_a$ and $U\in(\bar{\mathfrak{M}}_a)_{p.i.}$, then $\phi_a^0(z)U=Uz$.

It is enough to prove the equation for a projection $z\in\bar{\mathfrak{Z}}_0\bar{s}_a$. We have

$$s_0((Uz)^*(Uz))=zs_0(U^*U)z=s_0(U^*U)\wedge z.$$

Since ϕ_a^0 is a lattice automorphism, we have

$$s_0((Uz)(Uz)^*)=s_0(UU^*)\wedge\phi_a^0(z).$$

Since $s_0(UU^*)U=U$, we have

$$Uz=s_0\{(Uz)(Uz)^*\}Uz=\phi_a^0(z)Uz.$$

Similarly we have

$$\phi_a^0(z)U=\phi_a^0(z)Us_0\{(\phi_a^0(z)U)^*(\phi_a^0(z)U)\}$$
$$=\phi_a^0(z)U\{(\phi_a^0)^{-1}\phi_a^0(z)\wedge s_0(U^*U)\}$$
$$=\phi_a^0(z)Uz=Uz.$$

Step 8. $\phi_a^0(z)=z$ if $z\in\bar{\mathfrak{Z}}\bar{s}_a$.

It is enough to prove it for a projection z.

Let $z\in\bar{\mathfrak{Z}}\bar{s}_a$, $s_0(U^*U)=z$. Then $Uz=U=zU$. Hence $s_0(UU^*)\leqq z$. Let $z-s_0(UU^*)\equiv e$, $(\phi_a^0)^{-1}e\equiv e'$. Let $u\in(\bar{\mathfrak{M}}_a)_{p.i.}$, $u^*u=e'$, $uu^*=e$. Since $z\geqq e$ and $z\in\mathfrak{Z}$, $u=zu=uz$. Hence $e'\leqq z$. On the other hand $e\perp\phi_a^0(z)$ implies $e'\perp z$. Hence $e'=0$. Hence $z=s_0(UU^*)=\phi_a^0(z)$.

Step 9. $\bar{\tau}_\alpha$ leaves $\bar{\mathfrak{Z}}_0\bar{s}_a$ invariant and commutes with ϕ_a^0.

Since U_α commutes with Δ_ρ, $\bar{\mathfrak{Z}}_0$ remains invariant (as a set) under $\bar{\tau}_\alpha$. $\bar{s}_a$ commutes with U_α as we have seen in Step 2. $\bar{\tau}_\alpha$ also leaves $(\bar{\mathfrak{M}}_a)_{p.i.}$ invariant as a set. We now have

$$\bar{\tau}_\alpha s_0(U^*U)=s_0((\bar{\tau}_\alpha U)^*(\bar{\tau}_\alpha U))$$
$$\bar{\tau}_\alpha s_0(UU^*)=s_0((\bar{\tau}_\alpha U)(\bar{\tau}_\alpha U)^*)$$

because $\mathfrak{M}_0$ is also invariant under $\bar{\tau}_\alpha$ as a set. This implies $\bar{\tau}_\alpha\phi_a^0=\phi_a^0\bar{\tau}_\alpha$.

Step 10. $(\Omega_\rho,\ \phi_a^0(z)\Omega_\rho)=(\Omega_\rho,\ z\Omega_\rho)$ for $z\in\bar{\mathfrak{Z}}_0\bar{s}_a$.

By Radon-Nikodym theorem, there exists a strictly positive selfadjoint operator A_a^0 affiliated with $\bar{\mathfrak{Z}}_0\bar{s}_a$ such that

$$(\Omega_\rho,\ \phi_a^0(z)\Omega_\rho)=(A_a^0\Omega_\rho,\ zA_a^0\Omega).$$

Since $\bar{\tau}_\alpha$ commutes with ϕ_a^0 and ρ is invariant under $\bar{\tau}_\alpha$, A_a^0 must be invariant under $\bar{\tau}_\alpha$, namely $A_a^0=\int\lambda\, de_\lambda$, $[e_\lambda,\ U_\alpha]=0$. By the same argument as for $\bar{s}_a$ in Step 2, we obtain $e_\lambda\in\bar{\mathfrak{Z}}$. Namely $(A_a^0)^2$ is the Radon-Nikodym derivative of $(\phi_a^0)^*\bar{\rho}$ by $\bar{\rho}$ relative to $\bar{\mathfrak{Z}}$ where $\bar{\rho}=\rho\circ\pi^{-1}$. Since $\phi_a^0=1$ on $\bar{\mathfrak{Z}}\bar{s}_a$, we have $A_a^0=1$.

Step 11. $(UU^*)^\natural=e^a\phi_a^0\{(U^*U)^\natural\}$ for $U\in(\bar{\mathfrak{M}}_a)_{p.i.}$.
Let $z\in\bar{\mathfrak{Z}}_0\bar{s}_a$. Then

$$\begin{aligned}(\Omega_\rho, U^*Uz\Omega_\rho) &= (\Omega_\rho, U^*\phi_a^0(z)U\Omega_\rho) && \text{(by Step 7)}\\ &= (\Omega_\rho, U^*\phi_a^0(z)j_\rho(U^*)\Omega_\rho)e^{-a/2} && \text{(by } U^*\Omega_\rho\in\mathfrak{H}_{-a})\\ &= (j_\rho(U)\Omega_\rho, U^*\phi_a^0(z)\Omega_\rho)e^{-a/2}\\ &= (\Omega_\rho, UU^*\phi_a^0(z)\Omega_\rho)e^{-a}\\ &= (\Omega_\rho, (UU^*)^\natural\phi_a^0(z)\Omega_\rho)e^{-a} && \text{(by Lemma 7)}\\ &= (\Omega_\rho, \{(\phi_a^0)^{-1}(UU^*)^\natural\}z\Omega_\rho)e^{-a} && \text{(by Step 10).}\end{aligned}$$

Hence

$$(U^*U)^\natural = e^{-a}(\phi_a^0)^{-1}\{(UU^*)^\natural\}.$$

Step 12. There exists $U_a\in(\mathfrak{M}_a)_{p.i.}$ such that $U_aU_a^*=s_a$ if $a>0$.

The proof is the same as Step 5 of Section 3, except the inequality is now proved using Step 11 as follows:

$$\begin{aligned}(s_a-U_0^*U_0)^\natural &= s_a-e^{-a}(\phi_a^0)^{-1}\{(UU^*)^\natural\}\\ &\geqq e^{-a}(\phi_a^0)^{-1}\{(s_a-UU^*)^\natural\}\\ &\geqq e^{-a}(\phi_a^0)^{-1}\{(U'U'^*)^\natural\}=(U'^*U')^\natural.\end{aligned}$$

Step 13. There exists $N_a\in(\bar{\mathfrak{M}}_a)_{p.i.}$ such that $N_aN_a^*+N_a^*N_a=\bar{s}_a$, $(N_a^*N_a)^\natural=(1+e^a)^{-1}\bar{s}_a$.

For any $T\in\bar{s}_a\bar{\mathfrak{Z}}_0$, $0\leqq T\leqq 1$, there exists a projection $e_T\in(\bar{\mathfrak{M}}_0\bar{s}_a)$ such that $(e_T)^\natural=T$. Let $e_{(0)}$ be a projection in $(\bar{\mathfrak{M}}_0\bar{s}_a)$ such that $(e_{(0)})^\natural=(1+e^{-a})^{-1}\bar{s}_a$. Let $U_{(0)}=e_{(0)}U_a$. Then $U_{(0)}U_{(0)}^*=e_{(0)}$ and hence

$$(U_{(0)}^*U_{(0)})^\natural=e^{-a}(\phi_a^0)^{-1}\{e_0^\natural\}=(1+e^a)^{-1}\bar{s}_a=(\bar{s}_a-e_{(0)})^\natural.$$

There exists $u\in(\bar{\mathfrak{M}}_0\bar{s}_a)_{p.i.}$ such that

$$uu^*=U_{(0)}^*U_{(0)},\ u^*u=\bar{s}_a-e_{(0)}.$$

Setting $U_{(0)}u=N_a$, we have $N_a\in(\bar{\mathfrak{M}}_a)_{p.i.}$,

$$N_a^*N_a=u^*u=\bar{s}_a-e_{(0)},\qquad N_aN_a^*=U_{(0)}U_{(0)}^*=e_{(0)}.$$

Hence $N_a^*N_a+N_aN_a^*=\bar{s}_a$. We have $(N_a^*N_a)^\natural=\bar{s}_a-(e_{(0)})^\natural=(1+e^a)^{-1}\bar{s}_a$.

Step 14. $z\in\bar{\mathfrak{Z}}_0\bar{s}_a$ commutes with $\pi_\rho(\mathfrak{A})_a\equiv\bar{\mathfrak{M}}_a\cap\pi_\rho(\mathfrak{A})$.

Let $Q, Q'\in\pi_\rho(\mathfrak{A})_a$.

Then $Q\bar{\tau}_\alpha^{-1}(N_a^*)\in\bar{\mathfrak{M}}_0$, it commutes with z. We have

$$(Q'\Omega_\rho, Q\bar{\tau}_\alpha^{-1}(N_a^*)z\bar{\tau}_\alpha^{-1}(N_a)\Omega_\rho)$$
$$=(Q'\Omega_\rho, zQ\bar{\tau}_\alpha^{-1}(N_a^*N_a)\Omega_\rho).$$

The functional $f(Q)=(\Omega_\rho, QN_a^*N_a\Omega_\rho)$ satisfies

$$\begin{aligned} f(Q)&=(\Omega_\rho, QN_a^*J_\rho\Delta_\rho^{1/2}N_a^*\Omega) \\ &=(\Omega_\rho, QN_a^*j_\rho(N_a^*)\Omega_\rho)e^{-a/2} \\ &=(j_\rho(N_a)\Omega_\rho, QN_a^*\Omega_\rho)e^{-a/2} \\ &=(N_a^*\Omega_\rho, QN_a^*\Omega_\rho)e^{-a}\geqq 0 \end{aligned} \tag{5.1}$$

for $Q\in\bar{\mathfrak{M}}$, $Q\geqq 0$ and

$$f(w)=(\Omega_\rho, w(N_a^*N_a)^\natural\Omega_\rho)=(1+e^a)^{-1}(\Omega_\rho, \bar{s}_aw\Omega_\rho)$$

for $w\in\mathfrak{Z}_0$.

Since the functional $(\Omega_\rho, \bar{s}_aQ\Omega_\rho)=(\Omega_\rho, \bar{s}_aQ\bar{s}_a\Omega_\rho)$ is invariant under $\bar{\tau}_\alpha$ and since $Q'^*zQ\bar{s}_a$ is weakly τ_α-central in $\bar{\mathfrak{M}}_0$, we have by Lemma 9 of [3],

$$(Q'\Omega_\rho, zQ\bar{\tau}_\alpha^{-1}(N_a^*N_a)\Omega_\rho)$$
$$=(\Omega_\rho, \bar{\tau}_\alpha\{Q'^*zQ\bar{s}_a\}(N_a^*N_a)\Omega_\rho)$$
$$\to(1+e^a)^{-1}(\Omega_\rho, Q'^*zQ\bar{s}_a\Omega_\rho).$$

On the other hand, we have $Q^*Q'\Omega_\rho=j_\rho(Q^*Q')^*\Omega_\rho$ by $Q^*Q'\in\bar{\mathfrak{M}}_0$. Hence

$$(Q'\Omega_\rho, Q\bar{\tau}_a^{-1}(N_a^*)z\bar{\tau}_a^{-1}(N_a)\Omega_\rho)=(N_a\Omega_\rho, \bar{\tau}_\alpha(j_\rho(Q^*Q')z)N_a\Omega_\rho).$$

The $*$ subalgebra generated by $\pi_\rho(\mathfrak{A})_0$ and $\bar{\mathfrak{Z}}_0$ is strongly $\bar{\tau}_\alpha$ central in $\bar{\mathfrak{M}}_0$. We now prove that this algebra is strongly dense in $\bar{\mathfrak{M}}_0$. Let $Q\in\bar{\mathfrak{M}}_0$, $\Phi_i\in\mathfrak{H}$, $i=1,\dots, n$ and $\varepsilon>0$. Let $\tilde{f}$ be a C^∞ function such that its support is in $[-a_1/2, a_1/2]$ and $\tilde{f}(0)=1$. Let $f(\lambda)=(2\pi)^{-1}\int \tilde{f}(p)e^{-ip\lambda}dp$, $Q(f)=\int(\bar{\tau}(t)Q)f(t)dt$ and $c=\int|f(\lambda)|\,d\lambda(\geqq\tilde{f}(0)=1)$. Since Ω_ρ is separating for $\bar{\mathfrak{M}}_0$, there exists $Q_i\in\bar{\mathfrak{M}}_0'$ such that $\|\Phi_i-Q_i\Omega_\rho\|<(4\|Q\|c)^{-1}\varepsilon$. Since $\mathfrak{A}$ is weakly dense in $\mathfrak{M}$, the unit ball of $\pi_\rho(\mathfrak{A})$ is strongly dense in the unit ball of $\bar{\mathfrak{M}}$ and there exists $Q_\varepsilon\in\pi_\rho(\mathfrak{A})$ satisfying $\|Q_\varepsilon\|\leqq\|Q\|$ and $\|(Q_\varepsilon-Q)\Omega_\rho\|<(2c\max\|Q_i\|)^{-1}\varepsilon$. Then $\|\{Q_\varepsilon(f)-Q(f)\}\Omega_\rho\|<(2\max\|Q_i\|)^{-1}\varepsilon$. Since $Q\in\bar{\mathfrak{M}}_0$, we have $Q(f)=Q$. Due to the assumed support of $\tilde{f}$, $Q_\varepsilon(f)\Omega_\rho\in\mathfrak{H}_0$ and hence $\{\bar{\tau}(t)Q_\varepsilon(f)\}\Omega_\rho=Q_\varepsilon(f)\Omega_\rho$. Hence $Q_\varepsilon(f)\in\pi_\rho(\mathfrak{A})_0$. We have

$$\|(Q_\varepsilon(f)-Q(f))\Phi_i\|\leqq(\|Q_\varepsilon(f)\|+\|Q\|)\|\Phi_i-Q_i\Omega_\rho\| \\ +\|Q_i\|\|(Q_\varepsilon(f)-Q(f))\Omega_\rho\|<\varepsilon.$$

Hence $\pi_\rho(\mathfrak{A})_0$ is dense in $\bar{\mathfrak{M}}_0$.

We can now use Lemma 1 of [3] to obtain

$$\begin{aligned}&(N_a\Omega_\rho,\ \bar{\tau}_\alpha(j_\rho(Q^*Q')z)N_a\Omega_\rho)\\ &\rightarrow(1+e^a)^{-1}(\Omega_\rho,\ \bar{s}_a j_\rho(Q^*Q')z\Omega_\rho)\\ &=(Q'\Omega_\rho,\ Qz\bar{s}_a\Omega_\rho)(1+e^a)^{-1}.\end{aligned}$$

We now have

$$(Q'\Omega_\rho,\ [z,\ Q]\bar{s}_a\Omega_\rho)=0.$$

By Lemma 5 of [3], $Q'\Omega_\rho$ is dense in $\mathfrak{H}_a$. Since $[z,\ Q]\bar{s}_a\Omega_\rho\in\mathfrak{H}_a$, we have

$$[z,\ Q]\bar{s}_a\Omega_\rho=0.$$

Since $\bar{s}_a\in\mathfrak{Z}$, $[z,\ Q]\bar{s}_a=[z\bar{s}_a,\ Q]=[z,\ Q]$ by assumption $z\in\mathfrak{Z}_0\bar{s}_a$. Since Ω_ρ is separating $[z,\ Q]=0$.

Step 15. $\bar{\mathfrak{Z}}_0=\bar{\mathfrak{Z}}$.

By the same proof as that of Lemma 5, $\bigcup_a\pi_\rho(\mathfrak{A})_a$ is total in $\bar{\mathfrak{M}}$. Since $Q\Omega_\rho\in\mathfrak{H}_a$ for $Q\in\bar{\mathfrak{M}}_a$, $(1-\bar{s}_a)Q\Omega_\rho=0$ and hence $(1-\bar{s}_a)Q=0$ for $Q\in\pi_\rho(\mathfrak{A})_a\subset\bar{\mathfrak{M}}_a$. Hence $Q\in\pi_\rho(\mathfrak{A})_a$ commutes with $(1-\bar{s}_a)z$ trivially and with $\bar{s}_a z$ by Step 14 if $z \in \bar{\mathfrak{Z}}_0$. Hence $z \in \bar{\mathfrak{Z}}_0$ commutes with $\pi_\rho(\mathfrak{A})_a$ for any a and hence belongs to $\bar{\mathfrak{Z}}$.

Step 16. $r_\infty(P_n M) = S_{x_n}$.

Since the spectrum of modular operator for $P_n \Omega$ is S_{x_n}, $x_n = e^{-a_n} = \phi_n(1)^{\frac{1}{2}}$, we have by [6]

$$r_\infty(P_n M) \subset S_{x_n}.$$

We now show that $P_n M$ has the property L'_{λ_n} with $\lambda_n = (1+x_n)^{-1} x_n$ which shows

$$r_\infty(P_n M) = S_{x_n}$$

by [1], if the space is separable.

The same computation as (5.1) shows

$$(\Omega_\rho, Q N_a \Omega_\rho) = (\Omega_\rho, N_a Q \Omega_\rho) e^{-a} \tag{5.2}$$

for all $Q \in \mathfrak{M}$. Since ρ is τ_α invariant, we have

$$\begin{aligned}(\Omega_\rho, Q \bar{\tau}_\alpha^{-1}(N_a) \Omega_\rho) &= (\Omega_\rho, \bar{\tau}_\alpha(Q) N_a \Omega_\rho) \\ &= e^{-a}(\Omega_\rho, N_a \bar{\tau}_\alpha(Q) \Omega_\rho) = e^{-a}(\Omega_\rho, \bar{\tau}_\alpha^{-1}(N_a) Q \Omega_\rho).\end{aligned}$$

Therefore

$$(1-\lambda_n)(\Omega_\rho, Q \bar{\tau}_\alpha^{-1}(N_{a_n}) \Omega_\rho) = \lambda_n (\Omega_\rho, \bar{\tau}_\alpha^{-1}(N_{a_n}) Q \Omega_\rho).$$

For any $Q \in \pi_\rho(\mathfrak{A})$, let

$$Q(g_n) = \int \bar{\tau}_\rho(t) Q g_n(t) \, dt$$

where g_n is given in the proof of Lemma 5. It belongs to $\pi_\rho(\mathfrak{A})$ and $\lim_{n\to\infty} Q(g_n) = Q$. Furthermore

$$\bar{\tau}_\rho(z) Q(g_n) = \int \bar{\tau}_\rho(t) Q g(n(t-z)) n \, dt$$

for real z and the right hand side has an analytic continuation to all complex z as $\pi_\rho(\mathfrak{A})$-valued function. We have

$$Q(g_n)\Omega_\rho = J_\rho \Delta_\rho^{1/2} Q(g_n)^* \Omega_\rho = j_\rho(\bar{\tau}_\rho(i/2)Q(g_n))^* \Omega_\rho$$

which can be proven by analytic continuation.

Let Ψ_j, $j=1,\dots,k$ be a finite collection of vectors in $\mathfrak{H}_\rho$. Then there exist $Q_j \in \pi_\rho(A)$ and n for any given $\varepsilon>0$, such that $Q_j' = Q_j(g_n)$ satisfies

$$\|\Psi_j\| \|\Psi_j - Q_j'\Omega_\rho\| < \varepsilon/4, \quad \|Q_j'\Omega_\rho\| \leqq \|\Psi_j\|. \tag{5.3}$$

Then

$$\|[N_a, \bar{\tau}_\alpha(Q_j')]\Omega_\rho\|^2 = (\Omega_\rho, N_a^* \bar{\tau}_\alpha(X) N_a \Omega_\rho),$$

$$X \equiv Q_j'^* Q_j' - Q_j'^* j_\rho(\bar{\tau}_\rho(i/2)Q_j')^* - j_\rho(\bar{\tau}_\rho(i/2)Q_j')Q_j'$$

$$+ j_\rho(\bar{\tau}_\rho(i/2)Q_j') j_\rho(\bar{\tau}_\rho(i/2)Q_j')^*.$$

If $z \in \bar{\mathfrak{Z}}$, then

$$\begin{aligned}
\omega_{N_a\Omega_\rho}(z) &= (\Omega_\rho, N_a^* N_a z \Omega_\rho) \\
&= e^{-a}(\Omega_\rho, N_a N_a^* z \Omega_\rho) \qquad \text{(by (5.2))} \\
&= (1+e^a)^{-1}(\Omega_\rho, \bar{s}_a z \Omega_\rho) \\
&= (1+e^a)^{-1} \omega_{\bar{s}_a \Omega_\rho}(z).
\end{aligned}$$

Since $\omega_{\bar{s}_a\Omega_\rho}$ is $\bar{\tau}_\alpha$ invariant, we have by Lemma 1 of [3],

$$\lim_\alpha \omega_{N_a\Omega_\rho}(\bar{\tau}_\alpha(X)) = (1+e^a)^{-1}(\Omega_\rho, \bar{s}_a X \Omega_\rho) = 0.$$

Hence there exists α such that $N_\varepsilon' \equiv \bar{\tau}_\alpha^{-1}(N_{a_n})$ satisfies

$$\begin{aligned}
&\|\Psi_j\| \|N_\varepsilon' Q_j' \Omega_\rho - Q_j' N_\varepsilon' \Omega_\rho\| \\
&\qquad = \|\Psi_j\| \|[N_{a_n}, \tau_\alpha(Q_j')]\Omega_\rho\| < \varepsilon/2,
\end{aligned} \tag{5.4}$$

$$\begin{aligned}
&\|\Psi_j\| \|N_\varepsilon'^* Q_j' \Omega_\rho - Q_j' N_j'^* \Omega\| \\
&\qquad = \|\Psi_j\| \|[N_{a_n}^*, \tau_\alpha(Q_j')]\Omega_\rho\| < \varepsilon/2.
\end{aligned} \tag{5.5}$$

where the second inequality is obtained in a similar manner as the first.

Let $\omega_1 \cdots \omega_k$ be normal states of $P_n\mathfrak{M}$. Since $\pi_\rho(P_n)\Omega_\rho$ is cyclic and separating for $P_n\mathfrak{M}$ on $\pi_\rho(P_n)\mathfrak{H}_\rho$, there exists $\Psi_j \in \pi_\rho(P_n)\mathfrak{H}_\rho$ such that

$\omega_{\Psi j}\circ\pi_\rho=\omega_j$. For $Q\in P_n\mathfrak{M}$ and $N_\varepsilon=P_n\pi_\rho^{-1}N'_\varepsilon$, we have from (5.2) ~ (5.5)

$$|\lambda_n\omega_j(N_\varepsilon Q)-(1-\lambda_n)\omega_j(QN_\varepsilon)|$$

$$=|\lambda_n\omega_{\Psi j}(N'_\varepsilon\pi_\rho(Q))-(1-\lambda_n)\omega_{\Psi_j}(\pi_\rho(Q)N'_\varepsilon)|\leqq\varepsilon||Q||,$$

$j=1,\dots,k$. N_ε is a partial isometry in $P_n\mathfrak{M}$ satisfying

$$N_\varepsilon^* N_\varepsilon+N_\varepsilon N_\varepsilon^*=P_n.$$

Hence it also satisfies $N_\varepsilon^2=0$. This proves the property L'_{λ_n}.

Step 17. $r_\infty(P_0\mathfrak{M})=r_\infty(P\mathfrak{M}_0)=\{1\}$.

By the proof of Step 16 applied to $a=0$, $\mathfrak{M}_0$ satisfies the property $L'_{1/2}$. Hence $P\mathfrak{M}_0$ with any central projection P satisfies the same property. Since $r_\infty(\mathfrak{N})\subset\{1\}$ for any finite von Neumann algebra $\mathfrak{N}$[4], we have $r_\infty(P_0\mathfrak{M})=r_\infty(P_n\mathfrak{M}_0)=\{1\}$ if the space is separable by [1].

Q.E.D.

§6. Discussions

For a von Neumann algebra $\mathfrak{M}$, the relation $r_\infty(\mathfrak{M})=S(\mathfrak{M})$ implies that $\mathfrak{M}$ is r_∞-pure where $S(\mathfrak{M})$ is Connes S set [5]. Namely, for any central projection p of $\mathfrak{M}$,

$$r_\infty(\mathfrak{M}p)\supset r_\infty(\mathfrak{M}),$$

$$S(\mathfrak{M}p)\subset S(\mathfrak{M}),$$

$$r_\infty(\mathfrak{M}p)\subset S(\mathfrak{M}p),$$

where the second inclusion is because Δ_ρ for $\mathfrak{M}p$ is always a restriction of $\Delta_{\bar\rho}$ for $\mathfrak{M}$ for some $\bar\rho$. Hence

$$r_\infty(\mathfrak{M}p)=S(\mathfrak{M}p)=r_\infty(\mathfrak{M}).$$

The decomposition $\mathfrak{M}=\Sigma P_n\mathfrak{M}$ in Theorem 2 is a partial central decomposition according to asymptotic ratio set into r_∞ pure parts [7].

By exactly the same method as the proof of Theorem 2, we can analyze a von Neumann algebra $\mathfrak{M}$ with a cyclic and separating vector Ω, such that $\log \Delta$ has exclusively an isolated point spectrum and each $\mathfrak{H}_a$ is cyclic for $\mathfrak{M}$. The last assumptions imply that each $\mathfrak{H}_a$ is separating for $\mathfrak{M}$ because $J\mathfrak{H}_a = \mathfrak{H}_{-a}$ and replaces Step 2 of the proof of Theorem 2. We can proceed up to Step 8 without any further assumption. However Step 10 no longer holds and hence one finds a formula

$$(U^*U)^\natural = e^{-a}(A_a^0)^2(\phi_a^0)^{-1}((UU^*)^\natural).$$

If we make a further assumption that $e^{-a}(A_a^0)^2 \leqq 1$, then we can complete the analysis and we obtain the same conclusion as Theorem 1 except that ϕ_n no longer satisfies $\phi_n(z) = \phi_n(1)z$ and $M(P_n\mathfrak{M}_0, \phi_n)$ corresponds to the case $A_{\phi_n} = e^{an/2}$ in Theorem 3.

If $e^{-a}(A_a^0)^2 \leqq 1$ does not hold, we are left with an isomorphism from a subalgebra of $\mathfrak{M}$ onto another subalgebra of $\mathfrak{M}$. It will be of interest to generalize $M(\mathfrak{F}, \phi)$ for such ϕ.

The construction of $M(\mathfrak{F}, \phi)$ in Section 2 can be generalized to the case where $\mathfrak{M}$ has a commutative semigroup G of injective endomorphisms, which we shall briefly sketch. This situation is relevant to $R_x \otimes R_y \sim R_\infty$ when $\log x/\log y$ is irrational. We assume that

$$\phi(\mathfrak{F}) = \phi(1)\mathfrak{F}\phi(1), \qquad \phi \in G.$$

If ϕ_1, ϕ_2 and ϕ are injective endomorphisms of $\mathfrak{M}$, then $\phi \circ \phi_1 = \phi \circ \phi_2$ implies $\phi_1 = \phi_2$. Hence a commutative semigroup G of injective endomorphisms of $\mathfrak{M}$ has an envelopping group $\bar{G}$ such that $\bar{G} \supset G$ and G generates $\bar{G}$. Elements in $\bar{G}$ is a pair (ϕ_a, ϕ_b) of elements $\phi_a, \phi_b \in G$ with an equivalence relation $(\psi\phi_a, \psi\phi_b) = (\psi'\phi_a, \psi'\phi_b)$ for any $\psi, \psi' \in G$, where we include an identity mapping 1 of $\mathfrak{F}$ in G. The multiplication in $\bar{G}$ is

$$(\phi_a, \phi_b)(\psi_a, \psi_b) = (\phi_a\psi_a, \phi_b\psi_b)$$

and $(\phi, 1)$ is identified with $\phi \in G$. $1 = (\phi, \phi)$ is the identity in $\bar{G}$.

For each $g \in \bar{G}$, we make a fixed choice $\phi_a(g)$ and $\phi_b(g)$ such that $g = (\phi_a(g), \phi_b(g))$, where $\phi_a(1) = \phi_b(1) = 1$ and $\phi_a(g^{-1}) = \phi_b(g)$, $\phi_b(g^{-1}) = \phi_a(g)$ for convenience sake.

The space $\mathfrak{H}$, on which $M(\mathfrak{F}, G)$ is to be defined, is spanned by mutually orthogonal subspaces

$$\mathfrak{H}_g = p_g \mathfrak{K}$$

where p_g is a partially isometric mapping from $\mathfrak{K}$ into $\mathfrak{H}$ such that

$$E(g) \equiv p_g^* p_g = \phi_a(g)(1)\phi_b(g)'(1), \tag{6.1}$$

where $\phi'(y) = j_\Psi \phi j_\Psi(y)$ as before.

In section 2, we have $G = \{\phi^n, n=0, 1, \ldots\}$, $\bar{G}$ is the additive group of integers, $\phi_a(n) = \phi^n$ for $n \geqq 0$, $\phi_a(n) = 1$ for $n \leqq 0$, $\phi_b(n) = 1$ for $n \geqq 0$ and $\phi_b(n) = \phi^{|n|}$ for $n \leqq 0$.

Faithful representations of $\mathfrak{F}$ and $\mathfrak{F}'$ are defined by

$$\pi(x) = \Sigma\, p_g \phi_a(g)(x) p_g^*, \qquad x \in \mathfrak{F}, \tag{6.2}$$

$$\pi'(y) = \Sigma\, p_g \phi_b(g)'(y) p_g^*, \qquad y \in \mathfrak{F}'. \tag{6.3}$$

A_ϕ is defined by (2.1) for each $\phi \in G$ and V of Lemma 1 is denoted by $V(\phi)$. It satisfies

$$V(\phi)^* V(\phi) = 1,\ V(\phi) V(\phi)^* \equiv e(\phi) = \phi(1)\phi'(1), \tag{6.4}$$

$$V(\phi) Q = \phi(Q) V(\phi),\ V(\phi)^* \phi(Q) = Q V(\phi)^*,\ Q \in \mathfrak{F}, \tag{6.5}$$

$$V(\phi) Q' = \phi'(Q') V(\phi),\ V(\phi)^* \phi'(Q') = Q' V(\phi)^*,\ Q' \in \mathfrak{F}', \tag{6.6}$$

$$[J_\Psi, V(\phi)] = 0, \tag{6.7}$$

$$V(\phi_1) V(\phi_2) = V(\phi_1 \phi_2),\ V(1) = 1. \tag{6.8}$$

From (6.4) and (6.8), we have

$$\begin{aligned} V(\phi_1) V(\phi_2)^* &= V(\phi_2)^* V(\phi_2) V(\phi_1) V(\phi_2)^* \\ &= V(\phi_2)^* V(\phi_1) e(\phi_2). \end{aligned} \tag{6.9}$$

Operators $U(g)$ and $U'(g)$ are defined by

$$U(g) = \sum_{g'} p_{gg'} V(\phi_b(g'))^* V(\phi_b(gg')) p_{g'}^*, \qquad g \in \bar{G}, \tag{6.10}$$

$$U'(g^{-1}) = \sum_{g'} p_{gg'} V(\phi_a(g'))^* V(\phi_a(gg')) p_{g'}^*, \quad g \in \bar{G}. \tag{6.11}$$

They satisfy

(6.12) $$U(g)^*=U(g^{-1}),\ U'(g^{-1})^*=U'(g),$$

(6.13) $$U(g)U((1,\phi))=U(g(1,\phi)),\ U(1)=1,$$

(6.14) $$U(g)\pi(x)U((\phi,1))=U(g(\phi,1))\pi(\phi(x)),\ x\in\mathfrak{F},$$

(6.15) $$U'(g)U'((1,\phi))=U'(g(1,\phi)),\ U'(1)=1,$$

(6.16) $$U'(g)\pi'(y)U'((\phi,1))=U'(g(\phi,1))\pi'(\phi'(y)),\ y\in\mathfrak{F}'.$$

Taking adjoint and using (6.12), we also have

(6.13)′ $$U((\phi,1))U(g)=U((\phi,1)g),$$

(6.14)′ $$U((1,\phi))\pi(x)U(g)=\pi(\phi(x))U((1,\phi)g),$$

(6.15)′ $$U'((\phi,1))U'(g)=U'((\phi,1)g),$$

(6.16)′ $$U'((1,\phi))\pi'(y)U'(g)=\pi'(\phi'(y))U'((1,\phi)g).$$

The von Neumann algebras are defined by

(6.17) $$M(\mathfrak{F},G)=\{\pi(\mathfrak{F}),\ U(\bar{G})\}'',$$

(6.18) $$M'(\mathfrak{F},G)=\{\pi'(\mathfrak{F}'),\ U'(\bar{G})\}''.$$

Then

(6.19) $$M(\mathfrak{F},G)'=M'(\mathfrak{F},G)$$

and the vector $\Omega(\mathfrak{F},G)\equiv p_1\Psi$ is cyclic and separating for $M(\mathfrak{F},G)$ with modular operator and modular conjugation operator given by

(6.20) $$\Delta(\mathfrak{F},G)=\Sigma p_g\{\phi_a(g)A_{\phi_a(g)}\}^2\{j_\Psi(\phi_b(g)A_{\phi_b(g)})\}^{-2}p_g^*,$$

(6.21) $$J(\mathfrak{F},G)=\Sigma p_{g^{-1}}J_\Psi p_g^*.$$

If $A_\phi>1$ for all $\phi\in G$, then the set $M(\mathfrak{F},G)_0$ of $Q\in M(\mathfrak{F},G)$ commuting with $\Delta(\mathfrak{F},G)$ is $\pi(\mathfrak{F})$ and its center is all $z\in\mathfrak{F}_c$ such that $\phi(z)=z\phi(1)$ for all $\phi\in G$.

Sketch of proof We shall present proofs of computations later. $J(\mathfrak{F},G)$ is an antiunitary involution (the choice $\phi_a(g^{-1})=\phi_b(g)$ and $\phi_b(g^{-1})=\phi_a(g)$

simplifies the expression of J) and satisfies

$$J(\mathfrak{F}, G)\pi(x)J(\mathfrak{F}, G)=\pi'(j_{\Psi}(x)),\ x\in\mathfrak{F}, \tag{6.22}$$

$$J(\mathfrak{F}, G)U(g)J(\mathfrak{F}, G)=U'(g),\ g\in\bar{G}, \tag{6.23}$$

$$J(\mathfrak{F}, G)\Delta(\mathfrak{F}, G)J(\mathfrak{F}, G)=\Delta(\mathfrak{F}, G)^{-1}. \tag{6.24}$$

We have $M'(\mathfrak{F}, G)\subset M(\mathfrak{F}, G)'$ from

$$[\pi(x), \pi'(y)]=0,\ x\in\mathfrak{F},\ y\in\mathfrak{F}', \tag{6.25}$$

$$[U(g), \pi'(y)]=0,\ y\in\mathfrak{F}',\ g\in\bar{G}, \tag{6.26}$$

$$[\pi(x), U'(g)]=0,\ x\in\mathfrak{F},\ g\in\bar{G}, \tag{6.27}$$

$$[U(g_1), U'(g_2)]=0,\ g_1\in\bar{G},\ g_2\in\bar{G}. \tag{6.28}$$

$\Omega(\mathfrak{F}, G)$ is cyclic for $M(\mathfrak{F}, G)$ because

$$U((\psi, 1))\pi(\mathfrak{F})U((1, \phi))\Omega(\mathfrak{F}, G)\supseteqq p_{(\psi, \phi)}\mathfrak{F}V(\phi_b\{(\psi, \phi)\})\Psi \tag{6.29}$$

is dense in $p_{(\psi, \phi)}\mathfrak{K}$. By applying $J(\mathfrak{F}, G)$ and using (6.22) and (6.23), we see that $\Omega(\mathfrak{F}, G)$ is cyclic for $M'(\mathfrak{F}, G)$ and hence is separating for $M(\mathfrak{F}, G)$.

By setting $S=J(\mathfrak{F}, G)\Delta(\mathfrak{F}, G)^{1/2}$, we have

$$\begin{aligned} &SU((\psi, 1))\pi(x)U((1, \phi))\Omega(\mathfrak{F}, G) \\ &\quad =U((\phi, 1))\pi(x^*)U((1, \psi))\Omega(\mathfrak{F}, G). \end{aligned} \tag{6.30}$$

In any monomial of $\pi(\mathfrak{F})$ and $U(G)$, we make the following reordering. First factor any $U(g)$ as $U((\psi, \phi))=U((\psi, 1))U((1, \phi))$ by (6.13). Bring all $U((1, \phi))$ to the right using $(6.14)'$ with $g=1$ and $(6.14)'$ with $x=1$, where $U((1, \phi)g)$ is again decomposed. Similarly bring all $U((\psi, 1))$ to the left using (6.14) with $g=1$. Collect all $U((1, \phi))$ into one using (6.13) in the form $U((1, \phi_1))U((1, \phi_2))=U((1, \phi_1\cdot\phi_2))$. Similarly collect all $U((\Psi. 1))$ into one by using $(6.13)'$. We then see that $U((\psi, 1))\pi(x)$ $U((1, \phi))$ are total in $M(\mathfrak{F}, G)$. As we shall show after the following computations, $U((G, 1))\pi(\mathfrak{F})U((1, G))\Omega(\mathfrak{F}, G)$ contains a total set of analytic vectors for $\Delta(\mathfrak{F}, G)^{1/2}$. Hence (6.30) shows that $J(\mathfrak{F}, G)$ and $\Delta(\mathfrak{F}, G)$ are modular conjugation operator and modular operator for $\Omega(\mathfrak{F},G)$.

From (6.22), (6.23) and $M'(\mathfrak{F}, G)\subset M(\mathfrak{F}, G)'$, we have (6.19). The assertion for the case $A_\phi>1$ is proved in the same way as the proof of Theorem 3.

Computations. We present proof of those formulas which require more complicated computations.

Formula (6.13). Setting $\phi_1=\phi_b(g')$, $\phi_2=\phi_b((1, \phi)g')$, $\phi_3=\phi_b(g(1, \phi)g')$ and $\psi=\phi_a((1, \phi)g')$, we have

(6.31)
$$\begin{aligned}&U(g)U((1, \phi))\\&=\sum_{g'} p_{g(1,\phi)g'}V(\phi_2)^*V(\phi_3)\phi_2'(1)\psi(1)V(\phi_1)^*V(\phi_2)p_{g'}^*\\&=\sum_{g'} p_{g(1,\phi)g'}V(\phi_1\phi_2)^*V(\phi_3)e(\phi_1)(\phi_1\phi_2)'(1)\phi_1\psi(1)V(\phi_2)p_{g'}^*,\end{aligned}$$

where we have used (6.1), (6.5), (6.6), (6.9) and (6.8). $e(\phi_1)$ is absorbed into $(\phi_1\phi_2)'(1)(\phi_1\psi)(1)$ because $\phi_1(1)\phi_1(\psi(1))=\phi_1\psi(1)$ and $\phi_1'(1)\phi_1'(\phi_2'(1))=(\phi_1\phi_2)'(1)$.

There exist ξ and $\eta\in G$ such that

(6.32)
$$\eta\phi_a((1,\phi)g')=\xi\phi_a(g'),\ \eta\phi_b((1, \phi)g')=\xi\phi_b(g')\phi.$$

Then

$$\xi\eta\phi_b((1, \phi)g')\phi_a(g')(1)=\xi\eta\phi_a((1, \phi)g')\phi_b(g')(\phi(1))$$

which implies $\phi_2\phi_a(g')(1)=\phi_1\psi(\phi(1))$. Since $\phi_a(g')(1)p_{g'}^*=p_{g'}^*$, $V(\phi_2)\phi_a(g')(1)=\phi_2\phi_a(g')(1)V(\phi_2)$ and $\phi_1\psi(1)\phi_1\psi(\phi(1))=\phi_1\psi(\phi(1))$, $\phi_1(\psi(1))$ is absorbed into $p_{g'}^*$ in (6.31). Similarly $p_{g(1,\phi)g'}\phi_3'(1)=p_{g(1,\phi)g'}$, $V(\phi_1\phi_2)^*V(\phi_3)(\phi_2\phi_1)'(1)=\phi_3'(1)V(\phi_1\phi_2)^*V(\phi_3)$ and hence $(\phi_1\phi_2)'(1)$ is absorbed into $p_{g(1,\phi)g'}$. Using $V(\phi_2)^*V(\phi_2)=1$,

$$U(g)U((1, \phi))=\sum_{g'} p_{g(1,\phi)g'}V(\phi_1)^*V(\phi_3)p_{g'}^*=U(g(1, \phi)).$$

Formula (6.14). Setting $\phi_1=\phi_b(g')$, $\phi_2=\phi_b((\phi, 1)g')$, $\phi_3=\phi_b(g(\phi, 1)g')$ and $\psi=\phi_a((\phi, 1)g')$, we have

$$U(g)\pi(x)U((\phi,1))$$

$$=\sum_{g'} p_{g(\phi,1)g'} V(\phi_2)^* V(\phi_3)\psi(x)\phi_2'(1) V(\phi_1)^* V(\phi_2) p_{g'}^*$$

$$=\sum_{g'} p_{g(\phi,1)g'} V(\phi_1\phi_2)^* V(\phi_3) e(\phi_1)\phi_1\psi(x)(\phi_1\phi_2)'(1) V(\phi_2) p_{g'}^*.$$

As before, $e(\phi_1)$ is absorbed into $\phi_1\psi(x)(\phi_1\phi_2)'(1)$ and $(\phi_1\phi_2)'(1)$ is then absorbed into $p_{g(\phi,1)g'}$. By an equation similar to (6.32), we have

$$\phi_1\psi(x)=\phi_2\phi_a(g')\phi(x).$$

Hence

$$U(g)\pi(x)U((\phi,1))=\sum_{g'} p_{g(\phi,1)g'} V(\phi_1)^* V(\phi_3)\phi_a(g')\phi(x) p_{g'}^*,$$

$$=U(g(\phi,1))\pi(\phi(x)).$$

Formula (6.26). Setting $\phi_1=\phi_b(g')$ and $\phi_2=\phi_b(gg')$, we have

$$[U(g),\pi'(y)]=\sum_{g'} p_{gg'}[V(\phi_1)^* V(\phi_2)E(g')\phi_1'(y)$$

$$-\phi_2'(y)E(gg')V^*(\phi_1)V(\phi_2)]p_{g'}=0$$

where $E(g')$ and $E(gg')$ are absorbed into $p_{g'}$ and $p_{gg'}$ respectively.

Formula (6.28). Setting $\phi_1=\phi_b(g')$, $\psi_1=\phi_a(g')$, $\phi_2=\phi_b(g_2^{-1}g')$, $\psi_2=\phi_a(g_2^{-1}g')$, $\phi_3=\phi_b(g_1g')$, $\psi_3=\phi_a(g_1g')$, $\phi_4=\phi_b(g_1g_2^{-1}g')$ and $\psi_4=\phi_a(g_1g_2^{-1}g')$, we have

$$[U(g_1),U'(g_2)]=\sum p_{g_1g_2^{-1}g'}[V(\phi_2)^* V(\phi_4)\phi_2'(1)\psi_2(1)V(\psi_1)^* V(\psi_2)$$

$$-V(\psi_3)^* V(\psi_4)\phi_3'(1)\psi_3(1)V(\phi_1)^* V(\phi_3)]p_{g'}^*.$$

Since $\psi_2(1)V(\psi_1)^* V(\psi_2)=V(\psi_1)^* V(\psi_2)\psi_1(1)$ and $\psi_1(1)p_{g'}^*=p_{g'}^*$, $\psi_2(1)$ is absorbed into $p_{g'}^*$. Likewise, $\phi_3'(1)$ is absorbed into $p_{g'}^*$, $\phi_2'(1)$ and $\psi_3(1)$ are absorbed into $p_{g_1g_2^{-1}g'}$. By (6.9), we have

$$[U(g_1),U'(g_2)]=\sum p_{g_1g_2^{-1}g'}[V(\phi_2\psi)^* V(\phi_4)e(\psi_1)V(\psi_2)$$

$$-V(\psi_3\phi_1)^* V(\psi_4)e(\phi_1)V(\phi_3)]p_{g'}^*.$$

Since $\psi_1(1)p^*_{g'}=p^*_{g'}$, $V(\psi_2)\psi_1(1)=\psi_2\psi_1(1)V(\psi_2)$ and $\psi_1(1)\psi_1(\psi_2(1))=\psi_1\psi_2(1)$, $\psi_1(1)$ in $e(\psi_1)$ is absorbed into $p^*_{g'}$. Likewise, $\phi'_1(1)$ in $e(\phi_1)$ is absorbed into $p^*_{g'}$, $\psi'_1(1)$ in $e(\psi)$ and $\phi_1(1)$ in $e(\phi_1)$ are absorbed into $p_{g_1g_2^{-1}g'}$. We now have

$$[U(g_1),\ U'(g_2)]=\Sigma\, p_{g_1g_2^{-1}g'}[V(\phi_2\psi_1)^*V(\phi_4\psi_2)-V(\psi_3\phi_1)^*V(\psi_4\phi_3)]p^*_{g'}.$$

Let

$$\eta_1\phi_2=\xi_1\phi_b(g_2^{-1})\phi_1,\ \eta_1\psi_2=\xi_1\phi_a(g_2^{-1})\psi_1,$$

$$\eta_2\phi_4=\xi_2\phi_b(g_2^{-1})\phi_3,\ \eta_2\psi_4=\xi_2\phi_a(g_2^{-1})\psi_3.$$

Then

$$\begin{aligned}\eta_1\eta_2\psi_2\phi_4\phi_1\psi_3&=\xi_1\xi_2\phi_a(g_2^{-1})\phi_b(g_2^{-1})\phi_3\psi_1\phi_1\psi_3\\&=\eta_1\eta_2\phi_2\psi_4\psi_1\phi_3.\end{aligned}$$

Hence $\psi_2\phi_4\phi_1\psi_3=\phi_2\psi_4\psi_1\phi_3$, which implies

$$\begin{aligned}V(\phi_2\psi_1)^*V(\phi_4\psi_2)&=V(\phi_2\psi_1\phi_1\psi_3)^*V(\phi_1\psi_3\phi_4\psi_2)\\&=V(\phi_1\psi_3\phi_2\psi_1)^*V(\phi_2\psi_1\psi_4\phi_3)\\&=V(\phi_1\psi_3)^*V(\psi_4\phi_3).\end{aligned}$$

Hence we have (6.28).

Formula (6.29) Setting $\phi_1=\phi_b((1,\phi))$, $\psi_1=\phi_a((1,\phi))$, $\phi_2=\phi_b((\psi,\phi))$ and $\psi_2=\phi_a((\psi,\phi))$, we have

$$U((\psi,1))\pi(x)U((1,\phi))\Omega(\mathfrak{F},G)= \qquad (6.33)$$

$$p_{(\psi,\phi)}V(\phi_1)^*V(\phi_2)\phi'_1(1)\psi_1(x)V(\phi_1)\psi$$

where $\psi_1(1)$ has already been absorbed into $\psi_1(x)$ and $\phi'_1(1)$ can be absorbed into $p_{(\psi,\phi)}$. Since $\phi_1=\psi_1\phi$ (which follows from (6.32) with $g'=1$) we have

$$\begin{aligned}V(\phi_1)^*V(\phi_2)\psi_1(x)V(\phi_1)&=V(\phi)^*\phi_2(x)V(\phi)V(\phi_2)\\&=\phi^{-1}(\phi(1)\phi_2(x)\phi(1))V(\phi_2)\end{aligned} \qquad (6.34)$$

where we have used

$$V(\phi)^* x V(\phi) = V(\phi)^* \phi(1) x \phi(1) V(\phi)$$
$$= \phi^{-1}(\phi(1) x \phi(1)) V(\phi)^* V(\phi) = \phi^{-1}(\phi(1) x \phi(1)).$$

Let

$$\eta\phi_a((\psi, \phi)) = \xi\psi, \ \eta\phi_b((\psi, \phi)) = \xi\phi.$$

Then $\xi\eta\psi_2\phi = \xi\eta\psi\phi_2$ and hence $\psi_2\phi = \psi\phi_2$. Hence $\phi_2(\mathfrak{F}) \supset \psi\phi_2(\mathfrak{F})$ implies

$$\phi^{-1}(\phi(1)\phi_2(\mathfrak{F})\phi(1)) = \phi^{-1}(\phi(1)\phi_2(1)\mathfrak{F}\phi_2(1)\phi(1))$$
$$\supset \phi^{-1}(\phi(1)(\psi\phi_2)(1)\mathfrak{F}(\phi\phi_2)(1)\phi(1)) = \psi_2(1)\phi^{-1}(\phi(1)\mathfrak{F}\phi(1))\phi_2(1)$$
$$= \psi_2(1)\mathfrak{F}\phi_2(1).$$

Hence

$$U((\psi, 1))\pi(\mathfrak{F})U((1, \phi))\Omega(\mathfrak{F}, G)$$
$$\supseteqq p_{(\psi,\phi)}\psi_2(1)\mathfrak{F}\phi_2(1)V(\phi_2)\Psi$$
$$= p_{(\psi,\phi)}\mathfrak{F}V(\phi_2)\Psi.$$

Since A_{ϕ_2} is positive definite, support of $\phi_2(A_{\phi_2}) = \phi_2(1)$. Since $J_\Psi\phi_2(1)\Psi = \phi_2(1)\Psi$, $\mathfrak{F}V(\phi_2)\Psi = \mathfrak{F}\phi_2(A_{\phi_2})\Psi$ is dense in the closure of

$$\mathfrak{F}\phi_2(1)\Psi = \mathfrak{F}j_\Psi(\phi_2(1))\Psi = \phi_2'(1)\mathfrak{F}\Psi$$

which is $\phi_2'(1)\mathfrak{R}$. Therefore (6.29) is dense in $p_{(\psi,\phi)}\mathfrak{R}$.

Formula (6.30) By (6.33) and (6.34), we have

$$U((\psi, 1))\pi(x)U((1, \phi))\Omega(\mathfrak{F}, G)$$
$$= p_{(\psi,\phi)}\phi^{-1}(\phi(1)\phi_2(x)\phi(1))\phi_2(A_{\phi_2})\Psi.$$

By multiplying $\Delta(\mathfrak{F}, G)^{1/2}$, we obtain

$$p_{(\psi,\phi)}\{\psi_2(A_{\psi_2})\}\phi^{-1}(\phi(1)\phi_2(x)\phi(1))\phi_2(A_{\phi_2})j_\Psi(\phi_2(A_{\phi_2}))^{-1}\Psi$$
$$= p_{(\psi,\phi)}\{\psi_2(A_{\psi_2})\}\phi^{-1}(\phi(1)\phi_2(x)\phi(1))\Psi.$$

Hence

$$SU((\psi, 1))\pi(x)U((1, \phi))\Omega(\mathfrak{F}, G)$$
$$= p_{(\phi,\psi)}\phi^{-1}(\phi(1)\phi_2(x^*)\phi(1))\psi_2(A_{\psi_2})\Psi$$
$$= p_{(\phi,\psi)}V(\phi)^*\phi_2(x^*)V(\phi\psi_2)\Psi.$$

Since $\psi_2\phi = \psi\phi_2$, we have

$$V(\phi)^*\phi_2(x^*)V(\phi) = V(\phi\psi_2)^*\psi_2\phi_2(x^*)V(\phi\psi_2)$$
$$= V(\psi\phi_2)^*\psi_2\phi_2(x^*)V(\psi\phi_2)$$
$$= V(\psi)^*\psi_2(x^*)V(\psi)$$
$$= V(\phi_b((1, \psi)))^*\psi_2\phi_a((1, \psi))(x^*)V(\phi_b((1, \psi)))$$

where we have used $\phi_b((1, \psi)) = \phi_a((1, \psi))\psi$ in the last equality. Hence

$$SU((\psi, 1))\pi(x)U((1, \phi))\Omega(\mathfrak{F}, G)$$
$$= p_{(\phi,\psi)}V(\phi_b((1, \psi)))^*V(\psi_2)\phi_a((1, \psi))(x^*)V(\phi_b((1, \psi)))\Psi$$
$$= U((\phi, 1))\pi(x^*)U((1, \psi))\Omega(\mathfrak{F}, G)$$
$$= \{U((\psi, 1))\pi(x)U((1, \phi))\}^*\Omega(\mathfrak{F}, G),$$

where $\psi_2 = \phi_a((\psi, \phi)) = \phi_b((\phi, \psi))$ is used. Q.E.D.

We now prove that $U((G, 1))\pi(\mathfrak{F})U(1, G))\Omega(\mathfrak{F}, G)$ contains a total set of analytic vectors for $\Delta(\mathfrak{F}, G)^{1/2}$. Let $\psi, \phi \in G$, $\psi_2 = \phi_a((\psi, \phi))$, $\phi_2 = \phi_b((\psi, \phi))$. Let E_λ^χ denote the spectral projection of A_χ, $\chi \in G$, It is in the center of $\mathfrak{F}$. Let $p(\lambda) = \psi(E_\lambda^{\phi_2})$, $q(r) = \phi(1 - E_r^{\phi_2})$. Then $\lim_{\lambda \to +\infty} p(\lambda) = 1$, $\lim_{r \to +0} q(r) = 1$. Since $\phi_2\psi = \phi\psi_2$, we have $\phi_2(p(\lambda)) = \phi\{\psi_2(E_\lambda^{\phi_2})\}$. Hence

$$U((\psi, 1))\pi(p(\lambda)xq(r))U((1, \phi))\Omega(\mathfrak{F}, G)$$
$$= p_{(\psi,\phi)}\psi_2(E_\lambda^{\phi_2})\phi^{-1}(\phi(1)\phi_2(x)\phi(1))\phi_2(A_{\phi_2})\phi_2(1 - E_r^{\phi_2})\Psi$$
$$= p_{(\psi,\phi)}[\psi_2(E_\lambda^{\phi_2})j_\Psi\{\phi_2(1 - E_r^{\phi_2})\}]\phi^{-1}(\phi(1)\phi_2(x)\phi(1))\phi_2(A_{\phi_2})\Psi,$$

which is obviously an analytic vector for $\Delta(\mathfrak{F}, G)^{1/2}$ for $\lambda < +\infty, r > 0$. Hence we have a total set of analytic vectors for $\Delta(\mathfrak{F}, G).^{1/2}$

Group-measure construction. If G is a commutative group of *-automorphisms, then we see that $M(\mathfrak{F}, G)$ defined above is unitarily equivalent to an ordinary group measure construction in the following manner.

$\bar{G}$ is now isomorphic to G with $g \in \bar{G}$ corresponding to $\hat{g} \equiv \phi_a(g)\phi_b(g)^{-1} \in G$. We have $\phi(1) = \phi'(1) = 1$, $p_g^* p_g = 1$ and $V(\phi)$, $\phi \in G$ is a unitary representation of G. We denote $\bar{V}(g) = V(\phi_a(g))V(\phi_b(g))^*(= V(\phi_a(g)\phi_b(g)^{-1}))$. Since $\phi(\mathfrak{F}_c) = \mathfrak{F}_c$, $\phi(A_\phi)$ is affiliated with $\mathfrak{F}_c$.

We define

$$W = \sum_g p_g V(\phi_a(g))^* p_g^* \tag{6.35}$$

which is a unitary operator on $\mathfrak{H}$. Then

$$W\Omega(\mathfrak{F}, G) = \Omega(\mathfrak{F}, G), \tag{6.36}$$

$$W\pi(x)W^* = \sum_g p_g x p_g^*, \ x \in \mathfrak{F}', \tag{6.37}$$

$$W\pi'(y)W^* = \sum_g p_g g^{-1}(y) p_g^*, \ y \in \mathfrak{F}', \tag{6.38}$$

$$WU(g)W^* = \sum_{g'} p_{g'g} \bar{V}(g)^* p_{g'}^*, \ g \in \bar{G}, \tag{6.39}$$

$$WU'(g^{-1})W^* = \sum_{g'} p_{gg'} p_{g'}^*, \ g \in \bar{G}, \tag{6.40}$$

$$WJ(\mathfrak{F}, G)W^* = \sum_g p_{g^{-1}} \bar{V}(g) J_\Psi p_g^*, \tag{6.41}$$

$$W\Delta(\mathfrak{F}, G)W^* = \sum_g p_g \Delta_g p_g^*. \tag{6.42}$$

Here $\Delta_g = A_g^2$ and satisfies

$$(\hat{g}(\Delta_g^{1/2})\Psi, \ \hat{g}(x\Delta_g^{1/2})\Psi) = (\Psi, \ x\Psi), x \in \mathfrak{F}. \tag{6.43}$$

$\bar{V}(g)$ is defined as a unitary operator satisfying

$$\bar{V}(g)x\Psi = \hat{g}(x\Delta_g^{1/2})\Psi. \tag{6.44}$$

The formulas on the right hand sides of (6.37)$\sim$(6.40) are usual group-measure construction (at least for a commutative $\mathfrak{F}$) and works even when $\bar{G}$ is non-commutative. Formulas (6.41) and (6.42) give modular conjugation and modular operators also for non-commutative $\bar{G}$.

Example. Let $R_x = \underset{g\in G}{\otimes}(R_g, \Omega_g)$ where G is a countable set for the moment, R_g is a type I_2 factor and Ω_g has a spectrum λ and $(1-\lambda)$ relative to R_g (independent of g) where $\lambda(1-\lambda)^{-1}=x$. Since the modular operator for $\Omega = \otimes\Omega_g$ has a spectrum at $S_x = \{0\}\cup\{x^n;\ n=0, \pm 1, \dots\}$ and R is asymptotically abelian relative to any one parameter non-compact shift of G, the condition $\mathfrak{Z}=\mathfrak{Z}_0$ is satisfied by Theorem 2 and $(R_x, \Omega)\sim(M(\mathfrak{F}, \phi), \Omega(\mathfrak{F}, \phi))$ for a hyperfinite finite factor $\mathfrak{F}$ and its endomorphism ϕ. The hyperfiniteness of $\mathfrak{F}$ is easily seen by expressing it by a group measure construction where the group is generated by an ascending sequence of finite groups.

Now let G be a group and $V(g)$, $g\in G$ is a unitary operator on $\underset{g'}{\otimes}(\mathfrak{H}_{g'}, \Omega_{g'})$ which shifts indices $g'\in G$ by left multiplication, namely

$$V(g)\pi_{g_1}(Q_1)\dots\pi_{g_n}(Q_n)\Omega = \pi_{gg_1}(Q_1)\dots\pi_{gg_n}(Q_n)\Omega,$$

where all R_g, Ω_g are identified with a single I_2 factor R_0 and a vector Ω_0, π_{g_k} is a natural representation of $R\sim R_{g_k}$ on $\otimes(\mathfrak{H}_g, \Omega_g)$ and $Q_k\in R_0$. Then $V(g)\Omega=\Omega$, $V(g)R_xV(g)^*=R_x$ and hence $V(g)(R_x)_0V(g)^*=(R_x)_0$ because $V(g)$ commutes with modular operator Δ for Ω. It also commutes with the modular conjugation J for Ω.

Consider $M(R_x, G)$ constructed in exactly the same way as $M(\mathfrak{F}, G)$ by (6.36)$\sim$(6.40) and $M(\mathfrak{F}, G)=(\pi(\mathfrak{F}), U(G))''$. Then (6.41) and (6.42) give modular conjugation operator and modular operator for $\Omega(R_x, G)$ where J_Ψ and Δ_g are to be replaced by J and Δ. In particular $\Delta(R_x, G)$ has a spectrum S_x.

The set $M(R_x, G)_0$ of modular invariant elements of $M(R_x, G)$ is $M((R_x)_0, G)$. An isometric operator $U\in R_x$ inducing an injective endomorphism ϕ such that $(R_x, \Omega)\sim(M((R_x)_0, \phi), \Omega((R_x)_0, \phi))$ also induces an injective endomorphism of $M(R_x, G)_0$ by

$$\pi(U)^*Q\pi(U)=\bar{\phi}(Q),\ Q\in M(R_x, G)_0.$$

The pair, $M(R_x, G)$ and $\Omega(R_x, G)$, is then unitarily equivalent to the pair,

$M(\overline{\mathfrak{F}}, \bar{\phi})$ with $\overline{\mathfrak{F}} \equiv M((R_x), G)_0 \sim M((R_x)_0, G)$ and $\Omega(\overline{\mathfrak{F}}, \bar{\phi})$. The case where G is a free group of two generators is given by Pukanszky. Since $M(R_x, G)$ for this case does not have property L its asymptotic ratio set is $\{0\}$.

Acknowledgement

The author would like to thank members of Department of Mathematics, Queen's University, where this work has been done, for helpful discussions and warm hospitality. The author is indebted to Dr. Nielsen for pointing out a connection with group measure construction and with Pukanszky's example.

The author learned after completion of this work that Takesaki [8] has investigated $M(\mathfrak{F}, \phi)$.

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