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On the Definition of C^*-algebras

By

Huzihiro ARAKI and George A. ELLIOTT*

Abstract

It is shown that conditions $\|xy\| \leqq \|x\| \|y\|$ and $\|x^*\| = \|x\|$ follow from the other axioms for C^*-algebras.

Our first result is

Theorem 1. *Let $\mathfrak{A}$ be a $*$-algebra with a complete linear space norm such that for all $x \in \mathfrak{A}$*

$$\|x^* x\| = \|x\|^2. \tag{1}$$

Then $\mathfrak{A}$ is a C^*-algebra.

Proof. Step (1). *For x, $y \in \mathfrak{A}$,*

$$\|xy\| \leqq 4\|x^*\| \|y\|. \tag{2}$$

From

$$4cdxy = \sum_{n=0}^{3} i^n(\bar{d}y^* + (-i)^n cx)(dy + i^n \bar{c} x^*) \tag{3}$$

we obtain (2) by setting $c = \|x^*\|^{-1}$, $d = \|y\|^{-1}$ (if $x \neq 0$, $y \neq 0$) and using the triangle inequality and (1).

Step (2). *For x, $y \in \mathfrak{A}$,*

$$\|xy\| \leqq 16\|x\| \|y\|. \tag{4}$$

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* Present address: Mathematics Institute, University of Copenhagen, Copenhagen, DENMARK.

We have

$$\|xy\|=\|(xy)^*(xy)\|\|xy\|^{-1} \quad (\text{by } (1))$$
$$=\{\|y^*(x^*xy)\|\|x^*xy\|^{-1}\}\{\|x^*xy\|\|xy\|^{-1}\}$$
$$\leqq 16\|y\|\|x\| \quad (\text{by } (2)).$$

Step (3). *Any norm-closed commutative sub-$*$-algebra of $\mathfrak{A}$ is a C^*-algebra.*

From $x^*x=xx^*$ and (1), we have

$$\|x^*\|=\|x\|. \tag{5}$$

From (1), we have

$$\|x\|=\|x^*x\|^{1/2}=\|(x^*x)^{2^n}\|^{2^{-(n+1)}}$$
$$=\|(x^{2^n})^*x^{2^n}\|^{2^{-(n+1)}}=\|x^{2^n}\|^{2^{-n}}.$$

Hence by (4)

$$\|xy\|=\lim\|(xy)^{2^n}\|^{2^{-n}} \tag{6}$$
$$\leqq\lim 16^{2^{-n}}\|x^{2^n}\|^{2^{-n}}\|y^{2^n}\|^{2^{-n}}=\|x\|\,\|y\|,$$

if $[x, y]=0$, $[x, x^*]=0$ and $[y, y^*]=0$.

Step (4). *The norm-closed sub-$*$-algebra $\mathfrak{A}(x)$ of $\mathfrak{A}$ generated by a selfadjoint $x\in\mathfrak{A}$ is a commutative C^*-algebra.*

Any maximal commutative sub-$*$-algebra of $\mathfrak{A}$ is a maximal commutative subalgebra (see [1], page 554), and hence is closed by (4). For any selfadjoint $x\in\mathfrak{A}$, there exists a maximal commutative sub-$*$-algebra of $\mathfrak{A}$ containing x by the axiom of choice. This is closed and is therefore a C^*-algebra by Step (3). Hence $\mathfrak{A}(x)$ is a C^*-algebra.

Step (5). *Call a normal $u\in\mathfrak{A}$ a "u-element" if*

$$u^*+u+u^*u=0. \tag{7}$$

Any $x\in\mathfrak{A}$ can be written as

(8) $$x=\Sigma\mu_j u_j,\quad \Sigma\mu_j=0$$

where u_j is a u-element and μ_j is a complex number.

Any $x\in\mathfrak{A}$ can be written as $x=x'+ix''$ where $x'=(x+x^*)/2$ and $x''=(x-x^*)/2i$ are selfadjoint.

The function $g(t)=(1-t^2)^{1/2}-1$ is continuous on $[-1, 1]$ and $g(0)=0$. Hence for any selfadjoint $y\in\mathfrak{A}$, $||y||\leqq 1$, there exists $g(y)\in\mathfrak{A}(y)$ such that

$$g(y)^2+y^2+2g(y)=0,$$

because $\mathfrak{A}(y)$ is a C^*-algebra. Then

$$u_\pm=g(x/||x||)\pm ix/||x||$$

are u-elements for any selfadjoint $x\in\mathfrak{A}$ and

(9) $$x=\mu_+u_++\mu_-u_-,\quad \mu_\pm=\mp i||x||/2.$$

Remark. u is a u-element of $\mathfrak{A}$ if and only if $e+u$ is unitary when an identity e is adjoined to $\mathfrak{A}$ algebraically.

Step (6). *Define*

(10) $$||x||'=\inf\{\Sigma|\mu_j|\}$$

where the infimum is taken over all decompositions (8). *Then*

(11) $$||xy||'\leqq||x||'\ ||y||',$$

(12) $$||x||\leqq 2||x||',$$

(13) $$||x||'\leqq||x|| \quad \text{if} \quad x^*=x.$$

If $x=\Sigma\mu_ju_j$, $y=\Sigma\nu_kv_k$, with $\Sigma\mu_j=\Sigma\nu_k=0$ and u_j and v_k u-elements of $\mathfrak{A}$, then

$$xy=\Sigma\mu_j\nu_k(u_j+v_k+u_jv_k),$$

$$(u_j+v_k+u_jv_k)^*(u_j+v_k+u_jv_k)=-(u_j+v_k+u_jv_k+u_j^*+v_k^*+v_k^*u_j^*)$$
$$=(u_j+v_k+u_jv_k)(u_j+v_k+u_jv_k)^*,$$

$$\Sigma \mu_j \nu_k = 0.$$

Hence we have (11).

If u is a u-element, we have $||u|| = ||u^*||$ by (5). By (1) and (7), we have

$$||u||^2 = ||u^* u|| = ||u + u^*|| \leqq 2||u||.$$

Hence $||u|| \leqq 2$ and

$$||x|| \leqq \Sigma |\mu_j| \, ||u_j|| \leqq 2\Sigma |\mu_j|$$

which implies (12).

(13) follows from (9) where $|\mu_+| + |\mu_-| = ||x||$.

Step (7). *For* x, $y \in \mathfrak{A}$,

$$||x\,y|| \leqq ||x|| \, ||y^*||. \tag{14}$$

Let

$$x_n = x_{n-1}^* x_{n-1}, \quad x_1 = x\,y. \tag{15}$$

Then

$$x_n = y^* \{(x^* x)(y y^*)\}^{2^{n-2}-1} (x^* x)\, y \qquad (n > 1), \tag{16}$$

$$||x_n||^{2^{-(n-2)}} = ||x_1^* x_1|| = ||x\,y||^2, \tag{17}$$

where (17) is due to (1).

By (16), (12) and (11), we have

$$\begin{aligned} ||x_n|| &\leqq 2||x_n||' \\ &\leqq 2||y^*||' \, ||y||' (||x^* x||')^{2^{n-2}} (||y y^*||')^{2^{n-2}-1} \end{aligned}$$

Substituting into (17) and taking the limit as $n \to \infty$, we have

$$||x\,y|| \leqq (||x^* x||')^{1/2} (||y y^*||')^{1/2}.$$

By (13) and (1), we obtain (14).

Step (8). *For* $x\in\mathfrak{A}$,

$$\|x^*\|=\|x\|. \tag{18}$$

By (1) and (14), where we substitute x^* for x and x for y,

$$\|x\|=\|x^*x\|^{1/2}\leqq(\|x^*\|^2)^{1/2}=\|x^*\|.$$

Substituting x^* for x, we have $\|x^*\|\leqq\|x\|$. Thus we have (18).

(14) and (18) prove that the given norm is an algebra norm; hence $\mathfrak{A}$ is a C^*-algebra. Q.E.D.

We can also prove the following:

Theorem 2. *Let* $\mathfrak{A}$ *be a* $*$*-algebra with a complete linear space norm such that for all* $x\in\mathfrak{A}$

$$\|x^*x\|=\|x^*\|\,\|x\|. \tag{19}$$

Suppose that $x\mapsto x^*$ *is norm-continuous. Then* $\mathfrak{A}$ *is a* C^**-algebra.*

Proof. (Part I). We prove that

$$\|x\|_c=\|x^*x\|^{1/2} \tag{20}$$

is a C^*-norm and satisfies

$$\|x\|\leqq 2\|x\|_c\leqq 4\|x\|. \tag{21}$$

Step (i). *For* $x,\ y\in\mathfrak{A}$,

$$\|xy\|\leqq(\|x\|^{1/2}\|y\|^{1/2}+\|x^*\|^{1/2}\|y^*\|^{1/2})^2. \tag{22}$$

Setting $c=(\|y\|\,\|y^*\|)^{1/2}$ and $d=(\|x\|\,\|x^*\|)^{1/2}$ in (3), we obtain (22) by (19).

Step (ii). *There exists* $k\geqq1$ *and* $l\geqq2$ *such that*

$$\|x^*\|\leqq k\|x\|, \tag{23}$$

$$\|x+x^*\|\leqq l\|x^*x\|^{1/2}. \tag{24}$$

By the assumption that $x \mapsto x^*$ is continuous, there exists $k \geqq 1$ satisfying (23). By substituting x^* for x, we also have $||x|| \leqq k||x^*||$. If we set $||x^*|| = a||x||$, we have $k \geqq a \geqq k^{-1}$. Hence

$$\begin{aligned} ||x + x^*|| &\leqq ||x|| + ||x^*|| = (a^{1/2} + a^{-1/2})||x x^*||^{1/2} \\ &\leqq l||x x^*||^{1/2} \end{aligned}$$

where $l = 2k^{1/2}$.

Step (iii). *For* $x, y \in \mathfrak{A}$,

$$||x y|| \leqq (1+k)^2 ||x|| \, ||y||. \tag{25}$$

This follows from (22) and (23).

Step (iv). *If* $x^* = x$, $y^* = y$ *and* $[x, y] = 0$, *then*

$$||x^n|| = ||x||^n, \tag{26}$$

$$||x y|| \leqq ||x|| \, ||y||. \tag{27}$$

Since $(x y)^* = x y$, we have

$$\begin{aligned} ||x y|| &= \lim ||x^{2^n} y^{2^n}||^{2^{-n}} \qquad \text{(by (19))} \\ &\leqq \lim (1+k)^{2^{1-n}} ||x^{2^n}||^{2^{-n}} ||y^{2^n}||^{2^{-n}} \qquad \text{(by (25))} \\ &= ||x|| \, ||y||. \end{aligned}$$

(27) implies $||x^n|| \leqq ||x||^n$. If $2^p > n$,

$$||x||^{2^p} = ||x^{2^p}|| \leqq ||x^n|| \, ||x^{2^p - n}|| \leqq ||x||^{2^p}.$$

Hence equality holds. Since $||x^{2^p - n}|| \leqq ||x||^{2^p - n}$ and $||x^n|| \leqq ||x||^n$, we have (26).

Step (v). *For normal* $x \in \mathfrak{A}$,

$$||x + x^*|| \leqq 2 \, ||x^* x||^{1/2}. \tag{28}$$

If $[x, x^*] = 0$, (19) implies

(29) $$\|(x^*)^{2^n}x^{2^n}\| = \|x^*x\|^{2^n}.$$

Let

$$a_n = \|x^{2^n} + (x^*)^{2^n}\|.$$

By (19), $\|y^2\| = \|y\|^2$ for $y^* = y$, hence

(30) $$a_n^2 \leqq a_{n+1} + 2\|x^*x\|^{2^n}$$

by (29). By (24) and (29),

(31) $$a_{n+1} \leqq l\|x^*x\|^{2^n}.$$

Define k_p recursively by

$$k_{p+1} = (2+k_p)^{1/2}, \ k_0 = l.$$

If

$$a_{m+1} \leqq k_p\|x^*x\|^{2^m}$$

holds, then by (30)

$$a_m \leqq k_{p+1}\|x^*x\|^{2^{m-1}}.$$

By induction starting from (31), we obtain

(32) $$a_0 = \|x + x^*\| \leqq k_{n+1}\|x^*x\|^{1/2}.$$

Since $k_0 \geqq 2$, we obtain $k_p \geqq 2$ recursively. We have

$$k_{p+1} - k_p = \{k_{p+1} + k_p\}^{-1}(2-k_p)(1+k_p) \leqq 0.$$

Hence $k_\infty = \lim k_p$ exists and satisfies

$$k_\infty^2 = 2 + k_\infty, \ k_\infty \geqq 2.$$

Hence $k_\infty = 2$. By taking the limit as $n \to \infty$ in (32), we obtain (28).

Step (vi). *Any norm-closed commutative sub-$*$-algebra $\mathfrak{B}$ of $\mathfrak{A}$ is a C^*-algebra relative to* (20).

Let $x \in \mathfrak{B}$, $y \in \mathfrak{B}$. Then by (28) and (27),

$$||x^*y+(x^*y)^*||\leqq 2||y^*x\ x^*y||^{1/2}=2||(x^*x)(y^*y)||^{1/2}$$
$$\leqq 2||x^*x||^{1/2}||y^*y||^{1/2}.$$

Hence

$$||(x+y)^*(x+y)||\leqq ||x^*x||+||y^*y||+||x^*y+(x^*y)^*||$$
$$\leqq (||x^*x||^{1/2}+||y^*y||^{1/2})^2;$$

i.e.

(33) $$||x+y||_c\leqq ||x||_c+||y||_c.$$

Evidently,

(34) $$||\lambda x||_c=|\lambda|\ ||x||_c,$$

(35) $$||x^*x||_c=||(x^*x)^2||^{1/2}=||x^*x||=||x||_c^2.$$

By (27), we have

$$||(xy)^*xy||=||x^*xy^*y||\leqq ||x^*x||\ ||y^*y||,$$

i.e.

(36) $$||xy||_c\leqq ||x||_c||y||_c.$$

By (28), we have

$$||x+x^*||\leqq 2||x||_c,$$
$$||x-x^*||=||(ix)+(ix)^*||\leqq 2||x||_c.$$

Hence

$$||x||\leqq 2||x||_c.$$

Since $||x^*||\ ||x||=||x||_c^2=||x^*||_c^2$,

$$||x||=||x^*||_c^2||x^*||^{-1}\geqq ||x^*||_c^2(2||x^*||_c)^{-1}=||x||_c/2.$$

This proves (21) (for $x\in\mathfrak{B}$).

(21) implies that $\mathfrak{B}$ is complete relative to $||\cdot||_c$. Hence by (33), (34),

(35) and (36), $\mathfrak{B}$ is a C^*-algebra relative to $\|\cdot\|_c$.

Step (vii). *For* $x=x^*\in\mathfrak{A}$, $\mathfrak{A}(x)$ *is a commutative* C^**-algebra relative to* $\|\cdot\|_c$.

The proof is the same as Step (4) where (25) is to be used in place of (4).

Step (viii). *Define* $\|x\|'$ *by* (10). *Then*

$$\|xy\|'\leqq\|x\|'\|y\|', \tag{37}$$

$$\|x\|\leqq 4\|x\|', \tag{38}$$

$$\|x\|'\leqq\|x\| \quad \textit{if } x^*=x. \tag{39}$$

Since u_j is normal, (28) and (7) imply

$$2\|u_j^*u_j\|^{1/2}\geqq\|u_j^*+u_j\|=\|u_j^*u_j\|.$$

By (21), $\|u_j\|\leqq 2\|u_j\|_c=2\|u_j^*\|_c\leqq 4\|u_j^*\|$. Hence

$$2\geqq\|u_j^*u_j\|^{1/2}=\|u_j^*\|^{1/2}\|u_j\|^{1/2}\geqq 2^{-1}\|u_j\|.$$

This implies (38). The rest is the same as Step (6).

Step (ix). *For* x, $y\in\mathfrak{A}$,

$$\|y^*x^*xy\|\leqq\|x^*x\|\,\|y^*y\|, \tag{40}$$

$$\|(x^*)^n x^n\|\leqq\|x^*x\|^n. \tag{41}$$

Let x_n be defined by (15). Then for $n>1$,

$$\|x_n\|^{2^{-(n-2)}}=\|x_1^*x_1\|,$$

$$\begin{aligned}\|x_n\|&\leqq 4\|x_n\|'\\ &\leqq 4\|y^*\|'\|y\|'(\|x^*x\|)^{2^{n-2}}(\|yy^*\|)^{2^{n-2}-1}\end{aligned}$$

where (38), (16), (37) and (39) are used. Hence we have (40).

Repeated use of (40) yields (41).

Step (x). $||x||_c$ *defined by* (20) *satisfies* (33), (34), (35), (36) *and* (21).

By using (41) in place of (29) in Step (v), we obtain

$$||x+x^*|| \leqq 2||x^*x||^{1/2} \tag{42}$$

for any $x \in \mathfrak{A}$. By (42) and (40), we obtain (33) and (36) in the same way as in Step (vi). (34) and (35) are immediate. The proof of (21) is the same as in Step (vi).

This completes Part I of the proof.

The following Lemmas which treat the case of finite-dimensional commutative $\mathfrak{A}$ are basic steps in Part II of the proof.

Lemma 1. *Let* $\mathfrak{A}$ *be a 2-dimensional complex vector space with a linear norm* $||x||$ *satisfying the following conditions*:

(a) *If* $x=(x_1, x_2)$ *with real* x_1 *and* x_2, *then*

$$||x|| = \sup\{|x_1|, |x_2|\}. \tag{43}$$

(b) *Let* $x^*=(\bar{x}_1, \bar{x}_2)$ *when* $x=(x_1, x_2)$. *Then*

$$||x^*||\,||x|| = \sup\{|x_1|^2, |x_2|^2\}. \tag{44}$$

Then (43) *holds for all* $x \in \mathfrak{A}$.

Proof. Let $\mathfrak{A}_1$ denote the unit ball of $\mathfrak{A}$ relative to the given norm and write

$$\mathfrak{A}(\theta) = \{(r_1, r_2); (r_1, r_2 e^{i\theta}) \in \mathfrak{A}_1\}.$$

Step (α). $\mathfrak{A}(\theta)$ *is convex, compact and contains a neighbouhood of the origin.*

Since $\mathfrak{A}_1$ is convex and compact, so is $\mathfrak{A}(\theta)$ for each θ. Since 0 is in the interior of $\mathfrak{A}_1$, 0 is in the interior of $\mathfrak{A}(\theta)$.

Step (β). *Each half line* $\{(r\cos\chi, r\sin\chi); r \in [0, \infty)\}$ *intersects* $\partial\mathfrak{A}(\theta)$ *at one and only one point* $r(\theta, \chi)$. *This is continuous in* (θ, χ). *Let*

$$\varphi(\theta, \chi) = r(\theta, \chi)/r(0, \chi).$$

Then for all integers n,

$$(45)\qquad \varphi(\theta, x)=\varphi(\theta, n\pi+x)=\varphi(n\pi+\theta, (-1)^n x);$$

$$(46)\qquad r(0, x)=\begin{cases}|\sec x| & \text{if } x\in[-\pi/4, \pi/4]\cup[3\pi/4, 5\pi/4],\\ |\operatorname{cosec} x| & \text{if } x\in[\pi/4, 3\pi/4]\cup[5\pi/4, 7\pi/4];\end{cases}$$

$$(47)\qquad \varphi(n\pi, x)=\varphi(\theta, n\pi/2)=1;$$

$$(48)\qquad \varphi(\theta, x)\,\varphi(-\theta, x)=1.$$

(45) follows from the definition of φ. (46) and (47) follow from the assumption (a). (48) follows from the assumption (b).

Step (γ). *Simultaneously for all* $x\in(0, \pi/2)$ *and* $\theta\in(0, \pi)$, *either* $\varphi(\theta, x)>1$ *or* $\varphi(\theta, x)=1$ *or* $\varphi(\theta, x)<1$.

Let $\varphi(\theta, x_0)=1$ for a fixed θ and a $x_0\in(0, \pi/4]$. By convexity, and $\varphi(\theta, 0)=1$, $\varphi(\theta, x)\leqq 1$ for $x\in[x_0, \pi/4]$ and $\varphi(\theta, x)\geqq 1$ for $x\in(0, x_0]$. Since $\varphi(-\theta, x_0)=1$ by (48), we also have $\varphi(-\theta, x)\leqq 1$ for $x\in[x_0, \pi/4]$ and $\varphi(-\theta, x)\geqq 1$ for $x\in(0, x_0]$. By (48), we have $\varphi(\theta, x)=1$ for $x\in(0,\pi/4]$. Similar argument starting from $\varphi(\theta, \pi/4)=1$ yields $\varphi(\theta, x)=1$ for $x\in[\pi/4, \pi/2)$. Similar argument holds when $x_0\in[\pi/4, \pi/2)$. Hence for each θ either $\varphi(\theta, x)>1$ or $\varphi(\theta, x)=1$ or $\varphi(\theta, x)<1$ simultaneously for all $x\in(0, \pi/2)$.

Let $\varphi(\theta_1, x)>1$ and $\varphi(\theta_2, x)\leqq 1$ for $\theta_1, \theta_2\in(0, \pi)$. Then $\varphi(-\theta_2, x)\geqq 1$ by (48). We may assume $x\in(0, \pi/4]$. With

$$x=(\varphi(\theta_1, x), \varphi(\theta_1, x)\tan x\, e^{i\theta_1})\in\mathfrak{A},$$

$$y=(\varphi(-\theta_2, x), \varphi(-\theta_2, x)\tan x\, e^{-i\theta_2})\in\mathfrak{A},$$

$$\alpha=\varphi(-\theta_2, x)\sin\theta_2\{\varphi(-\theta_2, x)\sin\theta_2+\varphi(\theta_1, x)\sin\theta_1\}^{-1},$$

$$\beta=\varphi(\theta_1, x)\sin\theta_1\{(\varphi(-\theta_2, x)\sin\theta_2+\varphi(\theta_1, x)\sin\theta_1\}^{-1},$$

we have $\|x\|=\|y\|=1$, $\alpha>0$, $\beta>0$ and $\alpha+\beta=1$. Hence $\|\alpha x+\beta y\|\leqq 1$. Since

$$\alpha x+\beta y=(\alpha\varphi(\theta_1, x)+\beta\varphi(-\theta_2, x), \xi)$$

for some real ξ, we have

$$\|\alpha x+\beta y\|\geqq\alpha\varphi(\theta_1, \chi)+\beta\varphi(-\theta_2, \chi)>1$$

which is a contradiction. Hence $\varphi(\theta, \chi)>1$ must hold simultaneously for all $\theta\in(0, \pi)$ and $\chi\in(0, \pi/2)$ if it holds at one such (θ, χ). Similar argument shows the same for $\varphi(\theta, \chi)<1$. If neither holds, then $\varphi(\theta, \chi)\equiv 1$.

Step (δ). *If* $\varphi(\theta, \chi)>1$ *for* $\theta\in(0, \pi)$ *and* $\chi\in(0, \pi/2)$, *then*

$$\varphi(\theta, \chi)\geqq 1+k\theta \tag{49}$$

where

$$k=(2\varphi(\theta_0, \chi)\sin\theta_0)^{-1}(\varphi(\theta_0, \chi)-1) \tag{50}$$

for any $\theta_0\in(0, \pi/4]$, $\chi\in(0, \pi/6]$ *and* $\theta\in(0, \theta_0]$.

For $\chi\in(0, \pi/4]$ and $\lambda\in[-1, 1]$ we have

$$\|(\varphi(\theta, \chi), \varphi(\theta, \chi)\tan\chi\, e^{i\theta})\|=1,$$

$$\|(1, \lambda)\|=1.$$

Hence for any $\alpha\geqq 0$, $\beta\geqq 0$, $\alpha+\beta=1$, we have

$$\|(\alpha\varphi(\theta, \chi)+\beta, \alpha\varphi(\theta, \chi)\tan\chi\, e^{i\theta}+\beta\lambda)\|\leqq 1. \tag{51}$$

Let

$$\lambda_1=\lambda_0\tan\chi,$$

$$\lambda_0=\beta^{-1}(\{(\alpha\varphi(\theta, \chi)+\beta)^2-\alpha^2\varphi(\theta, \chi)^2\sin^2\theta\}^{1/2}-\alpha\varphi(\theta, \chi)\cos\theta).$$

Then

$$\alpha\varphi(\theta, \chi)\tan\chi\, e^{i\theta}+\beta\lambda_1=(\alpha\varphi(\theta, \chi)+\beta)\tan\chi\, e^{i\theta'},$$

$$\tan\theta'=\alpha\varphi(\theta, \chi)\sin\theta(\alpha\varphi(\theta, \chi)\cos\theta+\beta\lambda_0)^{-1}.$$

For $\theta\in[0, \pi/4]$, we have $\lambda_0\in[1, \sqrt{2})$. Hence we have $\lambda_1\in[0, 1)$ if $\chi\in(0, \pi/6]$. Hence by (51) we must have

$$\alpha\varphi(\theta, \chi)+\beta=1+\alpha(\varphi(\theta, \chi)-1)\leqq\varphi(\theta', \chi).$$

As α varies from 0 to 1, θ' varies from 0 to θ. For fixed θ, we have

$$d\theta'/d\alpha=[\alpha^2\varphi(\theta, \varkappa)^2+\beta^2\lambda_0^2+2\alpha\beta\lambda_0\varphi(\theta, \varkappa)\cos\theta]^{-1}\lambda_0\varphi(\theta, \varkappa)\sin\theta$$

$$\leqq 2\varphi(\theta, \varkappa)\sin\theta,$$

using $\varphi(\theta, \varkappa)>1$, $\lambda_0\in[1, \sqrt{2})$, $\cos\theta>2^{-1/2}$, $\alpha+\beta=1$. Hence

$$\alpha\geqq(2\varphi(\theta, \varkappa)\sin\theta)^{-1}\theta'.$$

Denoting θ by θ_0 and θ' by θ, we obtain (49).

Step (ε). $\varphi(\theta_0, \varkappa)=1$ *in Step* (δ).

For $\varkappa\in(0, \pi/4]$ we have

$$\|(\varphi(\theta, \varkappa)e^{-i\theta}, \varphi(\theta, \varkappa)\tan\varkappa)\|=1,$$

$$\|(1, -1)\|=1.$$

Hence for any $\alpha>0$, $\beta>0$, $\alpha+\beta=1$, we have

$$\|(\alpha\varphi(\theta, \varkappa)e^{-i\theta}+\beta, \alpha\varphi(\theta, \varkappa)\tan\varkappa-\beta)\|\leqq 1.$$

Choose $\beta/\alpha=\varphi(\theta, \varkappa)\tan\varkappa$. Then we must have

$$|\alpha\varphi(\theta, \varkappa)e^{-i\theta}+\beta|\leqq 1.$$

On the other hand,

$$|\alpha\varphi(\theta, \varkappa)e^{-i\theta}+\beta|^2=(\alpha\varphi(\theta, \varkappa)+\beta)^2-2\alpha\beta\varphi(\theta, \varkappa)(1-\cos\theta)$$

$$\geqq(1+\alpha k\theta)^2-2\alpha\beta\varphi(\theta, \varkappa)(1-\cos\theta).$$

Since $\varphi(\theta, \varkappa)\to 1$ and $1-\cos\theta=O(\theta^2)$ as $\theta\to 0$, we have

$$(1+\alpha k\theta)^2-2\alpha\beta\varphi(\theta, \varkappa)(1-\cos\theta)>1$$

for sufficiently small θ if $k>0$. Since $k\geqq 0$, this implies $k=0$.

Steps (δ) and (ε) eliminate the possibility $\varphi(\theta, \varkappa)>1$ for $\theta\in(0, \pi)$ and $\varkappa\in(0, \pi/2)$. Similar argument eliminates the possibility $\varphi(\theta, \varkappa)>1$ for $\theta\in(-\pi, 0)$ and $\varkappa\in(0, \pi/2)$. Hence $\varphi(\theta, \varkappa)\equiv 1$; i.e. $\mathfrak{A}(\theta)=\mathfrak{A}(0)$. This proves Lemma 1. Q.E.D.

Lemma 2. *Let $\mathfrak{A}$ be an n-dimensional complex vector space with a linear norm $||x||$ satisfying the following conditions:*

(a) *If $x=(x_1,\dots,x_n)$ with real x_j, then*

$$||x||=\sup\{|x_j|\,;\ j=1,\dots,n\}. \tag{52}$$

(b) *Let $x^*=(\bar{x}_1,\dots,\bar{x}_n)$ when $x=(x_1,\dots,x_n)$. Then*

$$||x^*||\ ||x||=\sup\{|x_j|^2;\ j=1,\dots,n\}. \tag{53}$$

Then (52) *holds for all $x\in\mathfrak{A}$.*

Proof. With $\theta=(\theta_2,\dots,\theta_n)$ and $r=(r_2,\dots,r_n)$ write

$$\rho_\theta(r)=||(1,\ r_2e^{i\theta_2},\dots,r_ne^{i\theta_n})||. \tag{54}$$

Fix θ (with real θ_k) and let r_k vary from 0 to 1. Then the following properties hold.

(i) $\rho_\theta(r)=1$ if $r_k=0$ for all but one k (this follows from Lemma 1).

(ii) $\rho_\theta(r)\rho_{-\theta}(r)=1$ (this follows from condition (b)).

(iii) $\rho_\theta(r)$ is convex in r (this follows from the triangle inequality).

These properties have the following consequences.

(iv) Suppose that $\rho_\theta(r^a)=1$, $\rho_\theta(r^b)=1$. Then $\rho_\theta(\lambda r^a+(1-\lambda)r^b)=1$ for $0\leqq\lambda\leqq 1$. (By (ii) we have $\rho_{-\theta}(r^a)=1$, $\rho_{-\theta}(r^b)=1$, whence by (iii) $\rho_{-\theta}(\lambda r^a+(1-\lambda)r^b)\leqq 1$. Since by (iii) $\rho_\theta(\lambda r^a+(1-\lambda)r^b)\leqq 1$, by (ii) we have $\rho_\theta(\lambda r^a+(1-\lambda)r^b)=1$.)

(v) Suppose that $\rho_\theta(r^a)=1$, $\rho_\theta(\lambda r^a+(1-\lambda)r^b)=1$ with $0<\lambda<1$. Then $\rho_\theta(r^b)=1$. (By (iii), if $\rho_\theta(r^b)<1$ then $\rho_\theta(\lambda r^a+(1-\lambda)r^b)\leqq\lambda\rho_\theta(r^a)+(1-\lambda)\rho_\theta(r^b)<1$, whence $\rho_\theta(r^b)\geqq 1$. By (ii) we have also $\rho_{-\theta}(r^a)=1$, $\rho_{-\theta}(\lambda r^a+(1-\lambda)r^b)=1$, so that $\rho_{-\theta}(r^b)\geqq 1$. Hence by (ii) $\rho_\theta(r^b)=1$.)

We can now deduce $\rho_\theta(r)=1$ for all $r=(r_2,\dots,r_n)$ with $0\leqq r_k\leqq 1$. By (i) and (iv) $\rho_\theta(r)=1$ for r in the convex hull of the n points $(1,0,\dots,0),\dots,(0,\dots,0,1)$, i.e. in a neighbourhood of the origin. Hence by (v) $\rho_\theta(r)=1$ for all r with $0\leqq r_k\leqq 1$. This proves Lemma 2.

Q.E.D.

We are now ready to start Part II of the proof.

Proof (Part II). We prove that $||x||=||x||_c$.

Step (a). $\|x\| = \|x\|_c$ *for any commutative* $\mathfrak{A}$.

Let $y_1, \ldots, y_n \in \mathfrak{A}$ be such that $y_j \geqq 0$, $\|y_j\| = 1$ and $\|\Sigma y_i\| = 1$. Then

$$\|\Sigma \lambda_j y_j\| = \sup\{|\lambda_j|;\ j = 1, \ldots, n\} = \|\Sigma \lambda_j v_j\|_c. \tag{55}$$

This is seen as follows.

With the norm $\|\cdot\|_c$, $\mathfrak{A}$ is the C^*-algebra of all continuous functions vanishing at infinity on some locally compact Hausdorff space Ξ. Since $y_j(\xi) \geqq 0$ and $\sup_\xi \Sigma y_j(\xi) = 1$, we have

$$\sup_\xi |\Sigma \lambda_j y_j(\xi)| \leqq \sup_j |\lambda_j| \sup_\xi \Sigma y_j(\xi)$$
$$= \sup_j |\lambda_j|.$$

Since $\|y_j\| = 1$, there exists ξ_j such that $y_j(\xi_j) = 1$. Since $\sum_k y_k(\xi) \leqq 1$, we have $y_k(\xi_j) = 0$ for $k \neq j$. Hence

$$|\lambda_j| = |\Sigma \lambda_k y_k(\xi_j)| \leqq \sup_\xi |\Sigma \lambda_k y_k(\xi)|.$$

Thus

$$\|\Sigma \lambda_j y_j\|_c = \sup\{|\lambda_j|;\ j = 1, \ldots, n\}.$$

We now set $\|\Sigma \lambda_j y_j\| = \|\lambda\|$, $\lambda = (\lambda_1, \ldots, \lambda_n)$. If $\lambda^* = \lambda$, then $\|\lambda\| = \|\Sigma \lambda_j y_j\|_c = \sup_j |\lambda_j|$. We also have

$$\|\lambda^*\|\ \|\lambda\| = \|(\Sigma \lambda_j y_j)^*\|\ \|\Sigma \lambda_j y_j\| = \|\Sigma \lambda_j y_j\|_c^2 = \sup_j |\lambda_j|^2.$$

Therefore Lemma 2 is applicable and (55) holds for all λ.

The proof for a commutative $\mathfrak{A}$ is completed if we show that $\Sigma \lambda_j y_j$ as described above are dense in $\mathfrak{A}$.

Let $y \in \mathfrak{A}$ and $\varepsilon > 0$ be given. There exists a finite open minimal covering $\{\mathcal{O}_n\}_{n=0,1,\ldots,N}$ of Ξ such that the $\mathcal{O}_n$ for $n > 0$ are relatively compact in Ξ, and $|y(\xi) - \lambda_n| \leqq \varepsilon$ for $\xi \in \mathcal{O}_n$ for each n, where $\lambda_0 = 0$. For this open covering, there exists a partition of unity $y_n(\xi) \geqq 0$, $\Sigma y_n(\xi) = 1$ by continuous functions y_n vanishing outside of $\mathcal{O}_n$. We have $\|y - \sum_{n=1}^N \lambda_n y_n\|_c = \sup_\xi |\sum_{n=0}^N y_n(\xi)(y(\xi) - \lambda_n)| \leqq \varepsilon$. By minimality, $\|y_n\| = 1$. We also have $\|\Sigma y_n\| = 1$, $y_n \geqq 0$. This shows that $\Sigma \lambda_j y_j$ are dense in $\mathfrak{A}$.

Step (b). $||x||=||x||_c$ *if* $\mathfrak{A}$ *is separable.*

Let $\mathfrak{A}^{**}$ be the second dual of $\mathfrak{A}$ with respect to $||\cdot||$. Since the norm $||\cdot||_c$ on $\mathfrak{A}$ is equivalent to $||\cdot||$, it induces a norm $||\cdot||_c$ on $\mathfrak{A}^{**}$ equivalent to $||\cdot||$ on $\mathfrak{A}^{**}$. Since $||\cdot||_c$ is a C^*-algebra norm on $\mathfrak{A}$ the $*$-algebra structure of $\mathfrak{A}$ can be extended uniquely to $\mathfrak{A}^{**}$ so that $\mathfrak{A}^{**}$ is a W^*-algebra with respect to $||\cdot||_c$.

Suppose that $h\in\mathfrak{A}$, $h\geqq 0$. Since $\mathfrak{A}$ is assumed to be separable there is a countable approximate unit (e_n) for $\mathfrak{A}$ consisting of positive elements of norm $\leqq 1$. Set $h+\varepsilon\Sigma 2^{-n}e_n=h_\varepsilon$, $\varepsilon>0$. Then h_ε is strictly positive (i.e. for $0\leqq f\in\mathfrak{A}^*$, $f(h_\varepsilon)=0$ implies $f=0$; see [1]), and $||h_\varepsilon-h||\to 0$ as $\varepsilon\to 0$.

Denote by $\mathfrak{A}(h_\varepsilon)$ the sub-C^*-algebra of $\mathfrak{A}$ generated by h_ε. Then the second dual of $\mathfrak{A}(h_\varepsilon)$ with respect to $||\cdot||$ (resp. $||\cdot||_c$) can be identified with the $\sigma(\mathfrak{A}^{**}, \mathfrak{A}^*)$-closure $\mathfrak{A}(h_\varepsilon)^-$ of $\mathfrak{A}(h_\varepsilon)$ in $\mathfrak{A}^{**}$, normed by the restriction of $||\cdot||$ (resp. $||\cdot||_c$) from $\mathfrak{A}^{**}$. Since by Step (a) $||x||=||x||_c$ for $x\in\mathfrak{A}(h_\varepsilon)$, we have $||x||=||x||_c$ for $x\in\mathfrak{A}(h_\varepsilon)^-$. Moreover, because h_ε is strictly positive the unit of $\mathfrak{A}^{**}$ is contained in $\mathfrak{A}(h_\varepsilon)^-$. Therefore

$$||e^{ih_\varepsilon}||=||e^{ih_\varepsilon}||_c=1.$$

By continuity, we have

$$||e^{ih}||=1.$$

Suppose that $0\leqq h\in\mathfrak{A}^{**}$. Since $\mathfrak{A}^-=\mathfrak{A}^{**}$, h is the strong limit of a net (h_α) in $\mathfrak{A}$ with $0\leqq h_\alpha\leqq||h||$ (there exists a net (x_α) in $\mathfrak{A}$ with $x_\alpha=x_\alpha^*$, $||x_\alpha||\leqq||h^{1/2}||$ and $x_\alpha\to h^{1/2}$ strongly; set $h_\alpha=x_\alpha^2$). Then e^{ih} is the strong limit of e^{ih_α}, therefore the $\sigma(\mathfrak{A}^{**}, \mathfrak{A}^*)$-limit. Since $||e^{ih_\alpha}||=1$ we have

$$||e^{ih}||\leqq 1.$$

Any unitary $u\in\mathfrak{A}^{**}$ can be written as $u=e^{ih}$ with $0\leqq h\in\mathfrak{A}^{**}$. (If $u=\int_0^{2\pi}e^{i\theta}\mathrm{d}E_\theta$, take $h=\int_0^{2\pi}\theta\mathrm{d}E_\theta$.) Therefore for u unitary in $\mathfrak{A}^{**}$,

$$||u||\leqq 1.$$

By [6], for arbitrary $x\in\mathfrak{A}^{**}$,

$$||x||_c=\inf\Sigma|\lambda_j|,$$

where the infimium is taken over all decompositions

$$x=\Sigma\lambda_j u_j,\ u_j \text{ unitary in } \mathfrak{A}^{**}.$$

Since $\|u_j\|\leqq 1$ we have

$$\|x\|\leqq\Sigma|\lambda_j|$$

for any such decomposition. Hence $\|x\|\leqq\|x\|_c$. Substituting x^* for x we also have $\|x^*\|\leqq\|x\|_c$. For $x\in\mathfrak{A}$ we have $\|x\|\,\|x^*\|=\|x\|_c^2$; hence $\|x\|=\|x\|_c$.

Step (c). $\|x\|=\|x\|_c$ *for a general* $\mathfrak{A}$.

For given $x\in\mathfrak{A}$, consider the sub-C^*-algebra $\mathfrak{A}(x, x^*)$ of $\mathfrak{A}$ generated by x and x^*. It is separable and therefore $\|x\|=\|x\|_c$ by Step (b).

Q.E.D.

Problem 1. In Theorem 1 (Theorem 2) is it enough to assume $\|x^*x\|=\|x\|^2$ ($\|x^*x\|=\|x^*\|\,\|x\|$) only for normal x? (Cf. [2], [3], [4].)

Problem 2. In Theorem 2, is it necessary to assume the continuity of $x\mapsto x^*$? (If multiplication is continuous then continuity of $x\mapsto x^*$ can be deduced. Indeed, by the spectral radius theorem (which is now available) and (26), $\|x\|=\rho(x)$ whenever x is a limit of selfadjoint elements, so that if $x_n\rightarrow y$ with $x_n=x_n^*$ and $y=-y^*$ then

$$\|x_n+y\|=\rho(x_n+y)=\rho((x_n+y)^*)=\rho(x_n-y)=\|x_n-y\|;$$

$$y=0.$$

This shows that the set of selfadjoint elements is closed, whence by the closed graph theorem $x\mapsto x^*$ is continuous.)

Acknowledgement

In an informal conversation at the Conference in Ergodic Theory, June 19–23, 1972, Professor Doran of Texas Christian University has asked the question: whether the condition $\|xy\|\leqq\|x\|\,\|y\|$ is necessary in the axioms of a C^*-algebra. The first named author is greatly indebted to

Professor Doran for inviting him to the Conference in Ergodic Theory and for drawing his attention to this question.

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T. W. Palmer has communicated to the authors the following proof of Theorem 1 and Part I in the proof of Theorem 2:

For Theorem 1, if $uu^{**}=u^*u=u+u^*$, then

$$\|u\|^2=\|u^*u\|=\|u+u^*\|\leqq\|u\|+\|u^*\|,$$

$$\|u^*\|^2\leqq\|u\|+\|u^*\|,$$

whence $\|u\|\leqq 2$. For Theorem 2, if $uu^*=u+u^*$, then

$$\|u\|\|u^*\|=\|uu^*\|=\|u+u^*\|\leqq\|u\|+\|u^*\|,$$

$$k^{-1}\|u\|\leqq\|u^*\|\leqq k\|u\|,$$

whence $\|u\|\leqq k(1+k)$. These computations establish the boundedness in norm of the set of u-elements of $\mathfrak{A}$. As soon as the continuity of multiplication is proved, we see that the norm

$$|x|=\sup\{\|\lambda x+xy\|;\ |\lambda|+\|y\|=1,\ \lambda\in C,\ y\in\mathfrak{A}\}$$

is equivalent to the given norm and is an algebra norm. By Corollary 12 of [7], $\mathfrak{A}$ is a C^*-algebra relative to the norm $\|\cdot\|'$ and we obtain

$$\|x^*x\|=\lim_n\|(x^*x)^{2^n}\|^{2^{-n}}=\lim_n(\|(x^*x)^{2^n}\|')^{2^{-n}}=(\|x\|')^2.$$

Alternatively, we can use the continuity of the involution (in Theorem 1, (1) and (4) imply $\|x^*\|\leqq 16\|x\|$) as well as the continuity of the multiplication. A selfadjoint square root y of $1-x^2$ can be obtained as a power series when $x=x^*$ and $\|x\|<1$. Writing $x=u_+-u_-$, $u_\pm=(y\pm ix)/(2i)$, we have $\|x\|'\leqq\|x\|$ for $x=x^*$. Then the proof of Theorem 1 can be completed by Steps 7 and 8, and Part I of the proof of Theorem 2 can be completed by Steps (ix) and (x). This alternative procedure does not use the theorem of Berkson and Glickfeld.

Z. Sebestyén has communicated to the authors a negative answer to Problem 1, and also a slightly different way of shortening the proofs of Theorem 1 and Part I of the proof of Theorem 2.

The latter involves using the result of [8] and [9] that a Banach algebra with an involution such that $\|x^*x\|\geqq C\|x^*\|\,\|x\|$ for some $C>0$

is isomorphic (as an involutive algebra) to a C^*-algebra.

The counterexample to Problem 1 is the involutive algebra of all bounded operators on a Hilbert space of dimension $\geqq 2$, with the norm $\|\cdot\|_1$ defined by

$$\|x\|_1=\frac{1}{2}\left(\|x\|+\left\|\frac{1}{2}(x^*x+xx^*)\right\|^{1/2}\right)$$

where $\|\cdot\|$ is the usual operator norm. The subadditivity of $\|\cdot\|_1$ follows from the inequality

$$\|xy^*+y^*x+x^*y+yx^*\|^2\leqq 4\|x^*x+xx^*\|\|y^*y+yy^*\|,$$

which in turn follows from the discriminant condition for

$$(\lambda x+y)(\lambda x+y)^*+(\lambda x+y)^*(\lambda x+y)\geqq 0,\ \lambda\in R.$$

For every normal x

$$\|x^*x\|_1=\|x\|_1^2=\|x^*\|_1\|x\|_1.$$

If x_0 is the operator defined by

$$x_0\xi_1=\xi_2,\ x_0\xi_2=\frac{1}{2}\xi_1,\ x_0\xi_n=0 \qquad \text{for} \qquad n>2$$

where $\xi_1, \xi_2, \ldots$ is an orthonormal basis, then

$$\|x_0\|_1=\frac{1}{2}\left(1+\sqrt{\frac{5}{8}}\right)<1=\|x_0\|.$$

The norm $\|\cdot\|_1$ does satisfy

$$\|x^*x\|_1=\|x^*x\|=\|x^*\|\,\|x\|\geqq\|x^*\|_1\|x\|_1.$$