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On Types over von Neumann Subalgebras and the Dye Correspondence

By

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The family of abelian projections of von Neumann algebras plays an important role in the theory of types (discrete and continuous). An abelian projection E of a von Neumann algebra $\mathscr{A}$ is defined by the requirement that each projection P in $\mathscr{A}$ dominated by E is written as $P=QE$ for some Q in the center $\mathscr{Z}$ of $\mathscr{A}$. We have taken an interest in generalizing this definition of an abelian projection by using a general von Neumann subalgebra of $\mathscr{A}$ in place of the center $\mathscr{Z}$.

In [1], we defined a projection abelian over a von Neumann subalgebra (see Definition 3 below) and proved that some elementary properties of abelian projections are preserved under the generalization. Those results lead us to a natural generalization of continuous von Neumann algebras. In [2], we have extended the definition of continuous von Neumann algebras (Definition 10 below) and proved that a von Neumann subalgebra $\mathscr{B}$ of a von Neumann algebra $\mathscr{A}$ contained in the center has a useful property relative to an expectation (that is, $\mathscr{B}$ is a strong Maharam subalgebra of $\mathscr{A}$ in the sense of Definition 15) if $\mathscr{A}$ is continuous over $\mathscr{B}$ (Theorem A).

In this note, we shall define a von Neumann algebra $\mathscr{A}$ discrete over a von Neumann subalgebra $\mathscr{B}$. We prove that $\mathscr{A}$ splits into the direct sum of the part continuous over $\mathscr{B}$ and the part discrete over $\mathscr{B}$ if $\mathscr{B}$ is contained in the center of $\mathscr{A}$ (Theorem 14 in §2).

When a von Neumann subalgebra $\mathscr{B}$ of a von Neumann algebra $\mathscr{A}$ satisfies some conditions, $\mathscr{B}$ is discrete (resp. continuous) over the center of $\mathscr{A}$ if and only if $\mathscr{A}$ is discrete (resp. continuous) in the usual sense (Theorem 22 in §3).

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In §4, we shall apply theorems in §3 to the crossed product $G\otimes\mathscr{A}$ of a von Neumann algebra $\mathscr{A}$ by a freely acting automorphism group G of $\mathscr{A}$ with the following results:

One of main results in Dye's paper [5] is that if $\mathscr{A}$ is a non-atomic abelian von Neumann algebra, then there is a 1:1 correspondence between full subgroups of the full group $[G]$ determined by G and intermediate von Neumann subalgebras of $G\otimes\mathscr{A}$. Haga-Takeda have extended, in their paper [7], this result to a σ-finite finite von Neumann algebra $\mathscr{A}$. On the other hand, Dye has introduced the types (I and II) for a freely acting automorphism group on an abelian von Neumann algebra, and shown that the correspondence in the above conserves the type; if an intermediate von Neumann algebra is discrete (resp. continuous), then the corresponding subgroup is of type I (resp. type II).

We shall define types (discrete type and continuous type) for automorphism groups of von Neumann algebras in Definition 23. This definition is a generalization of that due to Dye. And we shall show that the correspondence of intermediate von Neumann subalgebras of $G\otimes\mathscr{A}$ and full subgroups of the full group $[G]$ determined by G, due to Haga-Takeda, preserves the type in the sense of Dye (Theorem 24 in §4).

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We shall use the terminologies and notations due to Dixmier [3] throughout this note without further explanations.

2. In the sequel, let $\mathscr{A}$ be a von Neumann algebra and $\mathscr{B}$ a von Neumann subalgebra of $\mathscr{A}$. Denote by $\mathscr{B}^c$ the relative commutant $\mathscr{B}'\cap\mathscr{A}$ of $\mathscr{B}$ in $\mathscr{A}$, $\mathscr{B}^p$ the set of all projections of $\mathscr{B}$ and $\bar{E}$ the $\mathscr{B}$-support of $E\in\mathscr{A}^p$, that is,

$$\bar{E}=\inf\ \{P\in\mathscr{B}^p;\ P\geqq E\}.$$

Lemma 1. *Let $(E_\iota)_{\iota\in I}$ be a family of projections in $\mathscr{A}$, then*

$$\overline{\sup_\iota E_\iota}=\sup_\iota\overline{E_\iota}.$$

Proof. By the definition, $\sup_\iota \overline{E_\iota} \geqq \sup_\iota E_\iota$, so $\sup_\iota \overline{E_\iota} \geqq \overline{\sup_\iota E_\iota}$. Conversely $\overline{\sup_\iota E_\iota} \geqq \overline{E_\iota}$, which implies that $\overline{\sup_\iota E_\iota} \geqq \sup_\iota \overline{E_\iota}$. Therefore $\overline{\sup_\iota E_\iota} = \sup_\iota \overline{E_\iota}$.

For completeness we include a proof of the following Lemma, which is proved in [1].

Lemma 2. *If $\bar{E}P$ is a projection for $E \in (\mathscr{B}^c)^p$ and $P \in \mathscr{B}^p$, then*

$$\overline{EP} = \bar{E}P.$$

Proof. $\overline{EP} \leqq \bar{E}P$ is clear by the definition. If $\overline{EP} \neq \bar{E}P$, then there exist two projections Q and R in $\mathscr{B}$ such that

$$\bar{E}P \gneqq Q \geqq EP$$

and that

$$\bar{E}(I-P) \geqq R \geqq E(I-P).$$

Therefore $Q+R \in \mathscr{B}^p$ satisfies $\bar{E} \gneqq Q+R \geqq E$, which contradicts the definition of $\bar{E}$.

The following definition is introduced in [1] as a generalization of the notion of abelian projections.

Definition 3. A projection $E \in \mathscr{A}$ is called *abelian over* $\mathscr{B}$ if $E \in \mathscr{B}^c$ and for every projection $P \in \mathscr{A}$ with $P \leqq E$, there exists a projection $Q \in \mathscr{B}$ such that $P = QE$.

Lemma 4. *Let E be a projection in $\mathscr{A}$ abelian over $\mathscr{B}$. A projection F in $\mathscr{B}^c$ is abelian over $\mathscr{B}$ if $F \leqq E$.*

Proof. Let P be a projection in $\mathscr{A}$ with $P \leqq F$. Then there exists a $Q \in \mathscr{B}^p$ such that $P = QE$ because $P \leqq E$ and E is abelian over $\mathscr{B}$. Hence

$$P = PF = QEF = QF.$$

For a general form of Lemma 4, see [1; Lemma 3].

The following lemma, proved in [2], gives an alternative algebraic definition of projections abelian over $\mathscr{B}$.

Lemma 5. *$E \in \mathscr{A}^p$ is abelian over $\mathscr{B}$ if and only if $E \in \mathscr{B}^c$ and*

$\mathscr{A}_E=\mathscr{B}_E$.

Now, we shall extend the definition of discrete von Neumann algebras as the following:

Definition 6. A von Neumann algebra $\mathscr{A}$ is called *discrete over* $\mathscr{B}$ if there exists an $E\in\mathscr{A}^p$ which is abelian over $\mathscr{B}$ and $\bar{E}=I$.

Remark. Let $\mathscr{Z}$ be the center of a von Neumann algebra $\mathscr{A}$. Then if $\mathscr{A}$ is discrete over $\mathscr{Z}$, $\mathscr{A}$ is discrete in the usual sense cf. [3]. If $\mathscr{A}$ is an abelian von Neumann algebra and discrete over $\mathscr{B}$, then $\mathscr{B}$ is called by Dye [4], a type I subalgebra; cf. Corollary 12 below.

Example. Let $\mathscr{B}$ be a von Neumann algebra and $\mathscr{C}$ a discrete factor, then $\mathscr{B}\otimes\mathscr{C}$ is discrete over $\mathscr{B}\otimes I$.

In fact, let P be a minimal projection in $\mathscr{C}$. Clearly $I\otimes P$ is abelian over $\mathscr{B}\otimes I$ and $\mathscr{B}\otimes I$-support of $I\otimes P=I\otimes I$. Therefore, $\mathscr{B}\otimes\mathscr{C}$ is discrete over $\mathscr{B}\otimes I$.

Theorem 7. *Let $\mathscr{B}$ be a von Neumann subalgebra of $\mathscr{A}$ contained in the center $\mathscr{Z}$ of $\mathscr{A}$. Then $\mathscr{A}$ is discrete over $\mathscr{B}$ if and only if each nonzero projection in $\mathscr{B}$ dominates a nonzero projection abelian over $\mathscr{B}$.*

Proof. If each nonzero projection in $\mathscr{B}$ dominates a nonzero projection abelian over $\mathscr{B}$, by Zorn's lemma, we have a maximal orthogonal family $(G_\iota)_{\iota\in I}$ of nonzero projections in $\mathscr{B}$ satisfying the following properties: for any $\iota\in I$, there exists an $E_\iota\in\mathscr{A}^p$ which is abelian over $\mathscr{B}$ and $\bar{E}_\iota=G_\iota$. Put $G=\Sigma_\iota G_\iota$. If $G\neq I$, there exists a nonzero $F\in\mathscr{A}^p$ which is abelian over $\mathscr{B}$ and $F\leqq I-G$ because $I-G\in\mathscr{B}^p$. Since $\bar{F}\leqq I-G$, $\bar{F}$ is orthogonal to each G_ι, which contradicts the maximality of $(G_\iota)_{\iota\in I}$. Therefore, we have $G=I$. Put $E=\Sigma_\iota E_\iota$. Since $(E_\iota)_{\iota\in I}$ is an orthogonal family, $\bar{E}=\overline{\Sigma_\iota E_\iota}=\Sigma_\iota G_\iota=G=I$ by Lemma 1. On the other hand $\mathscr{A}_E=\Sigma_\iota\oplus\mathscr{B}_{E_\iota}=\mathscr{B}_E$ because E_ι is abelian over $\mathscr{B}$, $(G_\iota)_{\iota\in I}$ is an orthogonal family of projections in $\mathscr{B}$ with $\Sigma_\iota G_\iota=I$ and $\mathscr{B}$ is contained in the center of $\mathscr{A}$ (cf. [3. p.22]). Hence E is abelian over $\mathscr{B}$. Thus there exists an $E\in\mathscr{A}^p$ which is abelian over $\mathscr{B}$ and $\bar{E}=I$.

Conversely, suppose that there exists a projection E which is abelian over $\mathscr{B}$ and $\bar{E}=I$. For a nonzero $P\in\mathscr{B}^p$, put $Q=EP$. Then $Q\leqq P$ and

$\bar{Q}=\overline{EP}=\bar{E}P=P\neq 0$ by Lemma 2, so $Q\neq 0$. Since $\mathscr{B}$ is contained in the center of $\mathscr{A}$, $Q\in\mathscr{B}^c$. Therefore, by Lemma 4, Q is abelian over $\mathscr{B}$. Thus each nonzero projection in $\mathscr{B}$ dominates a nonzero projection abelian over $\mathscr{B}$.

Proposition 8. *Let $\mathscr{B}$ be a von Neumann subalgebra of a von Neumann algebra $\mathscr{A}$ containing the center of $\mathscr{A}$. Assume that $\mathscr{A}$ is discrete over $\mathscr{B}$.*

(1) *$\mathscr{A}$ is discrete if and only if $\mathscr{B}$ is discrete;*

(2) *$\mathscr{A}$ is continuous if and only if $\mathscr{B}$ is continuous;*

(3) *$\mathscr{A}$ is a factor if and only if $\mathscr{B}$ is a factor;*

(4) *If $\mathscr{A}$ is a hyperfinite factor, then $\mathscr{B}$ is a hyperfinite factor.*

Proof. If $\mathscr{A}$ is discrete over $\mathscr{B}$, there exists an $E\in(\mathscr{B}^c)^p$ with $\mathscr{A}_E=\mathscr{B}_E$ and $\bar{E}=I$. Since $E\in\mathscr{B}'$ and the $\mathscr{B}\cap\mathscr{B}'$-support of $E=I$, it follows that $\mathscr{B}\cong\mathscr{B}_E=\mathscr{A}_E$ [3. p. 19, Prop. 2]. Therefore, (4) and the "only if'" parts of (1), (2) and (3) are obvious (see, for instance [3]). Conversely, $\mathscr{A}'\cong\mathscr{A}'_E=\mathscr{B}'_E$ because $E\in\mathscr{A}$ and the central support of $E=I$ by $\mathscr{B}\supset$ the center of $\mathscr{A}$. If $\mathscr{B}$ satisfies the condition of (1) (2) (3) respectively, then $\mathscr{B}'$ does so (see, for instance [3]). Therefore $\mathscr{A}$ does so.

Proposition 9. *Let $\mathscr{A}$ be a σ-finite infinite factor. If $\mathscr{A}$ is discrete over $\mathscr{B}$, then $\mathscr{A}$ is isomorphic to $\mathscr{B}\otimes\mathscr{L}(\mathfrak{K})$ for some Hilbert space $\mathfrak{K}$, where $\mathscr{L}(\mathfrak{K})$ is the factor of all bounded operators on $\mathfrak{K}$.*

Proof. If $\mathscr{A}$ is discrete over $\mathscr{B}$, there exists an $E\in(\mathscr{B}^c)^p$ with $\mathscr{A}_E=\mathscr{B}_E$ and $\bar{E}=I$. Since $\mathscr{A}$ is a σ-finite infinite factor, it follows that $\mathscr{A}$ is spatially isomorphic to $\mathscr{A}_E\otimes\mathscr{L}(\mathfrak{K})$ for some Hilbert space $\mathfrak{K}$ [6. Lemma 3. 12]. On the other hand, the $\mathscr{B}\cap\mathscr{B}'$-support of $\bar{E}=I$, so $\mathscr{B}_E$ is isomorphic to $\mathscr{B}$ by the condition that $E\in\mathscr{B}'$ [3. p. 19]. Therefore $\mathscr{A}$ is isomorphic to $\mathscr{B}\otimes\mathscr{L}(\mathfrak{K})$.

Remark. Proposition 9 is extended to the following:

If a von Neumann algebra $\mathscr{A}$ is discrete over a von Neumann subalgebra $\mathscr{B}$ containing the center of $\mathscr{A}$, then $\mathscr{A}$ is isomorphic to $(\mathscr{B}\otimes\mathscr{L}(\mathfrak{K}))_E$, for some Hilbert space $\mathfrak{K}$ and a projection E in $\mathscr{B}\otimes\mathscr{L}(\mathfrak{K})$.

The proof is clear from [3; p. 58, Theorem 3].

The following definition is introduced in [2].

Definition 10. A von Neumann algebra $\mathscr{A}$ is called *continuous over* a von Neumann subalgebra $\mathscr{B}$ if $\mathscr{A}$ contains no nonzero projections abelian over $\mathscr{B}$.

Proposition 11. *Let $\mathscr{B}$ be a von Neumann subalgebra of $\mathscr{A}$. If $\mathscr{A}$ is discrete (resp. continuous) over $\mathscr{B}$, then $\mathscr{A}_E$ is discrete (resp. continuous) over $\mathscr{B}_E$ for each nonzero $E \in (\mathscr{B} \cap \mathscr{B}')^p$.*

Proof. By the definition 10, it is clear that if $\mathscr{A}$ is continuous over $\mathscr{B}$, then $\mathscr{A}_E$ is continuous over $\mathscr{B}_E$ for every nonzero $E \in (\mathscr{B} \cap \mathscr{B}')^p$. If $\mathscr{A}$ is discrete over $\mathscr{B}$, there exists a $F \in (\mathscr{B}^c)^p$ which is abelian over $\mathscr{B}$ and $\bar{F} = I$. For each nonzero $E \in (\mathscr{B} \cap \mathscr{B}')^p$, $G = FE$ is abelian over $\mathscr{B}$ by Lemma 4 and

$$\bar{G} = \overline{FE} = \bar{F}E = E$$

by Lemma 2, so that $\mathscr{A}_E$ is discrete over $\mathscr{B}_E$.

Corollary 12. *If $\mathscr{A}$ is an abelian von Neumann algebra, then $\mathscr{A}$ is discrete over a von Neumann subalgebra $\mathscr{B}$ if and only if $\mathscr{B}$ is a type I subalgebra of $\mathscr{A}$ in the sense of Dye [4], that is, each nonzero projection in $\mathscr{A}$ dominates a nonzero projection abelian over $\mathscr{B}$.*

Proof. By Theorem 7, the "if" part is clear. Conversely, if there exists a nonzero projection which does not dominate any nonzero projection abelian over $\mathscr{B}$, then we have a nonzero $E \in \mathscr{B}^p$ such that $\mathscr{A}_E$ is continuous over $\mathscr{B}_E$ (cf. [4. p. 124]). By proposition 11, this is a contradiction.

Proposition 13. *Let $\mathscr{B}$ be a von Neumann subalgebra contained in the center of $\mathscr{A}$ and $(E_\iota)_{\iota \in I}$ be an orthogonal family of projections in $\mathscr{B}$ with $\Sigma_\iota E_\iota = I$. If $\mathscr{A}_{E_\iota}$ is discrete (resp. continuous) over $\mathscr{B}_{E_\iota}$ for each ι, then $\mathscr{A}$ is discrete (resp. continuous) over $\mathscr{B}$.*

Proof. Assume that $\mathscr{A}_{E_\iota}$ is discrete over $\mathscr{B}_{E_\iota}$ for each ι. Let F_ι be a projection abelian over $\mathscr{B}$ and $\bar{F}_\iota = E_\iota$, then by the proof of Theorem 7

$F=\Sigma_\iota F_\iota$ is abelian over $\mathscr{B}$ and $\bar{F}=\Sigma_\iota E_\iota=I$. Therefore $\mathscr{A}$ is discrete over $\mathscr{B}$.

Assume that $\mathscr{A}_{E_\iota}$ is continuous over $\mathscr{B}_{E_\iota}$ for each ι. If $\mathscr{A}$ is not continuous over $\mathscr{B}$, then there exists a nonzero $F\in(\mathscr{B}^c)^p$ abelian over $\mathscr{B}$. Let E_k be a projection in $(E_\iota)_{\iota\in I}$ such that $FE_k\neq 0$. FE_k is a nonzero projection abelian over $\mathscr{B}$ by Lemma 4, which contradicts the assumption that $\mathscr{A}_{E_k}$ is continuous over $\mathscr{B}_{E_k}$.

Theorem 14. *Assume that $\mathscr{B}$ is contained in the center of $\mathscr{A}$. Then there exists a unique $E\in\mathscr{B}^p$ satisfying the following*:

(1) *$\mathscr{A}_E$ is discrete over $\mathscr{B}_E$,*

and

(2) *$\mathscr{A}_{I-E}$ is continuous over $\mathscr{B}_{I-E}$.*

Proof. If $\mathscr{A}$ is not continuous over $\mathscr{B}$, then there exists a nonzero projection in $\mathscr{A}$ abelian over $\mathscr{B}$. Put $E=\sup_\iota\bar{E}_\iota$, where each E_ι is a nonzero projection abelian over $\mathscr{B}$. It follows that $E\in\mathscr{B}^p$ and $\mathscr{A}_E$ is discrete over $\mathscr{B}_E$. In fact, take a nonzero $P\in\mathscr{B}^p$ with $P\leqq E$, then we have $PE_\iota\neq 0$ for some ι, which implies $E\geqq PE_\iota\neq 0$ and $PE_\iota\in(\mathscr{B}^c)^p$ because $\mathscr{B}$ is abelian. Since $E_\iota\geqq PE_\iota$, PE_ι is abelian over $\mathscr{B}$ by Lemma 4. Thus by Theorem 7 $\mathscr{A}_E$ is discrete over $\mathscr{B}_E$. If $\mathscr{A}_{I-E}$ is not continuous over $\mathscr{B}_{I-E}$, there exists a nonzero $G\in\mathscr{A}^p$ which is abelian over $\mathscr{B}$ and $G\leqq I-E$. By the definition of E, we have $G\leqq\bar{G}\leqq E$, which implies that $G=0$. This is a contradiction. Therefore $\mathscr{A}_{I-E}$ is continuous over $\mathscr{B}_{I-E}$.

Assume that, for $G\in\mathscr{B}^p$, $\mathscr{A}_G$ is discrete over $\mathscr{B}_G$ and $\mathscr{A}_{I-G}$ is continuous over $\mathscr{B}_{I-G}$, then there exists a $Q\in\mathscr{A}^p$ which is abelian over $\mathscr{B}$ and $\bar{Q}=G$. By the definition of $E, G\leqq E$. If $E\neq G$, then $E-G$ dominates a nonzero projection abelian over $\mathscr{B}$. On the other hand, $E-G\leqq I-G$. Therefore $I-G$ dominates a nonzero projection abelian over $\mathscr{B}$, which contradicts the assumption that $\mathscr{A}_{I-G}$ is continuous over $\mathscr{B}_{I-G}$. Thus $E=G$, which shows the uniqueness of E.

Remark. As a corollary of Theorem 14, we have the following ([3. p. 121, Cor. 1]); there exists a unique central projection E such that $\mathscr{A}_E$ is discrete and that $\mathscr{A}_{I-E}$ is continuous.

3. Let $\mathscr{B}$ be a subalgebra of a von Neumann algebra $\mathscr{A}$. A positive linear mapping e of $\mathscr{A}$ onto $\mathscr{B}$ is called an *expectaion* of $\mathscr{A}$ onto $\mathscr{B}$ if e satisfies the following conditions;

(i) $$I^e = I$$

and

(ii) $$(AB)^e = A^e B \text{ for all } A \in \mathscr{A} \text{ and for all } B \in \mathscr{B}$$

(cf. [10] and [14]).

The main result in [2] is that concerning e-strong Maharam subalgebras:

Definition 15. Let e be a normal expectation of $\mathscr{A}$ onto $\mathscr{B}$. $\mathscr{B}$ is called an *e-strong Maharam subalgebra* of $\mathscr{A}$ if for any $P \in A^p$ and any $B \in \mathscr{B}$ such that $0 \leqq B \leqq P^e$ there exists a $Q \in \mathscr{A}^p$ such that $Q \leqq P$ and that $Q^e = B$.

Theorem A. ([2. Corollary 11]). *Let $\mathscr{B}$ be a von Neumann subalgebra of a von Neumann algebra $\mathscr{A}$ contained in the center of $\mathscr{A}$ and e a normal expectation of $\mathscr{A}$ onto $\mathscr{B}$. If $\mathscr{A}$ is continuous over $\mathscr{B}$, then $\mathscr{B}$ is an e-strong Maharam subalgebra of $\mathscr{A}$.*

In this section, we shall discuss two von Neumann algebras $\mathscr{C}$ and $\mathscr{A}$ satisfying the following conditions;

(*) $$\mathscr{C} \supset \mathscr{A} \text{ and } \mathscr{Z} = \mathscr{A} \cap \mathscr{A}' \supset \mathscr{Z}_{\mathscr{C}} = \mathscr{C} \cap \mathscr{C}',$$

or

(**) $$\mathscr{C} \supset \mathscr{A} \text{ and } \mathscr{Z} = \mathscr{A} \cap \mathscr{A}' = \mathscr{C} \cap \mathscr{A}'.$$

Remark that if $\mathscr{Z} = \mathscr{C} \cap \mathscr{A}'$ then $\mathscr{Z} \supset \mathscr{Z}_{\mathscr{C}}$.

Theorem 16. *Let $\mathscr{C}$ and $\mathscr{A}$ be two von Neumann algebras satisfying the condition (*). If $\mathscr{C}$ is finite and discrete, then $\mathscr{A}$ is discrete over $\mathscr{Z}_{\mathscr{C}}$.*

Proof. If $\mathscr{A}$ is not discrete over $\mathscr{Z}_{\mathscr{C}}$, then there exists by Theorem 14 a nonzero projection F in $\mathscr{Z}_{\mathscr{C}}$ such that $\mathscr{A}_F$ is continuous over $(\mathscr{Z}_{\mathscr{C}})_F$. $\mathscr{C}_F$ and $\mathscr{A}_F$ satisfy the condition (*) and $\mathscr{C}_F$ is finite and discrete because $F \in (\mathscr{Z}_{\mathscr{C}})^p$. So we shall assume that $\mathscr{A}$ is continuous over $\mathscr{Z}_{\mathscr{C}}$.

Since $\mathscr{C}$ is discrete, there exists a projection E in $\mathscr{C}$ such that $\mathscr{C}_E=(\mathscr{Z}_{\mathscr{C}})_E$ and $\mathscr{Z}_{\mathscr{C}}$-support of $E=I$. Since $\mathscr{C}$ is finite, there exists a normal faithful expectation e of $\mathscr{C}$ onto $\mathscr{Z}_{\mathscr{C}}$, that is, canonical natural mapping of $\mathscr{C}$ (cf. [3]). On the other hand, $\mathscr{Z}\supset\mathscr{Z}_{\mathscr{C}}$ and $\mathscr{A}$ is continuous over $\mathscr{Z}_{\mathscr{C}}$. Therefore, by Theorem A $\mathscr{Z}_{\mathscr{C}}$ is an e-strong Maharam subalgebra of $\mathscr{A}$, that is, there exists an $F\in\mathscr{A}^p$ with $E^e=F^e$. Applying the comparability theorem to E and F, we have a $G\in(\mathscr{Z}_{\mathscr{C}})^p$ such that $EG\prec FG$ and that $E(I-G)\prec F(I-G)$. If $EG\gtrsim FG$, then $E^eG\neq F^eG$ because e is positive and faithful, which contradicts the property that $E^e=F^e$. Hence $EG\sim FG$. Similarly, $E(I-G)\sim F(I-G)$.

Therefore $E\sim F$. Since E is abelian over $\mathscr{Z}_{\mathscr{C}}$, F is abelian over $\mathscr{Z}_{\mathscr{C}}$ cf. [1. Lemma 3]. So that $\mathscr{C}_F=(\mathscr{Z}_{\mathscr{C}})_F$, which implies $\mathscr{A}_F=(\mathscr{Z}_{\mathscr{C}})_F$. Thus $\mathscr{A}$ contains a nonzero projection F abelian over $\mathscr{Z}_{\mathscr{C}}$. This is a contradiction.

Corollary 17. *Let $\mathscr{C}$ and $\mathscr{A}$ be two von Neumann algebras satisfying the condition* $(*)$. *If $\mathscr{C}$ is finite and $\mathscr{A}$ is continuous over $\mathscr{Z}_{\mathscr{C}}$, then $\mathscr{C}$ is continuous.*

Proof. If $\mathscr{C}$ is not continuous, there exists a nonzero projection E in $\mathscr{Z}_{\mathscr{C}}$ such that $\mathscr{C}_E$ is discrete by Theorem 14. Then $\mathscr{C}_E$ and $\mathscr{A}_E$ satisfy the conditions of Theorem 16. Therefore $\mathscr{A}_E$ is discrete over $(\mathscr{Z}_{\mathscr{C}})_E$. Due to Proposition 11, this contradicts with the assumption that $\mathscr{A}$ is continuous over $\mathscr{Z}_{\mathscr{C}}$.

Lemma 18. *Let $\mathscr{C}$ and $\mathscr{A}$ be two von Neumann algebras satisfying the condition* $(**)$. *If $\mathscr{A}$ is discrete, then $\mathscr{A}=\mathscr{C}\cap\mathscr{Z}'$.*

Proof. Let $\mathscr{B}$ be a von Neumann algebra generated by $\mathscr{C}'$ and $\mathscr{Z}$, then $\mathscr{B}=(\mathscr{C}\cap\mathscr{Z}')'$ and $\mathscr{A}'\supset\mathscr{B}\supset\mathscr{Z}$. If $\mathscr{A}$ is discrete, then $\mathscr{A}'$ is discrete, cf. [3, p. 123, Theorem 1.].

Therefore $\mathscr{B}$ is normal in $\mathscr{A}'$ that is,

$$\mathscr{B}^{cc}=(\mathscr{B}'\cap\mathscr{A}')'\cap\mathscr{A}'=\mathscr{B},$$

cf. [3, p. 307, exercise 13]. In fact, if $\mathscr{A}'$ is discrete, then $\mathscr{A}'$ is isomorphic to a von Neumann algebra $\mathscr{D}$ such that $\mathscr{D}'$ is abelian, cf. [3]. Let $\mathscr{B}\subset\mathscr{A}'$ be isomorphic to $\mathscr{B}_1\subset\mathscr{D}$. $\mathscr{Z}\subset\mathscr{A}'$ is isomorphic to $\mathscr{D}\cap\mathscr{D}'=\mathscr{D}'$.

Hence $\mathscr{B}_1 \supset \mathscr{D}'$. Therefore

$$\mathscr{B}_1^{cc} = (\mathscr{B}_1' \cap \mathscr{D})' \cap \mathscr{D} = \mathscr{B}_1 \cap \mathscr{D} = \mathscr{B}_1.$$

Therefore $\mathscr{B}^{cc} = \mathscr{B}$.

On the other hand,

$$\begin{aligned}\mathscr{B}^{cc} &= (\mathscr{B}' \cap \mathscr{A}')' \cap \mathscr{A}' = (\mathscr{C} \cap \mathscr{Z}' \cap \mathscr{A}')' \cap \mathscr{A}' \\ &= (\mathscr{Z}' \cap \mathscr{Z})' \cap \mathscr{A}' = \mathscr{Z}' \cap \mathscr{A}' \\ &= \mathscr{A}'.\end{aligned}$$

Hence $\mathscr{B} = \mathscr{A}'$, that is $\mathscr{A} = \mathscr{B}' = \mathscr{C} \cap \mathscr{Z}'$.

Lemma 19. *Let $\mathscr{C}$ and $\mathscr{A}$ be two von Neumann algebras satisfying the condition* (**). *If $\mathscr{A}$ is discrete and $\mathscr{C}$ is continuous, then $\mathscr{A}$ is continuous over $\mathscr{Z}_{\mathscr{C}}$.*

Proof. If $\mathscr{A}$ is not continuous over $\mathscr{Z}_{\mathscr{C}}$, then there exists a nonzero E in $\mathscr{A}^p$ such that $\mathscr{A}_E = (\mathscr{Z}_{\mathscr{C}})_E$, which implies $\mathscr{A}_E = \mathscr{Z}_E = (\mathscr{Z}_{\mathscr{C}})_E$.

Since $E \in \mathscr{A} \subset \mathscr{C} \cap \mathscr{Z}'$, it follows that

$$\mathscr{C}_E = \mathscr{C}_E \cap (\mathscr{Z}_{\mathscr{C}})_E' = \mathscr{C}_E \cap \mathscr{Z}_E' = (\mathscr{C} \cap \mathscr{Z}')_E.$$

On the other hand, $\mathscr{A}$ is discrete, and so $\mathscr{C} \cap \mathscr{Z}' = \mathscr{A}$ by Lemma 18.

Therefore $\mathscr{C}_E = \mathscr{A}_E$, so that $\mathscr{C}_E = (\mathscr{Z}_{\mathscr{C}})_E$. Thus $\mathscr{C}$ contains a nonzero abelian projection, which is a contradiction.

Theorem 20. *Let $\mathscr{C}$ and $\mathscr{A}$ be two von Neumann algebras satisfying the condition* (**). *If $\mathscr{C}$ is continuous, then $\mathscr{A}$ is continuous over $\mathscr{Z}_{\mathscr{C}}$.*

Proof. There exists a projection E in $\mathscr{Z}$ such that $\mathscr{A}_E$ is discrete and that $\mathscr{A}_{I-E}$ is continuous. Since $\mathscr{Z}_{\mathscr{C}}$ is abelian, it follows that $\mathscr{A}_{I-E}$ is continuous over $(\mathscr{Z}_{\mathscr{C}})_{I-E}$, cf. for instance [2. Example 4].

On the other hand, $\mathscr{A}_E$ and $\mathscr{C}_E$ satisfy the conditions of Lemma 19 because $E \in \mathscr{Z} = \mathscr{C} \cap \mathscr{A}'$. Therefore $\mathscr{A}_E$ is continuous over $(\mathscr{Z}_{\mathscr{C}})_E$. So that by an analogy with the proof of Proposition 13, we have $\mathscr{A}$ continuous over $\mathscr{Z}_{\mathscr{C}}$.

Corollary 21. *Let $\mathscr{C}$ and $\mathscr{A}$ be two von Neumann algebras satisfying the conditon* (**). *If $\mathscr{A}$ is discrete over $\mathscr{Z}_{\mathscr{C}}$, then $\mathscr{C}$ is discrete.*

Proof. If $\mathscr{C}$ is not discrete, there exists a nonzero $E \in (\mathscr{Z}_{\mathscr{C}})^p$ such that $\mathscr{C}_E$ is continuous, by Theorem 14. Then $\mathscr{C}_E$ and $\mathscr{A}_E$ satisfy the conditions of Theorem 20 because $E \in \mathscr{Z}_{\mathscr{C}} \subset \mathscr{C} \cap \mathscr{A}'$. Therefore $\mathscr{A}_E$ is continuous over $(\mathscr{Z}_{\mathscr{C}})_E$, which contradicts the assumption that $\mathscr{A}$ is discrete over $\mathscr{Z}_{\mathscr{C}}$ by Proposition 11.

Theorem 22. *Let $\mathscr{C}$ and $\mathscr{A}$ be two von Neumann algebras satisfying the condition* (**). *Assume that $\mathscr{C}$ is finite. Then $\mathscr{C}$ is discrete* (*resp. continuous*) *if and only if $\mathscr{A}$ is discrete* (*resp. continuous*) *over $\mathscr{Z}_{\mathscr{C}}$.*

Proof. If $\mathscr{C}$ is discrete, $\mathscr{A}$ is discrete over $\mathscr{Z}_{\mathscr{C}}$ by Theorem 16. If $\mathscr{C}$ is continuous, $\mathscr{A}$ is continuous over $\mathscr{Z}_{\mathscr{C}}$ by Theorem 20. If $\mathscr{A}$ is discrete over $\mathscr{Z}_{\mathscr{C}}$, $\mathscr{C}$ is discrete by corollary 21. And if $\mathscr{A}$ is continuous over $\mathscr{Z}_{\mathscr{C}}$, $\mathscr{C}$ is continuous by corollary 17.

4. Let G be a countable group of (*-) automorphisms of a von Neumann algebra $\mathscr{A}$. An automorphism α of $\mathscr{A}$ is called *freely acting* on $\mathscr{A}$ when

$$AB = B^{\alpha} A \qquad \text{for all } B \in \mathscr{A}$$

implies

$$A = 0$$

([9]). G is called *freely acting* on $\mathscr{A}$ if $g \neq 1$ (the unit element) in G is freely acting on $\mathscr{A}$.

In this section, we are concerned with a finite von Neumann algebra $\mathscr{A}$ with a faithful normal G-invariant trace φ.

Now, we shall review briefly the concept of crossed product ([11] and [12]). Denote by $\Sigma_{g \in G}\, g \otimes A_g$ an operator valued function on G where $A_g \in \mathscr{A}$ is the value of the function at $g \in G$. Let $\mathscr{D}$ be the set of all operator valued functions on G such that $A_g = 0$ up to a finite subset of G. Then $\mathscr{D}$ is a linear space with the usual operation of the addition and scaler multiplication, and becomes a *-algebra by the following operations;

$$(\Sigma_{g\in G}\, g\otimes A_g)(\Sigma_{h\in G} h\otimes B_h)=\Sigma_{g,h\in G}\, gh\otimes A_g B_h^{g^{-1}}$$

and

$$(\Sigma_{g\in G}\, g\otimes A_g)^*=\Sigma_{g\in G}\, g^{-1}\otimes A_g^{*g}.$$

And φ is extended to a faithful trace $\tilde{\varphi}$ on $\mathscr{D}$ by

$$\tilde{\varphi}(\Sigma_{g\in G}\, g\otimes A_g)=\varphi(A_1).$$

Let $\mathfrak{H}$ be the representation space of $\mathscr{A}$ by φ (cf. for instance [3]), then $G\otimes\mathfrak{H}$, in the sense of Umegaki [14], is the representation space of $\mathscr{D}$ by $\tilde{\varphi}$. Define operators $I\otimes A$ and $U_g(g\in G,\ A\in\mathscr{A})$ on $G\otimes\mathfrak{H}$ by

$$I\otimes A(\Sigma_{h\in G} h\otimes B_h)=\Sigma_{h\in G} h\otimes AB_h$$

and

$$U_g(\Sigma_{h\in G} h\otimes B_h)=\Sigma_{h\in G}\, gh\otimes B_h^{g-1},$$

for any $\Sigma_{h\in G} h\otimes B_h\in\mathscr{D}$, where $\mathscr{D}$ is considered as a dense linear subset of $G\otimes\mathfrak{H}$. Then U_g is a unitary operator with

$$U_g^*(I\otimes A)U_g=I\otimes A^g.$$

Hereafter, we shall identify $I\otimes A$ with A since $\mathscr{A}$ is isomorphic to $I\otimes\mathscr{A}$.

The crossed product $G\otimes\mathscr{A}$ is the weak closure of $\mathscr{D}$ on $G\otimes\mathfrak{H}$, where $\mathscr{D}$ is now considered as a $*$-algebra of operators on $G\otimes\mathfrak{H}$, that is, a von Neumann algebra generated by $\mathscr{A}$ and $\{U_g\colon g\in G\}$. Then $G\otimes\mathscr{A}$ is a finite von Neumann algebra with a faithful normal trace $\tilde{\varphi}$.

Haga-Takeda [7] have extended the definition of full group (due to Dye) as in the below and proved Theorem B.

For two automorphisms α and β of $\mathscr{A}$, let $F(\alpha,\beta)$ be the maximum central projection such that $\alpha^{-1}\beta$ is an inner automorphism on $\mathscr{A}_{F(\alpha,\beta)}$ (cf. [7] or [9]). Consider the set

$$[G]=\{\text{automorphism } \alpha \text{ of } \mathscr{A}\colon \sup_{g\in G} F(\alpha,g)=I\}.$$

Then each $\alpha\in[G]$ induces the unitary U_α of $G\otimes\mathscr{A}$ with

$$U_\alpha^* A U_\alpha=A^\alpha \qquad \text{for} \quad A\in\mathscr{A},$$

$[G]$ forms a group containing G and G is called *full* if $[G]=G$. By the *fixed algebra* of G, we mean the algebra

$$\mathscr{Z}(G)=\{A\in\mathscr{A};\ A^g=A \quad \text{for all} \quad g\in G\}.$$

A subalgebra of $G\otimes\mathscr{A}$ containing $\mathscr{A}$ is called an *intermediate* von Neumann algebra of $G\otimes\mathscr{A}$.

Theorem B ([7. Theorem 2]). *Let G be a countable group of automorphisms freely acting on $\mathscr{A}$. Suppose that $\mathscr{A}$ is a finite von Neumann algebra with a faithful normal G-invariant trace. Then the lattice of all intermediate von Neumann algebras $\mathscr{C}$ of $G\otimes\mathscr{A}$ and the lattice of all full subgroups K of $[G]$ are isomorphic by associating with each full subgroup K the intermediate von Neumann subalgebra $\mathscr{C}$*

$$\mathscr{C}=\textit{the von Neumann algebra generated by } [U_\alpha;\ \alpha\in K]$$

and with each intermediate von Neumann subalgebra $\mathscr{C}$ the full subgroup K

$$K=\{\alpha\in[G];\ U_\alpha\in\mathscr{C}\}.$$

Now, we shall extend the definition of types of automorphism groups as follows:

Definition 23. A full subgroup K of $[G]$ is called to be *discrete type* (resp. *continuous type*) if $\mathscr{A}$ is discrete (resp. continuous) over the fixed algebra $\mathscr{Z}(K)$ of K.

Remark that the fixed algebra of a full group is contained in the center of $\mathscr{A}$ because a full group contains all inner automorphisms of $\mathscr{A}$.

Mixed types can occur, but by Theorem 14 a full group K can be devided into purely discrete type and continuous type parts. That is, for the projection $E\in\mathscr{Z}(K)$ in Theorem 14, each of $\mathscr{A}_E$ and $\mathscr{A}_{I-E}$ reduces K, so that K splits into the direct sum K_E+K_{I-E} of two groups, the first a discrete type of automorphisms of $\mathscr{A}_E$, the second a continuous type of automorphisms of $\mathscr{A}_{I-E}$. The summands are obviously uniquely determined.

Theorem 24. *Let G be a countable group of automorphisms freely acting on $\mathscr{A}$. Suppose that $\mathscr{A}$ is a finite von Neumann algebra with a faithful normal G-invariant trace. Let $\mathscr{C}$ be an intermediate von Neumann* *algebra of* $G\otimes\mathscr{A}$ *and* K *a full subgroup of* $[G]$ *which corresponds to* $\mathscr{C}$ *in the sense of Theorem B. Then* $\mathscr{C}$ *is discrete (resp. continuous) if and only if* K *is discrete type (resp. continuous type).*

Proof. By the assumption of G and $\mathscr{A}$, $G\otimes\mathscr{A}$ is finite, and so $\mathscr{C}$ is finite. Furthermore, by [8. Lemma 4.1], we have $\mathscr{C}\cap\mathscr{A}'=\mathscr{Z}=\mathscr{A}\cap\mathscr{A}'$.

Therefore $\mathscr{C}$ and $\mathscr{A}$ satisfy the conditions of Theorem 22. On the other hand, $\mathscr{Z}(K)=\mathscr{Z}_{\mathscr{C}}$, the center of $\mathscr{C}$ by [8. Corollary 4.3]. Hence we have this theorem.

In [8], Haga has proved the Dye correspondence in the different form from Theorem 24.

Corollary 25. ([5. Proposition 6.1]). *Suppose that* $\mathscr{A}$ *is abelian, that* G *is freely acting automorphism group on* $\mathscr{A}$ *and that* $\mathscr{A}$ *has a faithful normal G-invariant trace. Then the correspondence of intermediate von Neumann subalgebras of* $G\otimes\mathscr{A}$ *and the full subgroups of* $[G]$ *in the sense of Theorem B (that is due to Dye [5]) conserves the type.*

Especially assume that $\mathscr{A}$ is continuous in Theorem 24, then $\mathscr{A}$ is continuous over each abelian von Neumann subalgebra of $\mathscr{A}$. Therefore by Theorem 24, we have the following corollary.

Corollary 26. *Let* G *be a countable group of automorphisms freely acting on* $\mathscr{A}$. *Suppose that* $\mathscr{A}$ *is a finite continuous von Neumann algebra with a faithful normal G-invariant trace. Then each intermediate von Neumann subalgebra of* $G\otimes\mathscr{A}$ *is continuous.*

5. Prof. Y. Nakagami pointed out the following variants of Proposition 9.

Proposition A. *Let* $\mathscr{A}$ *be a finite discrete factor. If* $\mathscr{A}$ *is discrete over a von Neumann subalgebra* $\mathscr{B}$, *then* $\mathscr{A}$ *is isomorphic to* $\mathscr{B}\otimes\mathscr{L}(\mathfrak{K})$ *for some Hilbert space* $\mathfrak{K}$.

Proof. If $\mathscr{A}$ is discrete over $\mathscr{B}$, then $\mathscr{B}$ is a discrete factor by Proposition 8 and finite. Therefore, $\mathscr{A}$ is isomorphic to $\mathscr{B}\otimes\mathscr{L}(\mathfrak{K})$ for some Hilbert space $\mathfrak{K}$.

Proposition B. *Let $\mathscr{A}$ be a von Neumann algebra discrete over a von Neumann subalgebra $\mathscr{B}$ containing the center $\mathscr{Z}$ of $\mathscr{A}$. If $\mathscr{A}$ is properly infinite, then $\mathscr{A}$ is isomorphic to $\Sigma\oplus(\mathscr{B}_{E_\iota}\otimes\mathscr{L}(\mathfrak{H}_\iota))$, where E_ι runs over a partition of I in $\mathscr{Z}$ and $\mathfrak{H}_\iota$ is a Hilbert space for each ι.*

Corollary. *Let $\mathscr{A}$ be a properly infinite factor discrete over a von Neumann subalgebra $\mathscr{B}$, then $\mathscr{A}$ is isomorphic to $\mathscr{B}\otimes\mathscr{L}(\mathfrak{H})$ for some Hilbert space $\mathfrak{H}$.*

Poof of Proposition B. At the first, we shall show that there exists a nonzero $E\in\mathscr{Z}^p$ such that $\mathscr{A}_E$ is isomorphic to $\mathscr{B}_E\otimes\mathscr{L}(\mathfrak{H})$ for some Hilbert space $\mathfrak{H}$.

Since $\mathscr{A}$ is discrete over $\mathscr{B}$, there exists a $F\in(\mathscr{B}^C)^p$ such that $\bar{F}=I$ and that $\mathscr{A}_F=\mathscr{B}_F$. Since there exists a projection P in $\mathscr{Z}$ such that EP is finite and that $F(I-P)$ is properly infinite in $\mathscr{A}$, we may assume that F is finite or properly infinite. Let $\{F_\iota;\ \iota\in I\}$ be a maximal family of equivalent and mutually orthogonal projections in $\mathscr{A}$ such that $F=F_\iota$ for some $\iota\in I$. Then there exists a nonzero central projection E in $\mathscr{A}$ satisfying

$$(I-\Sigma_{\iota\in I}F_\iota)E\prec FE.$$

If F is finite and I is finite, then the central projection

$$E=\Sigma_{\iota\in I}F_\iota E+(I-\Sigma_{\iota\in I}F_\iota)E$$

is finite, which contradicts that $\mathscr{A}$ is properly infinite. If F is properly infinite, for each F_ι, there exists a family of countable projections in $\mathscr{A}$ which are equivalent to F_ι. Therefore we may choose I as an infinite set. Since

$$E=\Sigma_{\iota\in I}F_\iota E+(I-\Sigma_{\iota\in I}F_\iota)E\prec\Sigma_{\iota\in I}F_\iota E\leqq E,$$

it follows that $E\sim\Sigma_{\iota\in I}F_\iota E$. Therefore $\mathscr{A}_E$ is spatially isomorphic to $\mathscr{A}_{FE}\otimes\mathscr{L}(\mathfrak{H})$ for some Hilbert space $\mathfrak{H}$ with dim. $\mathfrak{H}=$card. I. Since $\overline{FE}=E$, $\mathscr{B}_E$ is isomorphic to $\mathscr{B}_{FE}=\mathscr{A}_{FE}$. Thus there exists a nonzero projection E in $\mathscr{Z}$ such that $\mathscr{A}_E$ is isomorphic to $\mathscr{B}_E\otimes\mathscr{L}(\mathfrak{H})$.

Let $(E_\iota)_{\iota\in I}$ be a maximal orthogonal family of projections in $\mathscr{Z}$ such that for each ι, $E_\iota\neq0$ and that $\mathscr{A}_{E_\iota}$ is isomorphic to $\mathscr{B}_{E_\iota}\otimes\mathscr{L}(\mathfrak{H}_\iota)$ for some

Hilbert space $\mathfrak{H}_\iota$. If $\Sigma_{\iota\in I}E_\iota \neq I$, then $G=I-\Sigma_{\iota\in I}E_\iota$ is a nonzero projection in $\mathscr{Z}$. By the assumption, $\mathscr{A}_G$ is properly infinite and $\mathscr{A}_G$ is discrete over $\mathscr{B}_G$ by Proposition 11. Therefore, there exists a nonzero projection Q in $\mathscr{Z}$ such that $Q\leqq G$ and that $\mathscr{A}_Q$ is isomorphic to $\mathscr{B}_Q\otimes\mathscr{L}(\mathfrak{K})$ for some Hilbert space $\mathfrak{K}$, which contradicts the maximality of $\{E_\iota\}$. Hence $\Sigma_{\iota\in I}E_\iota=I$, that is, $\mathscr{A}$ is isomorphic to $\Sigma_{\iota\in I}\oplus\mathscr{A}_{E_\iota}=\Sigma_{\iota\in I}\oplus(\mathscr{B}_{E_\iota}\otimes\mathscr{L}(\mathfrak{H}_\iota))$.

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