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A Polynomial Subalgebra of the Cohomology of the Steenrod Algebra

By

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1. Introduction

Let A be the mod 2 Steenrod algebra and $H^{**}(A) = \text{Ext}_A^{**}(Z_2, Z_2)$ its cohomology. The ultimate aim in studying $H^{**}(A)$ is the long-standing problem of computing the homotopy groups of spheres via the Adams spectral sequence [1]. $H^{s,t}(A)$ has been computed up to certain values of $t-s$ by Adams [4], Ivanovskiĭ [6] (see also: International Congress of Mathematicians, Moscow 1966), Liulevicius [8], May [12, 13], Tangora [21]. It is of interest to know any "systematic" phenomena in $H^{**}(A)$. In this direction a polynomial "wedge" subalgebra of $H^{**}(A)$ has been obtained by Mahowald and Tangora [9]. Also: Margolis, Priddy and Tangora proved in [10] that the Mahowald-Tangora "wedge" subalgebra is repeated every 45 stems, under the action of a specific "periodicity" operator. The present writer has described shortly in [24] a polynomial subalgebra of $H^{**}(A)$ generated by d_0, e_0, g. This subalgebra will be described here in more detail. The basic technique is to study $H^{**}(A)$ by studying $H^{**}(B)$ for a suitable subalgebra B of A. This technique is due to Adams [3]. It has also been used by Margolis, Priddy and Tangora [10]. Moreover G.W. Whitehead [22] shows that, using the Adams technique, one can obtain many polynomial subalgebras of $H^{**}(A)$.

The present paper is organized as follows: In Section 2 we state the main theorem and sketch its proof. The detailed proof involves the Adams technique and the construction of the generators d_0, e_0, g. These constructions use known relations between the classes h_i (see Adams [2], Novikov [19]); they also use Steenrod $\cup_i$-products in $F(A^*)$. In Section 3 we describe briefly these cup-i-products and in Section 4 we give the

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actual construction of the generators d_0, e_0, g. These and other generators can also be constructed with the aid of explicit resolutions (see [11], [12], [22]).

The contents of this paper constitute a major part of my doctoral thesis [25] at the University of Manchester, written under the supervision of Professor Adams. I wish to express my sincere thanks to Professor Adams for his suggestions and helpful ideas and for his constant interest and encouragement.

2. The Main Theorem

Let A be the mod 2 Steenrod algebra and $H^{**}(A)$ its cohomology. The following theorem is our main result.

Theorem. *$H^{**}(A)$ contains a polynomial subalgebra generated by the elements d_0, e_0, g. of dimensions* (4, 18), (4, 21), (4, 24) *respectively, subject to the single relation $e_0^2 = d_0 g$. That is the elements $e_0^i d_0^j g^k$ with $i=0,1$, $j \geqq 0$, $k \geqq 0$ are linearly independent.*

The line of proof is as follows: Let B be the exterior subalgebra of A generated by $Sq^{0,1}$ and $Sq^{0,2}$. Then $H^{**}(B)$ is a polynomial algebra on two generators, namely $x = \{[\xi_2]\}$ and $y = \{[\xi_2^2]\}$ in Milnor's [18] notation. B has a basis consisting of the elements $Sq^{0,j}$, where $0 \leqq j \leqq 3$. Hence B^* has a basis consisting of the elements ξ_2^j with $0 \leqq j \leqq 3$. The inclusion map i: $B \to A$ induces a map i^{**}: $H^{**}(A) \to H^{**}(B)$. The proof depends on showing that $i^{**} d_0 = x^2 y^2$, $i^{**} e_0 = x y^3$, $i^{**} g = y^4$. This will show that the elements $e_0^i d_0^j g^k$ with $i=0$, 1, $j \geqq 0$, $k \geqq 0$ are linearly independent. To obtain the relation $e_0^2 = d_0 g$ we observe that, by the above argument e_0^2 and $d_0 g$ are both nonzero elements of $H^{8,42}(A) \cong Z_2$ (see May [12], Appendix A). Thus $e_0^2 = d_0 g$. This proves the theorem. It remains only to sketch how to compute the effect of i^{**} on d_0, e, g. The inclusion i: $B \to A$ induces a known map i^*: $A^* \to B^*$ of the dual algebras and a map $F(i^*)$: $F(A^*) \to F(B^*)$ of the cobar construction. Now $F(i^*)$ maps the basis elements of B^* to themselves and every other element to zero. From the explicit construction of cocycles $\tilde{d}_0$, $\tilde{e}_0$, $\tilde{g}$ representing d_0, e_0, g respectively it will follow that: $\{F(i^*)\tilde{d}_0\}=x^2y^2$, $\{F(i^*)\tilde{e}_0\}=xy^3$ and $\{F(i^*)\tilde{g}\}=y^4$, which completes the proof of the theorem, having in mind that $H^{4,18}(A)=H^{4,21}(A)=H^{4,24}(A)\cong Z_2$. Actually we use the following Massey products:

$$d_0=<h_2^2,\ h_0,\ h_2^2,\ h_0>$$

$$e_0=<h_3^2,\ h_0^2,\ h_1,\ h_0>$$

$$g=<h_3^2h_0,\ h_0,\ h_1,\ h_2>.$$

3. Cup-i-products

Let A be a connected cocommutative Hopf algebra (over Z_2, for simplicity) and $H(A)$ its cohomology; for instance - the mod 2 Steenrod algebra is such a Hopf algebra (see Milnor [18]). Let $F(A^*)$ be the Adams cobar construction. It follows from Adams work (see e.g. [2]) that there exist maps

$$F(A^*)\otimes F(A^*)\xrightarrow{\cup_i}F(A^*),\qquad i=0,\ 1,\ 2\ldots$$

which have most of the properties of the Steenrod cup-i-products (see Steenrod [20]). The above cup-i-products induce Steenrod squares

$$Sq^i\colon\ H^{s,t}(A)\longrightarrow H^{s+i,2t}(A),$$

which enjoy most of the properties of their topological analogues. (Cartan formula, Adem relation etc.) These operations and their applications have been studied by many authors. (See: Adams [2], Ivanovskiĭ [6], Liulevicius [7], May [14], Milgram [15, 16, 17], Novikov [19]). The present writer has obtained explicit formulae for these cup-i-products in 1965, which appeared in his M.Sc. thesis [23], written under the supervision of Professor Adams. The detailed contents of [23] will appear elsewhere. Here we quote from [23, 25] the explicit formulae that we will need. More precisely: Let $x=[\alpha_1|\cdots|\alpha_p]$, $y=[\beta_1|\cdots|\beta_q]$ be two cochains in $F(A^*)$. Then $x\cup_i y$ is described in terms of appropriate iterated diagonals of A^*, as follows:

1st case: $i=2t$ (even). The general summand of $x\cup_i y$ is:

$$[\alpha_1|\cdots|\alpha_{r_0}|\alpha_{r_0+1}^{(1)}\beta_1|\cdots|\alpha_{r_0+1}^{(s_1-s_0-1)}\beta_{s_1-1}|\alpha_{r_0+2}\beta_{s_1}^{(1)}|\cdots|\alpha_{r_1}\beta_{s_1}^{(r_1-r_0-1)}|$$

$$\cdots|\alpha_{r_{t-1}+1}^{(1)}\beta_{s_{t-1}+1}|\cdots|\alpha_{r_{t-1}+1}^{(s_t-s_{t-1}-1)}\beta_{s_t-1}|\alpha_{r_{t-1}+2}\beta_{s_t}^{(1)}|\cdots|\alpha_{r_t}\beta_{s_t}^{(r_t-r_{t-1}-1)}$$

$$|\beta_{s_t+1}|\cdots|\beta_q].$$

Actually $x\cup_i y$ is a sum of terms of this form depending on indices j_0, $j_1,\ldots, j_i$ subject to the relations:

$$0\leqq j_0<j_1<\cdots<j_i\leqq p+q-i;\ t+\sum_{0\leqq k\leqq i}(-1)^k j_k=p.$$

The indices r_0, $r_1,\ldots, r_t$ and s_0, $s_1,\ldots, s_t$ are given by the equations:

$$r_m=m+\sum_{0\leqq k\leqq 2m}(-1)^k j_k,\ \ s_m=m+\sum_{0\leqq k\leqq 2m-1}(-1)^{k+1} j_k$$

for $m=0, 1,\ldots, t$.

Also: $s_0=0$, $r_0=j_0$ and $r_t=p$.

2nd case: $i=2t-1$ (odd). The general summand of $x\cup_i y$ is:

$$[\alpha_1|\cdots|\alpha_{r_0}|\alpha_{r_0+1}^{(1)}\beta_1|\cdots|\alpha_{r_0+1}^{(s_1-s_0-1)}\beta_{s_1-1}|\alpha_{r_0+2}\beta_{s_1}^{(1)}|\cdots|\alpha_{r_1}\beta_{s_1}^{(r_1-r_0-1)}|$$

$$\cdots|\alpha_{r_{t-2}+2}\beta_{s_{t-1}}^{(1)}|\cdots|\alpha_{r_{t-1}}\beta_{s_{t-1}}^{(r_t-r_{t-1}-1)}|\alpha_{r_{t-1}+1}^{(1)}\beta_{s_{t-1}+1}|\cdots|\alpha_{r_{t-1}+1}^{(s_t-s_{t-1}-1)}\beta_{s_t-1}|$$

$$\alpha_{r_{t-1}+2}|\cdots|\alpha_p].$$

In this case $x\cup_i y$ is a sum of terms of this form depending on indices j_0, $j_1,\ldots, j_i$ subject to the relations:

$$0\leqq j_0<j_1<\cdots<j_i\leqq p+q-i;\ \ t+\sum_{0\leqq k\leqq i}(-1)^{k+1} j_k=q+1.$$

the indices $r_0, r_1,\ldots, r_t$ and s_0, $s_1,\ldots, s_t$ are given by the equations:

$$r_m=m+\sum_{0\leqq k\leqq 2m}(-1)^k j_k;\ s_m=\sum_{0\leqq k\leqq 2m-1}(-1)^{k+1} j_k$$

for $m=0, 1,\ldots, t-1$.

Also: $s_0=0$, $r_0=j_0$ and $s_t=q+1$.

Examples: For $x=[\alpha_1|\cdots|\alpha_p]$, $y=[\beta_1|\cdots|\beta_q]$ we have:

$x\cup y=[\alpha_1|\cdots|\alpha_p|\beta_1|\cdots|\beta_q]$. We write xy for $x\cup y$.

$$x \cup_1 y = \sum_{0 \leqq j \leqq p-1} |\alpha_1|\cdots|\alpha_j|\alpha_{j+1}^{(1)}\beta_1|\cdots|\alpha_{j+1}^{(q)}\beta_q|\alpha_{j+2}|\cdots|\alpha_p].$$

$$x \cup_2 y = \sum[\alpha_1|\cdots|\alpha_j|\alpha_{j+1}^{(1)}\beta_1|\cdots|\alpha_{j+1}^{(k-j)}\beta_{k-j}|\alpha_{j+2}\beta_{k-j+1}^{(1)}|\cdots|\alpha_p\beta_{k-j+1}^{(l-k)}|$$

$$\beta_{k-j+2}|\cdots|\beta_q].$$

This summation is taken over all indices j, k, l such that: $0 \leqq j < k < l \leqq p+q-2$; $j-k+l=p-1$.

If δ is the coboundary of $F(A^*)$ then we have the usual coboundary formulae for cochains:

$$\delta(x \cup y) = \delta x \cup y + x \cup \delta y \qquad \text{and}$$

$$\delta(x \cup_i y) = \delta x \cup_i y + x \cup_i \delta y + x \cup_{i-1} y + y \cup_{i-1} x \qquad \text{for } i > 0.$$

We also have Hirsch formulae:

$$(x_1 \ldots x_n) \cup_1 z = \sum_{1 \leqq k \leqq n-1} x_1 \ldots x_{k-1}(x_k \cup_i z) x_{k+1} \ldots x_n \qquad \text{in } F(A^*).$$

$$e.g. \qquad (xy) \cup_1 z = (x \cup_1 z) y + x (y \cup_1 z).$$

$$\langle x, y, x \rangle = (x \cup_1 x) y \qquad \text{in } H^{**}(A)$$

(modulo appropriate indeterminacies).

For the topological analogues of the last two equations see Hirsch [5].

4. The Generators d_0, e_0, g.

We will construct cocycles $\tilde{d}_0$, $\tilde{e}_0$, $\tilde{g}$ representing $d_0 = \langle h_2^2, h_0, h_2^2, h_0 \rangle$, $e_0 = \langle h_3^2, h_0^2, h_1, h_0 \rangle$, $g = \langle h_3^2 h_0, h_0, h_1, h_2 \rangle$ respectively.

It is not difficult to see that these quadruple Massey products are defined.

The cocycle $\tilde{d}_0$.

We may take:

$$\tilde{d}_0 = R \cup [\xi_1] + [\xi_1^4|\xi_1^4] \cup S + XX,$$

where:

$$R=[\xi_2^4|\xi_2^4|\xi_1]+[\xi_1^4|\xi_2^4+\xi_1^{12}|\xi_1];\quad S=[\xi_2^2+\xi_1^6|\xi_1^4];$$

$$X=[\xi_2^2|\xi_2+\xi_1^3]+[\xi_1^4|\xi_1^2\xi_2]+[\xi_3|\xi_1^2].$$

The cocycle $\tilde{e}_0$.

We may take:

$$\tilde{e}_0=R\cup[\xi_1]+[\xi_1^8|\xi_1^8]\cup S+XZ,$$

where now R, S are such that δR, δS give representative cocycles for $<h_3^2, h_0^2, h_1>$, $<h_0^2, h_1, h_0>$ respectively. Also X, Z are such that δX, δZ give representative cocycles for $h_3^2 h_0^2$, $h_1 h_0$ respectively.

The cocycle $\tilde{g}$.

We may take:

$$\tilde{g}=[\xi_1^8|\xi_1^8|\xi_1]\cup R+S\cup[\xi_1^4]+XZ.$$

Here R, S are such that δR, δS give representative cocycles for $<h_0, h_1, h_2>$, $<h_3^2 h_0, h_0, h_1>$ respectively and X, Z are such that δX, δZ give representative cocycles for $h_3^2 h_0^2$, $h_1 h_2$ respectively. By using the explicit formulae for the cup-i-products involved in these constructions it can be shown that:

$$F(i^*)\tilde{d}_0=[\xi_2|\xi_2|\xi_2^2|\xi_2^2],$$

$$F(i^*)\tilde{e}_0=[\xi_2|\xi_2^2|\xi_2^2|\xi_2^2],$$

$$F(i^*)\,\tilde{g}=[\xi_2^2|\xi_2^2|\xi_2^2|\xi_2^2],$$

as required.

The actual computations are lengthy and they are omitted here. Details can be found in [25], where g is described in another way, based on a proposition which is crucial for proving the Adams periodicity theorem [3].

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