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Transformations of Germs of Differentiable Functions through Changes of Local Coordinates

By

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§1. Introduction

We generalize the results in N. Levinson [3], [4] and J. Cl. Tougeron [8] which show that some germs of differentiable (or analytic) functions are transformed through changes of coordinates into polynomials in one (or two) variable with coefficients which are germs in the other variables (§3).

H. Whitney [9] has shown that $xy(y-x)(y-(3+t)x)(y-\gamma(t)x)$ (where γ is a transcendental function and $\gamma(0)=4$) cannot be transformed into any polynomial through analytic changes of coordinates (locally at the origin), and we prove this function cannot be transformed even through differentiable changes of coordinates (§4).

In view of Thom's Principle, that is, for a germ of an analytic function f having 0 as a topologically isolated singularity, the variety $f^{-1}(0)$ determines the function f, we study particularly whether a germ $f\phi$ (such that $\phi(0)>0$) can be transformed into f (§5). As a corollary of the sequence we obtain a sufficient condition for a germ of a differentiable function in two variables to be transformed into a germ of an analytic function (§6).

In Appendix we show canonical forms of germs which have un point de naissance or un point critique du type queue d'aronde which is due to Cerf [2].

The method of proofs is to use almost all theorems in Malgrange [6].

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§2. Definitions

According to Malgrange [6] we denote respectively by $\mathcal{O}_n$ (or $\mathcal{O}(x)$),

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$\mathscr{E}_n$ (or $\mathscr{E}(x)$) the rings of germs at 0 in $\mathbf{R}^n$ of real analytic and C^∞-functions, and by $\mathscr{F}_n$ (or $\mathscr{F}(x)$) the ring of formal power series in n indeterminates over $\mathbf{R}$. One has a mapping T: $\mathscr{E}_n \to \mathscr{F}_n$ (Taylor expansion at 0). We regard $\mathscr{O}_n \subset \mathscr{F}_n$. We denote by $\mathfrak{m}(\mathscr{O}_n)$ (resp. $\mathfrak{m}(\mathscr{E}_n)$) the maximal ideal of $\mathscr{O}_n$ (resp. $\mathscr{E}_n$). As $\mathscr{O}_n$ and $\mathscr{F}_n$ are unique factorization rings, for any $f \in \mathscr{O}_n$ (resp. $\in \mathscr{E}_n$) f (resp. Tf) can be factorized. So $\mathscr{O}_n^\circ$ (resp. $\mathscr{E}_n^\circ$) denotes the set of all germs f in $\mathscr{O}_n$ (resp. $\mathscr{E}_n$) such that f (resp. Tf) is not 0, is in $\mathfrak{m}(\mathscr{O}_n)$ (resp. $\mathfrak{m}(\mathscr{E}_n)$) and has no multiple factors. Adif_n (resp. Dif_n) denotes the set of analytic local diffeomorphisms (resp. local diffeomorphisms of C^∞-class) around 0 in $\mathbf{R}^n$. We have

(1) for any $\tau \in \mathrm{Adif}_n$(resp. Dif_n) and $f \in \mathscr{O}_n^\circ$ (resp. $\mathscr{E}_n^\circ$) $f \circ \tau \in \mathscr{O}_n^\circ$ (resp. $\mathscr{E}_n^\circ$),

(2) $\mathscr{O}_n \cap \mathscr{E}_n^\circ = \mathscr{O}_n^\circ$ (Zariski-Nagata).

For any analytic set F in Ω (where $\Omega \ni a$, Ω open in $\mathbf{R}^n$), the germ of F at a is called an analytic germ at a, and denoted by F_a. To an analytic germ F, we make it correspond the ideal $I(F) \subset \mathscr{O}_n$ of germs of analytic functions which are zero on F.

We say $f(\in \mathscr{E}_n)$ is *flat* at 0 if $Tf=0$.

§3. Generalizations of [4], [8]

The next lemma is similar to the lemma at p. 33 in [7], and the method of the proof is the same.

Lemma 1. *Let f, g be in $\mathscr{E}_n$ (resp. $\mathscr{O}_n$) and $a_i(x, t)$ $(i=l+1,\dots, n)$ germs at $0 \times [0, 1]$ in $\mathbf{R}^n \times \mathbf{R}$ of C^∞-functions (resp. analytic functions). Assume that*

$$\text{as germs at } 0\times[0,1]\ f(x)-g(x)=\sum_{i=l+1}^{n} a_i(x,t)\left(\frac{\partial f}{\partial x_i}t+\frac{\partial g}{\partial x_i}(1-t)\right),$$

$$a_i(0, t)=0.$$

Then there exists $\tau \in \mathrm{Dif}_n$ (resp. Adif_n) such that

$$f\circ\tau = g,$$

$$\tau(x)=(\tau_1(x),\dots,\tau_n(x))=(x_1,\dots,x_l,\tau_{l+1}(x),\dots,\tau_n(x)).$$

Proof. Let X be a complete vector field on $\mathbf{R}^{n+1}$ such that the germ of X at $0\times[0,1]$ is $\frac{\partial}{\partial t}-\sum_{i=l+1}^{n} d_i\frac{\partial}{\partial x_i}$, and φ_t 1 parameter group of transformation defined by X. Then we have

$$X_{(x,t')}F=\left.\frac{dF(\varphi_t(x,t'))}{dt}\right|_{t=0}.$$

The assumptions give

$$X_{(x,t)}F=0,\quad \text{near}\quad 0\times[0,1]\quad \text{if}\quad F=ft+g(1-t)$$

$$\left.\begin{aligned}\varphi_t(0,t')&=(0,t+t'),\\ \varphi_t(x,t')&=(y,t+t').\end{aligned}\right]\quad \text{if } 0\leqq t+t'\leqq 1 \text{ and } x \text{ is near } 0$$

These show that $(ft+g(1-t))\circ\varphi_t(x,0)$ is a constant for any fixed x near 0, and the $\varphi_1(x,0)$ is a local diffeomorphism around $x=0$. This gives the result.

As a corollary of Lemma 1, we obtain the next lemma which is due to [8].

Lemma 2. *Let f, g be in $\mathfrak{m}^2(\mathscr{E}_n)$ (resp. $\mathfrak{m}^2(\mathscr{O}_n)$ such that $f-g$ is an element of the ideal generated by* $\frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_j}x_k\begin{pmatrix} i,j=l+1,\dots,n\\ k=1,\dots,n\end{pmatrix}$. *Then there exists $\tau\in\mathrm{Dif}_n$ (resp. Adif_n) such that*

$$f\circ\tau=g,$$

$$\tau(x)=(\tau_1(x),\dots,\tau_n(x))=(x_1,\dots,x_l,\tau_{l+1}(x),\dots,\tau_n(x)).$$

Proof. By the hypothesis, there exist $b_{i,j}\in\mathfrak{m}(\mathscr{E}_n)$ (resp. $\mathfrak{m}(\mathscr{O}_n)$) such that $f-g=\sum_{i,j=l+1}^{n} b_{i,j}\frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_j}$. We have

$$\frac{\partial(f-g)}{\partial x_i}=\sum_{j=l+1}^{n} c_{i,j}\frac{\partial g}{\partial x_j},\ \text{where } c_{i,j} \text{ is in } \mathfrak{m}(\mathscr{E}_n)\ (\text{resp. } \mathfrak{m}(\mathscr{O}_n)).$$

Set $\frac{\partial(f-g)}{\partial x_i}=\sum_{j=1+l}^{n} c_{i,j}\frac{\partial g}{\partial x_j}$ and $f-g=\sum_{i=l+1}^{n} d_i\frac{\partial g}{\partial x_i}$ where d_i is in $\mathfrak{m}(\mathscr{E}_n)$

(resp. $\mathfrak{m}(\mathcal{O}_n)$) in the equation in Lemma 1. Then it is enough to solve the equations

$$d_i = \sum_{j=l+1}^{n} a_j c_{i,j} t + a_i,\ l+1 \leqq i \leqq n,$$

$$a_i(0, t) = 0.$$

Let A be the matrix whose (i, j)-component is $c_{i,j}t$ and I the unit matrix. Then the equations above are

$$(d_i) = (A+I)(a_i) \text{ and } a_i(0, t) = 0.$$

We easily see that $A+I$ has the inverse B where components of B are germs at $0 \times [0, 1]$. This prove the lemma.

we use frequently the next lemma which is also shown in [8].

Lemma 3. *Let p be in $\mathscr{F}_n$ and $p(0)=0$. Then there exist an integer $k>0$ and $q_1, \ldots, q_n \in \mathscr{F}_n$ such that* $p^k = \sum_{i=1}^{n} q_i \frac{\partial p}{\partial x_i}$.

The next lemma is the result about differentiable functions, see [3] for analytic functions.

Lemma 4. *Let $g \in \mathscr{E}_n$ satisfy the following condition,*

$$\frac{\partial^m g}{\partial x_n^m}(0) \neq 0 \qquad \text{for some } m>0.$$

Then there are f in $\mathscr{E}_{n-1}[x_n]$ (polynomials in x_n with coefficients in $\mathscr{E}(x_1, \ldots, x_{n-1})$) and τ in Dif_n *such that*

$$f \circ \tau = g,$$

$$\tau(x) = (\tau_1(x), \ldots, \tau_n(x)) = (x_1, \ldots, x_{n-1}, \tau_n(x)).$$

Proof. The case $\frac{\partial g}{\partial x_n}(0) \neq 0$, this is trivial. Therefore, suppose $\frac{\partial g}{\partial x_n}(0) = 0$. Let

$$G=\left(\frac{\partial g}{\partial x_n}\right)^2 x_n,$$

then for some $m'>0$

$$\frac{\partial^{m'} G}{\partial x_n^{m'}}(0)\neq 0.$$

Applying Malgrange's preparation theorem [6] to G and g, we get

$$Q\in\mathscr{E}_n \ \ R_i\in\mathscr{E}(x_1,\dots,x_{n-1}) \qquad (i=0,\dots,p)$$

such that

$$g=QG+\sum_{i=0}^{p} R_i x_n^i.$$

Take $f=\sum_{i=0}^{p} R_i x_n^i$, then

$$g-f=QG.$$

If we apply Lemma 2 to f, g, then there exists $\tau=(x_1,\dots,x_{n-1},\tau_n(x))\in \mathrm{Dif}_n$ such that $f\circ\tau=g$.

Theorem 1. (1) *Let G be in $\mathscr{E}_{n-1}[x_n]$ (resp. $\mathscr{O}_{n-1}[x_n]$). Suppose the discriminant of G in x_n is not flat at* 0 *(resp. not* 0*). Then there exist F in $\mathscr{E}_{n-2}[x_{n-1}, x_n]$ (polynomials in x_{n-1}, x_n with coefficients in $\mathscr{E}(x_1, \dots, x_{n-2})$) (resp. $\mathscr{O}_{n-2}[x_{n-1}, x_n]$) and τ in* Dif_n *(resp.* Adif_n*) such that*

$$F\circ\tau=G.$$

(2) *The statement in* (1) *remains valid for any $G\in\mathscr{E}_n$ (resp. $\mathscr{O}_n$) which satisfies the following conditions;*

$G=G_1\cdot G_2$ *for some* $G_1\in\mathscr{E}(x_1,\dots,x_{n-1})$ *(resp.* $\mathscr{O}(x_1,\dots,x_{n-1})$*)*

$G_2\in\mathscr{E}_n^\circ$*(resp.* $\mathscr{O}_n^\circ$*)*, such that

(i) *for some* $m>0$ $\quad \dfrac{\partial^m G_2}{\partial x_n^m}(0)\neq 0,$

(ii) $TG_1\neq 0$.

Proof. When $n=2$ and $G\in\mathscr{O}_n$ Part (2) of this theorem is proved in

[4]. And when $G_1=1$ in Part (2) this theorem is shown in [8]. It is enough to prove the case $G\in\mathfrak{m}^2(\mathscr{E})$ (resp. $\mathfrak{m}^2(\mathscr{O})$). If we apply Lemma 4 to G_2 in (2). Then there exist $f\in\mathscr{E}_{n-1}[x_n]$ and $\tau_1\in\mathrm{Dif}_n$ such that

$$f\circ\tau_1=G_2,$$

$$\tau_1=(x_1,\dots,x_{n-1},\tau_n(x)),$$

and we get

$$(G_1\cdot f)\circ\tau_1=(G_1\circ\tau_1)\cdot(f\circ\tau_1)=G_1\cdot G_2=G.$$

Here $G_1\cdot f$ satisfies the assumption of (1). So it is sufficient to prove (1). Let S be the ring of quotients of $\mathscr{F}_{n-1}[x_n]$ with respect to the ideal of those elements which vanish at the origin. Then by Proposition III 4. 10 in [6] $\mathscr{F}_n$ is faithfully flat over S. Hence from Lemma 3 there exist an integer $k>0$ and $p_1, q_1,\dots, q_n\in\mathscr{F}_{n-1}[x_n]$ such that $pTG^k=\sum_{i=1}^n q_iT\frac{\partial G}{\partial x_i}$ and $p(0)\neq 0$. Because of this, one sees that $T\left(\frac{\partial G}{\partial x_i}\right)^2$ $(i=1,\dots,n)$ have no common divisor p (where p is in $\mathscr{F}_{n-1}[x_n]$ and not in $\mathscr{F}(x_1,\dots,x_{n-1})$). Let K be the quotient field over $\mathscr{F}(x_1,\dots,x_{n-1})$, and $K[x_n]$ the polynomial ring over K. Then by the fact that $K[x_n]$ is a principal ideal ring, it is shown that the ideal generated by $T\left(\frac{\partial G}{\partial x_i}\right)^2$ $i=1,\dots,n$ in $K[x_n]$ is $K[x_n]$. Hence there exist $\phi_i\in\mathscr{F}_{n-1}[x_n]$ such that

$$\psi=\sum_{i=1}^n \phi_i\cdot T\left(\frac{\partial G}{\partial x_i}\right)^2 \text{ is not 0 and in } \mathscr{F}(x_1,\dots,x_{n-1}).$$

By the theorem of E. Borel, there exist $\Phi_i\in\mathscr{E}_{n-1}[x_n]$ $i=1,\dots,n$ such that $T\Phi_i=\phi_i$. And if we put

$$\Psi=\sum_{i=1}^n \Phi_i\left(\frac{\partial G}{\partial x_i}\right)^2,$$

then Ψ is in $\mathscr{E}(x_1,\dots,x_{n-1})$ and not flat at 0. It is easily seen that there exist $\tau_2\in\mathrm{Dif}(x_1,\dots,x_{n-1})$ and $\Phi'_{i,j}\in\mathscr{E}_{n-1}[x_n]$ such that

$$\text{for some } m>0 \quad \frac{\partial^m\Psi\circ\tau_2}{\partial x_{n-1}^m}(0)\neq 0,$$

$$\Psi\circ\tau_2=\sum_{i,j=1}^{n}\Phi'_{i,j}\frac{\partial G\circ\tau_2}{\partial x_i}\frac{\partial G\circ\tau_2}{\partial x_j}.$$

Put $G\circ\tau_2=\sum_{i=0}^{l} f_i x_n^i$, $(f_i\in\mathscr{E}(x_1,\ldots,x_{n-1}))$ and apply Malgrange's preparation theorem to $(\Psi\circ\tau_2)\cdot x_{n-1}$ and f_i then there exist $Q_i\in\mathscr{E}(x_1,\ldots,x_{n-1})$, $P_{i,j}\in\mathscr{E}(x_1,\ldots,x_{n-2})$ $(i=0,\ldots,l\ j=0,\ldots,k)$ such that

$$f_i=(\Psi\circ\tau_2)\cdot x_{n-1}\cdot Q_i+\sum_{j=0}^{k}P_{i,j}x_{n-1}^j \qquad (i=0,\ldots,l).$$

If we put

$$F=\sum_{i,j}P_{i,j}x_{n-1}^j x_n^i,$$

then $G\circ\tau_2-F$ is in the ideal generated by $\frac{\partial G\circ\tau_2}{\partial x_i}\frac{\partial G\circ\tau_2}{\partial x_j}x_{n-1}$, $i, j=1,\ldots,n$, and we can apply Lemma 2 to $G\circ\tau_2$, F. From this there exists $\tau\in\mathrm{Dif}_n$ such that $F\circ\tau=G$.

In the analytic case it is enough for the proof that we remind the fact that

(1) $\mathscr{F}_n$ is faithfully flat over $\mathcal{O}_n$

and (2) Weierstrass' preparation theorem.

Remark 1. Let f be in $\mathscr{E}_n^\circ$ and let g be in $\mathscr{E}_n$ and be flat at 0. Then $f+g$ is in $\mathscr{E}_n^\circ$ and so $f+g$ can be transformed into a polynomial in two variables. On the contrary, $G+g$ such that G takes the one form in Theorem 1 and g flat at 0 cannot be necessarily transformed into a polynomial in two variables. For example $f=x^2y\pm e^{-1/y^2}$ cannot be transformed not only into any polynomial but also into any analytic function (locally at 0). The reason is as follows. If f can be transformed into some element of $\mathcal{O}_2$, then $\left|\frac{\partial f}{\partial x}\right|+\left|\frac{\partial f}{\partial y}\right|$ satisfies the inequality of Łojasiewicz (Theorem IV 4.1 in [6]) on some neighborhood of 0 in $\mathbf{R}^2$. However it is easily seen that $\left|\frac{\partial f}{\partial x}\right|+\left|\frac{\partial f}{\partial y}\right|$ does not satisfy the inequality on any neighborhood of 0 in $\mathbf{R}^2$. It is a contradiction.

If $f(\in\mathfrak{m}(\mathscr{E}_n))$ can be transformed into an element of $\mathcal{O}_n$ (or a polynomial), then $f=\prod_{i=1}^{k}f_i^i$ for some elements $f_i\in\mathscr{E}_n$ where $\prod_{i=1}^{k}f_i\in\mathscr{E}_n^\circ$.

The converse of this is not always true. For example, $f=(x^2+y^2)\cdot(x+e^{-1/z^2})$ cannot be transformed into any element of $\mathcal{O}_3$. For the proof, see [5].

But we have a partial converse

Theorem 2. *For any $G\in\mathscr{E}_2$ (resp. $\mathcal{O}_2$) such that $G=g_1\cdot g_2^2$, $g_1\cdot g_2\in\mathscr{E}_2^\circ$ (resp. $\mathcal{O}_2^\circ$), there exists $\tau\in\mathrm{Dif}_2$ (resp. Adif_2) such that $G\circ\tau$ is a polynomial.*

Proof. It is enough to prove the case $G\in\mathfrak{m}^2(\mathscr{E}_2)$ (resp. $\mathfrak{m}^2(\mathcal{O}_2)$). Because of the hypothesis on $g_1\cdot g_2$, one sees that g_2 is in $\mathscr{E}_2^\circ$. Applying Theorem 1 to g_2, we see that g_2 is transformed into a polynimial, so we may assume that g_2 is a polynomial. From Lemma 3 it is easily seen that the greatest common measure of $T\left(\frac{\partial G}{\partial x}\right)^2$ and $T\left(\frac{\partial G}{\partial y}\right)^2$ is g_2^2. If p, $q\,(\in\mathscr{F}_2)$ have no common divisor, then the height of the ideal generated by p, q is 2, hence the ideal contains x^n, y^n for some $n>0$. From this the ideal generated by $T\left(\frac{\partial G}{\partial x}\right)^2$ and $T\left(\frac{\partial G}{\partial y}\right)^2$ contains $x^n g_2^2$, $y^n g_2^2$ for some $n>0$. Consider a map $u:\mathscr{E}_2\to\mathscr{E}_2$ defined by

$$u(f)=f\circ\left(\left(\frac{\partial G}{\partial x}\right)^2\Big/g_2^2,\ \left(\frac{\partial G}{\partial y}\right)^2\Big/g_2^2\right),$$

and $\hat{u}:\mathscr{F}_2\to\mathscr{F}_2$ defined by

$$\hat{u}(f)=f\circ\left(T\left(\frac{\partial G}{\partial x}\right)^2\Big/g_2^2,\ T\left(\frac{\partial G}{\partial y}\right)^2\Big/g_2^2\right).$$

If we use the terminology and Proposition III 1.6 in [6], then we see that $\hat{u}$ being quasi-finite, hence u is guasi-finite, that is, for some $m'>0$ the ideal $\mathfrak{p}$ generated by $\left(\frac{\partial G}{\partial x}\right)^2$ and $\left(\frac{\partial G}{\partial y}\right)^2$ contains $g_2^2\mathfrak{m}(\mathscr{E}_2)^{m'}$. From this there exists a polynomial f such that, $fg_2^2-g_1g_2^2\in\mathfrak{p}\mathfrak{m}(\mathscr{E}_2)$.
Put $fg_2^2=F$ and apply Lemma 2 to F, G, then there exists $\tau\in\mathrm{Dif}_2$ such that $F\circ\tau=G$.

In the analytic case, we must remind that $\mathscr{F}_n$ is faithfully flat over $\mathcal{O}_n$. The proof is similar to that in the differentiable case.

§4. The Example of Whitney

For any transcendental function ν *(such that* $\nu(0)=4$*), the germ of the function* $f=xy(y-x)(y-(3+t)x)(y-\nu(t)x)$ *at* 0 *in* $\mathbf{R}^3$ *cannot be transformed into any polynomial even through differentiable changes of coordinates.*

Proof. Suppose

(i) g is a polynomial in three variables with real coefficients,

(ii) τ is a diffeomorphism of C^∞ class of a neighborhood U of 0 in $\mathbf{R}^3$ onto a neighborhood of 0 in $\mathbf{R}^3$, where $\tau(0)=0$,

(iii) $g\circ\tau=f$ on U.

Let V be the vanishing of f on U, then V consists of five analytic manifolds, hence the vanishing of g on $\tau(U)$ (that is, $\tau(V)$) consits of five differentiable manifolds. It is shown in [6] (Proposition VI 3.11)

"Let X_0 be an analytic germ at 0 in $\mathbf{R}^n$ with $\dim X_0=k$. Suppose that X_0 contains the germ V_0 of a C^∞ manifold of dimension k, then V_0 is the germ of an analytic manifold (which is then an irreducible component of X_0)."

From this and the hypothesis that g is a polynomial, one sees that each sheet of $\tau(V)$ is an analytic manifold. Let these sheets be $V_1,\dots,V_4$. For each $i(=1,\dots,4)$ the set $\mathfrak{p}_i$ of those polynomials which vanish on V_i is prime in the polynomial ring $\mathbf{R}[x_1, x_2, x_3]$ of 3-variables over $\mathbf{R}$. The reason is as follows. From the fact that the radical of $\mathfrak{p}_i$ is $\mathfrak{p}_i$ and that $\mathbf{R}[x_1, x_2, x_3]$ is noetherian, it is shown that $\mathfrak{p}_i=\bigcap_{j=1}^{k_i}\mathfrak{q}_{i,j}$ ($\mathfrak{q}_{i,j}$: prime) and the sum of the vanishing of $\mathfrak{q}_{i,j}(i=1,\dots,k_i)$ is the vanishing of $\mathfrak{p}_i$, especially the sum of the vanishing of $\mathfrak{q}_{i,j}(j=1,\dots,k_i)$ on V_i is V_i, hence $k_i=1$ and $\mathfrak{p}_i=\mathfrak{q}_{i,1}$ is prime. Let W_i be the vanishing of $\mathfrak{p}_i$ in $\mathbf{C}^3$. Then one can see easily that W_i is an algebraic variety of dim 2 and that $\mathfrak{p}_i\mathbf{C}[x_1, x_2, x_3]$ is 1 in height. From this and the fact that $\mathbf{C}[x_1, x_2, x_3]$ is integral over $\mathbf{R}[x_1, x_2, x_3]$, $\mathfrak{p}_i$ is a prime ideal of height 1 in $\mathbf{R}[x_1, x_2, x_3]$. Hence $\mathfrak{p}_i=p_i\mathbf{R}[x_1, x_2, x_3]$ for some $p_i\in\mathbf{R}[x_1, x_2, x_3]$, and the ideal of polynomials which vanish on $\tau(V)$ is $\prod_{i=i_1,\dots,i_j} p_i\mathbf{R}[x_1, x_2, x_3]$ (not always $j=4$). For the remainder of this proof, we only have to proceed as in [9].

S. Izumi pointed out the following.

Remark 2. Using Artin's theorem in [1] we can easily prove the fact above, moreover Proposition VI 3.11 in [6] which we use in the proof above also can be shown by the same theorem.

§5. Multiplication by Germs

Let f, g be in $\mathfrak{m}(\mathscr{E}_n)$ (resp. $\mathfrak{m}(\mathcal{O}_n)$), and $f^{-1}(0)=g^{-1}(0)$. Then here is a question whether there exists $\tau\in \mathrm{Dif}_n$ (resp. Adif_n) such that $f=g\circ\tau$. This section give an answer to this question in the case that f and g have a special relation such that $f=\varphi g$, $\varphi(0)>0$.

k in Lemma 3 is not always 1. For example, $p=x^6+x^4y^4+y^6$ cannot have a solution of the equation

$$p=\sum_{i=1}^{n} q_i\frac{\partial p}{\partial x_i}.$$

Definition. $\bar{\mathscr{E}}_n$ (resp. $\bar{\mathcal{O}}_n$) is the set of germs $P\in\mathfrak{m}(\mathscr{E}_n)$ (resp. $\mathfrak{m}(\mathcal{O}_n)$) which satisfy

$$P=\sum_{i=1}^{n} q_i\frac{\partial P}{\partial x_i}\qquad \text{for some } q_i\in\mathfrak{m}(\mathscr{E}_n)\ (\text{resp. } \mathfrak{m}(\mathcal{O}_n)).$$

Proposition 1. *For any* $g\in\bar{\mathscr{E}}_n$ (resp. $\bar{\mathcal{O}}_n$), $\phi\in\mathscr{E}_n$ (resp. $\mathcal{O}_n$) *(such that* $\phi(0)>0$), *there exists* $\tau\in\mathrm{Dif}_n$) (resp. Adif_n) *such that* $\phi\cdot g=g\circ\tau$.

Proof. Let's proceed as in the proof of Lemma 2. It is enough to have germs $a_i(x, t)$ which satisfy the following conditions;

$$a_i(0, t)=0, \tag{1}$$

$$(\phi-1)g=\sum_{i=1}^{n} a_i(x, t)\left(\frac{\partial(\phi-1)g}{\partial x_i}t+\frac{\partial g}{\partial x_i}\right). \tag{2}$$

By the hypothesis, there exist $b_j\in\mathfrak{m}(\mathscr{E}_n)$ (resp. $\mathfrak{m}(\mathcal{O}_n)$) such that

$$g=\sum_{j=1}^{n} b_j\frac{\partial g}{\partial x_j}.$$

From this, (2) becomes

$$\sum_j(\phi-1)b_j\frac{\partial g}{\partial x_j}=\sum_{i,j}a_i(x, t)b_j\frac{\partial\phi}{\partial x_i}\cdot\frac{\partial g}{\partial x_j}t+\sum_i a_i(x, t)\frac{\partial g}{\partial x_i}\{(\phi-1)t+1\}. \qquad (3)$$

If we get the following relation for all i,

$$(\phi-1)b_i=\sum_j a_j(x, t)b_i\frac{\partial\phi}{\partial x_j}t+a_i(x, t)\{(\phi-1)t+1\}. \qquad (4)$$

then (3) is established. Hence it is sufficient to find $a_i(x, t)$ which satisfy (1) and (4). Let A be the matrix whose (i, j)-component is $b_i\frac{\partial\phi}{\partial x_j}t$ and I the unit matrix. Then (4) is equal to

$$\begin{pmatrix}(\phi-1)b_1\\ \vdots\\ (\phi-1)b_n\end{pmatrix}=(A+\{(\phi-1)t+1\}I)\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}.$$

From $b_i\in\mathfrak{m}(\mathscr{E}_n)$ (resp. $\mathfrak{m}(\mathscr{E}_n)$), the determinant of $A+\{(\phi-1)t+1\}I$ does not vanish at any point of $0\times[0, 1]$, hence there is the inverse matrix B of $A+\{(\phi-1)t+1\}I$ (where all components of B are germs at $0\times[0, 1]$ of C^∞-functions (resp. analytic ones)). By $b_j(0)=0$, all components of $B\begin{pmatrix}(\phi-1)b_1\\ \vdots\\ (\phi-1)b_n\end{pmatrix}$ (that is, a_i) satisfy (1). The proposition follows.

The following is the converse of this result in the analytic case.

Proposition 2. *Suppose* $g\in\mathfrak{m}(\mathcal{O}_n)$ *satisfies the following condition; for any* $t\in[1, 2]$, *there exists* $\tau_t\in\mathrm{Adif}_n$ *such that* $tg=g\circ\tau_t$. *Then* $g\in\bar{\mathcal{O}}_n$.

Proof. For each integer $k(>1)$, T_k^n denotes the natural map from $(\mathcal{O}_n)^n$ to $(\mathcal{O}_n)^n/\mathfrak{m}^k(\mathcal{O}_n)\times(\mathcal{O}_n)^n$, $(\mathcal{O}_n)^n/\mathfrak{m}^k(\mathcal{O}_n)\times(\mathcal{O}_n)^n$ is a vector space of finite dimension over $\mathbf{R}$. Regard Adif_n as contained in $(\mathcal{O}_n)^n$, then the set of $T_k^n\tau_t(t\in[1, 2])$ is contained in a finite dimensional vector space over $\mathbf{R}$. Hence there exist $t_m\in[1, 2]$ $(m=1, 2,\dots)$ such that t_m and $T_k^n\tau_{t_m}$

converge. From these convergence one sees that when $m\to\infty$, $m'\to\infty$ then $\frac{t_m}{t_{m'}}$ and $T_k^n\tau_{t_m}\circ\tau_{t_{m'}}^{-1}$ converge respectively to 1 and T_k^nI (where I is the identity map). Hence from the beginning we may assume that t_m, $T_k^n\tau_{t_m}$ converge respectively to 1 and T_k^nI (when $m\to\infty$), and that

$$g\circ\tau_{t_m}=t_m g.$$

For sufficiently large m, and $1\leqq t\leqq t_m$, $\frac{t_m-t}{t_m-1}I+\frac{t-1}{t_m-1}\tau_{t_m}$ belongs to Adif_n. Let $f_m(x, t)=g\circ\left(\frac{t_m-t}{t_m-1}I+\frac{t-1}{t_m-1}\tau_{t_m}\right)$. Then $f_m(x, t)$ is a germ at $0\times[1, t_m]$ in $\mathbf{R}^n\times\mathbf{R}$ of analytic function, and satisfies

$$f_m(x, 1)=g,$$
$$f_m(x, t_m)=t_m g.$$

We have that

$$\frac{f_m(x,t_m)-f_m(x,1)}{t_m-1}-\frac{\partial f_m(x, 1)}{\partial t}\ \text{converges to 0 in}$$

$$\frac{\mathcal{O}_n}{\mathfrak{m}^k(\mathcal{O}_n)}\ \text{(when } m\to\infty).$$

This is because letting $f_m(x, t)=\sum_{l=0}^{\infty}h_{m,l}(x)(t-1)^l$ we have

$$h_{m,l}(x)\in\mathfrak{m}^l(\mathcal{O}_n),\ \text{and } h_{m,l}(x) \text{ converges in } \mathcal{O}_n/\mathfrak{m}^k(\mathcal{O}_n)\ \text{(when } m\to\infty).$$

From this we have

$$f_m(x, t)=\sum_{l=0}^{k-1}h_{m,l}(x)(t-1)^l \qquad \mathrm{mod}\ \mathfrak{m}^k(\mathcal{O}_n).$$

From the equations

$$\frac{f_m(x, t_m)-f_m(x, 1)}{t_m-1}=\frac{t_m g-g}{t_m-1}=g,$$

$$\frac{\partial f_m}{\partial t}(x, 1)=\sum_{i=1}^{n}\frac{\partial g}{\partial x_i}\left.\frac{\partial\left(\frac{t_m-t}{t_m}I+\frac{t}{t_m}\tau_{t_m}\right)_i}{\partial t}\right|_{t=1},$$

and the fact that $T_k\mathfrak{p}$ where $\mathfrak{p}$ is the ideal generated by $\frac{\partial g}{\partial x_i}x_j$ is a linear subspace of a finite dimensional vector space over $\mathbf{R}$, we see that g is contained in $\mathfrak{p}+\mathfrak{m}^k(\mathcal{O}_n)$. Because k is arbitrary, and by Theorem of Krull we see that g is centained in $\mathfrak{p}$.

Remark 3. In the same way as in the above proofs, we can prove that for any $f\in\mathfrak{m}(\mathcal{E}_n)$, the following two conditions are equivalent;

(1) for any $\phi\in\mathcal{E}_n$ (such that $\phi(0)>0$) there exists $\tau\in\mathrm{Dif}_n$ such that $T(f\cdot\phi)=T(f\circ\tau)$,

(2) there exists $q_i\in\mathcal{F}_n$ $(i=1,\dots,n)$ such that $q_i(0)=0$ $Tf=\sum_{i=1}^{n}q_iT\frac{\partial f}{\partial x_i}$.

§6. A Sufficient Condition

In this section we consider only the case $n=2$.

Lemma 5. *Let $f_i(i=1,\dots,n)$ be in $\mathcal{E}_2$ such that $g=\prod_{i=1}^{n}f_i$ is in $\mathcal{O}_2^\circ$. Then there exist $\phi_i\in\mathcal{E}_2$ $(i=1,\dots,n)$ such that each ϕ_if_i is analytic and $\prod_{i=1}^{n}\phi_i=1$, $\phi_i(0)=1$.*

Proof. Here we don't distinguish a function from the germ defined by its function.

We have the following facts

(i) $\mathcal{O}_n$ and $\mathcal{F}_n$ are unique factorization rings,

(ii) for any prime ideal $\mathfrak{p}$ of $\mathcal{O}_n$, $\mathfrak{p}\mathcal{F}_n$ is prime in $\mathcal{F}_n$ (Zariski-Nagata).

By these it is easily seen that we can assume Tf_i are prime in $\mathcal{F}_2$, and that there exist $p_i\in\mathcal{F}_2$ $(i=1,\dots,n)$ where $p_i(0)=1$ $\prod_{i=1}^{n}p_i=1$ $p_iTf_i=q_i\in\mathcal{O}_2$. Let ψ_i $(i=1,\dots,n)$ be in $\mathcal{E}_2$ such that $T\psi_i=p_i$ $(i=1,\dots,n)$. Then we see that $q_i-\psi_if_i$ are flat at 0.

For each i, the vanishing of f_i is the vanishing V_i of q_i. The reason is as follows. Any analytic germ X in $\mathbf{R}^2$ is described in the following form (by some linear transformation) (see p.57 in [6]);

"$p(x, y)$ is a distinguished polynomial in x with coefficients in $\mathcal{O}(y)$ and X is the vanishing of p."

Hence V_i is $\{0\}$ or a sum of curves. If V_i is a sum of curves and f_j vanishes on a subgerm $(\neq\{0\})$ of $V_i (j \neq i)$, then there is a sequence a_n $(\to 0)$ in V_i such that $f_j(a_n)=0$ $(n=1, 2\dots)$. By the fact that $q_j-\psi_j f_j$ is flat at 0, for any $N>0$ there is a neighborhood of 0 where we have

$$|q_j(x)-\psi_j f_j(x)| \leqq |x|^N.$$

From these we see that for sufficient large n

$$|q_j(a_n)| \leqq |a_n|^N. \tag{1}$$

Now, from the hypothesis that $\Pi q_j = g$ is in $\mathcal{O}_2^\circ$, one sees easily that the ideal (generated by q_i, q_j) is 2 in height, and that the ideal is contained in $I(V_i \cap V_j)$. Hence $I(V_i \cap V_j)$ is 2 in height, and $V_i \cap V_j = \{0\}$. It is shown in [6] (Corollary IV 4.4) that any two analytic sets X, Y are regularly situated (i.e. locally there exists a pair of constants $c>0$ and $\alpha>0$ such that for every x in X $d(x, Y) \geqq c\, d(x, X\cap Y)^\alpha$). From this there exists a pair of $c>0$ and $\alpha>0$ such that for sufficiently large n

$$d(a_n, V_j) \geqq c|a_n|^\alpha. \tag{2}$$

But, from the inequality of Łojasiewicz, we have locally for some constants $c'>0$, $\alpha'>0$

$$|q_j(x)| \geqq c'(d(x, V_j))^{\alpha'},$$

especially for sufficiently large n

$$|q_j(a_n)| \geqq c'(d(a_n, V_j))^{\alpha'}. \tag{3}$$

From (1), (2), (3), we get for suffiiciently large n

$$c'(d(a_n, V_j))^{\alpha'} \leqq \left(\frac{d(a_n, V_j)}{c}\right)^{\frac{N}{\alpha}}.$$

By the fact that N is arbitrary and $d(a_n, V_j)$ is not 0, this is a contradiction. Thus we have shown that

$$V_i \text{ is } \{0\},$$

or that

f_j does not vanish on any point $(\neq 0)$ of $V_i (j \neq i)$.
From the hypothesis $(\Pi f_i = g)$, the sum of the vanishing (of f_i on V_i) is V_i. Hence one sees that V_i is the vanishing of f_i.

On the other hand, any analytic set X in Ω (open in $\mathbf{R}^2$) is coherent (i.e. for any $a \in \Omega$ and for any finite system $\{h_i\}$ of generators of $I(X_a)$, there exists a neighborhood Ω' of a such that for any $b \in \Omega'$ h_i generate $I(X_b)$). The reason is, if $I(X_a)$ is 2 in height, then the above statement is trivial, if $I(X_a)$ is 1 in height, then $I(X_a)$ is principal and is generated by h (where h is in $\mathcal{O}_2^\circ$). From this we have near a

$$\left|\frac{\partial h}{\partial x}(b)\right| + \left|\frac{\partial h}{\partial y}(b)\right| \neq 0,\ b \neq a,$$

and one sees easily that near a $I(X_b)$ is generated by h.

In [6] Malgrange has shown the following theorem (VI 3.10): Let $K(X_0)$ be the ideal in $\mathscr{E}_n$ of C^∞ functions vanishing on X_0, then the following properties are equivalent,

(a) $K(X_0) = I(X_0)\mathscr{E}_n$,

(b) X_0 is coherent at 0.

From this, if $V_i \neq \{0\}$ then there exists $\phi_i' \in \mathscr{E}_2$ such that $f_i = q_i \phi_i'$.

Malgrange [6] also has proved the fact (Theorem VI 1.1′): Let Ω be an open set in $\mathbf{R}^n$ and $h_1, \dots, h_p$ analytic functions in Ω. Let $\phi \in \mathscr{E}(\Omega)$ (the set of C^∞ functions defined on Ω). Then ϕ can be written in the form $\phi = \sum_{i=1}^{p} h_i \psi_i$ (with $\psi_i \in \mathscr{E}(\Omega)$) if and only if for any $a \in \Omega$, the Taylor expansion $T_a \phi$ of ϕ at a belongs to the ideal generated by $T_a h_i$ in $\mathscr{F}_n$. From this, if $V_i = 0$ then there exists $\phi_i' \in \mathscr{E}_2$ such that $f_i = q_i \phi_i'$. Since $\Pi f_i = \Pi q_i$, $p_i(0) = 1$. Hence $\Pi \phi_i' = 1$ and $\phi_i'(0) = 1$. Thus the lemma is proved.

Remark 4. In the lemma above we see that an analytic set in $\mathbf{R}^2$ is coherent. But the one in $\mathbf{R}^3$ is not always coherent. The counter example is the "umbrella" (p. 95 in [6]).

Lemma 6. *Let f be such that $f = \prod_{i=1}^{k} f_i^i$ (where $\prod_{i=1}^{k} f_i \in \mathscr{E}_2^\circ$). Then there exist $\tau \in \mathrm{Dif}_2$ and $\phi \in \mathscr{E}_2 - \mathfrak{m}(\mathscr{E}_2)$ such that $\phi(f \circ \tau)$ is analytic.*

Proof. Applying Theorem 1 to $g=\prod_{i=1}^{k} f_i$, we get $\tau\in\mathrm{Dif}_2$ such that $g\circ\tau$ is analytic. From Lemma 5 there exist $\phi_i\in\mathscr{E}_2$ $(i=1,\dots,k)$ such that $\phi_i(f_i\circ\tau)$ are analytic and $\prod_{i=1}^{k}\phi_i=1$, $\phi_i(0)=1$. Hence $\prod_{i=1}^{k}\{\phi_i^i\cdot(f_i\circ\tau)^i\}$ is analytic and equal to $(\prod_{i=1}^{k}\phi_i^i)(f\circ\tau)$.

Combining this and Proposition 1, we have

Theorem 3. *For any* $f\in\bar{\mathscr{E}}_2$ *such that* $f=\prod_{i=1}^{k} f_i^i$, $\prod_{i=1}^{k} f\in\mathscr{E}_2^\circ$ *there exist* $\tau\in\mathrm{Dif}_2$ *such that* $f\circ\tau$ *is analytic.*

Remark 5. Let $f\in\mathscr{E}_2$ be such that for some $q_i\in\mathfrak{m}(\mathscr{F}_n)$, $Tf=\sum_{i=1}^{n} q_i T\dfrac{\partial f}{\partial x_i}$, then by Remark 3 there exist $\tau\in\mathrm{Dif}_2$ and $\psi\in\mathscr{E}_2$ such that ψ is flat at 0 and $f\circ\tau+\psi$ is analytic.

§7. Appendix: A Proof to Cerf's Result

Let us recall the following concept introduced by Cerf [2].

Let W be a C^∞-manifold, f a C^∞-function defined on W, and c a critical point of f. One calls *condimension of critical point* c the codimension of the ideal generated by the germs of the first partial derivatives of f at c in the ring of germs of C^∞-function: $W\to\mathbf{R}$ zero at c.

Critical points of codimension 0, 1, or 2 are called respectively *points critiques du type de Morse*, *points de naissance*, and *points critiques du type queue d'aronde.*

We present a proof to the following Cerf's result (shown at p. 23 in [2]).

Proposition 3. *If* $f\in\mathfrak{m}(\mathscr{E}_n)$ *has a critical point of codimension* 1 (*resp.* 2) *at* 0, *then there exists* $\tau\in\mathrm{Dif}_n$ *such that*

$$f\circ\tau=-x_1^2-\cdots-x_i^2+x_{i+1}^2+\cdots+x_{n-1}^2+x_n^3,$$

$$(resp.=-x_1^2-\cdots-x_i^2+x_{i+1}^2+\cdots+x_{n-1}^2\pm x_n^4).$$

Proof. The proof of the case of codim 2 follows in the same way as its of codim 1, so we prove only the case of codim 1.

Let $\mathfrak{p}(f)$ be the ideal generated by the germs of the first partial derivatives of f in $\mathscr{E}_n$. We have

$$\mathfrak{m}/\mathfrak{p}=\mathbf{R}^1, \qquad \text{mod } m^2.$$

Hence, after some linear transformation, we can assume

$$\frac{\partial f}{\partial x_i}\in\mathfrak{m}-\mathfrak{m}^2, \qquad i=1,\dots, n-1,$$

$$\frac{\partial f}{\partial x_n}\in\mathfrak{m}^2-\mathfrak{m}^3.$$

Therefore there exist g_0, g_1 and $g_2\in\mathscr{E}(x_1,\dots, x_{n-1})$ and $h\in\mathscr{E}_n-\mathfrak{m}$ such that

$$f=g_0+g_1x_n+g_2x_n^2+hx_n^3,$$

$$g_0\text{: of Morse type,}$$

$$g_1\in\mathfrak{m}^2(\mathscr{E}_{n-1}),$$

$$g_2\in\mathfrak{m}(\mathscr{E}_{n-1}).$$

Applying Morse's theorem to g_0, we have

$$g_0=-x_1^2-\cdots-x_i^2+x_{i+1}^2+\cdots+x_{n-1}^2.$$

Let $g_2=\sum_{i=1}^{n-1}a_ix_i+g_3$ such that a_i are constants and $g_3\in\mathfrak{m}^2$. Then, transforming x_i by $x_i\pm\frac{a_i}{2}x_n^2$, we only have to prove the case $g_2\in\mathfrak{m}^2$. Let us apply Lemma 2 to f and $G=g_0+hx_n^3$. We see easily that $\mathfrak{p}(G)$ is generated by $x_1,\dots, x_{n-1}$, and x_n^2, and that $\mathfrak{p}^2(G)\mathfrak{m}$ contains $g_1x_1+g_0x_n^2$. From Lemma 2 there exists $\tau\in\mathrm{Dif}_n$ such that $f\circ\tau=G$. Let $\tau'\in\mathrm{Dif}_n$ denote $(x_1,\dots, x_{n-1}, h^{1/3}x_n)$. Then $f\circ\tau\circ\tau'=-x_1^2-\cdots-x_i^2+x_{i+1}^2+\cdots+x_{n-1}^2+x_n^3$.

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