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On the Classification of $(n-2)$-connected $2n$-manifolds with Torsion Free Homology Groups

By

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Introduction and Summary

C.T.C. Wall investigated the classification problems of highly connected manifolds in his series of papers [18], [19], and [21]. But since then only a few investigations to classify the manifolds with lower connectedness have been made. As such investigations there are Wall [20], Tamura [17], and Ishimoto [6]. In this paper, we consider to classify $(n-2)$-connected $2n$-manifolds which have torsion free homology groups (equivalently, torsion free $(n-1)$-th homology groups) and are $(n-1)$-parallelizable. Such a manifold can be decomposed as a connected sum of an $(n-2)$-connected $2n$-manifold which has the vanishing n-th homology group and is $(n-1)$-parallelizable and an $(n-1)$-connected manifold. So that, our main problems are to classify the former $2n$-manifolds and to investigate the uniqueness of the decomposition. Firstly, we completely classify the handlebodies of $\mathscr{H}(2n+1, k, n+1)$ $(n\geqq 4)$ up to diffeomorphism, and then, using the results we consider to classify $(n-2)$-connected $2n$-manifolds which have vanishing n-th homology groups and are $(n-1)$-parallelizable up to diffeomorphism $\bmod \theta_{2n}$. Here, $\mathscr{H}(m, k, s)$ is the collection of handlebodies $W=D^m \bigcup_{\{f_i\}} \{\bigcup_{i=1}^{k} D_i^s \times D_i^{m-s}\}$ with the disjoint smooth imbeddings $f_i\colon \partial D_i^s \times D_i^{m-s} \to \partial D^m$, $i=1, 2,\cdots, k$. The uniqueness of the decomposition is also considered up to diffeomorphism $\bmod \theta_{2n}$. Throughout this paper, manifolds are connected, closed, and differentiable.

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Let M be an $(n-2)$-connected $2n$-manifold $(n \geqq 3)$ with torsion free homology groups. The *type* of M is defined as follows; Let ϕ: $H^{n-1}(M) \times H^{n-1}(M) \to Z_2$ be a symmetric bilinear form defined by $\phi(x, y) = < S_q^2 x_2 \cup y_2, [M]_2 >$, where x_2, y_2, and $[M]_2$ mean that they are considered in the Z_2-coefficient, and $[M]$ denotes the fundamental class of $H_{2n}(M)$. Let $\operatorname{rank} H_{n-1}(M) = k$ and we define the rank of ϕ by the rank of the corresponding matrix representation. Then, M is of *type* 0 if $\operatorname{rank} \phi = 0$ (that is, S_q^2: $H^{n-1}(M; Z_2) \to H^{n+1}(M; Z_2)$ is trivial), of *type* I if there exists an $x \in H^{n-1}(M)$ such that $\phi(x, x) \neq 0$ and $\operatorname{rank} \phi = k$, and of *type* II if $\phi(x, x) = 0$ for any $x \in H^{n-1}(M)$ and $\operatorname{rank} \phi = k$. M is of *type* (0+I) if there exists an $x \in H^{n-1}(M)$ such that $\phi(x, x) \neq 0$ and $\operatorname{rank} \phi < k$, and of *type* (0+II) if $\phi(x, x) = 0$ for any $x \in H^{n-1}(M)$ and $0 < \operatorname{rank} \phi < k$. M belongs to some type and the type is uniquely determined.

Now, we put the following hypothesis on M;

(H) It is $(n-1)$-parallelizable.

Remark. If $n = 0, 4, 6, 7 \pmod 8$, (H) is satisfied. If $n = 1, 5 \pmod 8$ and the $(n-1)/4$-th Pontryagin class is zero, (H) is satisfied. For π-manifolds and almost parallelizable manifolds, (H) is always satisfied.

Theorem 1. *Let M be an $(n-2)$-connected $2n$-manifold $(n \geqq 4)$ which has the vanishing n-th homology group and satisfies the hypothesis (H). Then, M has the representation* $\bmod \theta_{2n}$ *as shown in the following tables* 1, 2, *and* 3.

Here, A_α, B_β are the $(n-1)$-sphere bundles over $(n+1)$-spheres with the characteristic elements α, $\beta \in \pi_n(SO_n)$ respectively such that $\pi(\alpha) = 0$, $\pi(\beta) = 1$ for $\pi: \pi_n(SO_n) \to \pi_n(S^{n-1}) \cong Z_2$ $(n \geqq 4)$, the homomorphism induced by the projection. $\#$ denotes the connected sum operation, and for an integer $m \geqq 0$, mA_α, mB_β, and $m(S^{n+1} \times S^{n-1})$ denote the connected sum of m-copies of A_α, B_β, and $S^{n+1} \times S^{n-1}$, respectively. We put rank $H_{n-1}(M) = k$, and put $\operatorname{rank} \phi = q$ and $p = k - q$ if M is of type (0+I). p and q are the homotopy invariants of M.

The homotopy groups $\pi_n(SO_n)$ are given as follows (Kervaire [9], Paechter [13]);

$n(\geqq 3, \neq 6)$	$8s$	$8s+1$	$8s+2$	$8s+3$	$8s+4$	$8s+5$	$8s+6$	$8s+7$
$\pi_n(SO_n)$	$Z_2+Z_2+Z_2$	Z_2+Z_2	Z_4	Z	Z_2+Z_2	Z_2	Z_4	Z,

Table 1

$n(\geqq 4)$	Type 0
$4t-1$	$A_a\#(k-1)(S^{n+1}\times S^{n-1})$, $a\geqq 0$ $t=2 \Rightarrow a$; even $\geqq 0$
$n=4t$ (t; odd)	$A_{(a,0)}\#(k-1)(S^{n+1}\times S^{n-1})$, $a=0, 1$
$n=4t$ (t ; even)	$A_{(a,0,b)}\#(k-1)(S^{n+1}\times S^{n-1})$, $a, b=0, 1$ $A_{(1,0,0)}\#A_{(0,0,1)}\#(k-2)(S^{n+1}\times S^{n-1})$
$n=4t+1$ (t; odd)	$A_a\#(k-1)(S^{n+1}\times S^{n-1})$, $a=0, 1$
$n=4t+1$ (t; even)	$A_{(a,b)}\#(k-1)(S^{n+1}\times S^{n-1})$, $a, b=0, 1$ $A_{(1,0)}\#A_{(0,1)}\#(k-2)(S^{n+1}\times S^{n-1})$
$n=4t+2$ ($t\geqq 2$)	$A_a\#(k-1)(S^{n+1}\times S^{n-1})$, $a=0, 1, 2$
6	$k(S^7\times S^5)$

Table 2

$n(\geqq 4)$	Type I	Type $(0+\mathrm{I})$
$4t-1$	$t\geqq 3 \Rightarrow$ Nothing $t=2 \Rightarrow kB_c$, c; odd>0	$t\geqq 3 \Rightarrow$ Nothing $t=2 \Rightarrow p(S^8\times S^6)\#qB_c$, c; odd>0
$4t$ (t; odd)	$kB_{(0,1)}$	$p(S^{n+1}\times S^{n-1})\#qB_{(0,1)}$
$4t$ (t ; even)	$kB_{(0,1,0)}$, $kB_{(0,1,1)}$ $(k-1)B_{(0,1,0)}\#B_{(0,1,1)}$, $k\geqq 2$ $(k-2)B_{(0,1,0)}\#2B_{(0,1,1)}$, $k\geqq 3$	$p(S^{n+1}\times S^{n-1})\#qB_{(0,1,0)}$ $p(S^{n+1}\times S^{n-1})\#qB_{(0,1,1)}$ $p(S^{n+1}\times S^{n-1})\#(q-1)B_{(0,1,0)}\#B_{(0,1,1)}$, $q\geqq 2$ $p(S^{n+1}\times S^{n-1})\#(q-2)B_{(0,1,0)}\#2B_{(0,1,1)}$, $q\geqq 3$ $A_{(0,0,1)}\#(p-1)(S^{n+1}\times S^{n-1})\#qB_{(0,1,0)}$
$4t+1$	Nothing	Nothing
$4t+2$	Nothing	Nothing

Table 3

$n(\geqq 4)$	Type II	Type (0+II)
$4t-1$	$V\begin{pmatrix} d \\ 0 \end{pmatrix} \# (r-1) V\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $d \geqq 0$	$A_a \# (p-1)(S^{n+1} \times S^{n-1}) \# r V\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $a \geq 0$ $p(S^{n+1} \times S^{n-1}) \# V\begin{pmatrix} d \\ 0 \end{pmatrix} \# (r-1) V\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $d > 0$
	$t=2 \Rightarrow d$; even $\geqq 0$	$t=2 \Rightarrow a, d$; even, $a \geqq 0$, $d > 0$
$4t$ (t; odd)	$V\begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix} \# (r-1) V\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $d=0, 1$	$p(S^{n+1} \times S^{n-1}) \# V\begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix} \# (r-1) V\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $d=0, 1$ $A_{(1,0)} \# (p-1)(S^{n+1} \times S^{n-1}) \# r V\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
$4t$ (t; even)	$V\begin{pmatrix} d & 0 & 0 \\ d & 0 & 0 \end{pmatrix} \# (r-1) V\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $V\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & d \end{pmatrix} \# (r-1) V\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $V\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \# V\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & d \end{pmatrix} \# (r-2) V\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where $d=0, 1$.	$p(S^{n+1} \times S^{n-1}) \#$(Manifolds of type II). $A_{(1,0,0)} \# (p-1)(S^{n+1} \times S^{n-1}) \# V\begin{pmatrix} 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \# (r-1) V\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A_{(0,0,1)} \# (p-1)(S^{n+1} \times S^{n-1}) \# V\begin{pmatrix} d & 0 & 0 \\ d & 0 & 0 \end{pmatrix} \# (r-1) V\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A_{(1,0,1)} \# (p-1)(S^{n+1} \times S^{n-1}) \# V\begin{pmatrix} d & 0 & 0 \\ d & 0 & 0 \end{pmatrix} \# (r-1) V\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where $d=0, 1$. $A_{(1,0,0)} \# A_{(0,0,1)} \# (p-2)(S^{n+1} \times S^{n-1}) \# r V\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$4t+1$ $(t;\text{ odd})$	$V\begin{pmatrix} d \\ d \end{pmatrix} \# (r-1)V\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $d=0, 1$	$p(S^{n+1}\times S^{n-1}) \# V\begin{pmatrix} d \\ d \end{pmatrix} \# (r-1)V\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $d=0, 1$ $A_1 \# (p-1)(S^{n+1}\times S^{n-1}) \# rV\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$4t+1$ $(t;\text{ even})$	$V\begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix} \# (r-1)V\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $V\begin{pmatrix} 0 & 1 \\ 0 & d \end{pmatrix} \# (r-1)V\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $V\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \# V\begin{pmatrix} 0 & 1 \\ 0 & d \end{pmatrix} \# (r-2)V\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, where $d=0, 1$.	$p(S^{n+1}\times S^{n-1}) \#$(Manifolds of type II). $A_{(1,0)} \# (p-1)(S^{n+1}\times S^{n-1}) \# V\begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} \# (r-1)V\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $A_{(0,1)} \# (p-1)(S^{n+1}\times S^{n-1}) \# V\begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix} \# (r-1)V\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $A_{(1,1)} \# (p-1)(S^{n+1}\times S^{n-1}) \# V\begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix} \# (r-1)V\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, where $d=0, 1$. $A_{(1,0)} \# A_{(0,1)} \# (p-2)(S^{n+1}\times S^{n-1}) \# rV\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
$4t+2$ $(t\geqq 2)$	$V\begin{pmatrix} d \\ d \end{pmatrix} \# (r-1)V\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $d=0, 2$ $V\begin{pmatrix} 1 \\ 0 \end{pmatrix} \# (r-1)V\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$p(S^{n+1}\times S^{n-1}) \# V\begin{pmatrix} d \\ d \end{pmatrix} \# (r-1)V\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $d=0, 2$ $p(S^{n+1}\times S^{n-1}) \# V\begin{pmatrix} 1 \\ 0 \end{pmatrix} \# (r-1)V\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $A_a \# (p-1)(S^{n+1}\times S^{n-1}) \# rV\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $a=1, 2$
6	$rV\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$p(S^7\times S^5) \# rV\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

and $\pi_6(SO_6)=0$.

In Table 3, $2r=\operatorname{rank}\phi$ and $p=k-2r$. They are the homotopy invariants of M. Let $W\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$ be a handlebody $D^{2n+1}\underset{\{f_1,f_2\}}{\cup}\{D_1^{n+1}\times D_1^n\cup D_2^{n+1}\times D_2^n\}$ such that the link $f_1(\partial D_1^{n+1}\times 0)\cup f_2(\partial D_2^{n+1}\times 0)\subset\partial D^{2n+1}$ of the disjoint imbeddings $f_i\colon \partial D_i^{n+1}\times D_i^n\to\partial D_i^{2n+1}$, $i=1, 2$, has the non-zero linking element (Haefliger [5]) and the normal bundles of the spheres S_i^{n+1}, with hemispheres $D_i^{n+1}\times 0$ and D_i^{n+1} in D^{2n+1}, $i=1, 2$, have the characteristic elements $\alpha_1, \alpha_2\in\pi_n(SO_n)$ respectively such that $\pi(\alpha_1)=\pi(\alpha_2)=0$. In other words, $W\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$ is constructed as follows; let $\bar{A}$, $\bar{A}'$ be the n-disk bundles over $(n+1)$-spheres with characteristic elements α_1, α_2 respectively. Then, plumbing $\bar{A}$ and $\bar{A}'$ along a circles S^1 and then attaching a 3-cell with thickness D^{2n-2} to the boundary, $W\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$ is obtained (Ishimoto [7]). Let $V\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}=\partial W\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$. $V\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$ never has the homotopy type of the connected sum of the two $(n-1)$-sphere bundles over $(n+1)$-spheres. For an integer $m\geqq 0$, $mV\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$ denotes the connected sum of m-copies of $V\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$. If $\pi_n(SO_n)$ has the several direct summands, for example, if $\alpha_i=\alpha_i^1+\alpha_i^2$, $i=1, 2$, we denote $V\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$ by $V\begin{pmatrix}\alpha_1^1 & \alpha_1^2\\ \alpha_2^1 & \alpha_2^2\end{pmatrix}$.

Theorem 1′. *Let $\bar{A}_\alpha$, $\bar{B}_\beta$ be the n-disk bundles over $(n+1)$-spheres associated with A_α, B_β respectively. In the above tables if we replace $S^{n+1}\times S^{n-1}$, A_α, B_β, $V\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$, and $\sharp$ respectively by $S^{n+1}\times D^n$, $\bar{A}_\alpha$, $\bar{B}_\beta$, $W\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$, and the boundary connected sum operation $\natural$, then Table* 1, *Table* 2, *and Table* 3 *give the complete classification of handlebodies of $\mathscr{H}(2n+1, k, n+1)$ $(n\geqq 4)$ up to diffeomorphism.*

Theorem 2. *In Theorem* 1, *the representation of M is unique* $\bmod \theta_{2n}$ *in the following case when*

(i) *M is of type* 0, *or*

(ii) *M is of type* I *and $n\neq 4t$ (t; even), or*

(iii) *M is of type* (0+I) *and $n\neq 4t$ (t; even), or*

(iv) *M is of type* II *and $n=4t-1$ or* 6, *or*

(v) *M is of type* (0+II) *and $n=4t-1$ or* 6,

especially, in the above (i)–(v),

(vi) *when* $n=4t-1$ *or* 6.

Theorem 1 and Theorem 2 are obtained from the fact that $M=\partial W \bmod \theta_{2n}$, $W\in\mathscr{H}(2n+1, k, n+1)$, and by classifying W up to diffeomorphism. In [17], Tamura classified some $(n-2)$-connected $2n$-manifolds using Smale's decomposition theory although the uniqueness of the representations was not obtained except some cases. Our results of Theorem 1 and Theorem 2 include his results as manifolds of type 0 and furthermore we have the uniqueness of the representations. Our classifications are performed for $n\geqq 4$, at stable range in a sense. For $n=3$, Wall has classified the simply connected 6-manifolds which have torsion free homology groups and are 2-parallelizable, that is, $w_2=0$ ([20]).

Theorem 3. *Let M be an $(n-2)$-connected $2n$-manifold $(n\geqq 4)$ which has torsion free homology groups and satisfies the hypothesis (H). Then M is decomposed as $M=M_1\# M_2$, where M_1 is an $(n-2)$-connected $2n$-manifold which has the vanishing n-th homology group and satisfies the hypothesis (H), and M_2 is an $(n-1)$-connected $2n$-manifold. M_2 is always unique up to diffeomorphism* $\bmod \theta_{2n}$. *If $n=4t-1$ $(t\geqq 2)$ M_1 is also unique up to diffeomorphism* $\bmod \theta_{2n}$, *and the homotopy type of M_1- (a point) is always unique.*

Since there are much efforts on classifying $(n-1)$-connected $2n$-manifolds (Wall [18]), Theorem 3 is very important for our problems although it is not obvious whether M_1 is always unique up to diffeomorphism $\bmod \theta_{2n}$ or not.

From the above theorems we have

Theorem 4. *Let $n=4t-1$ $(t\geqq 2)$ and let M be an $(n-2)$-connected $2n$-manifold which have torsion free homology groups and is $(n-1)$-parallelizable if t is odd. Then M is decomposed as $M=M_1\# M_2$ uniquely* $\bmod \theta_{2n}$. *M_1 is a $2n$-manifold as shown in the above tables and is uniquely determined up to diffeomorphism* $\bmod \theta_{2n}$ *by* $S_q^2\colon H^{n-1}(M; Z_2)\to H^{n+1}(M; Z_2)$ *and the Pontrjagin class $P_t(M)$. M_2 is an $(n-1)$-connected $2n$-dimensional π-manifold and so, under the assumption that* $\mathrm{Arf}\, M_2=0$ *when $n=2^j-1$, diffeomorphic to* $S^n\times S^n\#\cdots\# S^n\times S^n \bmod \theta_{2n}$.

Theorem 4 is an extension of the result of Wall [20] to greater dimensions.

Corollary 5. *Let* $n=4t-1$ $(t\geqq 2)$ *and let* M_i, $i=1,2$, *be* $(n-2)$*-con nected* $2n$*-manifolds which have torsion free homology groups and are* $(n-1)$*-parallelizable if* t *is odd. If they are stably tangential homotopy equivalent, then they are diffeomorphic* $\bmod \theta_{2n}$ *under the assumption that the Arf invariants vanish for closed connected* $2n$*-dimensional* π*-manifolds when* $n=2^j-1$.

Corollary 6. *Let* $n=4t-1$ $(t\geqq 2)$ *and let* M *be an* $(n-2)$*-connected* $2n$*-dimensional* π*-manifold with torsion free homology groups. Then,* M *is uniquely represented* $\bmod \theta_{2n}$ *as*

$$M=\overbrace{S^{n+1}\times S^{n-1}\#\cdots\# S^{n+1}\times S^{n-1}}^{p}\#\overbrace{V\begin{pmatrix}0\\0\end{pmatrix}\#\cdots\# V\begin{pmatrix}0\\0\end{pmatrix}}^{r}\#\overbrace{S^n\times S^n\#\cdots\# S^n\times S^n}^{l},$$

where $2r=\operatorname{rank}\phi$, $p=\operatorname{rank}H_{n-1}(M)-2r$, $2l=\operatorname{rank}H_n(M)$, *and we make a similar assumption on the Arf invariant. (c.f. Ishimoto [6]).*

Directly from Theorem 1 and Theorem 3 we have,

Theorem 7. *Let* M *be an* $(n-2)$*-connected* $2n$*-manifold* $(n\geqq 4)$ *with torsion free homology groups of type I or of type* $(0+\mathrm{I})$. *If* $n=4t$ $(t;$ *odd*$\geqq 1)$, M *is uniquely represented* $\bmod \theta_{2n}$ *as*

$$M=\overbrace{S^{n+1}\times S^{n-1}\#\cdots\# S^{n+1}\times S^{n-1}}^{p}\#\overbrace{B_{(0,1)}\#\cdots\# B_{(0,1)}}^{q}\# M_2,$$

where $q=\operatorname{rank}\phi$, $p=\operatorname{rank}H_{n-1}(M)-q$, *and* M_2 *is an* $(n-1)$*-connected* $2n$*-manifold determined uniquely up to diffeomorphism* $\bmod \theta_{2n}$. *If* $n=4t+1$ *or* $4t+2$, $t\geqq 1$, *such manifolds* M *with non-trivial* $H_{n-1}(M)$ *do not exist.*

Theorem 8. *Let* M *be a* 4*-connected* 12*-manifold with torsion free homology groups. Then* M *is uniquely represented as*

$$M=\overbrace{S^7\times S^5\#\cdots\# S^7\times S^5}^{p}\#\overbrace{V\begin{pmatrix}0\\0\end{pmatrix}\#\cdots\# V\begin{pmatrix}0\\0\end{pmatrix}}^{r}\# M_2,$$

where $2r=\operatorname{rank}\phi$, $p=\operatorname{rank}H_5(M)-2r$, *and* M_2 *is a* 5*-connected* 12*-manifold* *completely determined by Theorem* 4 *of Wall* [18]. (*We note that* $\theta_{12}=0$).

Similarly we can consider to classify the simply connected $(2n+1)$-manifolds $(n\geqq 5)$ which have non-trivial homology groups only at dimensions 0, $n-1$, $n+2$, and $2n+1$ and are $(n-1)$-parallelizable. If S_q^2 is replaced by Adem's secondary cohomology operation, then similar arguments are applicable.

Part I. Classification of Handlebodies of $\mathscr{H}(2n+1, k, n+1)$

In Part I, we give the proof of Theorem 1′; we classify the handlebodies of $\mathscr{H}(2n+1, k, n+1)$, $n\geqq 4$, up to diffeomorphism using Wall's theory [19], and clarify the figures of the representative handlebodies.

Let $(H; \lambda, \alpha)$ be a triple consisting of a free abelian group H of finite rank, a bilinear form $\lambda\colon H\times H\to\pi_s(S^{m-s})$, $2m\geqq 3s+3$, $s\geqq 2$, and a map $\alpha\colon H\to\pi_{s-1}(SO_{m-s})$. We call it an $(H; \lambda, \alpha)$-system or simply an algebraic system if the following conditions are satisfied;

(1) $\lambda(y, x)=(-1)^s\lambda(x, y)$,

$\lambda(x, x)=S\pi\alpha(x)$,

(2) $\alpha(x+y)=\alpha(x)+\alpha(y)+\partial\lambda(x, y)$,

where $\pi\colon\pi_{s-1}(SO_{m-s})\to\pi_{s-1}(S^{m-s-1})$ is the homomorphism induced by the projection of SO_{m-s} to S^{m-s-1}, $S\colon\pi_{s-1}(S^{m-s-1})\to\pi_s(S^{m-s})$ is the suspension homomorphism, and $\partial\colon\pi_s(S^{m-s})\to\pi_{s-1}(SO_{m-s})$ is the boundary homomorphism in the homotopy exact sequence of the fibering $SO_{m-s}\to SO_{m-s+1}\to S^{m-s}$.

If there is no confusion we call an $(H; \lambda, \alpha)$-system briefly a system. The two systems $(H; \lambda, \alpha)$, $(H'; \lambda', \alpha')$ are isomorphic if and only if there exists an isomorphism $h\colon H\to H'$ such that $\lambda=\lambda'\circ(h\times h)$ and $\alpha=\alpha'\circ h$.

Wall's Theorem. *If* $s\geqq 2$, $2m\geqq 3s+3$, *diffeomorphism classes of handlebodies of* $\mathscr{H}(m, k, s)$ *correspond bijectively to isomorphism classes of* $(H; \lambda, \alpha)$-*systems with* rank $H=k$.

For a handlebody W of $\mathscr{H}(m, k, s)$, the corresponding algebraic system is defined by $H=H_s(W)$, Wall's pairing λ ([19]), and the map α assigning to each $x\in H_s(W)\cong\pi_s(W)$ the characteristic element $\alpha(x)$ of the normal bundle of the imbedded s-sphere which represents x.

1. Types of Handlebodies of $\mathscr{H}(2n+1, k, n+1)$

Let H be a free abelian group of rank k, and let $\phi\colon H\times H\to Z_2$ be a symmetric bilinear form.

Lemma 1.1. (i) *If there exists $x\in H$ such that $\phi(x, x)\neq 0$, then under some basis $\{e_1,\cdots, e_k\}$ of H, ϕ is represented by a $k\times k$ matrix*

$$(\phi(e_i, e_j))=\begin{pmatrix} 0 & & & & & \mathbf{0} \\ & \ddots^{\,p} & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & \ddots^{\,q} & \\ \mathbf{0} & & & & & 1 \end{pmatrix},$$

where $q\neq 0$ and p, q are independent of the choice of the base $\{e_1,\cdots, e_k\}$.

(ii) *If $\phi(x, x)=0$ for any $x\in H$, then under some basis $\{e_1,\cdots, e_k\}$ of H, ϕ is represented by a $k\times k$ matrix*

$$(\phi(e_i, e_j))=\begin{pmatrix} 0 & & & & & \mathbf{0} \\ & \ddots^{\,p} & & & & \\ & & 0 & & & \\ & & & \boxed{\begin{matrix}0&1\\1&0\end{matrix}} & & \\ & & & & \ddots^{\,r} & \\ \mathbf{0} & & & & & \boxed{\begin{matrix}0&1\\1&0\end{matrix}} \end{pmatrix},$$

where p, r are independent of the choice of the base $\{e_1,\cdots, e_k\}$.

The above two representations are exclusive.

Proof. Let $\{e_1,\cdots, e_k\}$ be a base of H and let $A=(a_{ij})=(\phi(e_i, e_j))$ be the symmetric $k\times k$ matrix with the components in Z_2. Then to exchange the base corresponds to multiply unimodular $k\times k$ matrices P, P^t with integer components to A from the left and from the right respectively. We show the lemma using a sequence of elementary row and column operations on A, performing the same operation on row and column (c.f. Artin [1]).

We show (ii). If there exists $a_{i1}\neq 0$ $(i\geqq 2)$ we may assume $a_{21}=a_{12}=1$. Then, adding the second row to the i-th row $(i\geqq 3)$ and subsequently the second column to the i-th column, all non-zero a_{i1}, a_{1i} $(i\geqq 3)$ are killed. Similarly all non-zero a_{i2}, a_{2i} $(i\geqq 3)$ can be killed. Repeating this and pushing out the zero columns to the left, we obtain the matrix of (ii). Since $2r=\operatorname{rank} A$, r is independent of the choice of the bases.

We show (i). We may assume $a_{11}=1$, and so all non-zero a_{i1}, a_{1i} $(i\geqq 2)$ can be killed. Let the resulting matrix be $A'=(a'_{ij})$. If there exists $a'_{ii}\neq 0$ $(i\geqq 2)$ we may assume $a'_{22}=1$ and the situation is quite similar. Repeating such operations till the diagonal elements of the rest are all zero and applying the operations of (ii) to the rest, we have the following matrix;

$$\begin{pmatrix} 0 & & & & & & & \mathbf{0} \\ & \ddots & & & & & & \\ & & 0 & & & & & \\ & & & 1 & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & & \\ & & & & & & \boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}} & \\ & & & & & & & \ddots \\ \mathbf{0} & & & & & & & & \boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}} \end{pmatrix},$$

where zero columns have been pushed out to the left. Now, the following reformations of the matrix will conclude the proof of (ii):

$$\begin{pmatrix} 1&0&0 \\ 0&0&1 \\ 0&1&0 \end{pmatrix} \xrightarrow{(2)+(1)} \begin{pmatrix} 1&0&1 \\ 0&0&1 \\ 1&1&0 \end{pmatrix} \xrightarrow{(3)+(1)} \begin{pmatrix} 1&1&1 \\ 1&0&1 \\ 1&1&0 \end{pmatrix} \xrightarrow{(1)+(2)} \begin{pmatrix} 1&0&1 \\ 0&1&0 \\ 1&0&0 \end{pmatrix} \xrightarrow{(1)+(3)} \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix},$$

where $(i)+(j)$ means the operation of the indicated row and column. Since $q=\operatorname{rank} A$, q is independent of the choice of the bases. This completes the proof.

Let H be a free abelian group of rank k, and let $\phi\colon H\times H\to Z_2$ be a symmetric bilinear form. We define the rank of ϕ by the rank of the representing matrix. We call ϕ to be of *type* 0 if $\operatorname{rank}\phi=0$, that is, $\phi(x, y)=0$ for all $x, y\in H$, of *type* I if there exists an $x\in H$ such that

$\phi(x, x)\neq 0$ and $\operatorname{rank}\phi=k$, and of *type* $(0+\mathrm{I})$ if there exists an $x\in H$ such that $\phi(x, x)\neq 0$ and $\operatorname{rank}\phi<k$. We also call ϕ to be of *type* II if $\phi(x, x)=0$ for any $x\in H$ and $\operatorname{rank}\phi=k$, and of *type* $(0+\mathrm{II})$ if $\phi(x, x)=0$ for any $x\in H$ and $0<\operatorname{rank}\phi<k$. The type of ϕ is uniquely determined and the corresponding matrix representation is given by (i) or (ii) of Lemma 1.1.

We define the *type* of an $(H; \lambda, \alpha)$-system of our case by that of the bilinear form λ. Isomorphic $(H; \lambda, \alpha)$-systems belong to the same type. We define the *type* of a handlebody $W\in\mathscr{H}(2n+1, k, n+1)$ by that of the corresponding $(H; \lambda, \alpha)$-system. Diffeomorphic handlebodies of $\mathscr{H}(2n+1, k, n+1)$ belong to the same type. In Part II, it will be shown that the type of a handlebody $W\in\mathscr{H}(2n+1, k, n+1)$ is determined by S_q^2: $H^{n-1}(\partial W; Z_2)\rightarrow H^{n+1}(\partial W; Z_2)$ and the cup product in $H^*(\partial W; Z_2)$.

2. Calculations of ∂ and π

The homotopy groups $\pi_n(SO_n)$ are given in the table of the introduction. Let ∂_n: $\pi_{n+1}(S^n)\rightarrow\pi_n(SO_n)$ be the boundary homomorphism in the homotopy exact sequence of the fibering $SO_n\rightarrow SO_{n+1}\rightarrow S^n$, and let π_n: $\pi_n(SO_n)\rightarrow\pi_n(S^{n-1})$ be the homomorphism induced by the projection of SO_n to S^{n-1}. By Kervaire [9] and Paechter [13], we know the following results, where 1 denotes the generator of $\pi_n(S^{n-1})\cong\pi_{n+1}(S^n)\cong Z_2 (n\geqq 4)$.

Lemma 2.1.

(i) $\partial_{4t-2}(1)=2\in Z_4$ *for* $t\geqq 3$, *and* $\partial_6(1)=0$.

(ii) $\partial_{4t-1}=0$ *for* $t\geqq 1$.

(iii) $\partial_{4t}\neq 0$ *for* $t\geqq 1$, *more precisely*,

$\partial_{8s}(1)=(1, 0, 0)\in Z_2+Z_2+Z_2$ *for* $s\geqq 1$, *and*

$\partial_{8s+4}(1)=(1, 0)\in Z_2+Z_2$ *for* $s\geqq 0$.

(iv) $\partial_{4t+1}\neq 0$ *for* $t\geqq 1$, *more precisely*,

$\partial_{8s+1}(1)=(1, 0)\in Z_2+Z_2$ *for* $s\geqq 1$, *and*

$\partial_{8s+5}(1)=1\in Z_2$ *for* $s\geqq 0$.

Lemma 2.2.

(i) $\pi_{4t-1}=0$ *for* $t\geqq 3$, *and if* $t=1, 2$,

$\pi_{4t-1}: Z\to Z_2$ *satisfies* $\pi_{4t-1}(1)=1$.

(ii) $\pi_{4t}\neq 0$ *for* $t\geqq 1$, *more precisely*,

$\pi_{8s}(1, 0, 0)=\pi_{8s}(0, 0, 1)=0$, $\pi_{8s}(0, 1, 0)=1$ *for* $s\geqq 1$,

and $\pi_{8s+4}(1, 0)=0$, $\pi_{8s+4}(0, 1)=1$, *for* $s\geqq 0$.

(iii) $\pi_{4t+1}=0$ *for* $t\geqq 1$.

(iv) $\pi_{4t+2}=0$ *for* $t\geqq 1$.

These results are known, except precise informations of ∂_{4t}, ∂_{4t+1}, and π_{4t}, by Kervaire [9] and using the homotopy exact sequence of the fibering $SO_{n-1}\to SO_n\to S^{n-1}$.

Since $\pi_{8s+5}(V_{m,m-8s-i})\cong\pi_{8s+4}(SO_{8s+i})$ (m is sufficiently large), by Paechter [13] we know the precise correspondence of suspension homomorphisms $\pi_{8s+4}(SO_{8s+3})\to\pi_{8s+4}(SO_{8s+4})\to\pi_{8s+4}(SO_{8s+5})$. So that, behavior of ∂_{8s+4}, π_{8s+4} is known from the homotopy exact sequences of the fiberings $SO_{8s+3}\to SO_{8s+4}\to S^{8s+3}$, $SO_{8s+4}\to SO_{8s+5}\to S^{8s+4}$. Similarly, using the splitting exact sequences (Kervaire [9])

$$0\to\pi_{8s+1}(V_{m,m-8s+i})\to\pi_{8s}(SO_{8s-i})\to\pi_{8s}(SO_m)\to 0$$

($i\leqq 1$, $s\geqq 1$, m is sufficiently large) and

$$0\to\pi_{8s+2}(V_{m,m-8s+i})\to\pi_{8s+1}(SO_{8s-i})\to\pi_{8s+1}(SO_m)\to 0$$

($i\leqq 3$, $s\geqq 1$, m is sufficiently large), by Paechter's computations [13], we know the precise correspondence of suspension homomorphisms $\pi_{8s}(SO_{8s-1})\to\pi_{8s}(SO_{8s})\to\pi_{8s}(SO_{8s+1})$, $\pi_{8s+1}(SO_{8s+1})\to\pi_{8s+1}(SO_{8s+2})$. So that, behavior of ∂_{8s}, π_{8s}, and ∂_{8s+1} is known from the homotopy exact sequences of the fiberings $SO_{n-1}\to SO_n\to S^{n-1}$, $n=8s, 8s+1, 8s+2$.

3. Classification of Handlebodies of Type 0

Let W be a handlebody of $\mathscr{H}(2n+1, k, n+1)$ and let $(H; \lambda, \alpha)$ be the corresponding algebraic system. Since W is of type 0 if and only if λ is trivial[(1)], classifying the handlebodies W of type 0 up to diffeomorphism comes to classifying the homomorphism $\alpha: H \to \pi_n(SO_n)$ up to equivalence, where the homomorphisms $\alpha_i: H \to \pi_n(SO_n)$, $i=1, 2$, are equivalent if and only if there exists an isomorphism $h: H \to H$ such that $\alpha_1 = h \circ \alpha_2$. We note that since $s\pi\alpha(x) = \lambda(x, x) = 0$, α maps H to $\operatorname{Ker} \pi$, $\pi: \pi_n(SO_n) \to \pi_n(S^{n-1})$.

If $W \in \mathscr{H}(2n+1, k, n+1)$ is a handlebody of type 0, any basis $\{u_1, u_2, \cdots, u_k\}$ of $H = H_{n+1}(W)$ gives a representation as $W = \bar{A}_{\alpha_1} \natural \bar{A}_{\alpha_2} \natural \cdots \natural \bar{A}_{\alpha_k}$, where $\natural$ means the boundary connected sum and $\bar{A}_{\alpha_i}$, $i=1, 2, \cdots, k$, are n-disk bundles over $(n+1)$-spheres with characteristic classes $\alpha_i = \alpha(u_i)$ such that $\pi(\alpha_i) = 0$ ([7]). For an integer $m \geqq 0$, $m(S^{n+1} \times D^n)$ denotes the boundary connected sum of m copies of $S^{n+1} \times D^n$.

Theorem 3.1. *The handlebodies W of type 0 of $\mathscr{H}(2n+1, k, n+1)$, $n \geqq 4$, are uniquely represented up to diffeomorphism as follows:*

(i) *If $n = 4t-1$ $(t \geqq 2)$, $W = \bar{A}_a \natural (k-1)(S^{n+1} \times D^n)$,*
where $a \in Z \cong \pi_{4t-1}(SO_{4t-1})$, $a \geqq 0$, especially $a \in 2Z$, $a \geqq 0$, if $t=2$.

(ii) *In the case when $n = 4t$ $(t \geqq 1)$,*
if $n = 8s+4$ $(s \geqq 0)$, $W = \bar{A}_{(a,0)} \natural (k-1)(S^{n+1} \times D^n)$,
where $(a, 0) \in Z_2 + Z_2 \cong \pi_{8s+4}(SO_{s+4})$, and
if $n = 8s$ $(s \geqq 1)$, $W = \bar{A}_{(a,0,b)} \natural (k-1)(S^{n+1} \times D^n)$,
or $W = \bar{A}_{(1,0,0)} \natural \bar{A}_{(0,0,1)} \natural (k-2)(S^{n+1} \times D^n)$,
where $(a, 0, b), (1, 0, 0), (0, 0, 1) \in Z_2 + Z_2 + Z_2 \cong \pi_{8s}(SO_{8s})$.

(iii) *In the case when $n = 4t+1$ $(t \geqq 1)$,*
if $n = 8s+5$ $(s \geqq 0)$, $W = \bar{A}_a \natural (k-1)(S^{n+1} \times D^n)$,
where $a \in Z_2 \cong \pi_{8s+5}(SO_{8s+5})$, and
if $n = 8s+1$ $(s \geqq 1)$, $W = \bar{A}_{(a,b)} \natural (k-1)(S^{n+1} \times D^n)$,
or $W = \bar{A}_{(1,0)} \natural \bar{A}_{(0,1)} \natural (k-2)(S^{n+1} \times D^n)$,
where $(a, b), (1, 0), (0, 1) \in Z_2 + Z_2 \cong \pi_{8s+1}(SO_{8s+1})$.

(iv) *In the case when $n = 4t+2$ $(t \geqq 1)$,*
if $t \geqq 2$, $W = \bar{A}_a \natural (k-1)(S^{n+1} \times D^n)$,
where $a = 0, 1, 2 \in Z_4 \cong \pi_{4t+2}(SO_{4t+2})$, and
if $t=1$, $W = k(S^7 \times D^6)$.

(1) Equivalently if and only if $S_q^2: H^{n-1}(\partial W; Z_2) \to H^{n+1}(\partial W; Z_2)$ is trivial (See Part II).

Proof. We consider the matrix representations of homomorphisms α: $H\rightarrow\pi_n(SO_n)$, where H is a free abelian group of rank k. By Lemma 2.2, $\mathrm{Ker}\,\pi$ are given as follows:

(1) $\mathrm{Ker}\,\pi_{4t-1}=\begin{cases} Z & \text{if } t\geqq 3,\\ 2Z & \text{if } t=1,2.\end{cases}$

(2) $\mathrm{Ker}\,\pi_{8s+4}=\{1,0)\}(\cong Z_2)\subset Z_2+Z_2$ $(s\geqq 0)$,

$\mathrm{Ker}\,\pi_{8s}=\{(1,0,0),(0,0,1)\}(\cong Z_2+Z_2)\subset Z_2+Z_2+Z_2$ $(s\geqq 1)$.

(3) $\mathrm{Ker}\,\pi_{8s+5}=Z_2$ $(s\geqq 0)$, $\mathrm{Ker}\,\pi_{8s+1}=Z_2+Z_2$ $(s\geqq 1)$.

(4) $\mathrm{Ker}\,\pi_{4t+2}=\begin{cases} Z_4 & \text{if } t\geqq 2,\\ 0 & \text{if } t=1.\end{cases}$

Let $\{u_1,\cdots,u_k\}$ be a basis of H and let $\alpha_i=\alpha(e_i)$. If $\mathrm{Ker}\,\pi=Z$ or $2Z$, by changing the basis, that is, by performing column operations to the $1\times k$ matrix $(\alpha_1,\alpha_2,\cdots,\alpha_k)$, α is represented as $(a,0,\cdots,0)$, where $a=G.C.D(\alpha_1,\alpha_2,\cdots,\alpha_k)>0$ or $a=0$ and a is independent of the choice of the base $\{u_1,\cdots,u_k\}$. If $\mathrm{Ker}\,\pi=Z_2$, similarly α is represented by some basis as $(a,0,\cdots,0)$, $a=0$, $1\in Z_2$. Let $\mathrm{Ker}\,\pi=Z_4$. If $\alpha(H)\subset\{0,2\}$, α has a representation as $(a,0,\cdots,0)$, $a=0$, $2\in Z_4$. If $\alpha(H)\not\subset\{0,2\}$, α is represented as $(1,0,\cdots,0)$. Let $\mathrm{Ker}\,\pi=Z_2+Z_2$ and $\alpha(u_j)=\alpha^1(u_j)+\alpha^2(u_j)$. We represent α by a $2\times k$ matrix $(\alpha^i(u_j))$. If $\alpha^1=0$, $\alpha^2\neq 0$, then α is represented as $\begin{pmatrix}00\cdots0\\10\cdots0\end{pmatrix}$, and if $\alpha^1\neq 0$, $\alpha=0$, then as $\begin{pmatrix}10\cdots0\\00\cdots0\end{pmatrix}$. If $\alpha^1\neq 0$, $\alpha^2\neq 0$, then α is represented as $\begin{pmatrix}10\cdots0\\10\cdots0\end{pmatrix}$ or $\begin{pmatrix}100\cdots0\\010\cdots0\end{pmatrix}$, and it is easily seen that such unimodular matrices L that $\begin{pmatrix}100\cdots0\\010\cdots0\end{pmatrix}=\begin{pmatrix}10\cdots0\\10\cdots0\end{pmatrix}L^t$ do not exsist. Thus, in any case of the above, the equivalence classes of the homomorphisms α correspond bijectively to the representations. This completes the proof.

4. Classification of Handlebodies of Type I

In this section we classify the handlebodies of type I of $\mathscr{H}(2n+1,k,n+1)$, $n\geqq 4$, up to diffeomorphism, that is, $(H;\lambda,\alpha)$-systems of type I with rank $H=k$ up to isomorphism.

Let $(H;\lambda,\alpha)$ be a system of type I. A base $\{v_1,\cdots,v_k\}$ of H is called to be *orthogonal* if $\lambda(e_i, e_j)=\delta_{ij}\in Z_2\cong\pi_{n+1}(S^n)$. If $\{v_1,\cdots, v_k\}$ is an orthogonal base of H, we may replace v_j by $v_j'=2lv_i+v_j$ $(i\neq j)$, where l is an integer; the new base is also orthogonal. An orthogonal base $\{v_1, \cdots, v_k\}$ of H gives a representation of the corresponding handlebody W as $W=\bar{B}_{\alpha_1}\natural\bar{B}_{\alpha_2}\natural\cdots\natural\bar{B}_{\alpha_k}$, where $\bar{B}_{\alpha_i}$, $i=1, 2,\cdots, k$, are n-disk bundles over $(n+1)$-spheres with the characteristic classes $\alpha_i=\alpha(v_i)$ such that $\pi(\alpha_i)=1$ ([7]). For an integer $m\geqq 0$, $m\bar{B}_\alpha$ denotes the bundary connected sum of m copies of $\bar{B}_\alpha$.

Theorem. 4.1. *Let $n=4t-1$ $(t\geqq 2)$. If $t\geqq 3$, the handlebodies of type I of $\mathscr{H}(2n+1, k, n+1)$ do not exist. If $t=2$, i.e. $n=7$, the handlebodies W of type* I *of $\mathscr{H}(2n+1, k, n+1)$ are uniquely represented up to diffeomorphism as $W=k\bar{B}_c$, where c is a positive odd integer of $\pi_7(SO_7)\cong Z$.*

To prove the theorem we use the following lemma.

Lemma 4.2. *Let H be a free abelian group of rank k, λ: $H\times H\rightarrow Z_2$ a symmetric bilinear form of type* I, *and α: $H\rightarrow Z$ a homomorphism which takes odd integers on some[(2)] orthogonal base. Then there exists such an orthogonal base $\{v_1, v_2,\cdots, v_k\}$ of H that $\alpha(v_1)=\alpha(v_2)=\cdots=\alpha(v_k)=c>0$, where c is an odd integer and independent of the choice of the bases.*

Proof. Let $\{v_1,\cdots, v_k\}$ be an orthogonal base of H and assume that $\alpha(v_i)$, $i=1, 2,\cdots k$, are odd integers. Let $\alpha(v_1)=\alpha(v_2)=\cdots=\alpha(v_{r-1})=a$ for some $r\geqq 2$. We show that by choosing some orthogonal base $(r-1)$ can be extended to r. Then, by repeating it we have the lemma.

Note that for given integers β, γ, $|\beta|<|\gamma|$, there exists an even integer $2l$ such that $\gamma=2l\beta+\gamma'$, $|\gamma'|\leqq|\beta|$. Let $\alpha_i=\alpha(v_i)$, $i=1, 2,\cdots, k$. If $|\alpha_1|<|\alpha_r|$, let $\alpha_r=2l\alpha_1+\alpha_r'$, $|\alpha_r'|\leqq|\alpha_1|$, and put $v_i'=v_i$ $(i\neq r)$, $v_r'=-2lv_r+v_r$. Then, $\alpha(v_1')=\alpha_1$, $\alpha(v_r')=\alpha_r'$, where α_r' is also an odd integer. If $|\alpha_1|>|\alpha_r|$, perform similarly. Apply similar way to the pair $(\alpha_1'=\alpha(v_1'), \alpha_r')$. Repeating this, we can perform Euclidean algorithm to the pair (α_1, α_r) by exchanging orthogonal bases until the residues are

(2) It is easily seen that 'some' induces 'any'.

equal up to sign.

Thus, replacing the base up to sign, there exists an orthogonal base $\{v_1', \cdots, v_k'\}$ of H such that $\alpha(v_1')=b$, $\alpha(v_2')=\cdots=\alpha(v_{r-1}')=a$, and $\alpha(v_r')=b$, where $b=G.C.D(\alpha_1=a, \alpha_r)>0$ and is an odd integer. Applying the similar way to the pairs $(\alpha(v_i'), \alpha(v_r'))$, $i=2, 3, \cdots, r-1$, we have an orthogonal base $\{v_1'', \cdots, v_k''\}$ of H such that $\alpha(v_1'')=\cdots=\alpha(v_r'')=b>0$. Thus $(r-1)$ is extended to r, and this completes the proof.

Proof of the theorem. If $(H; \lambda, \alpha)$ is a system of type I, there exists an orthogonal base $\{v_1, \cdots, v_k\}$ of H by Lemma 1.1. But, if $t\geqq 3$, $\lambda(v_i, v_i)=S\pi\alpha(v_i)=1$ contradicts to the fact that $\pi_{4t-1}=0$ (Lemma 2.2.). So that, there are no $(H; \lambda, \alpha)$-systems of type I, and therefore no handlebodies of type I, if $n=4t-1$ $t\geqq 3$. Let $t=2$. Since $\partial_{4t-1}=0$ for $t\geqq 1$ and $\alpha(x+y)=\alpha(x)+\alpha(y)+\partial\lambda(x, y)$, α: $H\to Z=\pi_7(SO_7)$ is a homomorphism. And, by Lemma 2.2, $\lambda(v_i, v_i)=S\pi\alpha(v_i)=1$ reduces $\alpha(v_i)$ to be odd. So that, we can apply Lemma 4.2. Then, c is an invariant of isomorphism classes of $(H; \lambda, \alpha)$-systems of type I, and two $(H; \lambda, \alpha)$-systems of type I are isomorphic if and only if they have the same value of c. Thus, c classifies $(H; \lambda, \alpha)$-systems of type I up to isomorphism and therefore handlebodies of type I up to diffeomorphism. This completes the proof.

Since $\pi_{4t+1}=\pi_{4t+2}=0$ for $t\geqq 1$, similarly we have

Theorem 4.3. *If $n=4t+1$ or $4t+2$, $t\geqq 1$, the handlebodies of type* I *of $\mathscr{H}(2n+1, k, n+1)$ do not exist.*

Let $n=4t$ $(t\geqq 1)$, and let $(H; \lambda, \alpha)$ be a system of type I. Since $S\pi\alpha(v_i)=\lambda(v_i, v_i)=1$ for any orthogonal base of H, $\alpha(v_i)$, $i=1, 2, \cdots, k$, are the elements of $\pi_n^{-1}(1)$. If $n=8s+4$ $(s\geqq 0)$, by Lemma 2.2, $\pi_{8s+4}^{-1}(1)$ consists of the two elements $(0, 1)$, $(1, 1)$ of $\pi_{8s+4}(SO_{8s+4})\cong Z_2+Z_2$, and so $\alpha(v_i)=(0, 1)$ or $(1, 1)$ for $i=1, 2, \cdots, k$. If $\alpha(v_i)=(1, 1)$, replace v_i by $-v_i$. Then, since $\partial_{8s+4}(1)=(1, 0)$, $\alpha(-v_i)=\alpha(v_i)+\partial\lambda(v_i, -v_i)=(0, 1)$. Therefore, there exists always an orthogonal base $\{v_1, \cdots, v_k\}$ of H such that $\alpha(v_1)=\alpha(v_2)=\cdots=\alpha(v_k)=(0, 1)$. This shows that if $n=8s+4$ $(s\geqq 0)$ there exists only one isomorphism class of $(H; \lambda, \alpha)$-systems of type I. Thus, we have

Theorem 4.4. *If $n=8s+4$ $(s\geqq 0)$, the handlebodies W of type* I *of*

$\mathscr{H}(2n+1, k, n+1)$ *are uniquely represented up to diffeomorphism as* $W=k\bar{B}_{(0,1)}$, $(0, 1)\in\pi_{8s+4}(SO_{8s+4})\cong Z_2+Z_2$.

If $n=8s$ $(s\geqq 1)$, by Lemma 2.2, $\pi_{8s}^{-1}(1)$ consists of the elements $(\gamma, 1, \delta)$ $(\gamma, \delta=0$ or $1)$ of $\pi_{8s}(SO_{8s})\cong Z_2+Z_2+Z_2$. Similarly to the above, if $\alpha(v_i)=(1, 1, 0)$ or $(1, 1, 1)$, then $\alpha(-v_i)=(0, 1, 0)$ or $(0, 1, 1)$ respectively. So that, if $n=8s$ $(s\geqq 1)$, any handlebody W of type I is represented as $W=r\bar{B}_{(0,1,0)}\natural(k-r)\bar{B}_{(0,1,1)}$, $0\leqq r\leqq k$.

Let $\alpha=(\alpha^1, \alpha^2, \alpha^3)$, $\alpha^i=p_i\circ\alpha$ $(i=1, 2, 3)$, where p_i is the projection of $\pi_{8s}(SO_{8s})\cong Z_2+Z_2+Z_2$ to the i-th direct summand. Then, since $\partial_{8s}(1)=(1, 0, 0)$, α^2, α^3 are homomorphisms and α^1 is a quadratic form over Z_2 with the associated bilinear form $\partial_{8s}\circ\lambda$. We call a $k\times k$ matrix L with integer components to be mod 2 *orthogonal* if $LL^t=E$ (mod 2).

Theorem 4.5. *If* $n=8s$ $(s\geqq 1)$, *the handlebodies* W *of type* I *of* $\mathscr{H}(2n+1, k, n+1)$ *are uniquely represented up to diffeomorphism as follows*:

(i) $W=k\bar{B}_{(0,1,0)}$, or

(ii) $W=k\bar{B}_{(0,1,1)}$, or

(iii) $W=(k-1)\bar{B}_{(0,1,0)}\natural\bar{B}_{(0,1,1)}$ $(k\geqq 2)$, or

(iv) $W=(k-2)\bar{B}_{(0,1,0)}\natural 2\bar{B}_{(0,1,1)}$ $(k\geqq 3)$,

where the characteristic elements belong to $\pi_{8s}(SO_{8s})\cong Z_2+Z_2+Z_2$.

Proof. The following assertions will complete the proof, where $\neq$ means that they are not diffeomorphic. Denote $\bar{B}_{(0,1,0)}$ by $\bar{B}$ and $\bar{B}_{(0,1,1)}$ by $\bar{B}'$.

Assertion 1. $k\bar{B}\neq(k-r)\bar{B}\natural r\bar{B}'$, $k\bar{B}'\neq(k-r)\bar{B}\natural r\bar{B}'$, $0<r<k$, *and* $k\bar{B}\neq k\bar{B}'$.

Proof. Let $(H; \lambda, \alpha)$ be the system of type I corresponding to $k\bar{B}$. If $k\bar{B}$ is diffeomorphic to $(k-r)\bar{B}\natural r\bar{B}'$ for some r, $0<r<k$, then there exist two orthogonal bases $\{v_1, v_2, \cdots, v_k\}$ and $\{v_1', v_2', \cdots, v_k'\}$ of H such that $\alpha(v_1)=\alpha(v_2)=\cdots=\alpha(v_k)=(0, 1, 0)$ and $\alpha(v_1')=\cdots=\alpha(v_{k-r}')=(0, 1, 0)$, $\alpha(v_{k-r+1}')=\cdots=\alpha(v_k')=(0, 1, 1)$. Since α^3: $H\to Z_2$ is a homomorphism,

this is a contradiction. We note that we have also proved $k\bar{B}\neq k\bar{B}'$. Let $(H;\ \lambda, \alpha)$ be the system of type I, corresponding to $k\bar{B}'$ and let $k\bar{B}'=(k-r)\bar{B}\natural r\bar{B}'$, $0<r<k$. Then, there exist two orthogonal bases $\{v_1, v_2, \cdots, v_k\}$ and $\{v_1', v_2', \cdots, v_k'\}$ of H such that $\alpha(v_1)=\cdots=\alpha(v_k)=(0, 1, 1)$ and $\alpha(v_1')=\cdots=\alpha(v_{k-r}')=(0, 1, 0)$, $\alpha(v_{k-r+1}')=\cdots=\alpha(v_k')=(0, 1, 1)$. Let $v_i'=\sum_{s=1}^{k} l_{is}v_s$, $i=1,2,\cdots, k$, and $L=(l_{ij})$ be the mod 2 orthogonal matrix. Then, for $1\leqq i\leqq k-r$, $0=\alpha^3(v_i')=\alpha^3(\sum_{s=1}^{k} l_{is}v_s)=\sum_{s=1}^{k} l_{is}$ (mod 2). But this contradicts to $\sum_{s=1}^{k} l_{is}=\sum_{s=1}^{k} l_{is}^2=1$ (mod 2).

Assertion 2. $(k-r)\bar{B}\natural r\bar{B}'=(k-r-2)\bar{B}\natural(r+2)\bar{B}'$, $0<r<k-2$, *where* '$=$' *means that they are diffeomorphic.*

Proof. We show that $\bar{B}\natural 3\bar{B}'=\bar{B}'\natural 3\bar{B}$. Let $(H;\ \lambda, \alpha)$, $(H';\ \lambda', \alpha)$ be the corresponding system of type I of $\bar{B}\natural 3\bar{B}'$, $\bar{B}'\natural 3\bar{B}$ respectively. Then, there exists an orthogonal base $\{v_1, v_2, v_3, v_4\}$ of H such that $\alpha(v_1)=(0, 1, 0)$ and $\alpha(v_2)=\alpha(v_3)=\alpha(v_4)=(0, 1, 1)$. Let

$$\begin{pmatrix} v_1' \\ v_2' \\ v_3' \\ v_4' \end{pmatrix}=\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

Then, $\{v_1', v_2', v_3', v_4'\}$ is a new orthogonal base of H since the 4×4 matrix is unimodular and mod 2 orthogonal. We have $\alpha^2(v_1')=\cdots=\alpha^2(v_4')=1$ and $\alpha^3(v_1')=1$, $\alpha^3(v_2')=\alpha^3(v_3')=\alpha^3(v_4')=0$. If $\alpha(v_1')=(1, 1, 1)$, or $\alpha(v_i')=(1, 1, 0)$, $2\leqq i\leqq 4$, replace v_1' by $-v_1'$ or v_i' by $-v_i'$. Then $\alpha(-v_1')=(0, 1, 1)$ and $\alpha(-v_i')=(0, 1, 0)$. Thus, there exists an orthogonal base $\{v_1', v_2', v_3', v_4'\}$ of H such that $\alpha(v_1')=(0, 1, 1)$ and $\alpha(v_2')=\alpha(v_3')=\alpha(v_4')=(0, 1, 0)$. This implies that $(H;\ \lambda, \alpha)$ is isomorphic to $(H';\ \lambda', \alpha')$, and therefore, $\bar{B}\natural 3\bar{B}'$ is diffeomorphic to $\bar{B}'\natural 3\bar{B}=3\bar{B}\natural\bar{B}'$.

Thus, if $0<r<k-2$, we have $(k-r)\bar{B}\natural r\bar{B}'=(k-r-3)\bar{B}\natural(3\bar{B}\natural\bar{B}')\natural(r-1)\bar{B}'=(k-r-3)\bar{B}\natural(\bar{B}\natural 3\bar{B}')\natural(r-1)\bar{B}'=(k-r-2)\bar{B}\natural(r+2)\bar{B}'$.

Assertion 3. $(k-1)\bar{B}\natural\bar{B}'\neq(k-2)\bar{B}\natural 2\bar{B}'$

Proof. Let $(H;\ \lambda, \alpha)$ be the system of type I corresponding to $(k-2)$

$\bar{B} \natural 2\bar{B}'$, and suppose that $(k-2)\bar{B} \natural 2\bar{B}'=(k-1)\bar{B} \natural \bar{B}'$. Then, there exist two orthogonal bases $\{v_1, v_2, \cdots, v_k\}$, $\{v'_1, v'_2, \cdots, v'_k\}$ of H such that $\alpha(v_1) =\alpha(v_2)=\cdots=\alpha(v_{k-2})=(0, 1, 0)$, $\alpha(v_{k-1})=\alpha(v_k)=(0, 1, 1)$ and $\alpha(v'_1)= \alpha(v'_2)=\cdots=\alpha(v'_{k-1})=(0, 1, 0)$, $\alpha(v'_k)=(0, 1, 1)$. There exists a mod 2 orthogonal matrix $L=(l_{ij})$ such that $(v'_1, v'_2, \cdots, v'_k)^t=L(v_1, v_2, \cdots, v_k)^t$. Since α^3: $H\to Z_2$ is a homomorphism, the above conditions on α insist L to be the following form:

$$L=\begin{pmatrix} \boxed{\quad L_1 \quad} & \begin{matrix} l_{1,k-1} \\ \vdots \end{matrix} & \begin{matrix} l_{1k} \\ \vdots \end{matrix} \\ l_{k-1,1}\cdots l_{k-1,k-1} & & l_{k-1,k} \\ l_{k1} \ \cdots \ l_{k,k-1} & & l_{kk} \end{pmatrix}, \quad \begin{matrix} l_{i,k-1}=l_{ik}(\mathrm{mod}\ 2) \qquad \text{if } i\leqq k-1, \\ \text{and } l_{k,k-1}=l_{kk}+1(\mathrm{mod}\ 2). \end{matrix}$$

Then, $LL^t=E\ (\mathrm{mod}\ 2)$ implies that L_1 is also a mod 2 orthogonal matrix. Let $\mathscr{L}'_i=(l_{i1}, l_{i2}, \cdots, l_{i,k-2})$ and $\mathscr{L}_i=(\mathscr{L}'_i, l_{i,k-1}, l_{i,k})$ the i-th row of L. Since $\mathscr{L}'_1, \mathscr{L}'_2, \cdots, \mathscr{L}'_{k-2}$ are linearly independent, $\mathscr{L}'_{k-1}=\sum_{i=1}^{k-2} c_i\mathscr{L}'_i\ (\mathrm{mod}\ 2)$, where c_i, $i=1,2,\cdots,k-2$, are integers. But, $0=(\mathscr{L}_{k-1}, \mathscr{L}_i)=((\mathscr{L}'_{k-1}, l_{k-1,k-1}, l_{k-1,k}), (\mathscr{L}'_i, l_{i,k-1}, l_{ik}))=(\mathscr{L}'_{k-1}, \mathscr{L}'_i)+l_{k-1,k-1}\cdot l_{i,k-1}+l_{k-1,k}\cdot l_{ik}=(\mathscr{L}'_{k-1}, \mathscr{L}'_i) =c_i\ (\mathrm{mod}\ 2)$, $i=1, 2, \cdots, k-2$. So that, $\mathscr{L}_{k-1}=(0, 0, \cdots, 0, l_{k-1,k-1}, l_{k-1,k})$ and $(\mathscr{L}_{k-1}, \mathscr{L}_{k-1})=l^2_{k-1,k-1}+l^2_{k-1,k}=l_{k-1,k-1}+l_{k-1,k}=0\ (\mathrm{mod}\ 2)$. This contradicts to $(\mathscr{L}_{k-1}, \mathscr{L}_{k-1})=1\ (\mathrm{mod}\ 2)$. Therefore, such a mod 2 orthogonal matrix as above do not exist. This completes the proof of Assertion 3.

Thus we have completed the proof of the theorem.

5. Classification of Handlebodies of Type (0+I)

In this section we classify the handlebodies of type (0+I) of $\mathscr{H}(2n+1, k, n+1)$, $n\geqq 4$, up to diffeomorphism, that is, $(H; \lambda, \alpha)$-systems of type (0+I) with rank $H=k$ up to isomorphism.

Let $(H; \lambda, \alpha)$ be a system of type (0+I) with rank $H=k$ and rank $\lambda =q$ $(0<q<k)$. By Lemma 1.1, there exists a base $\{u_1, \cdots, u_p; v_1, \cdots, v_q\}$ of H, $p+q=k$, such that $\lambda(u_i, u_j)=\lambda(u_i, v_j)=0$ and $\lambda(v_i, v_j)=\delta_{ij}\in Z_2 \cong\pi_{n-1}(S^n)$ for possible i, j. We call such a base to be *admissible*. α is a homomorphism on the subgroup generated by $\{u_1, \cdots, u_p\}$ and quadratic

form like on the subgroup generated by $\{v_1,\cdots, v_q\}$. If $\{u_1,\cdots, u_p;\ v_1,\cdots, v_q\}$ is an admissible base, we may replace u_i by $u_i'=u_i-2lv_j$, or v_i by $v_i'=v_i-lu_j$, where l is an integer; the new base is also admissible.

An admissible base $\{u_1,\cdots, u_p;\ v_1,\cdots, v_q\}$ of H gives a representation of the corresponding handlebody W as $W=\bar{A}_{\alpha_1}\natural\bar{A}_{\alpha_2}\natural\cdots\natural\bar{A}_{\alpha_p}\natural\bar{B}_{\beta_1}\natural\bar{B}_{\beta_2}\natural\cdots\natural\bar{B}_{\beta_q}$, where $\bar{A}_{\alpha_i}$, $\bar{B}_{\alpha_j}$ are n-disk bundles over $(n+1)$-spheres with characteristic elements $\alpha_i=\alpha(u_i)$, $\beta_j=\alpha(v_j)$, respectively, such that $\pi(\alpha_i)=0$, $\pi(\beta_j)=1$ ([7]). p and q are diffeomorphism invariants of W, more precisely, homotopy invariants of ∂W (See Part II).

Firstly, since $\pi_{4t-1}=0$ $(t\geqq 3)$ and $\pi_{4t+1}=\pi_{4t+2}=0$ $(t\geqq 1)$ we have the following, similarly to the section 4.

Theorem 5.1. *If $n=4t-1$ $(t\geqq 3)$, or $4t+1$ $(t\geqq 1)$, or $4t+2$ $(t\geq 1)$, the handlebodies of type $(0+\mathrm{I})$ of $\mathscr{H}(2n+1, k, n+1)$ do not exsist.*

If $n=4t-1$, $t=2$, we have

Theorem 5.2. *If $n=7$, the handlebodies W of type $(0+\mathrm{I})$ of $\mathscr{H}(2n+1, k, n+1)$ are uniquely represented up to diffeomorphism as follows:*

$$W=p(S^8\times D^7)\natural q\bar{B}_c,\quad p+q=k,$$

where $c>0$ is an odd integer of $\pi_7(SO_7)\cong Z$.

Proof. Let $(H;\ \lambda, \alpha)$ be a system of type $(0+\mathrm{I})$ with rank $H=k$. Since $\partial_{4t-1}=0$ $(t\geqq 1)$, the map α: $H\to Z=\pi_7(SO_7)$ is a homomorphism. For an admissible base $\{u_1, u_2,\geqq, u_p;\ v_1, v_2,\cdots, v_q\}$ $(p+q=k)$ of H, let $c=G.C.D(\alpha(u_1),\cdots, \alpha(u_p), \alpha(v_1),\cdots, \alpha(v_q))\geqq 0$. Then c is independent of the choice of admissible bases, that is, c is an isomorphism invariant of $(H;\ \lambda, \alpha)$-systems of type $(0+\mathrm{I})$, and therefore, a diffeomorphism invariant of handlebodies of type $(0+\mathrm{I})$.

Let $\{u_1, u_2,\cdots, u_p;\ v_1, v_2,\cdots, v_q\}$ be an admissible base of H. We may assume that $\alpha(u_1)=a\geqq 0$ (even), $\alpha(u_2)=\cdots=\alpha(u_p)=0$, and $\alpha(v_1)=\alpha(v_2)=\cdots=\alpha(v_q)=b>0$ (odd), (See Theorem 3.1. and Lemma 4.2.). If $a>b>0$, let $a=2lb+a_1$, $0\leqq|a_1|<b$, and put $u_1'=u_1-2lv_1$. Then $\alpha(u_1')=a-2lb=a_1$ (even). If $a_1<0$, replace u_1' by $-u_1'$. So that, we may assume $a_1\geqq 0$. If $b>a_1>0$, let $b=l_1a_1+b_1$, $0<b_1<a_1$, and put $v_i'=v_i-l_1u_1'$, $i=1, 2,\cdots, q$. Then $\alpha(v_i')=b-l_1a_1=b_1$ (odd), $i=1, 2,\cdots, q$.

Repeating this by Euclidean algorithm till $|a_s|=0$, we arrive at $c=(a, b)>0$. It is easily seen that $c=G.C.D.(\alpha(u_1),\cdots, \alpha(u_p), \alpha(v_1),\cdots, \alpha(v_q))$ and c is odd.

So that, there exists an admissible base $\{u_1,\cdots, u_p;\ v_1,\cdots, v_q\}$ of H such that $\alpha(u_1)=\cdots=\alpha(u_p)=0$ and $\alpha(v_1)=\cdots\alpha(v_q)=c>0$ (odd). Thus, two $(H;\ \lambda, \alpha)$-systems are isomorphic if and only if they have the same value of c. This completes the proof.

Theorem 5.3. *If* $n=8s+4$ $(s\geqq 0)$, *the handlebodies* W *of type* $(0+\mathrm{I})$ *of* $\mathscr{H}(2n+1, k, n+1)$ *are uniquely represented up to diffeomorphism as* $W=p(S^{n+1}\times D^n)\natural\, q\bar{B}_{(0,1)}$, *where* $p+q=k$ *and* $(0, 1)\in\pi_{8s+4}(SO_{8s+4})\cong Z_2+Z_2$.

Proof. Let $(H;\ \lambda, \alpha)$ be a system of type $(0+\mathrm{I})$ with $\operatorname{rank} H=k$ and $\operatorname{rank}\lambda=q$. By Theorem 3.1 and Theorem 4.4, there exists an admissible base $\{u_1,\cdots, u_p;\ v_1,\cdots, v_q\}$ of H such that $\alpha(u_1)=(a, 0)\in Z_2+Z_2$, $\alpha(u_2)=\cdots=\alpha(u_p)=(0,0)$ and $\alpha(v_1)=\cdots=\alpha(v_q)=(0, 1)$. If $\alpha(u_1)=(1, 0)$, let $u_1'=u_1+2v_1$. Then, since $\partial_{8s+4}(1)=(1, 0)$ by Lemma 2.1, $\alpha(u_1')=\alpha(u_1)+\alpha(2v_1)=\alpha(u_1)+2\alpha(v_1)+\partial\lambda(v_1, v_1)=(0, 0)$.

So that, if $n=8s+4$ $(s\geqq 0)$, for any $(H;\ \lambda, \alpha)$-system of type $(0+\mathrm{I})$ with $\operatorname{rank} H=k$ and $\operatorname{rank}\lambda=q$, there exists an admissible base $\{u_1,\cdots, u_p;\ v_1,\cdots, v_q\}$ of H such that $\alpha(u_1)=\cdots\alpha(u_p)=0$ and $\alpha(v_1)=\cdots=\alpha(v_q)=(0, 1)$. Therefore all such $(H;\ \lambda, \alpha)$-systems are isomorphic, and this completes the proof.

Theorem 5.4. *If* $n=8s$ $(s\geqq 1)$, *the handlebodies* W *of type* $(0+\mathrm{I})$ *of* $\mathscr{H}(2n+1, k, n+1)$ *are uniquely represented up to diffeomorphism as follows*:

(i) $W=p(S^{n+1}\times D^n)\natural\, q\bar{B}_{(0,1,0)}$, or

(ii) $W=p(S^{n+1}\times D^n)\natural\, q\bar{B}_{(0,1,1)}$, or

(iii) $W=p(S^{n+1}\times D^n)\natural\,(q-1)\bar{B}_{(0,1,0)}\natural\,\bar{B}_{(0,1,1)}$, $\quad q\geqq 2$, or

(iv) $W=p(S^{n-1}\times D^n)\natural\,(q-2)\bar{B}_{(0,1,0)}\natural\, 2\bar{B}_{(0,1,1)}$, $\quad q\geqq 3$, or

(v) $W=\bar{A}_{(0,0,1)}\natural\,(p-1)(S^{n+1}\times D^n)\natural\, q\bar{B}_{(0,1,0)}$, *where always* $p+q=k$ *and the characteristic elements belong to* $\pi_{8s}(SO_{8s}) \cong Z_2+Z_2+Z_2$.

Proof. Let $(H;\lambda,\alpha)$ be a system of type $(0+\mathrm{I})$ with rank $H=k$ and rank $\lambda=q$ $(0<q<k)$. Since $n=8s$ $(s\geqq 1)$, by Theorem 3.1 and preliminary notions of Theorem 4.5, there exists an admissible base $\{u_1,\cdots, u_p;\ v_1,\cdots, v_q\}$ of H such that $\alpha(v_1)=\cdots=\alpha(v_r)=(0,1,0)$, $\alpha(v_{r+1})=\cdots=\alpha(v_q)=(0,1,1)$, $0\leqq r\leqq q$, and

(a) $\alpha(u_1)=(a,0,b)$, $\alpha(u_2)=\cdots=\alpha(u_p)=(0,0,0)$,

or (b) $\alpha(u_1)=(1,0,0)$, $\alpha(u_2)=(0,0,1)$, $\alpha(u_3)=\cdots=\alpha(u_p)=(0,0,0)$.

In the case (a), replace u_1 by $u_1'=u_1+2v_1$ if $a=1$, then $\alpha(u_1')=(0,0,b)$ since $\partial_{8s}=(1,0,0)$, $s\geqq 1$. Furthermore, if $b=1$, replace v_{r+i} by $v_{r+i}'=u_1'+v_{r+i}$ for $i=1,2,\cdots,q-r$, then $\alpha(v_{r+i}')=(0,1,0)$. Thus (a) is reduced to the cases

(c) $\alpha(u_1)=\cdots=\alpha(u_p)=(0,0,0)$

$\alpha(v_1)=\cdots=\alpha(v_r)=(0,1,0)$, $\alpha(v_{r+1})=\cdots=\alpha(v_q)=(0,1,1)$,

$0\leqq r\leqq q$,

or (d) $\alpha(u_1)=(0,0,1)$, $\alpha(u_2)=\cdots=\alpha(u_p)=(0,0,0)$

$\alpha(v_1)=\cdots=\alpha(v_q)=(0,1,0)$.

The case (b) is similarly reduced to the case (d).

By Theorem 4.5, the cases (c) and (d) are reduced to the following five cases:

(1) $\alpha(v_1)=\cdots=\alpha(v_q)=(0,1,0)$,

(2) $\alpha(v_1)=\cdots=\alpha(v_q)=(0,1,1)$,

(3) $\alpha(v_1)=\cdots=\alpha(v_{q-1})=(0,1,0)$, $\alpha(v_q)=(0,1,1)$, $q\geqq 2$,

(4) $\alpha(v_1)=\cdots=\alpha(v_{q-2})=(0,1,0)$, $\alpha(v_{q-1})=\alpha(v_q)=(0,1,1)$, $q\geqq 3$,

where during (1)$\sim$(4), $\alpha(u_1)=\cdots=\alpha(u_p)=(0,0,0)$,

(5) $\alpha(u_1)=(0, 0, 1), \quad \alpha(u_2)=\cdots=\alpha(u_p)=(0, 0, 0),$

$\alpha(v_1)=\cdots=\alpha(v_q)=(0, 1, 0).$

Now, we prove that these five cases are independent. Let $\{u_1,\cdots, u_p; v_1,\cdots, v_q\}$, $\{u_1',\cdots, u_p'; v_1',\cdots, v_q'\}$ be admissible bases of H and assume that they satisfy (i), (j) respectively. Let $(u_1',\cdots, u_p', v_1',\cdots, v_q')^t=L(u_1,\cdots, u_p, v_1,\cdots, v_q)^t$, $L=(l_{ij})$, and let $L_1=(l_{ij})$ $(1\leqq i\leqq p,\ p+1\leqq j\leqq k)$, $L_2=(l_{ij})$ $(p+1\leqq i,\ j\leqq k)$. Then, $L_1L_1^t=0$ (mod 2) and $L_2L_2^t=E$ (mod 2). If $i, j=1, 2, 3, 4$, it is easily seen that there arise contradictions by quite similar way to that of Theorem 4.5. Let $j=5$ and $i\in\{1, 2, 3, 4\}$. If $i=1$, then $u_1'=\sum_{s=1}^{p} l_{is}u_s+\sum_{t=1}^{q} l_{i,p+t}v_t$ induces $\alpha^3(u_1')=0$, and this is a contradiction. If $i=2$, then $1=\alpha^3(u_1')=\sum_{t=1}^{q} l_{1,t+p}=\sum_{t=1}^{q} l_{1,t+p}^2$ (mod 2), and this contradicts $L_1L_1^t=0$ (mod 2). If $i=3$, then $0=\alpha^3(v_t')=l_{p+t,k}$ (mod 2), $t=1, 2,\cdots, q$, and this contradicts $L_2L_2^t=E$ (mod 2). If $i=4$, then $0=\alpha^3(v_t')=l_{p+t,k-1}+l_{p+t,k}$ (mod 2), $t=1, 2,\dots, q$, which contradicts $L_2L_2^t=E$ (mod 2).

Thus the $(H;\ \lambda, \alpha)$-systems corresponding to the above cases are independent up to isomorphism. This completes the proof.

6. Classification of Handlebodies of Type II

In this section, we classify the handlebodies of type II of $H(2n+1, k, n+1)$, $n\geqq 4$, up to diffeomorphism, that is, the $(H;\ \lambda, \alpha)$-systems of type II with rank $H=k$ up to isomorphism, where we note that $k=2r$.

Let $(H;\ \lambda, \alpha)$ be a system of type II with rank $H=2r$. A base $\{e_1, f_1,\cdots, e_r, f_r\}$ of H is called to be *symplectic* if $\lambda(e_i, e_j)=\lambda(f_i, f_j)=0$ and $\lambda(e_i, f_j)=\delta_{ij}\in Z_2\cong\pi_{n+1}(S^n)$ for $i, j=1, 2,\cdots, r$. If $\{e_1, f_1,\cdots, e_r, f_r\}$ is a symplectic base of H, $\alpha(e_i)$, $\alpha(f_j)$, $i, j=1, 2,\dots, r$, belong to $\mathrm{Ker}\,\pi_n$, π_n: $\pi_n(SO_n)\to\pi_n(S^{n-1})\cong Z_2$, and α is linear on the subgroups of H generated by $\{e_1,\cdots, e_r\}$, $\{f_1,\cdots, f_r\}$ respectively. If $\partial_n(1)\in\mathrm{Ker}\,\pi_n$, ∂_n: $\pi_{n+1}(S^n)\cong Z_2\to\pi_n(SO_n)$, then $\alpha(H)\subset\mathrm{Ker}\,\pi_n$.

Let $\{e_1, f_1,\cdots, e_r, f_r\}$ be a symplectic base of H. By the following transformations of bases we have new symplectic bases of H:

(t_0) Interchanging e_i with f_i or replacing e_i (or f_i) by $-e_i$ (or $-f_i$).

(t_1) Replacing e_i (or f_i) by $e_i'=e_i+lf_i$ (or $f_i'=le_i+f_i$).

(t_2) Replacing e_i by $e_i'=e_i+2l\cdot$ (the other basis element), or replacing f_i similarly.

Here, l are integers.

(t_3) Replacing e_i, f_i, e_j, and f_j, $i \neq j$, by $e_i'=e_i+e_j, f_i'=e_i+e_j+f_i$, $e_j'=f_i-f_j$, and $f_j'=e_j+f_i-f_j$ respectively.

For an $(H;\ \lambda, \alpha)$-system of type II with rank $H=2r$, a symplectic base $\{e_1, f_1, \cdots, e_r, f_r\}$ of H gives a representation of the corresponding handlebody W of type II as $W=W\begin{pmatrix}\alpha(e_1)\\ \alpha(f_1)\end{pmatrix}\natural\cdots\natural W\begin{pmatrix}\alpha(e_r)\\ \alpha(f_r)\end{pmatrix}$, where $W\begin{pmatrix}\alpha(e_i)\\ \alpha(f_i)\end{pmatrix}$ denotes a handlebody of $\mathscr{H}(2n+1, 2, n+1)$ with the algebraic system $(H_i;\ \lambda_i, \alpha_i)$ such that H_i has a base $\{e_i, f_i\}$ which is symplectic with respect to λ_i and $\alpha_i(e_i)=\alpha(e_i)$, $\alpha_i(f_i)=\alpha(f_i)$. ([7]). $W\begin{pmatrix}\alpha(e_i)\\ \alpha(f_i)\end{pmatrix}$ is neither diffeomorphic nor homeomorphic to the boundary connected sum of any two n-disk bundles over $(n+1)$-spheres. For an integer $m \geqq 0$, $mW\begin{pmatrix}a\\ b\end{pmatrix}$ denotes the boundary connected sum of m copies of $W\begin{pmatrix}a\\ b\end{pmatrix}$.

Lemma 6.1. *Let H be a free abelian group of rank k, λ: $H \times H \rightarrow Z_2$ a symmetric bilinear form of type* II, *and α: $H \rightarrow Z$* (or Z_2) *a homomorphism. Then, there exists a base $\{e_1, f_1, \cdots, e_r, f_r\}$ of H which is symplectic with respect to λ such that $\alpha(e_1)=d \geqq 0$, $\alpha(e_2)=\cdots=\alpha(e_r)=0$ and $\alpha(f_1)=\cdots=\alpha(f_r)=0$. Here, d is independent of the choice of such symplectic bases.*

Proof. Let $\{e_1, f_1, \cdots, e_r, f_r\}$ be a symplectic base of H with respect to λ. $\alpha(e_i), \alpha(f_j) \geqq 0$ may be always assumed by (t_0)-transformations. We can perform Euclidean algorithm to each pair $\{\alpha(e_i), \alpha(f_i)\}$ by (t_1)-transformations. Thus, interchanging e_i' with f_i' if necessary, there exists a symplectic base $\{e_1', f_1', \cdots, e_r', f_r'\}$ of H such that $\alpha(e_i') \geqq 0$, $\alpha(e_i')=(\alpha(e_i), \alpha(f_i))$ if $\alpha(e_i') \neq 0$, $i=1, 2, \cdots, r$, and $\alpha(f_j')=0$, $j=1, 2, \cdots, r$. Let $\alpha(e_i') \neq 0$ for $1 \leqq i \leqq r_0$ and $\alpha(e_i')=0$ for $r_0+1 \leqq i \leqq r$, under some exchanges of suffixes of $\{e_i', f_i'\}$, $i=1, 2, \cdots, r$, and assume that $\alpha(e_1')=\cdots=\alpha(e_{s-1}')=a>0$, $2 \leqq s \leqq r_0$. Then, using (t_2)-transformations, as in the proof of Lemma 4.2 $(s-1)$ can be extended to s, allowing that some of them may come to zero. Thus, there exists such a symplectic base $\{e_1', f_1', \cdots, e_r', f_r'\}$ of

H that $\alpha(e_1')=\cdots=\alpha(e_{r_1})=d>0$, $r_1\leqq r_0$, $\alpha(e_{r_1+1}')=\cdots=\alpha(e_r')=0$, and $\alpha(f_1')=\cdots=\alpha(f_r')=0$, where $d=G.C.D(\alpha(e_1), \alpha(f_1),\cdots, \alpha(e_r), \alpha(f_r))$. The last result is also valid if α has its values in Z_2.

In the above, if $\alpha(e_i')=\alpha(e_j')=d$, $\alpha(f_i')=\alpha(f_j')=0$ for two pairs $\{e_i', f_i'\}$, $\{e_j', f_j'\}$, replace them by $e_i''=e_i'-e_j'$, $f_i''=e_i'-e_j'+f_i'$, $e_j''=e_j'+f_i'-f_j'$, and $f_j''=f_i'-f_j'$ respectively. Then, $\alpha(e_i'')=\alpha(f_i'')=0$ and $\alpha(e_j'')=d$, $\alpha(f_j'')=0$. Similar way can be taken even if α has its values in Z_2. This completes the proof.

Theorem 6.2. *If $n=4t-1$ $(t\geqq 2)$, the handlebodies W of type* II *of $\mathscr{H}(2n+1, k, n+1)$ are uniquely represented up to diffeomorphism as*

$$W=W\begin{pmatrix} d \\ 0 \end{pmatrix}\natural(r-1)W\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad d\geqq 0,$$

where $k=2r$ and $d\in Z\cong\pi_{4t-1}(SO_{4t-1})$, especially $d\in 2Z=\mathrm{Ker}\,\pi_7$ if $t=2$.

Proof. Since $\partial_{4t-1}=0$ for $t\geqq 1$ by Lemma 2.1, α: $H\to\mathrm{Ker}\,\pi_{4t-1}$ is a homomorphism. By Lemma 2.2, $\mathrm{Ker}\,\pi_{4t-1}=\pi_{4t-1}(SO_{4t-1})\cong Z$ for $t\geqq 3$ and $\mathrm{Ker}\,\pi_7\cong 2Z$ if $t=2$. Then applying Lemma 6.1, any $(H;\lambda,\alpha)$-system of type II with $\mathrm{rank}\,H=2r$ has a symplectic base $\{e_1, f_1,\cdots, e_r, f_r\}$ such that $\alpha(e_1)=d\geqq 0$, $d\in\mathrm{Ker}\,\pi_{4t-1}$, $\alpha(e_2)=\cdots=\alpha(e_r)=0$, and $\alpha(f_1)=\cdots=\alpha(f_r)=0$. These $(H;\lambda,\alpha)$-systems are determined by d up to isomorphism, and this completes the proof.

Let $n=8s+5$ $(s\geqq 0)$ and $(H;\lambda,\alpha)$ be a system of type II. Then $\pi_{8s+5}(SO_{8s+5})\cong Z_2$ and ∂_{8s+5} is an isomorphism sicne $\partial_{8s+5}(1)=1$ $(s\geqq 0)$ by Lemma 2.1. So that, α is a quadratic form over Z_2 with the associated bilinear form $\partial\circ\lambda$, where a base of H is symplectic with respect to $\partial\circ\lambda$ if and only if it is symplectic with respect to λ. α is completely determined by the Arf invariant $c(\alpha)$.

Theorem 6.3. *If $n=8s+5$ $(s\geqq 0)$, the handlebodies W of type* II *of $\mathscr{H}(2n+1, k, n+1)$ are uniquely represented up to diffeomorphism as*

$$W=W\begin{pmatrix} d \\ d \end{pmatrix}\natural(r-1)W\begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $k=2r$ and $d=0, 1\in Z_2\cong\pi_{8s+5}(SO_{8s+5})$ according as the Arf invariants of the corresponding algebraic systems are equal to 0 *or* 1.

Proof. Applying the transformations (t_1), (t_3), any $(H;\ \lambda, \alpha)$-system of type II with rank $H=2r$ has a symplectic base $\{e_1, f_1, \dots, e_r, f_r\}$ such that $\alpha(e_1)=\alpha(f_1)=d\in Z_2\cong\pi_{8s+5}(SO_{8s+5})$, and $\alpha(e_2)=\cdots=\alpha(e_r)=\alpha(f_2)=\cdots=\alpha(f_r)=0$, where $d=0$ or 1 according as $c(\alpha)=0$ or 1. Since $c(\alpha)$ determines those $(H;\ \lambda, \alpha)$-systems up to isomorphism, this completes the proof.

Let $n=8s+4$ $(s\geqq 0)$ and $(H;\ \lambda, \alpha)$ be a system of type II. By Lemma 2.1 and Lemma 2.2, $\operatorname{Ker}\pi_{8s+4}=\{(1, 0)\}\cong Z_2$, $\partial_{8s+4}(1)=(1, 0)$, and so that $\alpha(H)\subset\operatorname{Ker}\pi_{8s+4}$. This means that the situation is quite similar to that when $n=8s+5$ $(s\geqq 0)$. Thus we have,

Theorem 6.4. *If $n=8s+4$ $(s\geqq 0)$, the handlebodies W of type* II *of $\mathscr{H}(2n+1, k, n+1)$ are uniquely represented up to diffeomorphism as*

$$W=W\begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix}\natural(r-1)W\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where $k=2r$ and $(d, 0)\in Z_2+Z_2\cong\pi_{8s+4}(SO_{8s+4})$ according as the Arf invariants of the corresponding algebraic systems are equal to 0 *or* 1.

Let $L=(l_{ij})$ be a $k\times k$ matrix $(k=2r)$ with integer components and let $\mathscr{L}_i$ be the i-th row of L. L is called to be mod 2 *symplectic* if $LJL=J \pmod 2$, where $J=\operatorname{diag}(U,\cdots, U)$, $U=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Define an inner product $(\mathscr{L}_i, \mathscr{L}_j)$ by $(\mathscr{L}_i, \mathscr{L}_j)=\sum_{s=1}^{r} l_{i,2s-1}\cdot l_{j,2s}+\sum_{s=1}^{r} l_{i,2s}\cdot l_{j,2s-1}$. Then L is mod 2 symplectic if and only if $(\mathscr{L}_{2i-1}, \mathscr{L}_{2j-1})=(\mathscr{L}_{2i}, \mathscr{L}_{2j})=0 \pmod 2$ and $(\mathscr{L}_{2i-1}, \mathscr{L}_{2j})=\delta_{ij} \pmod 2$ for $i, j=1, 2,\cdots, r$.

If $n=8s+1$ $(s\geqq 1)$, by Lemma 2.1 and Lemma 2.2, $\operatorname{Ker}\pi_{8s+1}=\pi_{8s+1}(SO_{8s+1})\cong Z_2+Z_2$ and $\partial_{8s+1}(1)=(1, 0)$. Let $(H;\ \lambda, \alpha)$ be a system of type II, and $\alpha=(\alpha^1, \alpha^2)$, $\alpha^i=p_i\circ\alpha$ $(i=1, 2)$, where p_i is the projection of Z_2+Z_2 to the i-th direct summand. Then, α^1 is a quadratic form over Z_2 with the associated bilinear form $p_1\circ\partial_{8s+1}\circ\lambda$, where $p_1\circ\partial_{8s+1}$ is an isomorphism. α^2 is a homomorphism since $p_2\circ\partial_{8s+1}$ is the zero homomorphism.

Theorem 6.5. *If $n=8s+1$ $(s\geqq 1)$, the handlebodies W of type* II *of $\mathscr{H}(2n+1, k, n+1)$, $k=2r$, are uniquely represented up to diffeomorphism* *as follows:*

$$\text{(i)}\quad \textit{If}\quad \alpha^2=0,\quad W=W\begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix}\natural(r-1)W\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\qquad d\in Z_2\ ,$$

where $d=c(\alpha^1)$, *the Arf invariant of* α^1.

(ii) *If* $c(\alpha^1)=0,\quad \alpha^2\neq 0,$

$$W=W\begin{pmatrix} 0 & 1 \\ 0 & d \end{pmatrix}\natural(r-1)W\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\qquad d\in Z_2,$$

where $d=\varepsilon(\alpha)$, *an invariant which is shown below.*

(iii) *If* $c(\alpha^1)\neq 0,\quad \alpha^2\neq 0,$

$$W=W\begin{pmatrix} 0 & 1 \\ 0 & d \end{pmatrix}\natural W\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\natural(r-2)W\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\qquad d\in Z_2,$$

where $d=\varepsilon(\alpha)$, *an invariant which is shown below.*

Here, $\alpha=(\alpha^1, \alpha^2)$ *is the map of the corresponding algebraic system* $(H;\ \lambda, \alpha)$.

Proof. Let $(H;\ \lambda, \alpha)$ be a system of type II with $\operatorname{rank} H=2r$ and $\{e_1, f_1,\dots, e_r, f_r\}$ be a symplectic base of H. By symplectic transformations, we simplify the matrix

$$A=\begin{pmatrix} \alpha^1(e_1) & \alpha^1(f_1) & \cdots & \alpha^1(e_r) & \alpha^1(f_r) \\ \alpha^2(e_1) & \alpha^2(f_1) & \cdots & \alpha^2(e_r) & \alpha^2(f_r) \end{pmatrix}.$$

We denote the simplifications by arrows and the symplectic transformations (t_i), $i=0, 1, 2, 3$.

$$\text{(i)}\quad \text{If } \alpha^2=0,\qquad A\xrightarrow{(t_1),\,(t_3)}\begin{pmatrix} 0\cdots 0 \\ 0\cdots 0 \end{pmatrix}\ \text{or}\ \begin{pmatrix} 1\,1\,0\cdots 0 \\ 0\,0\,0\cdots 0 \end{pmatrix}$$

according as $c(\alpha^1)=0, 1$.

(ii) If $c(\alpha^1)=0$ and $\alpha^2\neq 0$, by Lemma 6.1 and then by (t_1) and (t_3)-transformations $(l=1)$,

$$A\xrightarrow{(t_1)}\begin{pmatrix} \gamma\,\delta\,0\cdots 0 \\ 1\,0\,0\cdots 0 \end{pmatrix}\ (\gamma\cdot\delta=0)\quad\text{or}\quad \begin{pmatrix} 1\,1\,1\,1\,0\cdots 0 \\ 1\,0\,0\,0\,0\cdots 0 \end{pmatrix}.$$

But, we know the following:

$$\begin{pmatrix} \gamma & \delta \\ 1 & 0 \end{pmatrix} \xrightarrow{(t_1)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{(t_3)} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

So that, finally we have,

$$A \longrightarrow \begin{pmatrix} 0 & 0 \cdots 0 \\ 1 & \varepsilon \cdots 0 \end{pmatrix}, \ \varepsilon = 0,\ 1 \in Z_2.$$

Assertion 1. *The two systems* $(H;\ \lambda,\ \alpha)$, $(H';\ \lambda',\ \alpha')$ *which belong respectively to the cases* $\varepsilon = 0,\ 1$ *are not isomorphic.*

Proof. If they are isomorphic, there exist the two symplectic bases $\{e_1, f_1, \cdots, e_r, f_r\}$, $\{e'_1, f'_1, \cdots, e'_r, f'_r\}$ of H such that the values of α^1 and α^2 vansish on those basis elements except only $\alpha^2(e_1) = \alpha^2(e'_1) = \alpha^2(f'_1) = 1$. Let $L = (l_{ij})$ be the $k \times k$ matrix $(k = 2r)$ with integer components and let $(e'_1, f'_1, \cdots, e'_r, f'_r)^t = L(e_1, f_1, \cdots, e_r, f_r)^t$. Then L is unimodular and mod 2 symplectic, and satisfies the following conditions:

$$(1) \quad \sum_{s=1}^{r} l_{i,2s-1} \cdot l_{i,2s} = 0 \qquad (\text{mod } 2) \ (i = 1, 2, \cdots, k),$$

$$(2) \quad l_{i1} = \begin{cases} 1 \ (\text{mod } 2) & \text{if } i = 1, 2, \\ 0 \ (\text{mod } 2) & \text{if } i \geqq 3. \end{cases}$$

We show that there are no such mod 2 symplectic matrices L that satisfy the conditions (1) and (2).

L has the following form:

$$L = \left(\begin{array}{cc|c} 1 & l_{12} & l_{13} \cdots l_{1,2r} \\ 1 & l_{22} & l_{23} \cdots l_{2,2r} \\ \hline 0 & l_{32} & \\ \vdots & \vdots & L_1 \\ 0 & l_{2r,2} & \end{array} \right) \qquad (\text{mod } 2),$$

where $|L_1| \neq 0$ since L is mod 2 symplectic. Since $(\mathscr{L}_1, \mathscr{L}_j) = (\mathscr{L}_2, \mathscr{L}_j) = 0$

(mod 2) for $j=3, 4,\cdots, 2r$, $(l_{i3}, l_{i4},\cdots, l_{i,2r})$ $(i=1, 2)$ satisfies the following linear equations in mod 2:

$$\begin{cases} l_{33}x_4+l_{34}x_3+\cdots+l_{3,2r-1}x_{2r}+l_{3,2r}x_{2r-1}=l_{32} \\ l_{43}x_4+l_{44}x_3+\cdots+l_{4,2r-1}x_{2r}+l_{4,2r}x_{2r-1}=l_{42} \\ \qquad\cdots\cdots\cdots\cdots\cdots\cdots \\ \qquad\cdots\cdots\cdots\cdots\cdots\cdots \\ l_{2r,3}x_4+l_{2r,4}x_3+\cdots+l_{2r,2r-1}x_{2r}+l_{2r,2r}x_{2r-1}=l_{2r,2} \end{cases}$$

So that, we have $l_{1j}=l_{2j}$ (mod 2), $j=3, 4,\cdots, 2r$, and therefore $l_{12}=l_{22}$ (mod 2) by (1) and (2). This contradicts $|L|=1$, and this completes the proof of Assertion 1.

(iii) Let $c(\alpha^1)=1$ and $\alpha^2\neq 0$. Then, similarly we have,

$$A \longrightarrow \begin{pmatrix} 1\,1\,0\cdots0 \\ 1\,0\,0\cdots0 \end{pmatrix} \text{ or } \begin{pmatrix} \gamma\,\delta\,1\,1\,0\cdots0 \\ 1\,0\,0\,0\,0\cdots0 \end{pmatrix} \qquad (\gamma\cdot\delta=0)$$

$$\longrightarrow \begin{pmatrix} 1\,1\,0\cdots0 \\ 1\,0\,0\cdots0 \end{pmatrix}, \text{ or } \begin{pmatrix} 0\,0\,1\,1\,0\cdots0 \\ 1\,0\,0\,0\,0\cdots0 \end{pmatrix}, \text{ or } \begin{pmatrix} 0\,0\,1\,1\,0\cdots0 \\ 1\,1\,0\,0\,0\cdots0 \end{pmatrix}.$$

We also have,

$$\begin{pmatrix} 1\,1\,0\,0 \\ 1\,0\,0\,0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0\,0\,1\,1 \\ 0\,0\,1\,0 \end{pmatrix} \xrightarrow{(t_3)} \begin{pmatrix} 1\,0\,1\,1 \\ 1\,1\,0\,1 \end{pmatrix} \xrightarrow{(t_1)} \begin{pmatrix} 0\,0\,1\,1 \\ 0\,1\,0\,1 \end{pmatrix} \xrightarrow{(t_3)} \begin{pmatrix} 1\,0\,1\,1 \\ 0\,1\,0\,0 \end{pmatrix}$$

$$\xrightarrow{(t_1)} \begin{pmatrix} 0\,0\,1\,1 \\ 1\,1\,0\,0 \end{pmatrix}.$$

Thus, finally we have

$$A \longrightarrow \begin{pmatrix} 0\,0\,1\,1\,0\cdots0 \\ 1\,\varepsilon\,0\,0\,0\cdots0 \end{pmatrix}, \qquad \varepsilon=0, 1\in Z_2.$$

Assertion 2. *The two systems* $(H; \lambda, \alpha)$, $(H'; \lambda', \alpha')$ *which belong respectively to the cases* $\varepsilon=0, 1$ *are not isomorphic.*

Proof. The proof is quite similar to that of Assertion 1. Only it is required to correct the condition on α^1, that is, to replace the condition (1) by

$$(1)' \quad l_{i3}+l_{i4}+\sum_{s=1}^{r} l_{i,2s-1}\cdot l_{i,2s}=\begin{cases} 1 \pmod 2 & \text{if } i=3, 4, \\ 0 \pmod 2 & \text{if } i\neq 3, 4. \end{cases}$$

This completes the proof of Assertion 2.

Thus we have proved that the $(H;\ \lambda, \alpha)$-systems of every case of the above are independent up to isomorphism, and this completes the proof of the theorem. (We note that the matrix A is transposed in the representations of W.)

Let $n=8s$ $(s\geqq 1)$ and $(H;\ \lambda, \alpha)$ be a system of type II. By Lemma 2.1 and Lemma 2.2, $\operatorname{Ker}\pi_{8s}=\{(1, 0, 0), (0, 0, 1)\}\subset\pi_{8s}(SO_{8s})\cong Z_2+Z_2+Z_2$, $\partial_{8s}(1)=(1, 0, 0)\in\operatorname{Ker}\pi_{8s}$, and therefore $\alpha(H)\subset\operatorname{Ker}\pi_{8s}$. So that, if we identify $\operatorname{Ker}\pi_{8s}$ with the direct sum of the first and the third direct summand of $\pi_{8s}(SO_{8s})$, then $\operatorname{Ker}\pi_{8s}\cong Z_2+Z_2$, $\partial_{8s}=(1, 0)$, and $\alpha\colon H\to Z_2+Z_2$. This situation is quite similar to that of the case when $n=8s+1$ $(s\geqq 1)$. So we have,

Theorem 6.6. *If $n=8s$ $(s\geqq 1)$, the handlebodies W of type* II *of $\mathscr{H}(2n+1, k, n+1)$, $k=2r$, are uniquely represented up to diffeomorphism as follows;*

(i) *If* $\alpha^3=0$, $\quad W=W\begin{pmatrix} d\,0\,0 \\ d\,0\,0 \end{pmatrix}\natural(r-1)W\begin{pmatrix} 0\,0\,0 \\ 0\,0\,0 \end{pmatrix}, \qquad d\in Z_2,$

where $d=c(\alpha^1)$, the Arf invariant of α^1.

(ii) *If* $c(\alpha^1)=0$, $\quad \alpha^3\neq 0$,

$$W=W\begin{pmatrix} 0\,0\,1 \\ 0\,0\,d \end{pmatrix}\natural(r-1)W\begin{pmatrix} 0\,0\,0 \\ 0\,0\,0 \end{pmatrix}, \quad d\in Z_2, \quad d=\varepsilon(\alpha).$$

(iii) *If* $c(\alpha^1)\neq 0$, $\quad \alpha^3\neq 0$,

$$W=W\begin{pmatrix} 0\,0\,1 \\ 0\,0\,d \end{pmatrix}\natural W\begin{pmatrix} 1\,0\,0 \\ 1\,0\,0 \end{pmatrix}\natural(r-2)W\begin{pmatrix} 0\,0\,0 \\ 0\,0\,0 \end{pmatrix}, \quad d\in Z_2, \quad d=\varepsilon(\alpha).$$

Here, α is the map of the corresponding algebraic system $(H;\ \lambda, \alpha)$ and $\alpha=(\alpha^1, \alpha^2, \alpha^3)$ takes its values in $\pi_{8s}(SO_{8s})\cong Z_2+Z_2+Z_2$.

Let $n=4t+2$ $(t\geqq 1)$. Then, by Lemma 2.1 and Lemma 2.2, $\operatorname{Ker}\pi_{4t+2}=\pi_{4t+2}(SO_{4t+2})=Z_4$ if $t\geqq 2$, $\pi_6(SO_6)=0$, and $\partial_{4t+2}(1)=2\in Z_4$ if $t\geqq 2$.

But, in this case, the argument is simple.

Theorem 6.7. *If* $n=4t+2$ $(t\geqq 1)$, *the handlebodies* W *of type* II *of* $\mathscr{H}(2n+1, k, n+1)$, $k=2r$, *are uniquely represented up to diffeomorphism as follows*:

(i) *If* $\alpha(H)\subset\{0, 2\}\subset Z_4$, $t\geqq 2$,

$$W=W\begin{pmatrix} d \\ d \end{pmatrix}\natural(r-1)W\begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $d=c(\alpha)$, *the Arf invariant of* α.

(ii) *If* $\alpha(H)\not\subset\{0, 2\}$, $t\geqq 2$,

$$W=W\begin{pmatrix} 1 \\ 0 \end{pmatrix}\natural(r-1)W\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(iii) *If* $n=6$, $W=rW\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Here, α *is the map of the corresponding algebraic system* $(H; \lambda, \alpha)$.

Proof. Let $(H; \lambda, \alpha)$ be the system of type II with $\operatorname{rank} H=2r$, and let $\{e_1, f_1\cdots, e_r, f_r\}$ be a symplectic base of H. By (t_0)-transformations, we may assume that $\alpha(e_i), \alpha(f_i)=0$, or 1, or 2.

(i) If $\alpha(e_i), \alpha(f_i)\in\{0, 2\}\subset Z_4$ for all i,
then $\alpha(H)\subset\{0, 2\}\cong Z_2$ since $\partial_{4t+2}(1)=2\in Z_4$. So that the situation is quite similar to that of the case when $n=8s+4, 8s+5$ $(s\geqq 0)$.

(ii) Let $\alpha(H)\not\subset\{0, 2\}$. By (t_0) and (t_1)-transformations, it may be assumed that $(\alpha(e_i), \alpha(f_i))=(0,0)$, or $(1, 0)$, or $(2, 2)$, for all i. Since $\alpha(H)\not\subset\{0, 2\}$, there exists some $\{e_j, f_j\}$ such that $(\alpha(e_j), \alpha(f_j))=(1, 0)$. Using it we can kill $(\alpha(e_i), \alpha(f_i))=(2, 2)$ by (t_2)-transformations. Hence, we may assume that $(\alpha(e_i), \alpha(f_i))=(0, 0)$ or $(1, 0)$ for all i. If $(\alpha(e_i), \alpha(f_i))=(\alpha(e_j), \alpha(f_j))=(1, 0)$, $i\neq j$, we can kill a pair of them by performing a (t_3)-transformation and then a (t_1)-transformation and (t_0)-transformations. Thus, there exists a symplectic base $\{e_1', f_1', \cdots, e_r', f_r'\}$ of H such that $(\alpha(e_1'), \alpha(f_1'), \cdots, \alpha(e_r'), \alpha(f_r'))=(1, 0, 0\cdots, 0)$.

(iii) If $n=6$, then $\pi_6(SO_6)=0$. So that, the proof is clear.

This completes the proof.

7. Classification of Handlebodies of Type (0+II)

In this section we classify the handlebodies of type $(0+\mathrm{II})$ of $\mathscr{H}(2n+1, k, n+1)$, $n\geqq 4$, up to diffeomorphism, that is, the $(H;\lambda,\alpha)$-systems of type $(0+\mathrm{II})$ with $\operatorname{rank} H=k$ up to isomorphism.

Let $(H;\lambda,\alpha)$ be a system of type $(0+\mathrm{II})$ with $\operatorname{rank} H=k$ and $\operatorname{rank}\lambda=2r$ $(0<2r<k)$. By Lemma 1.1, there exists a base $\{u_1,\cdots,u_r;\, e_1,f_1,\cdots,e_r,f_r\}$ $(p+2r=k)$ of H such that $\lambda(u_i,u_j)=\lambda(u_i,e_j)=\lambda(u_i,f_j)=0$, $\lambda(e_i,e_j)=\lambda(f_i,f_j)=0$, and $\lambda(e_i,f_j)=\delta_{ij}\in Z_2\cong\pi_{n+1}(S^n)$ for possible i, j. We call such a base to be *admissible*. α is a homomorphism on the subgroups generated by $\{u_1,\cdots,u_p\}$, $\{e_1,\cdots,e_r\}$, and $\{f_1,\cdots,f_r\}$ respectively. The following transformations of an admissible base $\{u_1,\cdots,u_p;\, e_1,f_1,\cdots,e_r,f_r\}$ of H yield new admissible bases of H:

(u_1) Replacing e_i or f_i by $e_i'=e_i+lu_j$ or $f_i'=f_i+lu_j$ respectively.

(u_2) Replacing u_i by $u_i'=u_i+2le_j$ or u_i+2lf_j.

Here, l are integers.

Lemma 7.1. *Let $(H;\lambda,\alpha)$ be a system of type* $(0+\mathrm{II})$ *with* $\operatorname{rank} H=k$ *and* $\operatorname{rank}\lambda=2r$ $(0<2r<k)$. *Let $\{u_1,\cdots,u_p;\, e_1,f_1,\cdots,e_r,f_r\}$ be an admissible base of H and let $(u_1',\cdots,u_p';\, e_1',f_1',\cdots,e_r',f_r')^t=T(u_1,\cdots,u_p;\, e_1,f_1,\cdots,e_r,f_r)^t$, where $T=(t_{ij})$ is a unimodular matrix. Then, the base $\{u_i',\cdots,u_p';\, e_i',f_i',\cdots,e_r',f_r'\}$ is also admissible if and only if T has the form*

$$T=\begin{array}{c} \begin{array}{cc} \overbrace{}^{p} & \overbrace{}^{2r} \end{array} \\ \begin{pmatrix} M & 0 \\ * & L \end{pmatrix} \end{array} \begin{array}{l} \}p \\ \}2r \end{array} \qquad (\mathrm{mod}\ 2)$$

and L is mod 2 *symplectic.*

Proof. The proof is straightforward. We note that if $\{u_1',\dots,u_p';\, e_1',f_1',\cdots,e_r',f_r'\}$ is admissible, $\lambda(u_i',x)=0$ for any $x\in H$ and so $\lambda(u_i',e_j)=\lambda(u_i',f_j)=0$ (mod 2) for all i, j.

We call such matrices T that $T\cdot\operatorname{diag}(0,J)\cdot T^t=\operatorname{diag}(0,J)$ (mod 2) to be *admissible.* T is admissible if and only if it satisfies the above conditions.

Let $(H;\ \lambda, \alpha)$ be a system of type $(0+\mathrm{II})$ with rank $H=k$ and rank $\lambda=2r$ $(0<2r<k)$. An admissible base $\{u_1,\cdots, u_p;\ e_1, f_1,\cdots, e_r, f_r\}$ of H gives a representation of the corresponding handlebody W of type $(0+\mathrm{II})$ of $\mathscr{H}(2n+1, k, n+1)$, $n\geqq 4$, such as

$$W=\bar{A}_{\alpha(u_1)}\natural\bar{A}_{\alpha(u_2)}\natural\cdots\natural\bar{A}_{\alpha(u_p)}\natural W\begin{pmatrix}\alpha(e_1)\\ \alpha(f_1)\end{pmatrix}\natural W\begin{pmatrix}\alpha(e_2)\\ \alpha(f_2)\end{pmatrix}\natural\cdots\natural W\begin{pmatrix}\alpha(e_r)\\ \alpha(f_r)\end{pmatrix},$$

([7]), where $\bar{A}_{\alpha(u_i)}$ is an n-disk bundle over the $(n+1)$-sphere with the characteristic element $\alpha(u_i)\in\pi_n(SO_n)$ such that $\pi(\alpha(u_i))=0$ and $W\begin{pmatrix}\alpha(e_i)\\ \alpha(f_i)\end{pmatrix}$ is that defined in the section 6. p and r are diffeomorphism invariants of W, more precisely, homotopy invariants of ∂W (See Part II).

Theorem 7.2. *If $n=4t-1$ $(t\geqq 2)$, the handlebodies W of type $(0+\mathrm{II})$ of $\mathscr{H}(2n+1, k, n+1)$ are uniquely represented up to diffeomorphism as follows*:

$$\text{(i)}\quad W=\bar{A}_a\natural(p-1)(S^{n+1}\times D^n)\natural rW\begin{pmatrix}0\\ 0\end{pmatrix},\quad a\geqq 0,\quad p+2r=k,$$

where $a\in\pi_{4t-1}(SO_{4t-1})\cong Z$.

$$\text{(ii)}\quad W=p(S^{n+1}\times D^n)\natural W\begin{pmatrix}d\\ 0\end{pmatrix}\natural(r-1)W\begin{pmatrix}0\\ 0\end{pmatrix},\quad d>0,\quad p+2r=k,$$

where $d\in\pi_{4t-1}(SO_{4t-1})\cong Z$.

In (i) *and* (ii), *especially if* $t=2$, *then* a *and* d *are even.*

Proof. Let $(H;\ \lambda, \alpha)$ be a system of type $(0+\mathrm{II})$ with rank $H=k$ and rank $\lambda=2r$ $(0<2r<k)$. By Theorem 3.1 and Theorem 6.2 there is an admissible base $\{u_1,\cdots, u_p;\ e_1, f_1,\cdots, e_r, f_r\}$ of H, $p+2r=k$, such that $(\alpha(u_1),\cdots, \alpha(u_p);\ \alpha(e_1), \alpha(f_1),\cdots, \alpha(e_r), \alpha(f_r))=(a, 0,\cdots, 0;\ d, 0,\cdots, 0)$, where we may assume that a, $d\geqq 0$. If $a\leqq d$, let $d=la+d_1$, $0\leqq d_1<a$. If $a>d$, let $a=2ld+a_1$, $0\leqq|a_1|\leqq d$. Then, by transformations (u_1) and (u_2) we can perform the Euclidean algorithm to the pair (a, d).

Thus we may assume that there exists an admissible base $\{u_1,\cdots, u_p;\ e_1, f_1,\cdots, e_r, f_r\}$ of H, $p+2r=k$, such that $(\alpha(u_1),\cdots,\alpha(u_p);\ \alpha(e_1), \alpha(f_1), \cdots, \alpha(e_r), \alpha(f_r))=(a, 0,\cdots, 0;\ 0,\cdots, 0)$ or $(0,\cdots, 0;\ d, 0,\cdots, 0)$. So, we have the representations of handlebodies of type $(0+\mathrm{II})$ as in the theorem. We show that the two algebraic systems $(H;\ \lambda, \alpha)$, $(H';\ \lambda', \alpha')$ of type $(0+\mathrm{II})$ with the different representations of α and α' by such admissible bases as above are not isomorphic. We show it when $a=d>0$. In the other cases it is clear since α, α' are homomorphisms. If they are isomorphic there exist the admissible bases $\{u_1, \cdots, u_p;\ e_1, f_1, \cdots, e_r, f_r\}$, $\{u'_1, \cdots, u'_p;\ e'_1, f'_1, \cdots, e'_r, f'_r\}$ of H such that $(\alpha(u_1), \cdots, \alpha(u_p);\ \alpha(e_1), \alpha(f_1), \cdots, \alpha(e_r), \alpha(f_r))=(a, 0, \cdots, 0;\ 0, \cdots, 0)$ and $\alpha(u'_1), \cdots, \alpha(u'_p);\ \alpha(e'_1), \alpha(f'_1), \cdots, \alpha(e'_r), \alpha(f'_r))=(0, \cdots, 0;\ d, 0, \cdots, 0)$. Let $(u'_1, \cdots, u'_p;\ e'_1, f'_1, \cdots, e'_r, f'_r)^t = T(u_1, \cdots, u_p;\ e_1, f_1, \cdots, e_r, f_r)^t$, where T is a unimodular matrix. Then, by Lemma 7.1, $t_{i1}=0 \pmod 2$ since $\alpha(u'_i)=t_{i1}\cdot a=0 \pmod 2$, $i=1, 2, \ldots, r$. So that $|T|=0 \pmod 2$, and this is a contradiction. This completes the proof.

Theorem 7.3. *If $n=8s+5$ $(s\geqq 0)$, the handlebodies W of type $(0+\mathrm{II})$ of $\mathscr{H}(2n+1, k, n+1)$ are uniquely represented up to diffeomorphism as follows*:

$$\text{(i)}\quad W=p(S^{n+1}\times D^n)\natural\, W\begin{pmatrix} d \\ d \end{pmatrix}\natural\,(r-1)W\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad p+2r=k, \quad \text{or}$$

$$\text{(ii)}\quad W=\bar{A}_1\natural\,(p-1)(S^{n+1}\times D^n)\natural\, rW\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad p+2r=k,$$

where $\quad d, 1\in \pi_{8s+5}(SO_{8s+5})\cong Z_2$.

Proof. Let $(H;\ \lambda, \alpha)$ be a system of type $(0+\mathrm{II})$ with $\operatorname{rank} H=k$ and $\operatorname{rank}\lambda=2r$ $(0<2r<k)$. By Theorem 3.1 and Theorem 6.3, and applying the transformation (u_1), there exists an admissible base $\{u_1, \cdots, u_p;\ e_1, f_1, \cdots, e_r, f_r\}$ of H such that $(\alpha(u_1), \cdots, \alpha(u_p);\ \alpha(e_1), \alpha(f_1), \cdots, \alpha(e_r), \alpha(f_r))=(0, \cdots, 0;\ 0, \cdots, 0)$ or $(0, \cdots, 0;\ 1, 0, \cdots, 0)$ or $(1, 0, \cdots, 0;\ 0, \cdots, 0)$. We show that any two algebraic systems $(H;\ \lambda, \alpha)$, $(H';\ \lambda', \alpha')$ with different representations of α, α' of the above are not isomorphic. If α has the first or the second representation, α' has the third, and the two systems are isomorphic, then there arise the contradictions as in the proof of Theorem 7.2. If α has the first representation and α' has the second and the two systems are isomorphic, by Lemma 7.1, there exists a mod 2 symplectic $2r\times 2r$ matrix $L=(l_{ij})$ such that

$$\sum_{j=1}^{r} l_{i,2j-1}\cdot l_{i,2j}=\begin{cases} 1 \pmod 2 & \text{if } i=1, 2, \\ 0 \pmod 2 & \text{otherwise.}\end{cases}$$

But Arf invariant shows that such matrices can not exist. This completes the proof.

Let $n=8s+4$ $(s\geqq 0)$ and $(H;\ \lambda, \alpha)$ be a system of type $(0+\mathrm{II})$. By Lemma 2.1 and Lemma 2.2 $\mathrm{Ker}\,\pi_{8s+4}=\{(0, 0),\ (1, 0)\}\subset\pi_{8s+4}(SO_{8s+4})\cong Z_2+Z_2$, $\partial_{8s+4}(1)=(1, 0)\in\mathrm{Ker}\,\pi_{8s+4}$, and therefore $\alpha(H)\subset\mathrm{Ker}\,\pi_{8s+4}\cong Z_2$. So that, the situation is quite similar to that of the case when $n=8s+5$ $(s\geqq 0)$. Thus we have,

Theorem 7.4. *If $n=8s+4$ $(s\geqq 0)$, the handlebodies W of type $(0+\mathrm{II})$ of $\mathscr{H}(2n+1, k, n+1)$ are uniquely represented up to diffeomorphism as follows*:

(i) $W=p(S^{n+1}\times D^n)\natural W\begin{pmatrix} d & 0\\ d & 0\end{pmatrix}\natural(r-1)W\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}$, $\quad p+2r=k$, or

(ii) $W=\bar{A}_{(1,0)}\natural(p-1)(S^{n+1}\times D^n)\natural rW\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}$, $\quad p+2r=k$,

where $(d, 0), (1, 0)\in\pi_{8s+4}(SO_{8s+4})\cong Z_2+Z_2$.

Let $n=8s+1$ $(s\geqq 1)$. For an $(H;\ \lambda, \alpha)$-system of type $(0+\mathrm{II})$, let $\alpha=(\alpha^1, \alpha^2)$, $\alpha^i=p_i\circ\alpha$ $(i=1, 2)$, where p_i is the projection of $\pi_{8s+1}(SO_{8s+1})\cong Z_2+Z_2$ to the i-th component.

Theorem 7.5. *If $n=8s+1$ $(s\geqq 1)$, the handlebodies W of type $(0+\mathrm{II})$ of $\mathscr{H}(2n+1, k, n+1)$ are uniquely represented up to diffeomorphism as follows*:

(i) $W=p(S^{n+1}\times D^n)\natural W_1$, *where W_1 is a handlebody of type* II *of* $\mathscr{H}(2n+1, 2r, n+1)$, $p+2r=k$.

(ii) $W=\bar{A}_{(1,0)}\natural(p-1)(S^{n+1}\times D^n)\natural W\begin{pmatrix} 0 & d\\ 0 & 0\end{pmatrix}\natural(r-1)W\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}$, or

(iii) $W=\bar{A}_{(0,1)}\natural(p-1)(S^{n+1}\times D^n)\natural W\begin{pmatrix} d & 0\\ d & 0\end{pmatrix}\natural(r-1)W\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}$, or

$$\text{(iv)}\quad W=\bar{A}_{(1,1)}\natural(p-1)(S^{n+1}\times D^n)\natural W\begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix}\natural(r-1)W\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where $d=0, 1\in Z_2$ *and* $p+2r=k$.

$$\text{(v)}\quad W=\bar{A}_{(1,0)}\natural\bar{A}_{(0,1)}\natural(p-2)(S^{n+1}\times D^n)\natural rW\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\quad p+2r=k.$$

Proof. Let $(H;\ \lambda, \alpha)$ be a system of type $(0+\mathrm{II})$ with $\operatorname{rank} H=k$ and $\operatorname{rank}\lambda=2r$ $(0<2r<k)$. By Theorem 3.1 and Theorem 6.5 there exists an admissible base $\{u_1,\cdots, u_p;\ e_1, f_1,\cdots, e_r, f_r\}$ of H which satisfies a case $A_i\times C_j$ of the following: we put $A=\begin{pmatrix}\alpha^1(u_1)\cdots\alpha^1(u_p)\\ \alpha^2(u_1)\cdots\alpha^2(u_p)\end{pmatrix}$ and $C=\begin{pmatrix}\alpha^1(e_1) & \alpha^1(f_1) & \cdots & \alpha^1(e_r) & \alpha^1(f_r)\\ \alpha^2(e_1) & \alpha^2(f_1) & \cdots & \alpha^2(e_r) & \alpha^2(f_r)\end{pmatrix}$.

$$(A_0)\quad A=\begin{pmatrix} 0\cdots 0\\ 0\cdots 0\end{pmatrix}\qquad (C_0)\quad C=\begin{pmatrix} 0\cdots 0\\ 0\cdots 0\end{pmatrix}$$

$$(A_1)\quad A=\begin{pmatrix} 1 & 0\cdots 0\\ 0 & 0\cdots 0\end{pmatrix}\qquad (C_1)\quad C=\begin{pmatrix} 1 & 1 & 0\cdots 0\\ 0 & 0 & 0\cdots 0\end{pmatrix}$$

$$(A_2)\quad A=\begin{pmatrix} 0 & 0\cdots 0\\ 1 & 0\cdots 0\end{pmatrix}\qquad (C_2)\quad C=\begin{pmatrix} 0 & 0\cdots 0\\ 1 & 0\cdots 0\end{pmatrix}$$

$$(A_3)\quad A=\begin{pmatrix} 1 & 0\cdots 0\\ 1 & 0\cdots 0\end{pmatrix}\qquad (C_3)\quad C=\begin{pmatrix} 0 & 0 & 0\cdots 0\\ 1 & 1 & 0\cdots 0\end{pmatrix}$$

$$(A_4)\quad A=\begin{pmatrix} 1 & 0 & 0\cdots 0\\ 0 & 1 & 0\cdots 0\end{pmatrix}\qquad (C_4)\quad C=\begin{pmatrix} 1 & 1 & 0 & 0\cdots 0\\ 0 & 0 & 1 & 0\cdots 0\end{pmatrix}$$

$$(C_5)\quad C=\begin{pmatrix} 1 & 1 & 0 & 0 & 0\cdots 0\\ 0 & 0 & 1 & 1 & 0\cdots 0\end{pmatrix}$$

It is easily seen that the above cases $A_i\times C_j$ $(i=0, 1,\cdots, 4,\ j=0, 1,\cdots, 5)$ are reduced to the cases $A_0\times C_j$ $(j=0, 1,\ldots, 5)$, $A_1\times C_j$ $(j=0, 2)$, $A_2\times C_j$ $(j=0, 1)$, $A_3\times C_j$ $(j=0, 1)$, and $A_4\times C_0$, using (u_1)-transformations and (t_1)-, (t_2)-transformations in the symplectic part. We show that these caces are independent up to isomorphism, that is, there are no admissible transformations between any two of those cases. Then the proof will be completed.

It is judged, using Lemma 7.1, mostly by comparing the values of α^1 or α^2 on the corresponding admissible bases. Otherwise, it is judged by the Arf invariant and the ε-invariant; for example, if $A_3\times C_0$ is equivalent to $A_3\times C_1$, then as in the proof of Theorem 7.3, Arf invariant shows that there arises a contradiction. If, for example, $A_0\times C_2$ is equivalent to $A_0\times C_3$, then by Lemma 7.1 there exists a $2r\times 2r$ matrix $L=(l_{ij})$ consisting of integers such that L is mod 2 symplectic, $|L|=1 \pmod 2$, and satisfies that $l_{2i-1,1}=l_{2i,1}=\delta_{i1} \pmod 2$ and $\sum_{j=1}^{r} l_{2i-1,2j-1}\cdot l_{2i-1,2j}=\sum_{j=1}^{r} l_{2i,2j-1}\cdot l_{2i,2j}=0 \pmod 2$. But ε-invariant shows that such matrices can not exist. The others are similar to these cases. This completes the proof.

If $n=8s$ $(s\geqq 1)$, by Lemma 2.1 and Lemma 2.2 $\operatorname{Ker}\pi_{8s}=\{(1,0,0),(0,0,1)\}\subset\pi_{8s}(SO_{8s})\cong Z_2+Z_2+Z_2$ and $\partial_{8s}(1)=(1,0,0)\in\operatorname{Ker}\pi_{8s}$. So that $\alpha(H)\subset\operatorname{Ker}\pi_{8s}$ for $(H;\lambda,\alpha)$-systems and the situation is reduced to that of the above. Thus we have,

Theorem 7.6. *If $n=8s$ $(s\geqq 1)$, the handlebodies W of type $(0+\mathrm{II})$ of $\mathscr{H}(2n+1,k,n+1)$ are uniquely represented up to diffeomorphism as follows:*

(i) $W=p(S^{n+1}\times D^n)\natural W_1$, *where W_1 is a handlebody of type* II *of $\mathscr{H}(2n+1,2r,n+1)$, $p+2r=k$.*

(ii) $W=\bar{A}_{(1,0,0)}\natural(p-1)(S^{n+1}\times D^n)\natural W\begin{pmatrix}0&0&d\\0&0&0\end{pmatrix}\natural(r-1)W\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}$, or

(iii) $W=\bar{A}_{(0,0,1)}\natural(p-1)(S^{n+1}\times D^n)\natural W\begin{pmatrix}d&0&0\\d&0&0\end{pmatrix}\natural(r-1)W\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}$, or

(iv) $W=\bar{A}_{(1,0,1)}\natural(p-1)(S^{n+1}\times D^n)\natural W\begin{pmatrix}d&0&0\\d&0&0\end{pmatrix}\natural(r-1)W\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}$,

where $d=0,1\in Z_2$ and $p+2r=k$.

(v) $W=\bar{A}_{(1,0,0)}\natural\bar{A}_{(0,0,1)}\natural(p-2)(S^{n+1}\times D^n)\natural rW\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}$, $p+2r=k$.

Let $n=4t+2$ $(t\geqq 1)$. By Lemma 2.1 and Lemma 2.2, $\operatorname{Ker}\pi_{4t+2}=\pi_{4t+2}(SO_{4t+2})\cong Z_4$ if $t\geqq 2$, $\pi_6(SO_6)=0$, and $\partial_{4t+2}(1)=2\in Z_4$ if $t\geqq 2$. We have the following.

Theorem 7.7. *If $n=4t+2$ $(t\geqq 1)$, the handlebodies W of type $(0+\mathrm{II})$ of $\mathscr{H}(2n+1,k,n+1)$ are uniquely represented up to diffeomorphism as follows;*

(i) *If* $\alpha(H)\subset\{0, 2\}\subset Z_4$, $\quad t\geqq 2$,

$$W=p(S^{n+1}\times D^n)\natural W\begin{pmatrix} d \\ d \end{pmatrix}\natural(r-1)W\begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $d=0,\ 2\in Z_4$ *and* $p+2r=k$,

or $\quad W=\bar{A}_2\natural(p-1)(S^{n+1}\times D^n)\natural rW\begin{pmatrix} 0 \\ 0 \end{pmatrix},\qquad p+2r=k.$

(ii) *If* $\alpha(H)\not\subset\{0, 2\}\subset Z_4$, $\quad t\geqq 2$,

$$W=p(S^{n+1}\times D^n)\natural W\begin{pmatrix} 1 \\ 0 \end{pmatrix}\natural(r-1)W\begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or $\quad W=\bar{A}_1\natural(p-1)(S^{n+1}\times D^n)\natural rW\begin{pmatrix} 0 \\ 0 \end{pmatrix},\qquad p+2r=k.$

(iii) *If* $\ n=6,\ W=p(S^{n+1}\times D^n)\natural rW\begin{pmatrix} 0 \\ 0 \end{pmatrix},\qquad p+2r=k.$

Here, α is the map of the corresponding algebraic system $(H;\ \lambda, \alpha)$.

Proof. Let $(H;\ \lambda, \alpha)$ be a system of type $(0+\mathrm{II})$ with $\operatorname{rank} H=k$ and $\operatorname{rank}\lambda=2r$ $(0<2r<k)$, and let $t\geqq 2$. By Theorem 3.1 and Theorem 6.7 there exists an admissible base $\{u_1,\cdots, u_p;\ e_1, f_1,\cdots, e_r, f_r\}$ $(p+2r=k)$ of H which represents α as the matrix (A, C) of a case $A_i\times C_j$ of the following, where $A=(\alpha(u_1),\cdots, \alpha(u_p))$ and $C=(\alpha(e_1),\ \alpha(f_1),\cdots, \alpha(e_r),\ \alpha(f_r))$:

$(A_0)\quad A=(0,\cdots, 0),\qquad (C_0)\quad C=(0,\cdots, 0)$

$(A_1)\quad A=(1, 0,\cdots, 0),\qquad (C_1)\quad C=(1, 0,\cdots, 0)$

$(A_2)\quad A=(2, 0,\cdots, 0),\qquad (C_2)\quad C=(2, 2, 0,\cdots, 0).$

It is easily seen that the above cases $A_i\times C_j$ $(i, j=0, 1, 2)$ are reduced to the cases $A_0\times C_0$, $A_0\times C_1$, $A_0\times C_2$, $A_1\times C_0$, and $A_2\times C_0$, using (u_1)-transformations. We show that the representations of α in these cases are independent up to equivalence.

If $\alpha(H)\subset\{0, 2\}\cong Z_2$, α has the representation of $A_0\times C_0$ or $A_0\times C_2$ or $A_2\times C_0$. But since $\partial_{4t+2}(1)=2$ $(t\geqq 2)$ independence of those representations is known similarly as in the case when $n=8s+5$ $(s\geqq 0)$. If $\alpha(H)$

$\not\subset\{0, 2\}$, then equivalently $\alpha(H)=Z_4$, and α has the representation of $A_0\times C_1$ or $A_1\times C_0$. The independence of the two is easily seen by comparing the values of α on the corresponding admissible bases using Lemma 7.1. This completes the proof.

Part II. Classification of $(n-2)$-connected $2n$-manifolds

In this part, we consider to classify $(n-2)$-connected $2n$-manifolds with torsion free homology groups up to diffeomorphism $\bmod \theta_{2n}$. Since the results are listed up in the introduction, we give the proofs of the theorems.

8. Proof of Theorem 1

Lemma 8.1. *Let M be an $(n-2)$-connected $2n$-manifold $(n\geqq 3)$ which has the vanishing n-th homology group and satisfies the hypothesis (H) in the introduction. Then, $M=\partial W\#\Sigma$ for some handlebody $W\in \mathscr{H}(2n+1, k, n+1)$ and some homotopy $2n$-sphere Σ, where* $k=\text{rank}\ H_{n-1}(M)$.

Proof. This is obtained by Theorem 6.3 of [7], and also confer Theorem 2 of [20].

Let $W=D^{2n+1}\underset{\{f_i\}}{\cup}\{\bigcup_{i=1}^{k} D_i^{n+1}\times D_i^n\}$ be a handlebody of $\mathscr{H}(2n+1, k, n+1)$, $n\geqq 4$, where f_i: $\partial D_i^{n+1}\times D_i^n\to\partial D^{2n+1}=S^{2n}$, $i=1, 2,\cdots, k$, are disjoint imbeddings. Let $\lambda_{ij}\in Z_2\cong\pi_n(S^{n-1})$ $(n\geqq 4)$ be the linking element (Haefliger [5]) defined by $f_j(S_j^n\times 0)$ in $S^{2n}-f_i(S_i^n\times 0)$ if $i\neq j$, and defined by $S_i'^n$ in $S^{2n}-f_i(S_i^n\times 0)$ slightly moved from $f_i(S_i^n\times 0)$ if $i=j$. Let $\varepsilon_i\in H^{n-1}(\partial W;Z_2)$, $i=1, 2,\ldots, k$, be the canonical generators which are dual to the homology classes $(x_i\times S_i^{n-1})\in H_{n-1}(\partial W;\ Z_2)$, $x_i\in\partial D_i^{n+1}$, $S_i^{n-1}=\partial D_i^n$, respectively. Then we have,

Lemma 8.2. *$\lambda_{ij}=<S_q^2\varepsilon_i\cup\varepsilon_j, [\partial W]_2>$ for all i, j, where $[\partial W]_2$ is the mod 2 fundamental class of $H_{2n}(\partial W;\ Z_2)$.*

Proof. Let $Y=S^{2n}-\bigcup_{i=1}^{k}\text{Int} f_i(S_i^n\times D_i^n)$. Y is a deformation retract of $S^{2n}-\bigcup_{i=1}^{r} f_i(S_i^n\times 0)$ and $\partial W=Y\underset{\{f_i\}}{\cup}\{\bigcup_{i=1}^{k} D_i^{n+1}\times S_i^{n-1}\}$. Let $f_i'=f_i\,|\,S_i^n\times y_i$,

$y_i \in S_i^{n-1} = \partial D_i^n$. There exists a continuous map $h\colon \bigvee_{i=1}^k S_i^{n-1} \to Y$ which induces an isomorphism $h_*\colon \pi_n(\bigvee_{i=1}^k S_i^{n-1}) \to \pi_n(Y)$. So that there are isomorphisms

$$\pi_n(Y) \underset{\cong}{\xleftarrow{h_*}} \pi_n(\bigvee_{i=1}^k S_i^{n-1}) \cong \sum_{i=1}^k \pi_n(S_i^{n-1}) \underset{\cong}{\xrightarrow{H}} Z_2 + \cdots + Z_2 \qquad (n \geqq 4).$$

Then, the element (f_j') of $\pi_n(Y)$ correspond to an element (g_j) of $\pi_n(\bigvee_{i=1}^k S_i^{n-1})$, and if $(g_j) = (g_{1j}) + \cdots + (g_{kj})$, we have $H(g_{ij}) = \lambda_{ij}$ by the definition of linking elements. We have the following commutative diagram·

$$\begin{array}{ccc}
H^{n-1}(\partial W;\, Z_2) & \xrightarrow{\quad S_q^2 \quad} & H^{n+1}(\partial W;\, Z_2) \\
{\scriptstyle i^*}\downarrow{\scriptstyle \cong} & & {\scriptstyle i^*}\downarrow{\scriptstyle \cong} \\
H^{n-1}(Y \underset{\{f_i'\}}{\cup} \{\bigcup_{i=1}^k D_i^{n+1} \times y_i\};\, Z_2) & \xrightarrow{S_q^2} & H^{n+1}(Y \underset{\{f_i'\}}{\cup} \{\bigcup_{i=1}^k D_i^{n+1} \times y_i\};\, Z_2) \\
{\scriptstyle F^*}\downarrow{\scriptstyle \cong} & & {\scriptstyle F^*}\downarrow{\scriptstyle \cong} \\
H^{n-1}(\bigvee_{i=1}^k S_i^{n-1} \underset{\{g_i\}}{\cup} \{\bigcup_{i=1}^k D_i^{n+1}\};\, Z_2) & \xrightarrow{S_q^2} & H^{n+1}(\bigvee_{i=1}^k S_i^{n-1} \underset{\{g_i\}}{\cup} \{\bigcup_{i=1}^k D_i^{n+1}\};\, Z_2),
\end{array}$$

where $h \circ g_i = f_i'$ may be assumed and F is a continuous map such that $F| \bigvee_{i=1}^k S_i^{n-1} = h$ and $F| \bigcup_{i=1}^k D_i^{n+1} = \text{identity}$. Let $\{d_1, \cdots, d_k\}$ be the base of $H_{n+1}(\partial W; Z_2) \cong H_{n+1}(W; Z_2)$ corresponding to the canonical base of $H_{n+1}(W; Z_2)$, and let $\{\delta_1, \cdots, \delta_k\}$ be the dual base of $H^{n+1}(W; Z_2)$. Let $\{\alpha_1, \cdots, \alpha_k\}$, $\{\beta_1, \cdots, \beta_k\}$ be the canonical bases of $H^r(\bigvee_{i=1}^k S_i^{n-1} \underset{\{g_i\}}{\cup} \{\bigcup_{i=1}^k D_i^{n+1}\}; Z_2)$, $r = n-1$, $n+1$, respectively. Then, since $S_q^2 \alpha_i = \sum_{j=1}^k H(g_{ij}) \beta_j = \sum_{j=1}^k \lambda_{ij} \beta_j$, the diagram shows that $S_q^2 \varepsilon_i = \sum_{j=1}^k \lambda_{ij} \delta_j$. Therefore, $< S_q^2 \varepsilon_i \cup \varepsilon_j, [\partial W]_2 > = < S_q^2 \varepsilon_i, d_j > = < \sum_{j=1}^k \lambda_{ij} \delta_j, d_j > = \lambda_{ij}$, and this completes the proof.

Remark 1. If W is a handlebody of $\mathscr{H}(2n, k, n+1)$ $(n \geqq 6)$, we have a similar result by replacing S_q^2 by Adem's secondary cohomology operation.

Remark 2. We have shown the above lemma for $n \geqq 4$. Confer Wall [20] for $n = 3$.

Theorem 8.3. *For a handlebody W of $H(2n+1, k, n+1)$, $n \geqq 4$, Wall's pairing λ: $H_{n+1}(W) \times H_{n+1}(W) \to \pi_{n+1}(S^n) \cong Z_2$ is isomorphic to the pairing ϕ: $H^{n-1}(\partial W) \times H^{n-1}(\partial W) \to Z_2$ defined in the introduction.*

Proof. Let $\{e_1, \cdots, e_k\}$ be the canonical base of $H_{n+1}(W)$. Since $\lambda(e_i, e_j) = S\lambda_{ij}$ by Wall [19], using Lemma 8.2, the following commutative diagram completes the proof:

$$\begin{tikzcd} H_{n+1}(W) \times H_{n+1}(W) \arrow[r, "\lambda"] & Z_2 \cong \pi_{n+1}(S^n) \\ H_{n+1}(\partial W) \times H_{n+1}(\partial W) \arrow[u, "i_* \times i_*", "\cong"'] & \\ H^{n-1}(\partial W) \times H^{n-1}(\partial W) \arrow[u, "D \times D", "\cong"'] \arrow[r, "\phi"] & Z_2 \cong \pi_n(S^{n-1}), \arrow[uu, dashed, "\cong"', "S"] \end{tikzcd}$$

where S is the suspension isomorphism, i_* is the isomorphism induced by the inclusion map i, and D is the isomorphism of Poincaré duality.

Corollary 8.4. $W\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}$ *is neither diffeomorphic nor homeomorphic to the boundary connected sum of the two n-disk bundles over $(n+1)$-spheres* $(n \geqq 4)$.

Corollary 8.5. $V\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}$ *never has the homotopy type of the connected sum of the two $(n-1)$-sphere bundles over $(n+1)$-spheres* $(n \geqq 4)$.

Now, we have proved Theorem 1; by Lemma 8.1 and Theorem 8.3, the type of M coincides with that of the membrane $W \in \mathscr{H}(2n+1, k, n+1)$, $k = \operatorname{rank} H_{n-1}(M)$, and therefore the results of Part I serve the representation of $M \bmod \theta_{2n}$.

9. Proof of Theorem 2

Let M be an $(n-2)$-connected $2n$-manifold $(n \geqq 4)$ which has the vanishing n-th homology group and satisfies the hypothesis (H) in the introduction. Then, by Lemma 8.1 $M = \partial W \# \Sigma$, where $W \in \mathscr{H}(2n+1, k, n+1)$, $k = \operatorname{rank} H_{n-1}(M)$, and Σ is a homotopy $2n$-sphere. By Theorem 8.3 M is of type 0 if and only if W is of type 0. The uniqueness of the representation of M of type 0 is know by the following theorem and Theorem 3.1.

Theorem 9.1. *Let W_i, $i=1, 2$, be the handlebodies of type 0 of $\mathscr{H}(2n+1, k, n+1)$, $n \geqq 4$. If ∂W_1 is diffeomorphic to ∂W_2 mod θ_{2n}, then W_1 is diffeomorphic to W_2.*

Proof. Let $(H_i; \lambda_i, \alpha_i)$, $i=1, 2$, be the corresponding algebraic systems of W_i, $i=1, 2$, respectively. We show that they are isomorphic. Since λ_i, $i=1, 2$, are trivial, it is sufficient to show that there exists an isomorphism $h: H_1 \to H_2$ such that $\alpha_1 = \alpha_2 \circ h$. Since ∂W_1 is diffeomorphic to $\partial W_2 \bmod \theta_{2n}$, there exists a homeomorphism $g: \partial W_1 \to \partial W_2$ which is an almost diffeomorphism. Let $i_s: \partial W_s \to W_s$, $s=1, 2$, be inclusion maps, and let $h=(i_2)_* \circ g_* \circ (i_1)_*^{-1}$, where $(i_s)_*: H_{n+1}(\partial W_s) \to H_{n+1}(W_s)$, $s=1, 2$, $g_*: H_{n+1}(\partial W_1) \to H_{n+1}(\partial W_2)$ are isomorphisms. Let u_i, $i=1, 2, \cdots, k$, be a base of $H_1 = H_{n+1}(W_1)$. Then we may assume that $W_1 = \bar{A}_1 \natural \bar{A}_2 \natural \cdots \natural \bar{A}_k$, where $\bar{A}_i$, $i=1, 2, \cdots, k$, are D^n-bundles over $(n+1)$-spheres, the zero-cross section S_i^{n+1} of $\bar{A}_i$ represents u_i, and each $A_i = \partial \bar{A}_i$ has a cross section $\bar{S}_i^{n+1}$ which represents $(i_1)_*^{-1}(u_i)$. (See §3). Let $\beta_1(\bar{S}_i^{n+1})$, $\beta_2(g(\bar{S}_i^{n+1})) \in \pi_n(SO_{n-1})$ be the characteristic elements of the normal bundles of $\bar{S}_i^{n+1}$ in ∂W_1 and $g(\bar{S}_i^{n+1})$ in ∂W_2 respectively. We know that $\alpha_1(u_i) = S\beta_1(\bar{S}_i^{n+1})$ and moving $(i_2 \circ g)(\bar{S}_i^{n+1})$ slightly into the interior of W_2 the normal bundle of it has $S\beta_2(g(\bar{S}_i^{n+1}))$ as its characteristic element, where S: $\pi_n(SO_{n-1}) \to \pi_n(SO_n)$ is the suspension homomorphism.
Then we have $\alpha_2(h(u_i)) = S\beta_2(g(\bar{S}_i^{n+1})) = S\beta_1(\bar{S}_i^{n+1}) = \alpha_1(u_i)$, $i=1, 2, \cdots, k$. Since α_1, α_2 are homomorphism, we have $\alpha_2 \circ h = \alpha_1$. This completes the proof.

Lemma 9.2. *Let $n=4t-1$ $(t \geqq 2)$ and let W be a handlebody of $\mathscr{H}(2n+1, k, n+1)$ with the algebraic system $(H; \lambda, \alpha)$. We have the following commutative diagram:*

$$\begin{tikzcd} & Z \arrow[d, "\pm c"] \\ H = H_{n+1}(W) \arrow[ur, "\alpha"] \arrow[r, "{<P_t(W),\ >}"'] & Z \end{tikzcd}$$

where c is the number as

$$c=\begin{cases} 12 & \text{if } t=2, \\ 2(2t-1)! & \text{if } t \text{ is odd } \geqq 3, \\ (2t-1)! & \text{if } t \text{ is even } \geqq 4. \end{cases}$$

Proof. For $x \in H_{n+1}(W)$ represent it by an imbedded $(n+1)$-sphere, and let $\nu=(N, \rho, S^{n+1})$ be the normal bundle with the characteristic element $\alpha(x) \in Z \cong \pi_{4t-1}(SO_{4t-1})$. It is well known that $P_t(\nu)=\pm c\cdot\bar{\mu}$, where $\bar{\mu}$ is the fundamental class of $H^{n+1}(S^{n+1})$. (See Kervaire [8] and Tamura [16], [17]). So that, we have $P_t(N)=\rho^*(P_t(\nu))=\pm c\alpha(x)\cdot\bar{e}$ and therefore $<P_t(W), x> = <P_t(N), e> = \pm c\alpha(x)$, where $e \in H_{n+1}(N)$, $\bar{e} \in H^{n+1}(N)$ are those corresponding to μ, $\bar{\mu}$ respectively.

Let M be as above and assume that M has the two representations $M=\partial W_1 \# \Sigma_1=\partial W_2 \# \Sigma_2$, where $W_i \in \mathscr{H}(2n+1, k, n+1)$, Σ_i are homotopy $2n$-spheres, $i=1, 2$, and $k=\operatorname{rank} H_{n-1}(M)$. There exist homeomorphisms h_i: $\partial W_i \to \partial W_i \# \Sigma_i$, $i=1, 2$, covered by bundle maps between the tangent bundles (See Shiraiwa [15]). So that there exists a homeomorphism g: $\partial W_1 \to \partial W_2$ such that $g^*(\tau(\partial W_2))=\tau(\partial W_1)$. The following theorem will show the uniqueness of the representation of M in any type when $n=4t-1$ $(t \geq 2)$.

Theorem 9.3. *Let $n=4t-1$ $(t \geqq 2)$ and let $W_i, i=1, 2$, be the handlebodies of $\mathscr{H}(2n+1, k, n+1)$. If there exists a homotopy equivalence f: $\partial W_1 \to \partial W_2$ such that $f^*(P_t(\partial W_2))=P_t(\partial W_1)$, then W_1 is diffeomorphic to W_2.*

Proof. Let $(H_i; \lambda_i, \alpha_i)$ be the corresponding algebraic system of W_i, $i=1, 2$, and we show that they are isomorphic. Let $(i_s)_*$: $H_{n+1}(\partial W_s) \to H_{n+1}(W_s)$, $s=1, 2$, be the isomorphisms induced by inclusion maps and define an isomorphism h: $H_1=H_{n+1}(W_1) \to H_2=H_{n+1}(W_2)$ by $h=(i_2)_* \circ f_* \circ (i_1)_*^{-1}$. Let $\sigma_s=D^{-1} \circ (i_s)_*^{-1}$, $s=1, 2$, where D: $H^{n-1}(\partial W_s) \to H_{n+1}(\partial W_s)$, $s=1, 2$, are Poincaré duality isomorphisms. Then $\sigma_1=f^* \circ \sigma_2 \circ h$ and by Theorem 8.3, it is easily seen that $\lambda_1(x, y)=\phi_1(\sigma_1(x), \sigma_1(y))= \phi_1((f^* \circ \sigma_2 \circ h)x, (f^* \circ \sigma_2 \circ h)y)=\phi_2(\sigma_2(hx), \sigma_2(hy))=\lambda_2(h(x), h(y))$, where

$\phi_s: H^{n-1}(\partial W_s)\times H^{n-1}(\partial W_s)\to Z_2$, $s=1, 2$, are the bilinear forms defined in the introduction.

On the other hand, $\alpha_s=\frac{1}{c}<P_t(W_s), >$, $s=1, 2$, by Lemma 9.2. Since $P_t(\partial W_s)=(i_s)^*P_t(W_s)$, $s=1, 2$, and $P_t(\partial W_1)=f^*P_t(\partial W_2)$, we have $<P_t(W_1), x>=<P_t(W_2), h(x)>$ and so that $\alpha_1(x)=\alpha_2(h(x))$. This completes the proof.

Corollary 9.4. *Let $n=4t-1$ $(t\geq 2)$ and let W_i, $i=1, 2$, be handlebodies of $\mathscr{H}(2n+1, k, n+1)$. The following four are equivalent.*

(i) *∂W_1 is homeomorphic to ∂W_2.*

(ii) *∂W_1 is diffeomorphic to ∂W_2.*

(iii) *W_1 is homeomorphic to W_2.*

(iv) *W_1 is diffeomorphic to W_2.*

Proof. We may show only that (i) induces (iv). Let $h: \partial W_1\to\partial W_2$ be the homeomorphism. Since $H^{n+1}(W_i; Z)\to H^{n+1}(W_i; Q)$, $i=1, 2$, are injective, where Q is the group of rational numbers, by topological invariance of rational Pontryagin classes (Novikov [12]), we have $h^*(P_t(W_2))=P_t(W_1)$. So that it follows from Theorem 9.3.

By Lemma 8.1 we have

Corollary 9.5. *Let $n=4t-1$ $(t\geqq 2)$ and let M_i, $i=1, 2$, be $(n-2)$-connected $2n$-manifolds which have vanishing n-th homology groups and are $(n-1)$-parallelizable if t is odd. Then, if M_1 is homeomorphic to M_2, M_1 is diffeomorphic to M_2 mod θ_{2n}.*

The other results of Theorem 2 are obtained directly from Theorem 1.

10. Proof of Theorem 3

Let M be an $(n-2)$-connected $2n$-manifold $(n\geqq 4)$ which has torsion free homology groups and satisfies the hypothesis (H). Let $e_1,\cdots, e_l$ be a base of $H_n(M)$. Since the elements of $H_n(M)$ are spherical, each e_i can be represented by an imbedded n-sphere S_i^n, and we may assume that S_i^n,

$i=1, 2,\ldots, l$, intersect at points. Then, shrinking the intersections by an isotopy of M into a small neighborhood, we have a handlebody $W\in \mathscr{H}(2n, l, n)$ which represents the homology group $H_n(M)$. Since the matrix $(e_i\cdot e_j)$ is unimodular, ∂W is a homotopy $(2n-1)$-sphere. Let $N=M-\operatorname{Int}W$. Then N is an $(n-2)$-connected $2n$-manifold with $\partial N=\partial W$ such that $H_{n-1}(N)\cong H_{n-1}(M)$, $H_{n+1}(N)\cong H_{n+1}(M)$, and $H_n(N)=0$. Since $H_{n-1}(N)$ is free and N is $(n-1)$-parallelizable, we can kill $H_{n-1}(N)$ by surgery so that it does not affect that $H_n(N)=0$ (See Ishimoto [6]). So that ∂N bounds a contractible manifold and therefore it must be a standard sphere. Thus, by closing N, W, we have M_1, M_2 respectively such that $M=M_1\# M_2$, where M_1 is an $(n-2)$-connected $2n$-manifold which has the vanishing n-th homology group and is $(n-1)$-parallelizable, and M_2 is an $(n-1)$-connected $2n$-manifold.

To prove the uniqueness of M_2 we use the following lemma.

Lemma 10.1. *In the following splitting exact sequence*

$$0 \longrightarrow \operatorname{Ker} h \longrightarrow \pi_n(M) \xrightarrow{h} H_n(M) \longrightarrow 0,$$

$\operatorname{Ker} h$ *consists of the elements of order* 2, *where* h *is the Hurewicz homomorphism* $(n\geqq 4)$.

Proof. Let M^* be the space obtained from M by attaching n-cells e_i^n, $i=1, 2,\cdots, k$, $(k=\operatorname{rank} H_{n-1}(M))$ to kill $\pi_{n-1}(M)\cong H_{n-1}(M)$. Then M^* and (M^*, M) are $(n-1)$-connected, and we have the following diagram well known:

$$\begin{array}{ccccc}
\pi_{n+1}(M^*, M) & \xrightarrow{\partial} & \pi_n(M) & \xrightarrow{i'} & \pi_n(M^*) \\
 & & \downarrow{\scriptstyle h} & & \cong\downarrow{\scriptstyle h} \\
 & 0 \longrightarrow & H_n(M) & \longrightarrow & H_n(M^*),
\end{array}$$

where horizontal sequences are exact. So that we know that $\operatorname{Ker} h=\operatorname{Ker} i_*$ $=\operatorname{Im}\partial$, where $\pi_{n+1}(M^*, M)\cong\pi_{n+1}(\bigvee_{i=1}^{k} e_i^n, \bigvee_{i=1}^{k} S_i^{n-1})\cong\pi_n(\bigvee_{i=1}^{k} S_i^{n-1})\cong Z_2+\cdots+Z_2$ $(n\geqq 4)$.

Lemma 10.2. *Any imbedded* n-*sphere of* M *representing a torsion element of* $\pi_n(M)$ *has the trivial normal bundle* $(n\geqq 4)$.

Proof. Let $\beta\colon \pi_n(M)\to\pi_{n-1}(SO_n)$ be the map which associates to each n-sphere the characteristic element of the normal bundle. This is well defined since $n\geqq 4$. Let $\mu\colon \pi_n(M)\times\pi_n(M)\to Z$ be the pairing of intersection numbers after $h\times h$. Then, since $\beta(x+y)=\beta(x)+\beta(y)+\partial\mu(x, y)$, $\partial\colon \pi_n(S^n)\to\pi_{n-1}(SO_n)$ by Wall [19], we know that β is a homomorphism on $\operatorname{Ker} h$, the torsion subgroup of $\pi_n(M)$. So that we may prove the lemma for the generators of $\operatorname{Ker} h=\operatorname{Im}\partial$. Let S_i^{n-1} be the imbedded $(n-1)$-sphere of M which represents the basis element of $H_{n-1}(M)\cong\pi_{n-1}(M)$, $i=1, 2,\cdots, k$. $\operatorname{Im}\partial$ is generated by the essential maps from the n-sphere to S_i^{n-1} for some numbers of i. Since each S_i^{n-1} has the trivial normal bundle in M by the $(n-1)$-parallelizability of M, such essential maps can be represented by the imbedded n-spheres T_i^n in $S_i^{n-1}\times D_i^{n+1}$, the product neighborhood of S_i^{n-1}. We note that any n-sphere $S^n (n\geqq 4)$ imbedded in $S^{n-1}\times D^{n+1}\subset R^{2n}$ has the trivial normal bundle considering it in R^{2n} by Haefliger [4], and this completes the proof.

Let M have the two decompositions as $M=M_1\# M_2=M_1'\# M_2'$, and let $W=M_2-\operatorname{Int} D^{2n}$, $W'=M_2'-\operatorname{Int} D'^{2n}$. W, W' are the handlebodies of $\mathscr{H}(2n, l, n)$, $l=\operatorname{rank} H_n(M)$, with the algebraic systems $(H;\ \lambda, \alpha)$, $(H';\ \lambda', \alpha')$ respectively. We show that W and W' are diffeomorphic, that is, their associated algebraic systems are isomorphic.

For any $x\in H_n(M)$ take an element $y\in\pi_n(M)$ such that $h(y)=x$ and define $\beta'(x)$ by $\beta(y)$. Then, $\beta'\colon H_n(M)\to\pi_{n-1}(SO_n)$ is well defined by Lemma 10.1 and Lemma 10.2. Let i, i' be the inclusion maps of W, W' into M respectively. We have the following commutative diagram:

$$\begin{tikzcd}
\pi_n(W) \arrow[rr, "i_*"] \arrow[dd, "\cong"', "h"] \arrow[dr, "\alpha"] & & \pi_n(M) \arrow[dl, "\beta"'] \arrow[dr, "\beta"] \arrow[dd, "h"] & & \pi_n(W') \arrow[ll, "i_*"'] \arrow[dl, "\alpha'"'] \arrow[dd, "\cong"', "h"] \\
& \pi_{n-1}(SO_n) \arrow[rr, phantom, "="] & & \pi_{n-1}(SO_n) & \\
H_n(W) \arrow[rr, "i_*", "\cong"'] \arrow[ur, "\alpha"] & & H_n(M) \arrow[ul, "\beta'"'] \arrow[ur, "\beta'"] & & H_n(W') \arrow[ll, "i'_*"', "\cong"] \arrow[ul, "\alpha'"']
\end{tikzcd}$$

where $\alpha=\beta'\circ i_*$ and $\alpha'=\beta'\circ i_*'$ are known from the other commutative triangles and squares.

Let $g: H=H_n(W)\to H'=H_n(W')$ be the isomorphism defined by $g=(i'_*)^{-1}\circ i_*$. Then, we have $\alpha=\beta'\circ i_*=\alpha'\circ(i'_*)^{-1}\circ i_*=\alpha'\circ g$. On the other hand, let $\mu': H_n(M)\times H_n(M)\to Z$ be the intersection number pairings. Since λ, λ' are the intersection number pairings, we have $\lambda=\mu'\circ(i_*\times i_*)$, $\lambda'=\mu'\circ(i'_*\times i'_*)$, and so that $\lambda=\lambda'\circ(g\times g)$. Thus we have proved that W and W' are diffeomorphic, that is, M_2 and M'_2 are diffeomorphic mod θ_{2n}.

Now, we prove the uniqueness of M_1 when $n=4t-1$ $(t\geqq 2)$. Let $N=M_1-\mathrm{Int}\,D^{2n}$, $N'=M'_1-\mathrm{Int}\,D'^{2n}$, and $M=N\cup W=N'\cup W'$. Let η: $H_{n+1}(M_1)\to H_{n+1}(M'_1)$ be the isomorphism defined by the composition of the isomorphisms $H_{n+1}(M_1)\xleftarrow{\cong} H_{n+1}(N)\xrightarrow{\cong} H_{n+1}(M)\xleftarrow{\cong} H_{n+1}(N')\xrightarrow{\cong} H_{n+1}(M'_1)$, and let η': $H^{n-1}(M_1)\to H^{n-1}(M'_1)$ be the isomorphism defined by the composition of the isomorphisms $H^{n-1}(M_1)\xrightarrow{\cong} H^{n-1}(N)\xleftarrow{\cong} H^{n-1}(M)\xrightarrow{\cong} H^{n-1}(N')\xleftarrow{\cong} H^{n-1}(M'_1)$. Then it is easily seen that $\phi=\phi'\circ(\eta'\times\eta')$ and $<P_t(M_1),\ >=<P_t(M'_1),\ >\circ\eta$, where ϕ, ϕ' are the associated bilinear forms of M, M' respectively. So that using Theorem 8.3. and Lemma 9.2, we know that the membranes of M_1 and M'_1 by Lemma 8.1. are diffeomorphic, where we note that $P_t(M)$, $P_t(M')$ are induced from those of the membranes. Thus M_1 and M'_1 must be diffeomorphic mod θ_{2n}.

The uniqueness of M_1-* up to homotopy is known by the following.

Lemma 10.3. *Let M, M' be $(n-2)$-connected $2n$-manilolds $(n\geqq 4)$ with the vanishing n-th homology groups, and let ϕ, ϕ' be the bilinear forms of M, M' respectively. If* rank $H_{n-1}(M)=$ rank $H_{n-1}(M')$, rank $\phi=$ rank ϕ', *and M, M' belong to the same type, then $M-$(a point) has the homotopy type of $M'-$(a point).*

Proof. By Lemma 1.1, there exist the bases $\{\alpha_1,\cdots,\alpha_k\}$, $\{\alpha'_1,\cdots,\alpha'_k\}$ of $H^{n-1}(M)$, $H^{n-1}(M')$ respectively such that $\phi(\alpha_i,\alpha_j)=\phi(\alpha'_i,\alpha'_j)$ for all i, j, where $k=\mathrm{rank}\,H_{n-1}(M)$. Let $\{\beta_1,\cdots,\beta_k\}$ be the base of $H^{n+1}(M)$ such that $<\alpha_i\cup\beta_j,[M]>=\delta_{ij}$ for all i, j, and let $a_i\in H_{n-1}(M)$, $b_i\in H_{n+1}(M)$ be the dual elements of α_i, β_i respectively, $i=1,2,\cdots,k$. Then, using Smale decomposition, $M-*$ has the homotopy type of $\bigvee_{i=1}^{k} S_i^{n-1}\cup_{\{f_i\}}\{\bigcup_{i=1}^{k} D_i^{n+1}\}$, where f_i: $\partial D_i^{n+1}\to\bigvee_{i=1}^{k} S_i^{n-1}$, $i=1,2,\cdots,k$, are the attaching maps.

Similarly, let β'_i, a'_i and b'_i, $i=1, 2,\cdots, k$, be those of M'. $M'-*'$ has the homotopy type of $\bigvee_{i=1}^{k} S_i^{n-1} \cup_{\{f'_i\}} \{\bigcup_{i=1}^{k} D_i^{n+1}\}$, where $f'_i\colon \partial D_i^{n+1} \to \bigvee_{i=1}^{k} S_i^{n-1}$, $i=1, 2,\cdots, k$, are the attaching maps. Then, under the isomorphisms $\pi_n(\bigvee_{i=1}^{k} S_i^{n-1}) \cong \pi_n(S_1^{n-1})+\cdots+\pi_n(S_k^{n-1}) \xrightarrow[\cong]{H} Z_2+\cdots+Z_2$ $(n\geqq 4)$, $S_q^2(\alpha_i)_2 = \sum_{j=1}^{k} H(f_{ij})(\beta_j)_2$, where $(f_i)=(f_{1i})+\cdots+(f_{ki})$ and $(\ \ )_2$ means that they are considered in the Z_2-coefficient. Thus we have $\phi(\alpha_i, \alpha_j) = < S_q^2(\alpha_i)_2 \cup (\alpha_j)_2, [M]_2 > = \sum_{t=1}^{k} H(f_{it}) < (\beta_t)_2 \cup (\alpha_j)_2, [M]_2 > = H(f_{ij})$. Let $(f'_i)=(f'_{1i})+\cdots+(f'_{ki})$. Similarly we have $\phi'(\alpha'_i, \alpha'_j)=H(f'_{ij})$. So that $H(f_{ij})=H(f'_{ij})$, for all i, j, since $\phi(\alpha_i, \alpha_j)=\phi'(\alpha'_i, \alpha'_j)$, and therefore we know that f_i is homotopic to f'_i, $i=1, 2,\cdots, k$. Thus $M-*$ has the homotopy type of $M'-*$.

This completes the proof of Theorem 3.

11. Proof of Theorem 4 and Others

Let $n=4t-1$ $(t\geqq 2)$ and let M be an $(n-2)$-connected $2n$-manifold which has torsion free homology groups and is $(n-1)$-parallelizable if t is odd. Let $M=M_1 \# M_2$ be the decomposition of M, and let $M_1=\partial W_1$ (mod θ_{2n}), $W_1 \in \mathscr{H}(2n+1, k, n+1)$, $k=\operatorname{rank} H_{n-1}(M)$. Then, as is seen in the proof of the uniqueness of M_1 of Theorem 3, the type of W_1 is determined by $S_q^2\colon H^{n-1}(M;\ Z_2) \to H^{n+1}(M;\ Z_2)$. Similarly, using the results of Part I when $n=4t-1$ $(t\geqq 2)$, the figure of W_1 is determined by $P_t(M)$ by Lemma 9.2. So that M_1 is determined mod θ_{2n} by $S_q^2\colon H^{n-1}(M; Z_2) \to H^{n+1}(M;\ Z_2)$ and $P_t(M)$.

If $n=4t-1$, M_2 is a π-manifold since the obstructions vanish. On the structure of $(n-1)$-connected $2n$-dimensional π-manifolds, see [6] for example.

The corollaries and other results are known at once.

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