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Relative Hamiltonian for Faithful Normal States of a von Neumann Algebra

By

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Abstract

Let Ψ be a cyclic and separating vector for a von Neumann algebra $\mathfrak{M}$ and Δ_Ψ be its modular operator. For any elements $Q_1,\dots, Q_n$ in $\mathfrak{M}$ and complex numbers $z_1,\dots, z_n$ such that Re $z_j \geqq 0$ and Σ Re $z_j \leqq 1/2$, Ψ is shown to be in the domain of $\Delta_\Psi^{z_1} Q_1 \dots \Delta_\Psi^{z_n} Q_n$ and $\|\Delta_\Psi^{z_1} Q_1 \dots \Delta_\Psi^{z_n} Q_n \Psi\| \leqq \|Q_1\| \dots \|Q_n\| \|\Psi\|$.

A selfadjoint operator $h = h(\varphi/\psi) \in \mathfrak{M}$ is called a Hamiltonian of a faithful normal state φ of $\mathfrak{M}$ relative to another faithful state ψ of $\mathfrak{M}$ if vectors ξ_φ and ξ_ψ representing φ and ψ (in the canonical cone $V_\Psi^{1/4}$) is related by

$$\xi_\varphi = \sum_{n=0}^{\infty} (-1)^n \int_0^{1/2} \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \Delta_{\xi_\psi}^{t_n} h \Delta_{\xi_\psi}^{t_{n-1} - t_n} h \dots \Delta_{\xi_\psi}^{t_1 - t_2} h \xi_\psi.$$

The operator

$$u_t^{\varphi\psi} = \sum_{n=0}^{\infty} (-i)^n \int_0^{t} \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \sigma_{t_n}^{\psi}(h) \sigma_{t_{n-1}}^{\psi}(h) \dots \sigma_{t_1}^{\psi}(h)$$

is shown to be an intertwining unitary operator between modular automorphisms σ_t^ψ and σ_t^φ for states ψ and φ:

$$u_t^{\varphi\psi} \sigma_t^\psi(x) = \sigma_t^\varphi(x) u_t^{\varphi\psi}, \quad x \in \mathfrak{M}.$$

The relative hamiltonian $\mathrm{h}(\varphi/\psi)$ is unique for given states φ and ψ. It exists and satisfies $\log l_1 \geqq -\mathrm{h}(\varphi/\psi) \geqq \log l_2$ if $l_1^{1/2}\xi_\psi \geqq \xi_\varphi \geqq l_2^{1/2}\xi_\psi$, where $\Phi_1 \geqq \Phi_2$ means that $\Phi_1 - \Phi_2$ is in the canonical cone $V_\Psi^{1/4}$. In particular, if $l_1\psi \geqq \varphi \geqq l_2\psi$, then $\mathrm{h}(\varphi/\psi)$ exists and satisfies the above inequality.

The modular operators Δ_{ξ_φ} and Δ_{ξ_ψ} are related by

$$\log \Delta_{\xi_\psi} - \log \Delta_{\xi_\varphi} = \mathrm{h}(\varphi/\psi) - J\mathrm{h}(\varphi/\psi)J$$

where J is the common modular conjugation operator for ξ_φ and ξ_ψ. The chain rule $\mathrm{h}(\varphi_1/\varphi_2) + \mathrm{h}(\varphi_2/\varphi_3) = \mathrm{h}(\varphi_1/\varphi_3)$ is satisfied.

§1. Introduction

Let $\mathfrak{M}$ be a $*$-algebra of finite matrices and φ_0 be a faithful tracial state on $\mathfrak{M}$. Then every positive linear functional φ on $\mathfrak{M}$ is uniquely

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represented by a positive element ρ_φ of $\mathfrak{M}$ by

$$\varphi(x)=\varphi_0(\rho_\varphi x), \quad x\in\mathfrak{M}. \tag{1.1}$$

The corresponding modular automorphism σ_t^φ is given by

$$\sigma_t^\varphi(x)=\rho_\varphi^{it}x\rho_\varphi^{-it}, \quad x\in\mathfrak{M}. \tag{1.2}$$

The L_2-norm $\|x\|_2\equiv\varphi_0(x^*x)^{1/2}$ makes $\mathfrak{M}$ a Hilbert space which we shall write $\mathfrak{H}$. The vector corresponding to $1\in\mathfrak{M}$ is written as ξ_0.

To each $\varphi\in\mathfrak{M}_*^+$, there corresponds a canonical vector $\xi_\varphi=\rho_\varphi^{1/2}\xi_0$ such that the expectation functional ω_{ξ_φ} by the vector ξ_φ is φ. The set of ξ_φ is a selfdual convex cone $\mathfrak{M}^+\xi_0$ in $\mathfrak{H}$, which has been denoted as V_{ξ_0} or $V_{\xi_0}^{1/4}$ in [2]. The modular conjugation operator J for ξ_φ is common for all faithful $\varphi\in\mathfrak{M}_*^+$ and is given by

$$Jx\xi_0=x^*\xi_0. \tag{1.3}$$

The modular operator Δ_{ξ_φ} for ξ_φ is given by

$$\Delta_{\xi_\varphi}x\xi_0=\rho_\varphi x\rho_\varphi^{-1}\xi_0. \tag{1.4}$$

In statistical mechanics, $h_\varphi=-\log\rho_\varphi$ for a faithful $\varphi\in\mathfrak{M}_*^+$ is called a Hamiltonian, σ_{-t}^φ is called a time translation automorphism and φ is called the Gibbs state for this Hamiltonian (with an inverse temperature $\beta=1$). We shall be concerned with a Hamiltonian of a faithful state φ relative to another faithful state ψ defined by

$$\mathrm{h}(\varphi/\psi)=h_\varphi-h_\psi \tag{1.5}$$

The original h_φ is $\mathrm{h}(\varphi/\varphi_0)$. Since a tracial state is not available for a purely infinite von Neumann algebra, we are forced to work with the relative Hamiltonian in the general case.

From (1.2), we immediately see that the unitary operator

$$u_t^{\varphi\psi}=\rho_\varphi^{it}\rho_\psi^{-it} \tag{1.6}$$

intertwines modular automorphisms of φ and ψ:

$$u_t^{\varphi\psi}\sigma_t^\psi(x)=\sigma_t^\varphi(x)u_t^{\varphi\psi}, \quad x\in\mathfrak{M}. \tag{1.7}$$

The operator in (1.6) has the following perturbation expansion

(1.8)
$$u_t^{\varphi\psi}=\sum_{n=0}^{\infty}(-i)^n\int_0^t dt_1\int_0^{t_1}dt_2\ldots\int_0^{t_{n-1}}dt_n\sigma_{t_n}^{\psi}(h)\ldots\sigma_{t_1}^{\psi}(h)$$
$$=\sum_{n=0}^{\infty}(-i)^n\int_0^t dt_1\int_0^{t_1}dt_2\ldots\int_0^{t_{n-1}}dt_n\sigma_{t_1}^{\varphi}(h)\ldots\sigma_{t_n}^{\varphi}(h)$$

where $h=\mathrm{h}(\varphi/\psi)$.

For two faithful states ψ and φ in $\mathfrak{M}_*^+$, there exists a unique $\mathrm{A}(\varphi/\psi)\in\mathfrak{M}$ such that $\mathrm{A}(\varphi/\psi)\xi_\psi=\xi_\varphi$. It is the Radon-Nikodym derivative satisfying the chain rule [2] and is given by

(1.9)
$$\mathrm{A}(\varphi/\psi)=\rho_\varphi^{1/2}\rho_\psi^{-1/2},$$

which has the following perturbation expansion:

(1.10)
$$\xi_\varphi=\mathrm{A}(\varphi/\psi)\xi_\psi=\sum_{n=0}^{\infty}(-1)^n\int\cdots\int d\lambda_1\ldots d\lambda_n\Delta_{\xi_\psi}^{\lambda_1}h\ldots\Delta_{\xi_\psi}^{\lambda_n}h\xi_\psi$$

where the integration is over the following simplex

(1.11)
$$\bar{I}_n^{1/2}\equiv\{(\lambda_1,\ldots,\lambda_n);\ \lambda_1\geqq 0,\ldots,\lambda_n\geqq 0,\ 1/2\geqq\lambda_1+\cdots+\lambda_n\}.$$

These expansion formulas are of the same form as the covariant perturbation expansion in an interaction picture used in quantum field theory and has been discussed in [3] in the Banach algebra context.

The purpose of the present paper is to show that the relative hamiltonian $\mathrm{h}(\varphi/\psi)$ exists for a certain class of the pair φ, ψ in $\mathfrak{M}_*^+$, it is unique and it satisfies (1.7), (1.10), the chain rule

(1.12)
$$\mathrm{h}(\varphi_1/\varphi_2)+\mathrm{h}(\varphi_2/\varphi_3)=\mathrm{h}(\varphi_1/\varphi_3),$$

and

(1.13)
$$\log\Delta_{\xi_\varphi}-\log\Delta_{\xi_\psi}=\mathrm{h}(\psi/\varphi)-j_\Psi(h(\psi/\varphi)).$$

§2. Multi-variable three-line theorem

The following is an immediate generalization of the three-line theorem of Doetsch to the case of many complex variables.

Theorem 2.1. *Let* $\mathrm{f}(z)$ *be a function of* n *complex variables* $z=$

$(z_1 \ldots z_n)$ *satisfying the following two conditions*:

(i) $f(z)$ is holomorphic in the tube

$$T(B)=\{z;\ \mathrm{Im}\ z\in B\} \tag{2.1}$$

where B is an open convex set in R^n.

(ii) $f(z)$ is continuous and bounded in the closure $\overline{T}(B)=T(\bar{B})$.

Let

$$g(y)=\sup_x |f(x+iy)|,\quad y\in\bar{B}. \tag{2.2}$$

Then $\log g(y)$ is a convex function of $y\in\bar{B}$.

Proof. We have to prove the inequality

$$\log g(\lambda y_a+(1-\lambda)y_b)\leqq\lambda\log g(y_a)+(1-\lambda)\log g(y_b) \tag{2.3}$$

for $0\leqq\lambda\leqq 1$, $y_a\in\bar{B}$ and $y_b\in\bar{B}$.

If $g(y_0)=0$ at one point y_0, then $f(x+iy_0)=0$ for all $x\in R^n$ and hence $f(z)=0$ identically (by the edge of wedge theorem if $y_0\in\partial B$). In this case, $\log g(y)=-\infty$ for all $y\in\bar{B}$ and (2.3) holds. We now assume that $f(z)\not\equiv 0$ and hence $g(y)\neq 0$.

For each y_a and y_b, we may restrict our attention to a compact convex subset of $\bar{B}$ containing y_a, y_b and a non-empty interior. Hence we may assume that $\bar{B}$ is compact without loss of generality.

First consider the case where $\lambda y_a+(1-\lambda)y_b\in B$ for $0<\lambda<1$. Consider a function of one complex variable z_0:

$$f_x(z_0)=f(i\{z_0 y_a+(1-z_0)y_b\}+x),\quad x\in R^n.$$

If $\mathrm{Re}\ z_0\in(0,1)$, then $f_x(z_0)$ is holomorphic by the tentative assumption, continuous and bounded in the closure. By Doetsch's three-line theorem, $\log\sup_{\sigma\in R}|f_x(\lambda+i\sigma)|$ is a convex function of λ in $[0,1]$. Since the supremum of a family of convex functions is convex, we have the convexity of

$$\sup_{x\in R^n}\log\sup_{\sigma}|f_x(\lambda+i\sigma)|=\log g(\lambda y_a+(1-\lambda)y_b)$$

in λ and hence (2.3).

For general points y_a and y_b in $\bar{B}$, we consider a sequence of points

y_a^n and y_b^n such that $\lim y_a^n = y_a$, $\lim y_b^n = y_b$ and $\lambda y_a^n + (1-\lambda) y_b^n \in B$ for $0 < \lambda < 1$ and all n. Instead of f, we first consider the function

$$\mathrm{f}_\beta(z) = \mathrm{f}(z)\exp(-\beta \sum z_j^2). \qquad (\beta > 0.)$$

For compact $\bar{B}$, $\mathrm{f}_\beta(z)$ tends uniformly to 0 as z tends to ∞ in $T(\bar{B})$. Let

$$\mathrm{g}_\beta(y) = \sup_x |\mathrm{f}_\beta(x+iy)|.$$

Then it is continuous in $y \in \bar{B}$. By what we have already proved we have

$$\log \mathrm{g}_\beta(\lambda y_a^n + (1-\lambda) y_b^n) \leqq \lambda \log \mathrm{g}_\beta(y_a^n) + (1-\lambda) \log \mathrm{g}_\beta(y_b^n).$$

By continuity, we have

$$\log \mathrm{g}_\beta(\lambda y_a + (1-\lambda) y_b) \leqq \lambda \log \mathrm{g}_\beta(y_a) + (1-\lambda) \log \mathrm{g}_\beta(y_b).$$

We can now complete the proof by showing that

$$\lim_{\beta \to +0} \mathrm{g}_\beta(y) = \mathrm{g}(y)$$

for each $y \in \bar{B}$. First we have

$$|\exp(-\beta \sum z_j^2)| \leqq \exp(\beta \sup_{y \in B} \sum y_j^2) \to 1$$

as $\beta \to 0$. ($\bar{B}$ is assumed to be compact at this stage of the proof.) Hence for sufficiently small β,

$$\mathrm{g}_\beta(y) \leqq \mathrm{g}(y) + \varepsilon,$$

for any given $\varepsilon > 0$. There exists x such that

$$|\mathrm{f}(x+iy)| \geqq \mathrm{g}(y) - \varepsilon/2.$$

For this x, we have $|\mathrm{f}_\beta(x+iy)| \geqq \mathrm{g}(y) - \varepsilon$ for sufficiently small β. Hence

$$\mathrm{g}(y) - \varepsilon \leqq \mathrm{g}_\beta(y) \leqq \mathrm{g}(y) + \varepsilon$$

for sufficiently small β for any given $\varepsilon > 0$. Q.E.D.

Remark. The convexity of $\log \mathrm{g}(y)$ implies the convexity of $\mathrm{g}(y)$.

By considering $\exp \mathrm{f}(z)$ instead of $\mathrm{f}(z)$, we also obtain the convexity of $\sup_x \mathrm{Re}\, \mathrm{f}(x+iy)$.

Corollary 2.2. *Let B in Theorem 2.1 be a simplex*

$$B=I_n^a \equiv \{y;\ y_1>0,\ldots,\ y_n>0,\ a>y_1+\cdots+y_n\},\ a>0.$$

Then $g(y)$ takes its maximum value in $\bar{B}$ at one of $(n+1)$ extremal points of $\bar{B}$: $\lambda^0 \equiv (0,\ldots,0)$, $\lambda^1 \equiv (a, 0,\ldots, 0)$, $\lambda^2 \equiv (0, a, 0,\ldots, 0),\ldots, \lambda^n \equiv (0,\ldots, 0, a)$.

Proof. Since $\log g(y)$ is convex, it takes its maximum value in a compact convex set $\bar{B}$ at one of its extremal points.

Q. E. D.

§3. Multiple KMS Property

Theorem 3.1. *Let Ψ be a cyclic and separating vector with a modular operator Δ_Ψ. It is in the domain of*

$$\Delta_\Psi^{iz_1} Q_1 \Delta_\Psi^{iz_2} Q_2 \ldots \Delta_\Psi^{iz_n} Q_n \equiv \mathrm{A}(z) \tag{3.1}$$

if $Q_k \in \mathfrak{M}$, $k=1,\ldots,n$, and $z=(z_1\ldots z_n)\in \mathrm{T}(-I_n^{1/2})$. The vector valued function $\mathrm{A}(z)\Psi$ of z is holomorphic in the tube $\mathrm{T}(-I_n^{1/2})$, strongly continuous and uniformly bounded in its closure $\overline{\mathrm{T}}(-I_n^{1/2})$:

$$\|\mathrm{A}(z)\Psi\| \leqq \|\Psi\| \|Q_1\| \cdots \|Q_n\|. \tag{3.2}$$

Proof. We prove Theorem by induction on n. When $n=0$, there is nothing to prove. Assume that Theorem is true for $n<m$ except that the strong continuity is temporarily replaced by the weak continuity and consider the case $n=m$.

Let $\Phi \in \mathrm{D}(\Delta_\Psi^{1/2})$. Then $\Phi \in \mathrm{D}(\Delta_\Psi^{iz_1})$ for $\mathrm{Im}\, z_1 \in [-1/2, 0]$. Consider

$$\mathrm{f}(z)=(\Phi_1, \Phi_2),$$

$$\Phi_1 = Q_1 \Delta_\Psi^{iz_2} Q_2 \Delta_\Psi^{iz_3} \ldots \Delta_\Psi^{iz_m} Q_m \Psi,$$

$$\Phi_2 = \Delta_\Psi^{-i\bar{z}_1} \Phi.$$

By Lemma 4 of [2], we have

$$\|\Phi_2\|^2 \leqq \|\Delta_\Psi^{1/2}\Phi\|^2 + \|\Phi\|^2.$$

Hence Φ_2 is a uniformly bounded and strongly continuous function of z_1 for $z_1 \in \overline{\mathrm{T}}(-I_1^{1/2})$. By inductive assumption, Φ_1 is a uniformly bounded and weakly continuous function of $(z_2, \dots, z_m) \in \overline{\mathrm{T}}(-I_{m-1}^{1/2})$. Hence $\mathrm{f}(z)$ is a uniformly bounded continuous function of $z = (z_1, \dots, z_n)$ in $\overline{\mathrm{T}}(-I_1^{1/2}) \times \overline{\mathrm{T}}(-I_{m-1}^{1/2})$, which contains $\overline{\mathrm{T}}(-I_m^{1/2})$. Since Φ_2 is holomorphic for $z_1 \in \mathrm{T}(-I_1^{1/2})$ and Φ_1 is holomorphic for $(z_2, \dots, z_m) \in \mathrm{T}(-I_{m-1}^{1/2})$, $\mathrm{f}(z)$ is holomorphic in $\mathrm{T}(-I_1^{1/2}) \times \mathrm{T}(-I_{m-1}^{1/2})$ by Hartogs' theorem.

We now consider $|\mathrm{f}(z)|$ when $\operatorname{Im} z$ is at one of extremal points of $-\bar{I}_m^{1/2}$. We have

$$|\mathrm{f}(x)| \leqq \|\Phi\| \|\Psi\| \|Q_1\| \cdots \|Q_m\|, \quad x \in R.$$

If $\operatorname{Im} z_j = 0$ except for $j = k$ and $\operatorname{Im} z_k = -1/2$, we have

$$\begin{aligned} \mathrm{f}(z) &= (Q_1(x_1) \cdots Q_{k-1}(x_1 + \cdots + x_{k-1}) \Delta_\Psi^{1/2} Q_k(x_1 + \cdots + x_k) \cdots \\ &\qquad\qquad Q_m(x_1 + \cdots + x_m)\Psi, \Phi) \\ &= (Q_1(x_1) \cdots Q_{k-1}(x_1 + \cdots + x_{k-1}) J_\Psi Q_m^*(x_1 + \cdots + x_m) \cdots \\ &\qquad\qquad Q_k^*(x_1 + \cdots + x_k)\Psi, \Phi) \end{aligned}$$

where $x_l = \operatorname{Re} z_l$ and $Q(t) = \Delta_\Psi^{it} Q \Delta_\Psi^{-it}$. Hence we have

$$|\mathrm{f}(z)| \leqq \|\Phi\| \|\Psi\| \|Q_1\| \cdots \|Q_m\|.$$

for all $z \in \overline{\mathrm{T}}(-I_m^{1/2})$. By Riesz theorem, there exists a vector $\Phi(z)$ such that $\mathrm{f}(z) = (\Phi(z), \Phi)$ and $\|\Phi(z)\| \leqq \|\Psi\| \|Q_1\| \cdots \|Q_m\|$. Hence Φ_1 is in the domain of Δ^{iz_1} and (3.2) holds for $n = m$.

Since $\mathrm{f}(z)$ is holomorphic in $\mathrm{T}(-I_m^{1/2})$ and continuous in, $\overline{\mathrm{T}}(-I_m^{1/2})$, the uniform boundedness implies that $\Phi(z) = \mathrm{A}(z)\Psi$ is weakly continuous in $\overline{\mathrm{T}}(-I_m^{1/2})$ and weakly holomorphic in $\mathrm{T}(-I_m^{1/2})$. Since the weak and strong holomorphy are the same due to Cauchy integral formula for polycircles, we have the desired properties for $n = m$.

To show the strong continuity from the weak continuity, it is enough to show the continuity of the norm. This follows from the next Theorem.

Q.E.D.

The following Theorem states a multiple KMS property and has been derived in somewhat different but equivalent context in [1], except for (4).

Theorem 3.2. *There exists a function* $\mathrm{F}(z)$ *of* $z=(z_1,\dots, z_n)$ *for a given cyclic and separating vector* Ψ *and operators* $Q_1,\dots, Q_{n+1}\in\mathfrak{M}$ *such that*

(1) $\mathrm{F}(z)$ *is holomorphic in* $z\in\mathrm{T}(-I_n^1)$,

(2) $\mathrm{F}(z)$ *is continuous in* $z\in\overline{\mathrm{T}}(-I_n^1)$,

(3) $\mathrm{F}(z)$ *is uniformly bounded in* $z\in\overline{\mathrm{T}}(-I_n^1)$:

$$|\mathrm{F}(z)|\leqq\|\Psi\|^2\|Q_1\|\cdots\|Q_{n+1}\|, \tag{3.3}$$

(4) *if* $z\in\overline{\mathrm{T}}(-I_n^1)$ *and*

$$\mathrm{Im}(z_1+\cdots+z_{k-1})\geqq-1/2,\quad \mathrm{Im}(z_k+\cdots+z_n)\geqq-1/2,$$

then $\mathrm{F}(z)=(\Phi_1, \Phi_2)$ *where*

$$\Phi_1=\Delta_\Psi^\alpha Q_k\Delta_\Psi^{iz_{k-1}}Q_{k-1}\dots\Delta_\Psi^{iz_1}Q_1\Psi,$$

$$\Phi_2=\Delta_\Psi^{-\alpha-i\bar{z}_k}Q_{k+1}^*\Delta_\Psi^{-i\bar{z}_{k+1}}Q_{k+2}^*\dots\Delta_\Psi^{-i\bar{z}_n}Q_{n+1}^*\Psi,$$

α *is any non-negative real number satisfying*

$$-\mathrm{Im}z_k\geqq\alpha\geqq0,$$

$$1/2+\mathrm{Im}(z_1+\cdots+z_{k-1})\geqq\alpha\geqq-1/2-\mathrm{Im}(z_k+\cdots+z_n),$$

(5) *if* $\mathrm{Im}\ z=0$, *then*

$$\mathrm{F}(z)=\omega_\Psi(Q_{n+1}Q_n(x_n)Q_{n-1}(x_{n-1}+x_n)\dots Q_1(x_1+\cdots+x_n)),$$

(6) *if* $\mathrm{Im}\ z_k=0$ except for $\mathrm{Im}\ z_j=-1$, *then*

$$\mathrm{F}(z)=\omega_\Psi(Q_j(x_j+\cdots+x_n)\dots Q_1(x_1+\cdots+x_n)Q_{n+1}\dots Q_{j+1}(x_{j+1}+\cdots+x_n)),$$

where $x_j=\mathrm{Re}\ z_j$ and $Q(t)=\Delta_\Psi^{it}Q\Delta_\Psi^{-it}$.

Proof. Let $Q_{j,\beta}=Q_j(f_\beta^G)$ where f_β^G is given by (3.11) of [2] and the notation $Q(f)$ is given by (3.7) of [2]. $Q_{j,\beta}(t)$ has an analytic continuation $Q_{j,\beta}(z)\in\mathfrak{M}$ for any complex number z and

$$\mathrm{F}^{\beta}(z) \equiv (Q_{n+1,\beta} \Delta_{\Psi}^{iz_n} Q_{n,\beta} \cdots \Delta_{\Psi}^{iz_1} Q_{1,\beta} \Psi, \Psi)$$
$$= (Q_{n+1,\beta} Q_{n,\beta}(z_n) \ldots Q_{1,\beta}(z_n + \cdots + z_1) \Psi, \Psi)$$

is an entire function of z. By Theorem 2.1, $|\mathrm{F}^{\beta}(z) - \mathrm{F}^{\beta'}(z)|$ for $z \in \mathrm{T}(-\bar{I}_n^1)$ is bounded by the maximum of $|\mathrm{F}^{\beta}(x - i\lambda^{(j)}) - \mathrm{F}^{\beta'}(x - i\lambda^{(j)})|$ for $j=0, 1, \ldots, n$ and real x where $\lambda^{(j)}$ is as in Corollary 2.2 in which we set $a=1$.

By usual KMS condition, we have

$$\mathrm{F}^{\beta}(x - i\lambda^{(j)}) = (Q_{j,\beta}(x_j + \cdots + x_n) \ldots Q_{1,\beta}(x_1 + \cdots + x_n) Q_{n+1,\beta} \cdots$$
$$Q_{j+1,\beta}(x_{j+1} + \cdots + x_n) \Psi, \Psi).$$

Since $\|Q_{j,\beta}\| \leqq \|Q_j\|$, $\lim_{\beta \to 0} Q_{j,\beta} = Q_j$ and $\lim_{\beta \to 0} Q_{j,\beta}^* = Q_j^*$, $|\mathrm{F}^{\beta} - \mathrm{F}^{\beta'}|$ converges to zero uniformly in z in any compact subset of $\mathrm{T}(-\bar{I}_n^1)$. Let $\mathrm{F}(z) = \lim_{\beta \to 0} \mathrm{F}^{\beta}(z)$. As the uniform limit of a continuous and holomorphic function, $F(z)$ is continuous on $\mathrm{T}(-\bar{I}_n^1)$ and holomorphic in $\mathrm{T}(-I_n^1)$. Since $|\mathrm{F}^{\beta}(z)| \leqq (\prod_j \|Q_{j,\beta}\|) \|\Psi\|^2 \leqq (\prod_j \|Q_j\|) \|\Psi\|^2$, (3) is also satisfied. Hence $\mathrm{F}(z)$ so constructed satisfies (1), (2), (3), (5) and (6). It remains to prove (4).

Consider

$$\mathrm{f}(z_1 \ldots z_{n+1}) = \mathrm{F}(z_1 \ldots z_{k-1}, z_k + z_{k+1}, \ldots z_{n+1}) - (\Psi_1, \Psi_2),$$
$$\Psi_1 = \Delta_{\Psi}^{iz_k} Q_k \Delta_{\Psi}^{iz_{k-1}} Q_{k-1} \ldots \Delta_{\Psi}^{iz_1} Q_1 \Psi,$$
$$\Psi_2 = \Delta_{\Psi}^{-i\bar{z}_{k+1}} Q_{k+1}^* \ldots \Delta_{\Psi}^{-i\bar{z}_{n+1}} Q_{n+1}^* \Psi,$$

in the domain

$$(z_1 \ldots z_k) \in \mathrm{T}(-I_k^{1/2}),\ (z_{k+1} \ldots z_{n+1}) \in \mathrm{T}(-I_{n-k+1}^{1/2}),$$
$$(z_1 \ldots z_k + z_{k+1} \ldots z_{n+1}) \in \mathrm{T}(-I_n^1).$$

$\mathrm{f}(z)$ is holomorphic in this domain.

If $\mathrm{Im}\,(z_1 + \cdots + z_{n+1}) \geqq -1/2$, then

$$(\Psi_1, \Psi_2) = (Q_{n+1} \Delta_{\Psi}^{iz_{n+1}} \ldots Q_{k+1} \Delta_{\Psi}^{i(z_{k+1}+z_k)} Q_k \ldots \Delta_{\Psi}^{iz_1} Q_1 \Psi, \Psi).$$

By the weak continuity of Theorem 3.1, $\mathrm{f}(z)$ is continuous in $z \in \overline{\mathrm{T}}(-I_{n+1}^{1/2})$. At $\mathrm{Im}\, z = 0$, we have $\mathrm{f}(z) = 0$. Hence $\mathrm{f}(z) = 0$ identically by edge of wedge theorem. This proves (4) for $z \in \overline{\mathrm{T}}(-I_n^1)$ satisfying

$$-\alpha + \mathrm{Im}(z_1 + \cdots + z_{k-1}) > -1/2, \ \mathrm{Im}(z_k + \cdots + z_n) + \alpha > -1/2.$$

For a fixed $z_1 \dots z_{k-1}$, we obtain (4) for $z \in \overline{\mathrm{T}}(-I_n^1)$ satisfying

$$-\alpha + \mathrm{Im}(z_1 + \cdots + z_{k-1}) > -1/2, \quad \mathrm{Im}(z_k + \cdots + z_n) + \alpha \geqq -1/2$$

by the weak continuity. For fixed $z_k \dots z_{n+1}$, we then obtain (4) for $z \in \overline{\mathrm{T}}(-I_n^1)$ satisfying

$$-\alpha + \mathrm{Im}(z_1 + \cdots + z_{k-1}) \geqq -1/2, \ \mathrm{Im}(z_k + \cdots + z_n) + \alpha \geqq -1/2$$

by weak continuity again. Q.E.D.

§4. Relative Hamiltonian

Proposition 4.1. *Let $h \in \mathfrak{M}$, $h^* = h$ and Ψ be a cyclic and separating vector. Then*

$$\Psi(h) = \sum_{n=0}^{\infty} \int \cdots \int \mathrm{d}t_1 \cdots \mathrm{d}t_n \Delta_\Psi^{t_n} h \Delta_\Psi^{t_{n-1}} h \dots \Delta_\Psi^{t_1} h \Psi \tag{4.1}$$

converges absolutely and uniformly over bounded h, where the integrations are over the region

$$\bar{I}_n^{1/2} = \{(t_1 \dots t_n);\ t_1 \geqq 0, \dots, t_n \geqq 0,\ t_1 + \cdots + t_n \leqq 1/2\}. \tag{4.2}$$

$\Psi(h)$ defined by (4.1) *is in V_Ψ, namely the closure of $\Delta_\Psi^{1/4} \mathfrak{M}^+ \Psi$* [2]. *If a sequence h_n converges to h strongly, then $\Psi(h_n)$ converges to $\Psi(h)$ strongly.*

Proof. By Theorem 3.1, we have

$$\|\Delta_\Psi^{t_n} h \Delta_\Psi^{t_{n-1}} h \cdots \Delta_\Psi^{t_1} h \Psi\| \leqq \|h\|^n \|\Psi\|.$$

Hence (4.1) is dominated by $\Sigma(n!)^{-1} \|h\|^n \|\Psi\| = e^{\|h\|} \|\Psi\|$ and hence converges absolutely and uniformly over a bounded set of h.

Since converging sequence is uniformly bounded and multiplication is continuous on a bounded set, we have

$$\lim \sigma_{t_1}^\psi(h_n) \dots \sigma_{t_m}^\psi(h_n) \Psi = \sigma_{t_1}^\psi(h) \dots \sigma_{t_m}^\psi(h) \Psi$$

for each $t_1 \dots t_m$. By Lebesgue dominated convergence theorem, we have the strong convergence (i.e. weak convergence plus the convergence of norm):

(4.3) $$\lim_n h_n(f_1)\dots h_n(f_m)\Psi = h(f_1)\dots h(f_m)\Psi$$

for any L_1 functions $f_1 \dots f_m$, where

$$h_n(f) = \int \sigma_t^\phi(h_n) f(t) \mathrm{d}t, \quad h(f) = \int \sigma_t^\phi(h) f(t) \mathrm{d}t.$$

We shall use f_β^G given by (3.11) of [2].

If $\mathrm{F}(z_1 \dots z_n)$ is a vector valued function holomorphic in $\{z; |z_j - z_j^0| \leqq \delta\}$ and bounded by A, then for $|z_j - z_j^0| \leqq \delta/2$, we have

(4.4) $$\|(\mathrm{d}/\mathrm{d}z_j)\mathrm{F}\| = \|(2\pi i)^{-1} \int_{|z-z_j^0|=\delta} (z - z_j)^{-2} \mathrm{F}(z_1 \dots z_{j-1} z z_{j+1} \dots z_n) \mathrm{d}z\|$$
$$\leqq 4A\delta^{-1}.$$

Hence $\|F(z) - F(z')\| \leqq 4A\delta^{-1}|z - z'|$ for $|z_j - z_j^0| \leqq \delta/2$, $|z_j' - z_j^0| \leqq \delta/2$. Since

$$\Delta_\Psi^{s_1} \sigma_{t_1}^\phi(h_n) \Delta_\Psi^{s_2} \sigma_{t_1+t_2}^\phi(h_n) \dots \Delta_\Psi^{s_m} \sigma_{t_1+\dots+t_m}^\phi(h_n)\Psi$$

is holomorphic in $z = (s_1 + it_1, \dots, s_m + it_m)$ for Re $z \in I_m^{1/2}$ and bounded uniformly for a bounded set of h_n, it has the equicontinuity in $(t_1, t_1 + t_2, \dots, t_1 + \dots + t_m)$ and hence

$$\lim_{\beta \to +0} \Delta_\Psi^{s_1} h_n(f_\beta^G) \dots \Delta_\Psi^{s_m} h_n(f_\beta^G)\Psi = \Delta_\Psi^{s_1} h_n \dots \Delta_\Psi^{s_m} h_n \Psi$$

strongly and uniformly over n for each fixed $(s_1 \dots s_m)$ in $I_m^{1/2}$. The same equation holds when h_n is replaced by h.

Furthermore,

$$\Delta_\Psi^{s_1} h_n(f_\beta^G) \dots \Delta_\Psi^{s_m} h_n(f_\beta^G)\Psi = h_n(f_1) \dots h_n(f_m)\Psi$$

where

$$f_j(t) = f_\beta^G(t + i(s_1 + \dots + s_j)).$$

Hence by (4.3) we have

$$\lim_{n\leftarrow 0} \Delta_\Psi^{s_1} h_n(f_\beta^G)\dots\Delta_\Psi^{s_m} h_n(f_\beta^G)\Psi = h(f_1)\dots h(f_m)\Psi$$
$$= \Delta_\phi^{s_1} h(f_\beta^G)\dots\Delta_\Psi^{s_m} h(f_\beta^G)\Psi.$$

Therefore, by taking the limit as $\beta\to+0$ and exchanging the order of limit in n and, β, we obtain

$$\lim_{n\to\infty} \Delta_\Psi^{s_1} h_n\dots\Delta_\Psi^{s_m} h_n\Psi = \Delta_\Psi^{s_1} h\dots\Delta_\Psi^{s_m} h\Psi$$

for each $(s_1\dots s_m)\in I_m^{1/2}$.

Since $\Delta_\Psi^{s_1} h_n\dots\Delta_\Psi^{s_m} h_n\Psi$ is uniformly bounded by $(\sup_n ||h_n||)^m||\Psi||$, we have

$$\lim_n \Psi(h_n) = \Psi(h)$$

by Lebesgue dominated convergence theorem.

We now show that $\Psi(h)\in V_\Psi$. Let $h_\beta = h(f_\beta^G)$. Then $||h_\beta||\leqq||h||$ and $\lim_{\beta\to+0} h_\beta = h$. If we show $\Psi(h_\beta)\in V_\Psi$, then $\Psi(h)=\lim \Psi(h_\beta)\in V_\Psi$.

The closure of $\Delta_\Psi^s h_\beta \Delta_\Psi^{-s}$ is given by

$$h_\beta(-is) = \int \Delta_\Psi^{it} h \Delta_\Psi^{-it} f_\beta^G(t+is)\mathrm{d}t \in M.$$

Hence

$$\Delta_\Psi^{t_n} h_\beta\dots\Delta_\Psi^{t_1} h_\beta\Psi = h_\beta(-it_n)\dots h_\beta(-i(t_1+\cdots+t_n))\Psi.$$

Changing integration variables to $s_1=t_1+\cdots+t_n$, $s_2=t_2+\cdots+t_n,\dots,$ $s_n=t_n$, we obtain

$$\text{(4.5)}\qquad \Psi(h_\beta) = \sum_{n=0}^{\infty}\int_0^{1/2}\mathrm{d}s_1\int_0^{s_1}\mathrm{d}s_2\dots\int_0^{s_{n-1}}\mathrm{d}s_n h_\beta(-is_n)\dots h_\beta(-is_1)\Psi$$
$$= \mathrm{Exp}_r\left(\int_0^{1/2};\ h_\beta(-is)\mathrm{d}s\right)\Psi$$

where the expansional Exp_r is defined in [3].

By formula (3.5) of [3], we have

$$\mathrm{Exp}_r\left(\int_0^{1/2};\ h_\beta(-is)\mathrm{d}s\right)$$
$$= \mathrm{Exp}_r\left(\int_0^{1/4};\ h_\beta(-is)\mathrm{d}s\right)\mathrm{Exp}_r\left(\int_0^{1/4};\ h_\beta(-i(s+1/4))\mathrm{d}s\right).$$

Since

$$h_{\beta}(-i(s_n+1/4))\dots h_{\beta}(-i(s_1+1/4))\Psi$$
$$=\Delta_{\Psi}^{1/2}h_{\beta}(-i(s_n-1/4))\dots h_{\beta}(-i(s_1-1/4))\Psi$$
$$=J_{\Psi}h_{\beta}(i(s_1-1/4))\dots h_{\beta}(i(s_n-1/4))\Psi,$$

we have

$$\mathrm{Exp}_r\Big(\int_0^{1/4};\ h_{\beta}(-i(s+1/4))\mathrm{d}s\Big)\Psi=J_{\Psi}\mathrm{Exp}_l\Big(\int_0^{1/4};\ h_{\beta}(i(s-1/4))\mathrm{d}s\Big)\Psi.$$

By formula (2.10) of [3], we have

$$\mathrm{Exp}_l\Big(\int_0^{1/4};\ h_{\beta}(i(s-1/4))\mathrm{d}s\Big)=\mathrm{Exp}_r\Big(\int_0^{1/4};\ h_{\beta}(-is)\mathrm{d}s\Big).$$

Hence

$$\Psi(h_{\beta})=Qj_{\Psi}(Q)\Psi\in V_{\Psi}$$

where

$$Q=\mathrm{Exp}_r\Big(\int_0^{1/4};\ h_{\beta}(-is)\mathrm{d}s\Big),\ j_{\Psi}(Q)=J_{\Psi}QJ_{\Psi}.$$

Q.E.D.

Definition 4.2. *Let $\varphi\in\mathfrak{M}_*^+$, $\psi\in\mathfrak{M}_*^+$ and ξ_{φ}, ξ_{ψ} be unique representatives of φ and ψ in V_{Ψ}. If $\xi_{\varphi}=\xi_{\psi}(-h)$, $h\in\mathfrak{M}$, $h=h^*$, then h is called a Hamiltonian of φ relative to ψ and denoted by* $\mathrm{h}(\varphi/\psi)$.

We shall prove the uniqueness in Lemma 4.6.

Remark. There is an unfortunate small discrepancy in the notation of mathematicians and physicists. As explained in Introduction, the time translation automorphism of physicists differs from the modular automorphisms of mathematicians by sign of the variable, which also causes a sign change in the statement of *KMS* condition. In the present definition, it would be simpler mathematically to call $-h$ as a relative hamiltonian, but we would like to avoid a further discrepancy in terminologies and we define the hamiltonian as it appears in statistical mechanics. In the present article, we follow mathematician's notation regarding modular automorphisms, MSK conditions and inner product of a Hilbert space.

Proposition 4.3. *Assume that* $\mathrm{h}(\varphi/\psi)$ *exists for faithful* φ *and* ψ *in* $\mathfrak{M}_*^+$. *Let* σ_t^φ *and* σ_t^ψ *be modular automorphisms for* φ *and* ψ. *Let*

$$u_t^{\varphi\psi}=\mathrm{Exp}_r\left(\int_0^t;\ -i\sigma_s^\psi(\mathrm{h}(\varphi/\psi))\mathrm{d}s\right), \tag{4.6}$$

$$\hat{u}_t^{\varphi\psi}=\mathrm{Exp}_l\left(\int_0^t;\ i\sigma_s^\psi(\mathrm{h}(\varphi/\psi))\mathrm{d}s\right). \tag{4.7}$$

Then

$$(u_t^{\varphi\psi})^*=\hat{u}_t^{\varphi\psi},\quad u_t^{\varphi\psi}\hat{u}_t^{\varphi\psi}=\hat{u}_t^{\varphi\psi}u_t^{\varphi\psi}=1. \tag{4.8}$$

$$u_t^{\varphi\psi}\sigma_t^\psi(x)=\sigma_t^\varphi(x)u_t^{\varphi\psi},\quad x\in\mathfrak{M}. \tag{4.9}$$

Proof. The first equation in (4.8) follows from definition. The second equation in (4.8) follows from formulas (2.14) and (2.15) of [3]. To prove (4.9), we consider

$$\varphi_\beta=\omega_{\xi_\psi(h_\beta)}$$

where $h=-\mathrm{h}(\varphi/\psi)$, $h_\beta=h(f_\beta^G)$ and f_β^G is given by (3.11) of [2]. we first prove (4.9) for φ_β instead of φ.

We have

$$u_t^{\varphi_\beta\psi}=\mathrm{Exp}_r\left(\int_0^t;\ ih_\beta(s)\mathrm{d}s\right),$$

$$\hat{u}_t^{\varphi_\beta\psi}=\mathrm{Exp}_l\left(\int_0^t;\ -ih_\beta(s)\mathrm{d}s\right),$$

where

$$h_\beta(z)=\int\sigma_s^\psi(h)f_\beta^G(s-z)\mathrm{d}s\in\mathfrak{M}. \tag{4.10}$$

We compare two functions

$$\begin{aligned}\mathrm{F}_1(t)&=\varphi_\beta(xu_t^{\varphi_\beta\psi}\sigma_t^\psi(y)\hat{u}_t^{\varphi_\beta\psi})\\&=\psi(A^*xA\sigma_t^\psi(k_t^{-1}yk_t)),\end{aligned} \tag{4.11}$$

and

(4.12)
$$F_2(t)=\varphi_\beta(u_t^{\varphi_\beta\psi}\sigma_t^\psi(y)\hat{u}_t^{\varphi_\beta\psi}x)$$
$$=\psi(\sigma_t^\psi(\hat{k}_t^{-1}y\hat{k}_t)A^*xA)$$

where $x, y\in\mathfrak{M}$

$$A=\mathrm{Exp}_r\left(\int_0^{1/2};\ h_\beta(-is)\mathrm{d}s\right),$$

$$k_t=\sigma_{-t}^\psi(\hat{u}_t^{\varphi_\beta\psi}A),$$

$$\hat{k}_t=\sigma_{-t}^\psi(\hat{u}_t^{\varphi_\beta\psi}(A^*)^{-1}),$$

and the second equalities of (4.11) and (4.12) are due to (4.5). We shall first prove that k_t and $\hat{k}_t$ have analytic continuations to the same entire function and $k_{t-i}=\hat{k}_t$.

By formulas (4.2) and (4.4) of [3], we have

$$\mathrm{Exp}_r\left(\int_0^{t+s};\ -ih_\beta(-\sigma)\mathrm{d}\sigma\right)$$

$$=\mathrm{Exp}_r\left(\int_0^t;\ -ih_\beta(-\sigma)\mathrm{d}\sigma\right)\sigma_{-t}^\psi\left\{\mathrm{Exp}_r\left(\int_0^s;\ -ih_\beta(-\sigma)\mathrm{d}\sigma\right)\right\}.$$

By formula (2.8) of [3], we have

$$\mathrm{Exp}_r\left(\int_0^t;\ -ih_\beta(-\sigma)\mathrm{d}\sigma\right)=\mathrm{Exp}_r\left(\int_0^1;\ -ith_\beta(-t\sigma)\mathrm{d}\sigma\right).$$

The right hand side is an entire function of t and its value at $t=\pm is$ is given by

$$\mathrm{Exp}_r\left(\int_0^1;\ -ith_\beta(-t\sigma)\mathrm{d}\sigma\right)=\mathrm{Exp}_r\left(\int_0^s;\ \pm h_\beta(\mp i\sigma)\mathrm{d}\sigma\right),$$

due to formula (2.8) of [3] where s is real positive. Hence the analytic continuation of $\mathrm{Exp}_r\left(\int_0^t;\ -ih_\beta(-\sigma)\,\mathrm{d}\sigma\right)$ to $z=t\pm is$ is given by

$$\mathrm{Exp}_r\left(\int_0^t;\ -ih_\beta(-\sigma)\mathrm{d}\sigma\right)\sigma_{-t}^\psi\left\{\mathrm{Exp}_r\left(\int_0^s;\ \pm h_\beta(\mp i\sigma)\mathrm{d}\sigma\right)\right\}$$

$$\equiv\mathrm{Exp}_r\left(\int_0^z;\ -ih_\beta(-\sigma)\mathrm{d}\sigma\right).$$

By formula (2.11) of [3], we have

$$\sigma^{\psi}_{-t}(\hat{u}^{\varphi_\beta\psi}_t)=\mathrm{Exp}_l\left(\int_0^t;\ -ih_\beta(\sigma-t)\mathrm{d}\sigma\right)$$

$$=\mathrm{Exp}_r\left(\int_0^t;\ -ih_\beta(-\sigma)\mathrm{d}\sigma\right).$$

Hence

$$k_t=\mathrm{Exp}_r\left(\int_0^{t+i/2};\ -ih_\beta(-\sigma)\mathrm{d}\sigma\right).$$

We also have

$$(A^*)^{-1}=\left(\mathrm{Exp}_l\left(\int_0^{1/2};\ h_\beta(i\sigma)\mathrm{d}\sigma\right)\right)^{-1}$$

$$=\mathrm{Exp}_r\left(\int_0^{1/2};\ -h_\beta(i\sigma)\mathrm{d}\sigma\right)$$

due to $h_\beta(-i\sigma)^*=h_\beta(i\sigma)$ and formulas (2.17), (2.14) and (2.15) of [3]. Hence

$$\hat{k}_t=\mathrm{Exp}_r\left(\int_0^{t-i/2};\ -ih_\beta(-\sigma)\mathrm{d}\sigma\right).$$

Therefore k_t and $\hat{k}_t$ have analytic continuations to the same entire function and $k_{t-i}=\hat{k}_t$.

Since $\mathrm{Exp}_r\left(\int_0^z;\ -ih_\beta(-\sigma)\mathrm{d}\sigma\right)$ has an inverse

$$\left\{\sigma^{\psi}_{-t}\mathrm{Exp}_l\left(\int_0^s;\ \mp h_\beta(\mp i\sigma)\mathrm{d}\sigma\right)\right\}\mathrm{Exp}_l\left(\int_0^t;\ ih_\beta(-\sigma)\mathrm{d}\sigma\right)$$

for all $z=t\pm is$, k_t^{-1} and $\hat{k}_t^{-1}$ have analytic continuations to the same entire function and $k_{t-i}^{-1}=\hat{k}_t^{-1}$.

Returning to $\mathrm{F}_1(t)$, it has an analytic continuation to $z=t-i/2$:

$$\mathrm{F}_1(t-i/2)=(\Delta_\Psi^{1/2}\sigma_t^{\psi}(k_z^{-1}yk_z)\Psi,\ (A^*xA)^*\Psi)$$

$$=(J_\Psi\sigma_t^{\psi}(k_z^{-1}yk_z)^*\Psi,\ (A^*xA)^*\Psi)$$

$$=(J_\Psi(A^*xA)^*\Psi,\ \sigma_t^{\psi}(k_z^{-1}yk_z)^*\Psi).$$

On the other hand $F_2(t)$ has an analytic continuation to $\bar{z}=t+i/2$:

$$F_2(t+i/2)=(A^*xA\Psi,\ \Delta_\Psi^{1/2}\sigma_t^\psi(\hat{k}_{\bar{z}}^{-1}y\hat{k}_{\bar{z}})^*\Psi)$$
$$=(J_\Psi(A^*xA)^*\Psi,\ \sigma_t^\psi(\hat{k}_{\bar{z}}^{-1}y\hat{k}_{\bar{z}})^*\Psi).$$

Since $k_z=\hat{k}_{\bar{z}}$, they coincide. Hence the $*$-automorphism

$$x\to u_t^{\varphi_\beta\psi}\sigma_t^\psi(x)\hat{u}_t^{\varphi_\beta\psi}$$

satisfies the KMS condition for φ_β. Hence it must be the modular automorphism of φ_β:

$$u_t^{\varphi_\beta\psi}\sigma_t^\psi(x)\hat{u}_t^{\varphi_\beta\psi}=\sigma_t^{\varphi_\beta}(x).$$

If $\beta\to+0$, h_β tends to h strongly. By Proposition 4.1, $\Psi(h_\beta)$ tends to $\Psi(h)$ strongly and hence φ_β tends to φ in norm. By Theorem 10 of [2], $\sigma_t^{\varphi_\beta}(x)$ tends to $\sigma_t^\varphi(x)$ strongly for $x\in\mathfrak{M}$.

On the other hand, $\sigma_{t_1}^\psi(h_\beta)\dots\sigma_{t_n}^\psi(h_\beta)$ converges strongly to $\sigma_{t_1}^\psi(h)\dots\sigma_{t_n}^\psi(h)$ and hence $u_t^{\varphi_\beta\psi}$ and $\hat{u}_t^{\varphi_\beta\psi}$ converge strongly to $u_t^{\varphi\psi}$ and $\hat{u}_t^{\varphi\psi}$ by Lebesgue dominated convergence theorem. Hence we have

$$u_t^{\varphi\psi}\sigma_t^\psi(x)\hat{u}_t^{\varphi\psi}=\sigma_t^\varphi(x),\ \ x\in\mathfrak{M}$$

which proves (4.9). Q.E.D.

Corollary 4.4. *If Ψ is cyclic and separating and $h\in\mathfrak{M}$, $h^*=h$, then $\Psi(h)$ is also cyclic and separating.*

Proof. By the proof of Proposition 4.3, $\Psi(h)$ satisfies KMS condition relative to the group of $*$-automorphisms

$$x\in\mathfrak{M}\to u_t\sigma_t^\psi(x)u_t^*=\hat{\sigma}_t(x)$$

where

$$u_t=\mathrm{Exp}_r\left(\int_0^t;\ i\sigma_s^\psi(h)\mathrm{d}s\right)$$

and the group property of the $*$-automorphisms follow from the cocycle equation

$$u_{t_1}\sigma_{t_1}(u_{t_2})=u_{t_1+t_2},$$

which is formula (4.2) of [3]. It is known that a KMS state is faithful if it is faithful on the center.

Let E be a central projection. Then $E\{\Psi(h)\}=(E\Psi)(Eh)$, where we restrict our attention to $E\mathfrak{M}$ and the restriction $E\Delta_\Psi$ of Δ_Ψ (which commutes with E) is $\Delta_{E\Psi}$. Since Ψ is separating $E\Psi\neq 0$ unless $E=0$.

We now prove $\Phi(h)\neq 0$ for any non-zero cyclic and separating Φ. We then have $E\Psi(h)\neq 0$ and hence $\Psi(h)$ is separating for M. Since $\Psi(h)\in V_\Psi$, $\Psi(h)$ is then also cyclic and the proof is complete.

To prove $\Phi(h)\neq 0$, we define

$$\Phi_z(h)\equiv\sum_{n=0}^{\infty}z^n\int_0^1 \mathrm{d}s_1\ldots\int_0^{s_{n-1}}\mathrm{d}s_n\Delta_\Phi^{s_n z}h\Delta_\Phi^{(s_{n-1}-s_n)z}\ldots\Delta_\Phi^{(s_1-s_2)z}h\Phi.$$

The integrand is continuous for $\mathrm{Re}\, z\in[0, 1/2]$, holomorphic for $\mathrm{Re}\, z\in(0, 1/2)$ and uniformly bounded by $\|h\|^n\|\Phi\|$ for $\mathrm{Re}\, z\in[0, 1/2]$. Hence $\Phi_z(h)$ is continuous for $\mathrm{Re}\, z\in[0, 1/2]$ and holomorphic for $\mathrm{Im}\, z\in(0, 1/2)$.

Next we prove the following formula:

$$\Phi_{z+it}(h)=u_t\Delta_\Phi^{it}\Phi_z(h)$$

where t is real,

$$u_t=\mathrm{Exp}_r\left(\int_0^t;\ i\sigma_s^\varphi(h)\mathrm{d}s\right)$$

and $\varphi=\omega_\Phi$. If this formula is proved, then $\Phi(h)=\Phi_{1/2}(h)=0$ implies $\Phi_{1/2+it}(h)=0$ and hence $\Phi_z(h)=0$ by the edge of wedge theorem. In particular $\Phi_0(h)=\Phi=0$. Since $\Phi\neq 0$ by assumption, we obtain $\Phi(h)\neq 0$.

Since u_t is a unitary operator strongly continuous in h and $\Phi_z(h)$ is also strongly continuous in h by the proof of Proposition 4.1, it is enough to prove the formula when h is replaced by $h_\beta=h(f_\beta^G)$. For h_β, the formula reduces to

$$\mathrm{Exp}_r\left(\int_0^1;\ (z+it)\sigma_{-i(z+it)s}^\varphi(h_\beta)\mathrm{d}s\right)$$

$$=\mathrm{Exp}_r\left(\int_0^t;\ i\sigma_s^\varphi(h_\beta)\mathrm{d}s\right)\sigma_t^\varphi\left\{\mathrm{Exp}_r\left(\int_0^1;\ z\sigma_{-izs}^\varphi(h)\mathrm{d}s\right)\right\},$$

which holds for $\mathrm{Re}\, z=0$ due to the formula (2.8) and (4.2) of [3] and hence for $\mathrm{Re}\, z\in[0, 1/2]$ by the edge of wedge theorem. Q.E.D.

Proposition 4.5. *If* $\Phi=\Psi(h)$, *then* $\Phi(-h)=\Psi$. *If* $\Phi_1=\Psi(h_1)$ *and* $\Phi_2=\Phi_1(h_2)$, *then* $\Phi_2=\Psi(h_1+h_2)$.

Proof. First we consider

$$h_{1\beta}=\int\Delta_{\Psi}^{it}h_1\Delta_{\Psi}^{-it}f_{\beta}^{G}(t)\mathrm{d}t,\ \Phi_{1\beta}=\Psi(h_{1\beta}),$$

$$h_{2\gamma\beta}=\int\Delta_{\Phi_{1\beta}}^{it}h_2\Delta_{\Phi_{1\beta}}^{-it}f_{\gamma}^{G}(t)\mathrm{d}t,\ \Phi_{2\beta\gamma}=\Phi_{1\beta}(h_{2\gamma\beta}).$$

Then we have

$$\Phi_{1\beta}=\mathrm{Exp}_r\left(\int_0^{1/2};\ h_{1\beta}(-is)\mathrm{d}s\right)\Psi,$$

$$\Phi_{2\beta\gamma}=\mathrm{Exp}_r\left(\int_0^{1/2};\ h_{2\gamma\beta}(-is)\mathrm{d}s\right)\Phi_{1\beta}$$

where $h_{1\beta}(z)$ and $h_{2\gamma\beta}(z)$ are analytic continuations of $h_{1\beta}(t)=\sigma_t^{\psi}(h_{1\beta})$ and $h_{2\gamma\beta}(t)=\sigma_t^{\varphi_{1\beta}}(h_{2\gamma\beta})$, and $\psi=\omega_{\Psi}$, $\varphi_{1\beta}=\omega_{\Phi_{1\beta}}$.

By Proposition 4.3, we have

$$\sigma_t^{\psi}(x)=u_t^{-1}\sigma_t^{\varphi_{1\beta}}(x)u_t,$$

$$u_t=\mathrm{Exp}_r\left(\int_0^t;\ ih_{1\beta}(s)\mathrm{d}s\right)=\mathrm{Exp}_r\left(\int_0^1;\ ith_{1\beta}(ts)\mathrm{d}s\right).$$

By analytic continuation of the right hand side, we obtain the analytic continuation of the left hand side:

$$\sigma_z^{\psi}(h_{2\gamma\beta})=\mathrm{Exp}_r\left(\int_0^1;\ izh_{1\beta}(zs)\mathrm{d}s\right)^{-1}h_{2\gamma\beta}(z)\mathrm{Exp}_r\left(\int_0^1;\ izh_{1\beta}(zs)\mathrm{d}s\right).$$

Hence

$$h_{2\gamma\beta}(-it)=\mathrm{Exp}_r\left(\int_0^t;\ h_{1\beta}(-is)\mathrm{d}s\right)\sigma_{-it}^{\psi}(h_{2\gamma\beta})\mathrm{Exp}_r\left(\int_0^t;\ h_{1\beta}(-is)\mathrm{d}s\right)^{-1}.$$

By formula (3.10) of [3],

$$\Phi_{2\gamma\beta}=\mathrm{Exp}_r\left(\int_0^{1/2};\ \{h_{1\beta}(-is)+\sigma_{-is}^{\psi}(h_{2\gamma\beta})\}\mathrm{d}s\right)\Psi.$$

$$=\Psi(h_{1\beta}+h_{2\gamma\beta}).$$

In the limit $\beta\to+0$, $h_{1\beta}$ tends to h_1, and hence $\Phi_{1\beta}$ tends to Φ_1 strongly. By the proof of Theorem 10 of [2], $\Delta_{\Phi_{1\beta}}^{it}$ tends to $\Delta_{\Phi_1}^{it}$ strongly (uniformly over bounded t) and hence

$$h_{2\gamma\beta}(z)=\int\Delta_{\Phi_{1\beta}}^{it}h_2\Delta_{\Phi_{1\beta}}^{-it}f_\gamma^G(t-z)\mathrm{d}t$$

tends to

$$h_{2\gamma}(z)=\int\Delta_{\Phi_1}^{it}h_2\Delta_{\Phi_1}^{-it}f_\gamma^G(t-z)\mathrm{d}t$$

which is an analytic continuation of

$$\sigma_t^{\varphi_1}(h_{2\gamma}),\ h_{2\gamma}=\int\Delta_{\Phi_1}^{it}h_2\Delta_{\Phi_1}^{-it}f_\gamma^G(t)\mathrm{d}t.$$

Therefore $h_{2\gamma\beta}$ tends to $h_{2\gamma}$ strongly and $\Phi_{1\beta}(h_{2\gamma\beta})$ tends strongly to $\Phi_1(h_{2\gamma})$. By Proposition 4.1, $\Psi(h_{1\beta}+h_{2\gamma\beta})$ tends to $\Psi(h_1+h_{2\gamma})$. Hence

$$\Phi_1(h_{2\gamma})=\Psi(h_1+h_{2\gamma}).$$

In the limit $\gamma\to+0$, $h_{2\gamma}$ tends to h_2 and hence by Proposition 4.1

$$\Phi_1(h_2)=\Psi(h_1+h_2).$$

By taking $h_2=-h_1$, we have $\Phi(-h)=\Psi$ when $\Phi=\Psi(h)$.

Q.E.D.

Proposition 4.6. *For given faithful ψ and $\varphi\in\mathfrak{M}_*^+$, $\mathrm{h}(\varphi/\psi)$ is unique if it exists.*

Proof. Any cone V_Ψ is related to any other cone $V_{\Psi'}$ by a unitary $u'\in\mathfrak{M}'$: $u'V_\Psi=V_{\Psi'}$. Hence the choice of V_Ψ does not affect the definition of $\mathrm{h}(\varphi/\psi)$. We fix one V_Ψ. Assume that

$$\Psi(h_1)=\Phi=\Psi(h_2),\ \omega_\Psi=\psi,\ \omega_\Phi=\varphi.$$

By Proposition 4.5,

$$\Psi(h_1-h_2)=\Phi(-h_2)=\Psi.$$

By Proposition 4.3, $\mathrm{Exp}_r\left(\int_0^t;\ i\sigma_s^\psi(h_1-h_2)\mathrm{d}s\right)$ must commute with all $\sigma_t^\psi(x)$,

$x\in\mathfrak{M}$ and hence is in the center of $\mathfrak{M}$. By differentiating by t at $t=0$, we see that h_1-h_2 is in the center of $\mathfrak{M}$. We then have $\sigma_t^{\psi}(h_1-h_2)=h_1-h_2$ and hence

$$\Psi(h_1-h_2)=\exp\{(h_1-h_2)/2\}\Psi=\Psi.$$

Since Ψ is separating, we have $h_1-h_2=0$ for selfadjoint h_1-h_2.

Q.E.D.

Proposition 4.7. $\hat{u}_t^{\varphi\psi}=u_t^{\varphi\psi}$.

Proof. By (4.9), we have

$$u_t^{\psi\varphi}=\mathrm{Exp}_r\Big(\int_0^t;\ u_s^{\varphi\psi}(-i\sigma_s^{\psi}(\mathrm{h}(\psi/\varphi)))(u_s^{\varphi\psi})^{-1}\mathrm{d}s\Big),$$

$$u_t^{\varphi\psi}=\mathrm{Exp}_r\Big(\int_0^t;\ -i\sigma_s^{\psi}(\mathrm{h}(\varphi/\psi))\mathrm{d}s\Big).$$

By formula (3.10) of [3], we have

$$u_t^{\varphi\psi}u_t^{\varphi\psi}=\mathrm{Exp}_r\Big(\int_0^t;\ -i\sigma_s^{\psi}(h(\psi/\varphi)+h(\varphi/\psi))\mathrm{d}s\Big).$$

By Proposition 4.5, the right hand side is 1 and hence

$$u_t^{\psi\varphi}=u_t^{\psi\varphi}u_t^{\varphi\psi}\hat{u}_t^{\varphi\psi}=\hat{u}_t^{\varphi\psi}.$$

Q.E.D.

Proposition 4.8. *Let* $\Phi=\Psi(-h)$, $h\in\mathfrak{M}$, $h^*=h$, $H_\Psi=-\log\Delta_\Psi, H_\Phi=-\log\Delta_\Phi$. *Then*

(4.13) $$H_\Phi=H_\Psi+h-j(h)$$

where $j(h)=J_\Psi hJ_\Psi=J_\Phi hJ_\Phi$. *For* $\omega_\Phi=\varphi$, $\omega_\Psi=\psi$,

(4.14) $$u_t^{\varphi\psi}\Delta_\Psi^{it}=\exp(-it(H_\Psi+h))=j(u_t^{\psi\varphi})\Delta_\Phi^{it},$$

(4.15) $$\Delta_\Phi^{-it}u_t^{\varphi\psi}=\exp(it(H_\Phi-h))=\Delta_\Psi^{-it}j(u_t^{\psi\varphi}),$$

(4.16) $$\Delta_\Phi^{it}=j(u_t^{\varphi\psi})u_t^{\varphi\psi}\Delta_\Psi^{it}=\Delta_\Psi^{it}j(u_{-t}^{\psi\varphi})u_{-t}^{\psi\varphi}.$$

Proof. By Remark to Proposition 16 of [3] and equation (4.6), we have the first equality of (4.14) where $H_\Psi + h$ is selfadjoint. By Proposition 4.7 and equation (4.8), we have

$$u_t^{\varphi\psi} = (u_t^{\psi\varphi})^* = \mathrm{Exp}_l\Big(\int_0^t; -i\sigma_s^\varphi(h)\mathrm{d}s\Big).$$

Hence by Remark to Proposition 16 of [3], we have the first equality of (4.15), where $H_\Phi - h$ is selfadjoint. ((4.15) can be obtained also from (4.14) by taking adjoint and interchanging φ and ψ.)

Consider

$$w = \Delta_\Phi^{-it} j(u_t^{\varphi\psi}) u_t^{\varphi\psi} \Delta_\Psi^{it}.$$

For $x \in \mathfrak{M}$ and $y \in \mathfrak{M}'$, we have

$$\begin{aligned} wxy = wxj(j(y)) &= \Delta_\Phi^{-it} j(u_t^{\varphi\psi}) u_t^{\varphi\psi} \sigma_t^\psi(x) j(\sigma_t^\psi\{j(y)\}) \Delta_\Psi^{it} \\ &= \Delta_\Phi^{-it}\, \sigma_t^\varphi(x) j(\sigma_t^\varphi\{j(y)\}) j(u_t^{\varphi\psi}) u_t^{\varphi\psi} \Delta_\Psi^{it} \\ &= xj(j(y))w = xyw. \end{aligned}$$

Hence $w \in \mathfrak{M} \cap \mathfrak{M}'$ $(=(\mathfrak{M} \cup \mathfrak{M}')')$. Obviously w is unitary. Since $V_\Psi = V_\Phi$ is invariant under multiplication of Δ_Ψ^{it}, Δ_Φ^{it} and $Qj(Q)$, $Q = u_t^{\varphi\psi} \in \mathfrak{M}$, we have $wV_\Psi \subset V_\Psi$. By the next Lemma, this implies $w \geqq 0$ and hence $w = 1$. Hence we have the first equality of (4.16). By taking adjoint and changing the sign of t, we obtain the second equality of (4.16). From $w = 1$, we also obtain second equalities of (4.14) and (4.15).

By (4.14), (4.15) and $w = 1$, we have

$$\begin{aligned} \exp(-it(H_\Psi + h)) = j(u_t^{\varphi\psi})^* \Delta_\Phi^{it} &= \{j(\Delta_\Phi^{-it} u_t^{\varphi\psi})\}^* \\ &= \exp(-it(H_\Phi + j(h))) \end{aligned}$$

where we have used the property $j(H_\Phi) = -H_\Phi$, which follows from $J_\Phi \Delta_\Phi J_\Phi = \Delta_\Phi^{-1}$. Since both $H_\Psi + h$ and $H_\Phi + j(h)$ are selfadjoint operators, we have

$$H_\Psi + h = H_\Phi + j(h). \tag{4.17}$$

Since $H_\Psi + h$ and $H_\Psi + h - j(h)$ have the same domain, which also coincides with the domain of H_Φ by (4.17), we have (4.13). Q.E.D.

Lemma 4.9. *If* $w\in\mathfrak{M}\cap\mathfrak{M}'$ *and* $wV_{\Psi}\subset V_{\Psi}$, *then* $w\geqq 0$.

Proof. Since $\mathfrak{M}\cap\mathfrak{M}'$ is commutative, w is normal. Let E be a spectral projection of w for an open set contained in the upper half complex plane. Since $E\in\mathfrak{M}\cap\mathfrak{M}'$, $Ej_{\Psi}(E)=EE^*=E$ by Lemma 3 of [2] and hence $EV_{\Psi}\subset V_{\Psi}$. Therefore, for any $\Phi\in V_{\Psi}$, we have $E\Phi\in V_{\Psi}$, $w\Phi\in V_{\Psi}$ and hence $\omega_{\Phi}(Ew)\geqq 0$. On the other hand, E is a spectral projection of w for an open set in the upper half plane and hence Im $\omega_{\Phi}(Ew)>0$ unless $E\Phi=0$. Therefore we have $E\Phi=0$ for any $\Phi\in V_{\Psi}$ and hence $E=0$. Similarly w can not have a spectrum in the lower half plane nor in the negative real axis. Hence $w\geqq 0$. Q.E.D.

Proposition 4.10. *If* $\Phi\leqq l^{1/2}\Psi$ (*i.e.* $l^{1/2}\Psi-\Phi\in V_{\Psi}$) *and if* $\mathrm{h}(\varphi/\psi)$ *exists for* $\omega_{\Phi}=\varphi$, $\omega_{\Psi}=\psi$ *then* $-\mathrm{h}(\varphi/\psi)\leqq\log l$.

Proof. For $h\in\mathfrak{M}$, $h^*=h$, we define

$$\Psi(t;h)=\sum_{n=0}^{\infty}\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\ldots\int_0^{t_{n-1}}\mathrm{d}t_n\Delta_{\Psi}^{t_n}h\Delta_{\Psi}^{t_{n-1}-t_n}\ldots\Delta_{\Psi}^{t_1-t_2}h\Psi,\qquad 0\leqq t\leqq 1/2. \tag{4.18}$$

$\Psi(h)$ defined earlier is $\Psi(1/2;h)$. We shall first prove the following formula:

$$\begin{aligned}\omega^t(x)&\equiv(\Delta_{\Psi}^{(1-t)/2}x\Psi,\ \Psi(t;h))\\&=(xJ_{\Psi}\Psi(t/2;h),\ J_{\Psi}\Psi(t/2;h)),\quad x\in\mathfrak{M}.\end{aligned}\tag{4.19}$$

First consider $h_{\beta}=h(f_{\beta}^{G})$ instead of h. Then

$$\Psi(t;h_{\beta})=\mathrm{Exp}_r\left(\int_0^t;\ h_{\beta}(-is)\mathrm{d}s\right)\Psi.$$

Hence

$$\begin{aligned}&(xJ_{\Psi}\Psi(t/2;h_{\beta}),\ J_{\Psi}\Psi(t/2;h_{\beta}))=(x\Psi,\ j_{\Psi}(k^*)j_{\Psi}(k)\Psi)\\&=(x\Psi,\ j_{\Psi}(k^*k)\Psi)=(x\Psi,\ \Delta_{\Psi}^{1/2}k^*k\Psi)\\&=(\Delta_{\Psi}^{(1-t)/2}x\Psi,\ \Delta_{\Psi}^{t/2}k^*k\Psi)\end{aligned}$$

where

$$k=\mathrm{Exp}_r\Big(\int_0^{t/2};\ h_\beta(-is)\mathrm{d}s\Big).$$

Since the closure of $\Delta_\Psi^{t/2}k^*\Delta_\Psi^{-t/2}$ is given by

$$\mathrm{Epx}_l\Big(\int_0^{t/2};\ h_\beta(i(s-t/2))\mathrm{d}s\Big)=\mathrm{Exp}_r\Big(\int_0^{t/2};\ h_\beta(-is)\mathrm{d}s\Big)=k,$$

due to formula (2.11) of [3], we obtain as the closure of $\Delta_\Psi^{t/2}k^*k\Delta_\Psi^{-t/2}$

$$\mathrm{Exp}_r\Big(\int_0^{t/2};\ h_\beta(-is)\mathrm{d}s\Big)\mathrm{Exp}_r\Big(\int_0^{t/2};\ h_\beta(-i(t/2+s))\mathrm{d}s\Big)$$

$$=\mathrm{Exp}_r\Big(\int_0^{t};\ h_\beta(-is)\mathrm{d}s\Big)$$

by formula (3.5) of [3]. Hence (4.19) is proved for $h=h_\beta$.

By taking the limit $\beta\to+0$, we obtain (4.19) by continuity of $\Psi(t;h)$ on h, which can be proved in exctly the same way as the continuity of $\Psi(h)$ in h.

Assume now $\Phi=\Psi(h)$, $\Phi\leqq l^{1/2}\Psi$ For a general $\alpha\in[0,1/2]$, the closure of $\Delta_\Psi^\alpha\mathfrak{M}^+\Psi$ is denoted by V_Ψ^α in [2]. By Theorem 3 (5) of [2], it is dual to $V_\Psi^{1/2-\alpha}$. By Taking $x\in\mathfrak{M}^+$ in (4.19), we obtain

$$\Psi(t;h)\in(V_\Psi^{1/2-t/2})'=V_\Psi^{t/2}. \tag{4.20}$$

We now prove

$$l^{2^{-n}}\Psi-\Psi(2^{-n};h)\in V_\Psi^{2^{-(n+1)}} \tag{4.21}$$

by induction on n. It is true for $n=1$ by our assumption.

Assume that

$$l^t\Psi-\Psi(t;h)\in V_\Psi^{t/2},\quad t\leqq 1/2.$$

Since $\Delta_\Psi^{(1-t)/2}x\Psi\in V_\Psi^{1/2-t/2}=(V_\Psi^{t/2})'$ for $x\in\mathfrak{M}^+$, we have

$$\omega^t(x)\leqq l^t(\Delta_\Psi^{(1-t)/2}x\Psi,\Psi)=l^t\psi(x),\quad x\in\mathfrak{M}^+.$$

Hence $\omega_\chi\leqq l^t\psi$ for $\chi=J_\Psi\Psi(t/2;h)$. Hence, there exists $y_t\in\mathfrak{M}'$ such that

$$J_\Psi\Psi(t/2;h)=y_t\Psi,\ \|y_t\|\leqq l^{t/2}.$$

Let $x_t = J_\Psi y_t J_\Psi \in \mathfrak{M}$.

By Theorem 3(2) of [2] and equation (4.20), we have

$$J_\Psi \Psi(t/2;\, h) = \Delta_\Psi^{(1-t)/2} \Psi(t/2;\, h).$$

Hence

$$\Delta_\Psi^{1/2} x_t^* \Psi = J_\Psi x_t \Psi = J_\Psi \Psi(t/2;\, h) = \Delta_\Psi^{(1-t)/2} x_t \Psi.$$

By Lemma 6 of [2], $\sigma_s^\psi(x_t)$ has an analytic continuation to Im $s \in [0, t/2)$ and $\|\sigma_s^\psi(x_t)\| \leqq \|x_t\| \leqq l^{t/2}$ for the analytic continuation. In particular, $0 \leqq \sigma_{it/4}^\psi(x_t) \leqq l^{t/2}$ where the positivity comes from $\Psi(t/2;\, h) \in V_\Psi^{t/4}$ and Theorem 3(7) of [2]. Hence

$$l^{t/2}\Psi - \Psi(t/2;\, h) = \Delta_\Psi^{t/4}(l^{t/2} - \sigma_{it/4}^\psi(x_t))\Psi \in V_\Psi^{t/4}.$$

This completes the inductive proof of (4.21).

We have

$$\lim_{n\to\infty} (l^{2^{-n}} - 1)2^n = \log l,$$

$$\lim_{n\to\infty} \{\Psi(2^{-n};\, h) - \Psi\} 2^n = h\Psi,$$

where the last equation is due to the estimate

$$\left\| \int_0^t \mathrm{d}t_1 \ldots \int_0^{t_{n-1}} \mathrm{d}t_n \Delta^{t_n} h \Delta^{t_{n-1}-t_n} \ldots \Delta^{t_1 - t_2} h\Psi \right\|$$

$$\leqq (n!)^{-1} t^n \|h\|^n \|\Psi\|, \quad 0 \leqq t \leqq 1/2$$

and

$$\lim_{t\to 0} t^{-1} \int_0^t \mathrm{d}s \Delta^s h\Psi = h\Psi.$$

Hence we have

$$(4.22) \qquad (\log l - h)\Psi = \lim_{n\to\infty} 2^n (l^{2^{-n}}\Psi - \Psi(2^{-n};\, h)).$$

Since $\|\Delta^{1/2-t} x\Psi\| \leqq 2\|x\| \|\Psi\|$ for $x \in \mathfrak{M}$ $t \in [0, 1/2]$, and $\Delta^{1/2-t} x\Psi$ is strongly continuous in $t \in [0, 1/2]$, we have

$$(\Delta^{1/2}x\Psi,\ \Phi)=\lim_n(\Delta^{2^{-1}-2^{-(n+1)}}x\Psi,\ \Phi_n)$$

whenever $||\Phi_n-\Phi||\rightarrow 0$. By (4.22) and (4.21), we have

$$(\Delta^{1/2}x\Psi,\ (\log l-h)\Psi)\geqq 0$$

for all $x\in\mathfrak{M}^+$. Hence $(\log l-h)\Psi\in V_\Psi^0$ and we have

$$\log l\geqq h.$$

Q.E.D.

Corollary 4.11. *If* $l_1^{1/2}\Psi\leqq\Phi\leqq l_2^{1/2}\Psi$, $\varphi=\omega_\Phi$, $\psi=\omega_\Psi$, *and* $\mathrm{h}(\varphi/\psi)$ *exists, then*

$$\log l_1\leqq-\mathrm{h}(\varphi/\psi)\leqq\log l_2. \tag{4.23}$$

Proof. By Proposition 4.10, we have $-\mathrm{h}(\varphi/\psi)\leqq\log l_2$. Since $\Psi\leqq l_1^{-1/2}\Phi$, we have $-\mathrm{h}(\psi/\varphi)\leqq-\log l_1$. Since $\mathrm{h}(\varphi/\psi)=-\mathrm{h}(\psi/\varphi)$, we have $\log l_1\leqq-\mathrm{h}(\varphi/\psi)$. Q.E.D.

Proposition 4.12. *For* $h\in\mathfrak{M}$, $h^*=h$, *a cyclic and separating vector* Ψ *is in the domain of* $\exp z(-H_\Psi+h)$ *for* $\mathrm{Re}\ z\in[0,1/2]$, *where* $H_\Psi=-\log\Delta_\Psi$, *the vector*

$$\Psi(z)\equiv\exp z(-H_\Psi+h)\Psi=\sum_{n=0}^{\infty}z^n\int_0^1 ds_1\int_0^{s_1}ds_2\ldots\int_0^{s_{n-1}}ds_n \tag{4.24}$$

$$\Delta_\Psi^{s_nz}h\Delta_\Psi^{(s_{n-1}-s_n)z}\ldots\Delta_\Psi^{(s_1-s_2)z}h\Psi$$

is holomorphic in z *for* $\mathrm{Re}\ z\in(0,1/2)$ *and strongly continuous in* z *for* $\mathrm{Im}\ z\in[0,1/2]$. *If* h_n *tends to* h *strongly then*

$$\lim_n\exp z(-H_\Psi+h_n)\Psi=\exp z(-H_\Psi+h)\Psi \tag{4.25}$$

strongly for $\mathrm{Re}\ z\in[0,1/2]$.

Proof. For $z=it$ with real t, we have

$$\sum_{n=0}^{\infty}z^n\int_0^1 ds_1\int_0^{s_1}ds_2\ldots\int_0^{s_{n-1}}ds_n\Delta_\Psi^{s_nz}h\Delta_\Psi^{(s_{n-1}-s_n)z}\ldots\Delta_\Psi^{(s_1-s_2)z}h\Psi$$

$$= \sum_{n=0}^{\infty} i^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \sigma^{\psi}_{t_n}(h) \dots \sigma^{\psi}_{t_1}(h) \Psi$$

$$= u_t^{\varphi\psi} \Psi = u_t^{\varphi\psi} \Delta_{\Psi}^{it} \Psi = \exp it(-H_{\Psi} + h) \Psi$$

where $\Phi = \Psi(h)$, $\varphi = \omega_{\Phi}$, $\psi = \omega_{\Psi}$ and Proposition 4.8 is used. Hence (4.24) holds for pure imaginary z.

If H is any selfadjoint operator and $e^{zH\Psi}$ with pure imaginary z has an "analytic continuation" $\Psi(z)$ holomorphic for Re $z \in (0, \delta)$ and continuous for Re $z \in [0, \delta]$, then Ψ is in the domain of e^{zH}, Re $z \in [0, \delta]$ and $\Psi(z) = e^{zH}\Psi$ due to the following argument:

Let $H = \int \lambda dE_{\lambda}$ and D be the union of ranges of $E_L - E_{-L}$ for all $L > 0$. D is a core of e^{zH} for any z. For each $\Phi \in D$, we have

$$(\Psi(z), \Phi) = (\Psi, e^{\bar{z}H} \Phi)$$

for pure imaginary z. Both sides are holomorphic in z for Re $z \in (0, \delta)$ and continuous in z for Re $z \in [0, \delta]$. Hence the equality holds for all z with Re $z \in [0, \delta]$ by the edge of wedge theorem. Since D is a core of $e^{\bar{z}H}$, the equality holds for all $\Phi \in D(e^{\bar{z}H})$. Hence $\Psi \in D(e^{zH})$ and $\Psi(z) = e^{zH}\Psi$.

Therefore we obtain Proposition if we show that the right hand side of (4.24) is holomorphic for Re $z \in (0, 1/2)$, is strongly continuous for Re $z \in [0, 1/2]$, and sequentially strongly continuous in h.

Due to (3.2), the sum in (4.24) converges uniformly in norm for Re$z \in [0, 1/2]$ and over a bounded set of h. Due to Theorem 3.1, the integrand in each term of (4.24) is holomorphic for Re $z \in (0, 1/2)$ and continuous for Re $z \in [0, 1/2]$. Since the integrand is dominated by $||h||^n ||\Psi||$ irrespective of s_k and z, we obtain the holomorphy of the integral by Fubini's theorem applied to Cauchy integral formula. We also obtain the strong continuity in z by Lebesgue dominated convergence theorem applied to the inner product with other vector and the norm. Hence we have holomorphy and continuity of the sum by the uniform convergence.

Exactly the same proof as Proposition 4.1 shows that the right hand side of (4.24) is sequentially strongly continuous in h.

Q.E.D.

Proposition 4.13. *Let $h \in \mathfrak{M}$, $h^* = h$, Ψ be a cyclic and separating* *vector*, $\Phi=\Psi(h)$, $\psi=\omega_\Psi$ *and* $\varphi=\omega_\Phi$. *Assume that* $l_1^{1/2}\Psi\geqq\Phi\geqq l_2^{1/2}\Psi$ *for some* l_1 *and* l_2. *Then there exists an invertible* $\mathrm{A}(z)\in\mathfrak{M}$ *for* $\mathrm{Re}\, z\in[0, 1/2]$ *such that* $\mathrm{A}(z)$ *is holomorphic for* $\mathrm{Re}\, z\in(0, 1/2)$, *strongly continuous for* $\mathrm{Re}\, z\in[0, 1/2]$, $\|\mathrm{A}(z)\|\leqq\max(l_1^{1/2}, 1)$, $\|\mathrm{A}(z)^{-1}\|\leqq\max(l_2^{-1/2}, 1)$, $\mathrm{A}(s+it)=u_t^{\varphi\psi}\sigma_t^\psi\{A(s)\}$, $\mathrm{A}(0)=1$, $\mathrm{A}(1/2)\Psi=\Phi$, $\Psi(z)=\mathrm{A}(z)\Psi$.

Proof. Consider $\Psi(z)$ of Proposition 4.12. We have

$$\Psi(s+it)=\exp it(-H_\Psi+h)\Psi(s)=u_t^{\varphi\psi}\Delta_\Psi^{it}\Psi(s) \tag{4.26}$$

due to (4.14). By Theorem 3 (8) of [2], there exists $A\in\mathfrak{M}$ such that $\Phi=A\Psi$, $\|A\|\leqq l_1^{1/2}$, $\|A^{-1}\|\leqq l_2^{-1/2}$. Hence

$$\Psi(it)=u_t^{\varphi\psi}\Psi,\ \Psi(1/2+it)=u_t^{\varphi\psi}\sigma_t^\psi(A)\Psi. \tag{4.27}$$

For a vector x and $Q\in\mathfrak{M}'$, consider

$$\mathrm{f}(z)=(\Psi(z), Q^*x).$$

Since $\|\Psi(z)\|\leqq e^{\|h\|}\|\Psi\|$, $\mathrm{f}(z)$ is uniformly bounded for $\mathrm{Re}\, z\in[0, 1/2]$. By Proposition 4.12, it is holomorphic for $\mathrm{Re}\, z\in(0, 1/2)$ and continuous for $\mathrm{Re}\, z\in[0, 1/2]$. Furthermore, by (4.27), we have the following bounds on the boundary lines:

$$\begin{aligned}|\mathrm{f}(it)|&=|(u_t^{\varphi\psi}\Psi, Q^*x)|=|(u_t^{\varphi\psi}Q\Psi, x)|\\&\leqq\|Q\Psi\|\|x\|,\\|\mathrm{f}(1/2+it)|&=|(u_t^{\varphi\psi}\sigma_t^\psi(A)\Psi, Q^*x)|=|(u_t^{\varphi\psi}\sigma_t^\psi(A)Q\Psi, x)|\\&\leqq\|A\|\|Q\Psi\|\|x\|.\end{aligned}$$

Hence we have

$$|\mathrm{f}(z)|\leqq\max(\|A\|, 1)\|Q\Psi\|\|x\|.$$

This implies the existence of operators $\mathrm{A}(z)$ such that

$$\mathrm{f}(z)=(\mathrm{A}(z)Q\Psi, x),\ \|\mathrm{A}(z)\|\leqq\max(l_1^{1/2}, 1)$$

due to Riesz theorem and $\|A\|\leqq l_1^{1/2}$.

Since $\mathrm{f}(z)=(Q\Psi(z), x)$, we have

$$Q\Psi(z)=\mathrm{A}(z)Q\Psi,\ Q\in\mathfrak{M}'.$$

Hence $[\mathrm{A}(z), Q']Q\Psi=0$ for any $Q, Q'\in\mathfrak{M}'$ and hence $\mathrm{A}(z)\in\mathfrak{M}$.

By interchanging the role of Ψ and Φ, we obtain $\mathrm{B}(z)\in\mathfrak{M}$ such that $\mathrm{B}(it)=u_t^{\psi\varphi}$ and $\|\mathrm{B}(z)\|\leqq\max(l_2^{-1/2}, 1)$. Since $\mathrm{A}(it)\mathrm{B}(it)=u_t^{\varphi\psi}u_t^{\psi\varphi}=1$, we have $\mathrm{A}(z)\mathrm{B}(z)=1$. Similarly $\mathrm{B}(z)\mathrm{A}(z)=1$. Hence $\|\mathrm{A}(z)^{-1}\|=\|\mathrm{B}(z)\|\leqq\max(l_2^{-1/2}, 1)$.

Since $\Psi(z)$ is strongly continuous for $\mathrm{Re}\ z\in[0, 1/2]$, we have the strong continuity of $\mathrm{A}(z)Q'\Psi=Q'\Psi(z)$ for all $Q'\in\mathfrak{M}$. Since $\mathrm{A}(z)$ is uniformly bounded, it is strongly continuous for $\mathrm{Re}\ z\in[0, 1/2]$.

Since $\Psi(z)$ is holomorphic for $\mathrm{Re}\ z\in(0, 1/2)$, we have

$$\mathrm{A}(z)Q'\Psi=Q'\Psi(z)=(2\pi i)^{-1}\int_\Gamma(z-z')^{-1}Q'\Psi(z')\mathrm{d}z'$$

$$=\left\{(2\pi i)^{-1}\int_\Gamma(z-z')^{-1}\mathrm{A}(z')\mathrm{d}z'\right\}Q'\Psi,$$

for any $Q'\in\mathfrak{M}'$ and any simple closed curve Γ, encircling the point z and contained in the strp $\{z;\ \mathrm{Re}\ z\in(0, 1/2)\}$. Hence $\mathrm{A}(z)$ is holomorphic for $\mathrm{Re}\ z\in(0, 1/2)$.

The equality $\mathrm{A}(s+it)=u_t^{\varphi\psi}\sigma_t^{\psi}\{\mathrm{A}(s)\}$ follows from (4.26) and $\Psi(z)=\mathrm{A}(z)\Psi$. Q.E.D.

Proposition 4.14. *In Proposition* 4.13, $\sigma_z^{\psi}(h)$ *has an analytic continuation to* $\mathrm{Im}\ z\in(-1/2, 1/2)$.

Proof. Since $\mathrm{A}(z)$ and $\mathrm{A}(z)^{-1}$ is holomorphic in z for $\mathrm{Re}\ z\in(0, 1/2)$, the operator

$$\mathrm{B}(z)=\mathrm{A}(z)^{-1}(\mathrm{d}/\mathrm{d}z)\mathrm{A}(z) \tag{4.28}$$

is also holomorphic for $\mathrm{Re}\ z\in(0, 1/2)$. For pure imaginary $z=it$, we have $\mathrm{A}(z)=u_t^{\varphi\psi}=\mathrm{Exp}_r\left(\int_0^t;\ i\sigma_s^{\psi}(h)\mathrm{d}s\right)$ and hence

$$\mathrm{B}(it)=\sigma_t^{\psi}(h).$$

Since $\mathrm{B}(z)$ is holomorphic for $\mathrm{Re}\ z\in(0, 1/2)$, $\mathrm{B}(-\bar{z})^*$ is holomorphic for $\mathrm{Re}\ z\in(-1/2, 0)$. We also have $\mathrm{B}(it)=\mathrm{B}(-\overline{it})^*$. By edge of wedge theorem, there exists an operator valued analytic function $\mathrm{h}(z)$ for $\mathrm{Re}\ z\in$

$(-1/2, 1/2)$ such that $h(z)=B(z)$ for $\mathrm{Re}\, z\in(0, 1/2)$, $h(z)=B(-\bar{z})^*$ for $\mathrm{Re}\, z\in(-1/2, 0)$ and $h(it)=\sigma_t^\psi(h)$. Q.E.D.

§5. Existence Proof (I)

Lemma 5.1. *Let* $\Phi\in V_\Psi$,

$$F(t)=2\pi\,\mathrm{sech}^2\, 2\pi t, \tag{5.1}$$

$$\Phi_F=\int_{-\infty}^{\infty}\Delta_\Psi^{it}\Phi F(t)dt. \tag{5.2}$$

Then Φ_F *is in* $D(\Delta_\Psi^{iz})$ *for* $\mathrm{Im}\, z\in(-1/4, 1/4)$, *it is in* V_Ψ *and satisfies*

$$\Phi=\int_{-1/4}^{1/4}\Delta_\Psi^{t}\Phi_F dt. \tag{5.3}$$

If $\Phi=Q\Psi$, $Q\in\mathfrak{M}$, *then* $\Phi_F=Q_F\Psi$ *where* $\psi=\omega_\Psi$ *and*

$$Q_F=\int_{-\infty}^{\infty}\sigma_t^\psi(Q)F(t)dt. \tag{5.4}$$

Proof. We have

$$\tilde{F}(u)\equiv\int e^{iut}F(t)dt=u(e^{u/4}-e^{-u/4})^{-1}. \tag{5.5}$$

Hence $\Phi_F=\tilde{F}(\log\Delta_\Psi)\Phi$ is in the domain of Δ_Ψ^{iz} for $\mathrm{Im}\, z\in(-1/4, 1/4)$ because $e^{\alpha u}\tilde{F}(u)$ is bounded for $|\mathrm{Re}\,\alpha|<1/4$. We also have

$$\int_{-1/4}^{1/4}\tilde{F}(u)e^{tu}dt=1.$$

Hence (5.3) holds.

Since V_Ψ is a convex cone invariant under Δ_Ψ^{it} and $F(t)>0$, we have $\Phi_F\in V_\Psi$. If $\Phi=Q\Psi$, then (5.4) follows from (5.2) due to $\Delta_\Psi^{-it}\Psi=\Psi$.
Q.E.D.

Lemma 5.2. *Let* $Q\in\mathfrak{M}$ *be such that* $Q\Psi\in V_\Psi$, $\sigma_t^\psi(Q)$ *has an "analytic continuation"* $\sigma_z^\psi(Q)\in\mathfrak{M}$ *for* $\mathrm{Im}\, z\in[-\delta, 0]$ *and* $\|\sigma_{-i\delta}^\psi(Q)-1\|\leqq L$ *where* δ *is any fixed number in* $(0, 1/8)$. *Let* Q_F *be given by* (5.4). *Then* $\sigma_t^\psi(Q_F)$

has an "analytic continuation" $\sigma_z^{\psi}(Q_{\mathrm{F}})\in M$ *for* $\operatorname{Im} z\in(-\delta-1/4,\ \delta+3/4)$,

(5.6) $$h_1\equiv\sigma_{i/4}^{\psi}(Q_{\mathrm{F}})-2=\sigma_{i/4}^{\psi}\{(Q-1)_{\mathrm{F}}\}$$

is in $\mathfrak{M}$, $h_1^*=h_1$, $\|h_1\|\leqq 2L$,

(5.7) $$\Psi_1\equiv\Psi(h_1)=\mathrm{Exp}_r\Big(\int_0^{1/2};\sigma_{-is}^{\psi}(h_1)\mathrm{d}s\Big)\Psi$$

and

(5.8) $$Q_1\equiv Q\mathrm{Exp}_l\Big(\int_0^{1/2};\ -\sigma_{-is}^{\psi}(h_1)\mathrm{d}s\Big)$$

satisfy $Q_1\Psi_1=Q\Psi$, $\sigma_z^{\psi}(Q_1)\in\mathfrak{M}$ *for* $\operatorname{Im} z\in[-\delta_1,0]$, *and*

$$\|\sigma_{-i\delta_1}^{\psi_1}(Q_1)-1\|\leqq(L^2+(1+L)L')e^{L/2},\quad \psi_1=\omega_{\Psi_1},$$

$$L'=(1/2)\{\pi L\log 2(\delta-\delta_1)\}^2\exp\{-\pi L\log 2(\delta-\delta_1)\},$$

where δ_1 *is any number in* $(0,\delta)$.

Proof. By Theorem 3(7) of [2], $Q\Psi\in V_{\Psi}$ implies that $\sigma_t^{\psi}(Q)$ has an "analytic continuation" $\sigma_z^{\psi}(Q)\in\mathfrak{M}$ for $\operatorname{Im} z\in[0,1/2]$ and $(\sigma_z^{\psi}(Q))^*=\sigma_{\bar z+i/2}^{\psi}(Q)$. ("Analytic continuation" here means a function continuous in the closed strip and holomorphic in the interior.) By assumption, we also have an "analytic continuation" $\sigma_z^{\psi}(Q)\in\mathfrak{M}$ for $\operatorname{Im} z\in[-\delta,0]$. By edge of wedge theorem, we have $\sigma_z^{\psi}(Q)\in\mathfrak{M}$ for $\operatorname{Im} z\in[-\delta,1/2+\delta]$, $(\sigma_z^{\psi}(Q))^*=\sigma_{\bar z+i/2}^{\psi}(Q)$ and

(5.9) $$\|\sigma_z^{\psi}(Q-1)\|\leqq\|\sigma_{-i\delta}^{\psi}(Q-1)\|\leqq L,\ \operatorname{Im} z\in[-\delta,1/2+\delta].$$

(see proof of Lemma 6 of [2].)

By (5.4), we have an analytic continuation

(5.10) $$\sigma_z^{\psi}(Q_{\mathrm{F}})=\int_{-\infty}^{\infty}\sigma_t^{\psi}(\sigma_{z_1}^{\psi}(Q))\mathrm{F}(t-z_2)\mathrm{d}t\in\mathfrak{M}$$

whenever $z=z_1+z_2$, $\operatorname{Im} z_1\in(-\delta,1/2+\delta)$ and $\operatorname{Im} z_2\in(-1/4,1/4)$ i.e. for $\operatorname{Im} z\in(-\delta-1/4,3/4+\delta)$. Since $Q_{\mathrm{F}}\Psi\in V_{\Psi}$, we have

$$\sigma_{i/4}^{\psi}(Q_{\mathrm{F}})\geqq 0$$

by Theorem 3 (7) of [2]. Hence $h_1^*=h_1$. Since $\tilde{\mathrm{F}}(0)=2$, $1_{\mathrm{F}}=2$ and we have the second equality of (5.6). By proof of Lemma 6(4) of [2] and the equality $\sigma^{\phi}_{i/2}(Q_F-2)=(Q_F-2)^*$, we have $\|h_1\|\leqq\|Q_F-2\|\leqq 2\|Q-1\|\leqq 2\|\sigma^{\phi}_{-i\delta}(Q-1)\|\leqq 2L$ where the third inequality follows from proof of Lemma 6(4) of [2] and $(\sigma^{\phi}_z(Q))^*=\sigma^{\phi}_{\bar{z}+i/2}(Q)$.

Let

$$Q_1'=\mathrm{Exp}_l\left(\int_0^{1/2};\ -\sigma^{\phi}_{-is}(h_1)\mathrm{d}s\right)-1+\int_0^{1/2}\sigma^{\phi}_{-is}(h_1)\mathrm{d}s. \tag{5.11}$$

By Lemma 5.1,

$$\left(\int_0^{1/2}\sigma^{\phi}_{-is}(h_1)\mathrm{d}s\right)\Psi=\int_{-1/4}^{1/4}\Delta^s_{\Psi}Q_F\Psi\mathrm{d}s-\Psi=(Q-1)\Psi.$$

Since Ψ is separating $\int_0^{1/2}\sigma^{\phi}_{-is}(h_1)\mathrm{d}s=Q-1$. Hence the definition (5.8) implies

$$Q_1=1-(Q-1)^2+QQ_1'. \tag{5.12}$$

We have

$$\|\sigma^{\phi}_z(Q_1)-1\|\leqq\|\sigma^{\phi}_z(Q)-1\|^2+\{1+\|\sigma^{\phi}_z(Q)-1\|\}\|\sigma^{\phi}_z(Q_1')\|,$$

$$\|\sigma^{\phi}_z(Q_1')\|\leqq\sum_{n=2}^{\infty}\int_0^{1/2}\mathrm{d}s_1\dots\int_0^{s_{n-1}}\mathrm{d}s_n\|\sigma^{\phi}_{z-is_1}(h_1)\|\cdots\|\sigma^{\phi}_{z-is_n}(h_1)\|$$

$$=\sum_{n=2}^{\infty}(n!)^{-1}\left\{\int_0^{1/2}\mathrm{d}s\|\sigma^{\phi}_{z-is}(h_1)\|\right\}^n.$$

By (5.10), (5.6) and (5.9), we have

$$\|\sigma^{\phi}_{z-is}(h_1)\|\leqq\int_{-\infty}^{\infty}\|\sigma^{\phi}_{z-is+i\mathrm{a}(s)+i/4}(Q-1)\|\,|\mathrm{F}(t+i\mathrm{a}(s))|\,\mathrm{d}t$$

$$\leqq L\int_{-\infty}^{\infty}|\mathrm{F}(t+i\mathrm{a}(s))|\,\mathrm{d}t=8\pi L\mathrm{a}(s)/\sin 4\pi\mathrm{a}(s)$$

if $\mathrm{a}(s)$ is chosen such that

$$\mathrm{Im}\ z-s+\mathrm{a}(s)+1/4\in[-\delta,1/2],\quad |\mathrm{a}(s)|<1/4.$$

If $\mathrm{Im}\ z\in[-\delta_1,0]$ and $\delta_1<\delta$, one can choose

$$\mathrm{a}(s)=(s-1/4)(1-2(\delta-\delta_1)).$$

For $|a(s)|<1/4$, we have $|a(s)(1/4-|a(s)|)/\sin 4\pi a(s)|\leqq 2^{-5}$ due to $|x/\sin x|\leqq\pi/2$ for $|x|\leqq\pi/2$ and $\sin(\pi-x)=\sin x$. Hence

$$\int_0^{1/2}\mathrm{d}s\|\sigma_{z-is}^{\psi}(h_1)\|\leqq(2-4(\delta-\delta_1))^{-1}\pi L|\log 2(\delta-\delta_1)|$$

$$\leqq\pi L|\log 2(\delta-\delta_1)|.$$

Since

$$\sum_{n=2}^{\infty}(n!)^{-1}x^n=e^x-1-x\leqq x^2e^x/2,\ x\geqq 0,$$

we have

$$\|\sigma_z^{\psi}(Q_1')\|\leqq L',\ \mathrm{Im}\ z\in[-\delta_1,\ 0].$$

By analytic continuation of (4.9), $\sigma_t^{\psi_1}(Q_1)$ has the following analytic continuation:

$$\sigma_z^{\psi_1}(Q_1)=\mathrm{Exp}_r\Big(\int_0^1;\ iz\sigma_{zs}^{\psi}(h_1)\mathrm{d}s\Big)\sigma_z^{\psi}(Q_1)\mathrm{Exp}_l\Big(\int_0^1;\ -iz\sigma_{zs}^{\psi}(h_1)\mathrm{d}s\Big).$$

By (5.6), (5.9) and $\int F(t)\mathrm{d}t=2$, we have $\|\sigma_{-is}^{\psi}(h_1)\|\leqq 2L$ for $0\leqq s\leqq\delta_1$ and hence $\|\sigma_{-i\delta_1}^{\psi_1}(Q_1)\|\leqq\|\sigma_{-i\delta_1}^{\psi}(Q_1)\|\ \exp 4\delta_1 L\leqq(L^2+(1+L)L')e^{L/2}$.

Finally, $Q_1\Psi_1=Q\Psi$ follows from formula (2.15) of [3].

Q.E.D.

Lemma 5.3. *Let $Q\in\mathfrak{M}$ be such that $Q\Psi\in V_{\Psi}$, $\sigma_z^{\psi}Q\in\mathfrak{M}$ for* $\mathrm{Im}\ z\in[-\delta, 0]$, $\delta\in(0, 1/8)$ *and* $\|\sigma_{-i\delta}^{\psi}Q-1\|\leqq L_0$,

(5.13) $$L_0\leqq(4\pi\log\delta)^{-2}$$

Then there exists $h\in\mathfrak{M}$, $h^*=h$ such that $Q\Psi=\Psi(h)$.

Proof. We fix $\delta_n=2^{-n}\delta$, $n=0, 1,\ldots$. By Lemma 5.2, we obtain a sequence of vectors Ψ_n, operators $Q_n\in\mathfrak{M}$, operators $h_n\in\mathfrak{M}$ and positive numbers L_n, $n=1, 2,\ldots$ such that $\Psi_n=\Psi_{n-1}(h_n)$, $\Psi_0=\Psi$, $h_n^*=h_n$, $\|h_n\|\leqq 2L_{n-1}$, $\omega_{\Psi_n}=\psi_n$, $\sigma_t^{\psi_n}Q_n$ has an analytic continuation $\sigma_z^{\psi_n}Q_n$ for $\mathrm{Im}\ z\in[-\delta_n, 0]$, $\|\sigma_{-i\delta_n}^{\psi_n}(Q_n)-1\|\leqq L_n$, $Q_n\Psi_n=Q\Psi(\in V_{\Psi})$, $L_n=(L_{n-1}^2+(1+L_{n-1})L_{n-1}')e^{L_{n-1}/2}$ and

(5.14) $$L'_{n-1}=(1/2)\{\pi L_{n-1}\log 2(\delta_{n-1}-\delta_n)\}^2\exp\{-\pi L_{n-1}\log 2(\delta_{n-1}-\delta_n)\}.$$

We prove that $L_n \leqq 2^{-n}L_0$ inductively. Assume that $L_{n-1}\leqq 2^{-(n-1)}L_0$. We have

$$\{\pi\log 2(\delta_{n-1}-\delta_n)\}^2 L_{n-1}\leqq \pi^2(\log\delta-(n-1)\log 2)^2 2^{-(n-1)}L_0$$

$$\leqq 2\pi^2(\log\delta)^2 L_0+2\pi^2(\log 2)^2(n-1)^2 2^{-(n-1)}L_0.$$

Since $n^2 2^{-n}\leqq 9/8$, we have $(\log 2)^2(n-1)^2 2^{-(n-1)}\leqq(\log\delta)^2$ and

$$\{\pi\log 2(\delta_{n-1}-\delta_n)\}^2 L_{n-1}\leqq 4\pi^2(\log\delta)^2 L_0\leqq 1/4$$

Hence we also have

$$\exp\{-\pi\log 2(\delta_{n-1}-\delta_n)L_{n-1}\}\leqq\exp(L_0^{1/2}/2)$$

Since $L_{n-1}\leqq L_0\leqq 2^{-4}$, we have

$$L_n/L_{n-1}=L_{n-1}+(1+L_{n-1})L'_{n-1}/L_{n-1}$$

$$\leqq(2^{-4}+(1+2^{-4})(1/8)\exp(1/8))e^{1/32}<1/2.$$

This proves $L_n\leqq 2^{-n}L_0$.

Since $\|h_n\|\leqq 2L_{n-1}$, $h=\sum h_n$ is norm convergent and hence $\lim \Psi_n=\Psi(h)$ by Lemmas 4.5 and 4.1. Since $Q_n\Psi_n\in V_\Psi=V_{\Psi_n}$, where $\Psi_n=\Psi(h_1+\cdots+h_n)$ is cyclic and separating by Corollary 4.4, $\sigma_t^{\psi_n}(Q_n)$ has an analytic continuation $\sigma_z^{\psi_n}(Q_n)$ for $\mathrm{Im}\, z\in(0, 1/2)$ and $(\sigma_z^{\psi_n}(Q_n))^*=\sigma_{\bar z+i/2}^{\psi_n}(Q_n)$. Hence we have the analytic continuation $\sigma_z^{\psi_n}(Q_n)$ for $\mathrm{Im}\, z\in[-\delta_n, 1/2+\delta_n]$ and by proof of Lemma 6 of [2],

$$\|Q_n-1\|\leqq\|\sigma_{-i\delta_n}^{\psi_n}(Q_n)-1\|\leqq L_n.$$

Hence $\lim\|Q_n-1\|=0$ and we have

$$Q\Psi=\lim Q_n\Psi_n=\Psi(h).$$

Q.E.D.

Proposition 5.4. *Let $Q\in\mathfrak{M}$ be such that $J_\Psi Q\Psi=Q\Psi$ and $\sigma_t^\psi(Q)$ has an "analytic continuation" $\sigma_z^\psi Q$ for* $\mathrm{Im}\, z\in[-1/2, 0]$. *Then there exists $h\in\mathfrak{M}$ such that $h^*=h$ and $e^Q\Psi=\Psi(h)$.*

Proof. By assumption, $Q\Psi = J_\Psi Q\Psi$ which implies $\Delta_\Psi^{-1/2}Q\Psi = Q^*\Psi$. Hence $\sigma_t^\psi(Q)$ has an "analytic continuation" $\sigma_z^\psi(Q)$ for Im $z\in[0, 1/2]$ by Lemma 6 of [2]. It satisfies $\sigma_{i/2}^\psi(Q)=Q^*$, which implies $\sigma_z^\psi(Q)^*= \sigma_{z+i/2}^\psi(Q)$ for Im $z=0$ and hence for Im $z\in[-1/2, 0]$ by an "analytic continuation". In particular

$$\sigma_{i/4}^\psi(Q)^*=\sigma_{i/4}^\psi(Q).$$

Let $\Phi_t=e^{tQ}\Psi$, $\varphi_t=\omega_{\Phi_t}$. By Theorem 3 (7) of [2], $\sigma_{i/4}^\psi(e^{tQ})=\exp t\sigma_{i/4}^\psi(Q)\geqq 0$ implies $\Phi_t\in V_\Psi$ for real t.

By Lemma 7 of [2], we have

$$\Delta_{\Phi_t}^{1/2}Q\Phi_t=e^{tQ}\sigma_{-i/2}^\psi(Q)e^{-tQ}\Phi_t.$$

By Lemma 6 of [2], $\sigma_z^{\varphi_t}(Q)$ has an "analytic continuation" for Im $z\in [-1/2, 0]$ and

$$||\sigma_z^{\varphi_t}(Q)||\leqq a=\max\{||Q||, e^{2t||Q||}||\sigma_{-i/2}^\psi(Q)||\}.$$

We now choose N such that

$$\exp(a/N)-1\leqq(4\log\delta)^{-2}$$

for a fixed $\delta\in(0, 1/8)$. Then

$$\begin{aligned}||\sigma_{-i\delta}^{\varphi_t}(e^{Q/N}-1)|| &= ||\exp\{\sigma_{-i\delta}^{\varphi_t}(Q)/N\}-1|| \\ &\leqq \exp\{||\sigma_{-i\delta}^{\varphi_t}(Q)||/N\}-1\leqq(4\log\delta)^{-2}.\end{aligned}$$

We can now apply Lemma 5.3 and find $h_n\in\mathfrak{M}$ for each integer $n\in[0, N]$ such that $\Phi_{n/N}=\Phi_{(n-1)/N}(h_n)$, $h_n^*=h_n$. Then

$$e^Q\Psi=\Phi_{N/N}=\Psi(\sum_{n=1}^N h_n).$$

Q.E.D.

Remark. Vectors $e^Q\Psi$ satisfying the condition of Proposition 5.4 are dense in V_Ψ, which can be seen as follows.

The vectors $\Delta_\Psi^{1/4}x\Psi$, $x\in\mathfrak{M}$, $x\geqq 0$ are dense in V_Ψ by definition. Furthermore

$$||\Delta^{1/4}(x-x')\Psi||^2 \leqq 2||(x-x')\Psi||^2$$

for $x-x'=(x-x')^* \in \mathfrak{M}$ by equation (3.13) of [2]. Let $x=\int \lambda dE_\lambda$, $x_L= x(E_L-E_{1/L})+(1/L)E_{1/L}+(1-E_L)$, $y_{L,\beta}=(\log x_L)(f_\beta^G)$. Then

$$\lim_{L\to\infty} \lim_{\beta\to+0} e^{y_{L,\beta}}\Psi = x\Psi$$

and hence $Q_{L,\beta}=\sigma_{-i/4}^{\Phi} y_{L,\beta}$ satisfy

$$\lim_{L\to\infty} \lim_{\beta\to+0} e^{Q_{L,\beta}}\Psi = \Delta_\Psi^{1/4} x\Psi.$$

$\sigma_z^\Phi Q_{L,\beta}$ is an entire function of z and $J_\Psi Q_{L,\beta}\Psi = Q_{L,\beta}\Psi$ due to $\sigma_{i/4}^\Phi(Q_{L,\beta}) = y_{L,\beta}=y_{L,\beta}^*$. Hence $Q_{L,\beta}$ satisfies the requirement for Q in Proposition 5.4.

§6. Existence Proof (II)

We use the technique introducted by Connes. (See [5].)

Lemma 6.1. *Let Ψ be a cyclic and separating vector for a von Neumann algebra $\mathfrak{M}$ on a Hilbert space $\mathfrak{H}$ and $h\in\mathfrak{M}$, $h^*=h\neq 0$. Let $\mathfrak{N}$ be a type I_2 factor on 4 dimensional space $\mathfrak{K}$, $\{u_{ij}\}$ and $\{u'_{ij}\}$ be matrix units of $\mathfrak{N}$ and $\mathfrak{N}'$, and $\{e_{ij}\}$ be an orthonormal basis of $\mathfrak{K}$ such that $u_{ij}e_{kl}=\delta_{jk}e_{il}$ and $u'_{ij}e_{kl}=\delta_{jl}e_{ki}$. Let $\hat{\mathfrak{M}}=\mathfrak{M}\otimes\mathfrak{N}$, $0<\lambda<1$ and*

(6.1) $$\chi_{h,\lambda}=\lambda^{1/2}\Psi\otimes e_{11}+(1-\lambda)^{1/2}\Psi(h)\otimes e_{22}.$$

Then $\chi_{h,\lambda}$ is a cyclic and separating vector of $\hat{\mathfrak{M}}$, the modular conjugation operator for $\chi_{h,\lambda}$ is $J_\Psi\otimes J_e$, $e=e_{11}+e_{22}$, and the modular operator Δ for $\chi_{h,\lambda}$ is given by

(6.2) $$\Delta(\sum_{ij}\Phi_{ij}\otimes e_{ij})=\sum_{ij}(\Delta_{ij}\Phi_{ij})\otimes e_{ij},$$

(6.3) $$\Delta_{11}=\Delta_\Psi,\ \Delta_{22}=\Delta_{\Psi(h)},$$

(6.4) $$\Delta_{12}=\{\lambda/(1-\lambda)\}\exp(-H_\Psi-j(h)),$$

(6.5) $$\Delta_{21}=\{(1-\lambda)/\lambda\}\exp(-H_\Psi+h).$$

Proof. Any $Q\in\hat{\mathfrak{M}}$ can be decomposed as $Q=\sum Q_{ij}u_{ij}$, $Q_{ij}=\sum_k u_{ki}Qu_{jk}\in\mathfrak{M}\otimes 1$. For $J=J_\Psi\otimes J_e$, we have

$$(\chi_{h,\lambda}, Qj(Q)\chi_{h,\lambda})$$
$$=\lambda(\Psi, Q_{11}j_\Psi(Q_{11})\Psi)+\lambda^{1/2}(1-\lambda)^{1/2}(\Psi, Q_{12}j_\Psi(Q_{12})\Psi(h))$$
$$+\lambda^{1/2}(1-\lambda)^{1/2}(\Psi(h), Q_{21}j_\Psi(Q_{21})\Psi)+(1-\lambda)(\Psi(h), Q_{22}j_\Psi(Q_{22})\Psi(h)).$$

Due to $\Psi\in V_\Psi$ and $\Psi(h)\in V_\Psi$, the right hand side is positive. Since $J=J_\Psi\otimes J_e$ obviously satisfies $J\hat{\mathfrak{M}}J=\hat{\mathfrak{M}}'$ $(=\mathfrak{M}'\otimes\mathfrak{N}')$, $J\chi_{h,\lambda}=\chi_{h,\lambda}$ and is an antiunitary involution, it is the modular conjugation operator for $\chi_{h,\lambda}$ by Theorem 1 of [2].

Since $\omega_{\chi_{h,\lambda}}((1\otimes u_{jj})Q)=\omega_{\chi_{h,\lambda}}(Q(1\otimes u_{jj}))$, $1\otimes u_{jj}$ commutes with Δ. Hence $1\otimes j_e(u_{kk})$ also commutes with Δ. Since $u_{jj}j_e(u_{kk})e_{il}=\delta_{ij}\delta_{kl}e_{jk}$, we have

$$\Delta^{1/2}\sum_{ij}\Phi_{ij}\otimes e_{ij}=\sum_{ij}\Delta_{ij}^{1/2}\Phi_{ij}\otimes e_{ij}.$$

For $Q=\Sigma Q_{ij}u_{ij}$ and $\Phi=Q\chi_{h,\lambda}=\Sigma\Phi_{ij}\otimes e_{ij}$, we have $\Delta^{1/2}Q\chi_{h,\lambda}=JQ^*\chi_{h,\lambda}$. Hence

$$\Delta_{11}^{1/2}Q_{11}\Psi=J_\Psi Q_{11}^*\Psi=\Delta_\Psi^{1/2}Q_{11}\Psi, \tag{6.6}$$

$$\Delta_{22}^{1/2}Q_{22}\Psi(h)=J_\Psi Q_{22}^*\Psi(h)=\Delta_{\Psi(h)}^{1/2}Q_{22}\Psi(h), \tag{6.7}$$

$$\Delta_{12}^{1/2}Q_{12}\Psi(h)=J_\Psi Q_{12}^*(1-\lambda)^{-1/2}\lambda^{1/2}\Psi, \tag{6.8}$$

$$\Delta_{21}^{1/2}Q_{21}\Psi=J_\Psi Q_{21}^*\lambda^{-1/2}(1-\lambda)^{1/2}\Psi(h). \tag{6.9}$$

Since $\hat{\mathfrak{M}}\chi_{h,\lambda}$ is a core of $\Delta^{1/2}$, $Q_{ij}\Psi_j(\Psi_1=\Psi,\ \Psi_2=\Psi(h))$ must be a core of $\Delta_{ij}^{1/2}$. (6.6) implies that two selfadjoint operators $\Delta_{11}^{1/2}$ and $\Delta_\Psi^{1/2}$ coincide on their core and hence must be equal. Similarly (6.7) implies $\Delta_{22}^{1/2}=\Delta_{\Psi(h)}^{1/2}$. Hence we have (6.3).

To prove (6.4) and (6.5), we first consider the case where h is replaced by $h_\beta=h(f_\beta^G)$. Then $\Psi(h_\beta)=A\Psi=j(A)\Psi$, $\Psi=A^{-1}\Psi(h_\beta)=j(A^{-1})\Psi(h_\beta)$ where $A\in\mathfrak{M}$, $A^{-1}\in\mathfrak{M}$ and $A=\mathrm{Exp}_r\left(\int_0^{1/2};\ \sigma_{-is}^\Psi(h_\beta)ds\right)$. We have

$$J_\Psi Q_{12}^* \Psi = \Delta_\Psi^{1/2} Q_{12} \Psi = \Delta_\Psi^{1/2} Q_{12} j_\Psi(A^{-1}) \Psi(h_\beta)$$
$$= \Delta_\Psi^{1/2} j_\Psi(A^{-1}) Q_{12} \Psi(h_\beta) = j_\Psi(\Delta_\Psi^{-1/2} A^{-1}) Q_{12} \Psi(h_\beta).$$

The inverse $A^{-1} = \mathrm{Exp}_l\left(\int_0^{1/2};\ -\sigma_{-is}^{\phi}(h_\beta)ds\right)$ is an analytic continuation of

$$(6.10) \qquad u_t^{\phi\varphi} = (u_t^{\varphi\phi})^* = \mathrm{Exp}_l\left(\int_0^t;\ -i\sigma_s^{\phi}(h_\beta)ds\right)$$
$$= \mathrm{Exp}_l\left(\int_0^{1/2};\ -i(2t)\sigma_{2ts}^{\phi}(h_\beta)ds\right)$$

to $t = -i/2$, where $\varphi = \omega_{\Psi(h_\beta)}$. By an analytic continuation of (4.15), we obtain

$$\Delta_\Psi^{-1/2} A^{-1} = \exp((H_\Psi - h_\beta)/2)$$

which holds on $A\mathrm{D}(\Delta_\Psi^{-1/2})$. (See second paragraph of proof of Proposition 4.12.) Hence we have

$$\Delta_{12}^{1/2} Q_{12} \Psi(h_\beta) = (\lambda/(1-\lambda))^{1/2} j_\Psi\{\exp((H_\Psi - h_\beta)/2)\} Q_{12} \Psi(h_\beta)$$
$$= (\lambda/(1-\lambda))^{1/2} \exp(-(H_\Psi + j(h_\beta))/2) Q_{12} \Psi(h_\beta).$$

Since $\mathfrak{M}\Psi(h)$ is a core of $\Delta_{12}^{1/2}$ and both $\Delta_{12}^{1/2}$ and $\exp(-(H_\Psi + j(h_\beta))/2)$ are selfadjoint, we have (6.4) for h_β.

Hence

$$\Delta_{12}^{it} = (\lambda/(1-\lambda))^{it} \exp(-it(H_\Psi + j(h_\beta)))$$
$$= (\lambda/(1-\lambda))^{it} j\{\exp(-it(H_\Psi - h_\beta))\}$$
$$= (\lambda/(1-\lambda))^{it} j\{u_t^{\varphi\phi} \Delta_\Psi^{it}\}.$$

As $\beta \to +0$, Δ^{it} for $\chi_{h_\beta,\lambda}$ tends strongly to Δ^{it} for $\chi_{h,\lambda}$ by Theorem 10 of [2], and $u_t^{\varphi\phi}$ for $\varphi = \omega_{\Psi(h_\beta)}$ tends strongly to $u_t^{\varphi\phi}$ for $\varphi = \omega_{\Psi(h)}$. Hence we have (6.4) for a general h.

From (6.9), we have

$$\Delta_{21}^{1/2} Q_{21} \Psi = ((1-\lambda)/\lambda)^{1/2} \Delta_\Psi^{1/2} A^* Q_{21} \Psi.$$

Since $A^* = \mathrm{Exp}_l\left(\int_0^{1/2};\ \sigma_{is}^{\phi}(h_\beta)ds\right)$ is an analytic continuation of (6.10) to

$t=i/2$, we obtain as before

$$\Delta_{\Psi}^{1/2}A^{*}=\exp(-(H_{\Psi}-h_{\beta})/2).$$

By taking the limit $\beta\rightarrow+0$, we obtain (6.5). Q.E.D.

Remark. If u' is a unitary operator in $\mathfrak{M}'$ and

$$\chi=\lambda^{1/2}\Psi\otimes e_{11}+(1-\lambda)^{1/2}u'\Psi(h)\otimes e_{22},$$

then

$$\hat{u}'=1\otimes j(u_{11})+u'\otimes j(u_{22})$$

is a unitary element of $\hat{\mathfrak{M}}'$ and $\chi=\hat{u}'\chi_{h,\lambda}$. Hence modular conjugation operator and modular operator for such χ are given by $\hat{u}'(J_{\Psi}\otimes J_{e})(\hat{u}')^{*}$ and $\hat{u}'\Delta(\hat{u}')^{*}$.

Lemma 6.2. *If $h_{n}\in\mathfrak{M}$, $h_{n}^{*}=h_{n}$, $\lim\Psi(h_{n})=\Phi$ (strongly), Φ is cyclic and separating and $l_{1}^{1/2}\Psi\geqq\Psi(h_{n})\geqq l_{2}^{1/2}\Psi$ for strictly positive l_{1} and l_{2} independent of n, then $h=$w-$\lim h_{n}$ exists and $\Phi=\Psi(h)$.*

Proof. Let $\chi_{n}\equiv\chi_{h_{n},\lambda}$ be defind as (6.1) and

$$\chi=\lambda^{1/2}\Psi\otimes e_{11}+(1-\lambda)^{1/2}\Phi\otimes e_{22}.$$

Then $\Phi=\lim\Psi(h_{n})$ implies $\chi=\lim\chi_{n}$. χ is cyclic and separating and by Theorem 10 of [2]

$$\lim_{n}\Delta_{\chi_{n}}^{it}=\Delta_{\chi}^{it},\ \lim_{n}\Delta_{\Psi(h_{n})}^{it}=\Delta_{\Phi}^{it} \tag{6.11}$$

where the convergence is in the strong operator topology and is uniform over a compact set of t.

Since $(\Delta_{\chi_{n}})_{21}^{it}\otimes u_{22}j(u_{11})=\Delta_{\chi_{n}}^{it}(1\otimes u_{22}j(u_{11}))$, and $(\Delta_{\chi})_{21}^{it}\otimes u_{22}j(u_{11})=\Delta_{\chi}^{it}(1\otimes u_{22}j(u_{11}))$, we have

$$\lim_{n}\exp it(-H_{\Psi}+h_{n})=\exp it(\log(\Delta_{\chi})_{21}) \tag{6.12}$$

by (6.5) where the convergence is uniform in t over a compact set.

By multiplying e^{-t} and integrating over $t\in[0,\infty)$, we obtain

(6.12′) $$\lim_n \{(-H_\Psi+h_n)+i\}^{-1}=\{\log(\Delta_\chi)_{21}+i\}^{-1}.$$

Hence, by subtracting $(-H_\Psi+i)^{-1}$, we have

$$\lim(-H_\Psi+i)^{-1}h_n\{(-H_\Psi+h_n)+i\}^{-1}$$
$$=(-H_\Psi+i)^{-1}-\{\log(\Delta_\chi)_{21}+i\}^{-1}.$$

By Corollary 4.11, $||h_n||$ is uniformly bounded. Hence we obtain

$$||(-H_\Psi+i)^{-1}h_n[\{(-H_\Psi+h_n)+i\}^{-1}-\{\log(\Delta_\chi)_{21}+i\}^{-1}]\xi||\to 0.$$

Hence $(-H_\Psi+i)^{-1}h_n\eta$ is strongly convergent for any η in the range of $\{\log\Delta_{21}+i\}^{-1}$, which is a dense set. Since $||(-H_\Psi+i)^{-1}h_n||$ is uniformly bounded, we have the existence of

$$\lim_n(-H_\Psi+i)^{-1}h_n\equiv h_0.$$

For $\xi_2\in \mathrm{D}((-H_\Psi+i)^*)$, we have

$$|(h_0\xi_1,\ (-H_\Psi+i)^*\xi_2)|=\lim_n|(h_n\xi_1,\xi_2)|\leqq\sup||h_n||\,||\xi_1||\,||\xi_2||.$$

Hence $h_0\xi_1\in \mathrm{D}((-H_\Psi+i))$ and $||(-H_\Psi+i)h_0||\leqq\sup||h_n||$. We have

$$\text{w-lim}\ h_n=h\equiv(-H_\Psi+i)h_0.$$

By (4.14), we have

(6.13) $$u_t^{\varphi_n\psi}=\{\exp it(-H_\Psi+h_n)\}\Delta_\Psi^{-it},$$

which is strongly convergent, uniformly in t over a compact set, due to (6.12), where $\varphi_n\equiv\omega_{\Psi(h_n)}$.

By Proposition 4.13, there exists $\mathrm{A}_n(z)\in\mathfrak{M}$ for $\mathrm{Re}\,z\in[0, 1/2]$ such that $\mathrm{A}_n(z)$ is holomorphic for $\mathrm{Re}\,z\in(0, 1/2)$, strongly continuous for $\mathrm{Re}\,z\in[0,1/2]$, $||\mathrm{A}_n(z)||\leqq\max(l_1^{1/2}, 1)$, $||\mathrm{A}_n(z)^{-1}||\leqq\max(l_2^{-1/2}, 1)$, $\mathrm{A}_n(s+it)=u_t^{\varphi_n\psi}\sigma_t^\psi\{\mathrm{A}_n(s)\}$, $\mathrm{A}_n(0)=1$ and $\mathrm{A}_n(1/2)\Psi=\Psi(h_n)$.

Since $\mathrm{A}_n(it)Q'\Psi=Q'$ $\{\exp it(-H_\Psi+h_n)\}\Psi$ and $A_n((1/2)+it)Q'\Psi=Q'u_t^{\varphi_n\psi}\sigma_t^\psi\{A_n(1/2)\}\Psi=Q'\{\exp it\,(-H_\Psi+h_n)\}\Psi(h_n)$ for $Q'\in\mathfrak{M}'$ are strongly convergent, uniformly over real t in a compact set, and since $||A_n((1/2)+it)||$ and $||A_n(it)||$ are uniformly bounded, $A_n((1/2)+it)$ and $A_n(it)$ are strongly convergent, uniformly over real t in a compact set.

Since $\mathrm{A}_n(z)$ is holomorphic for $\mathrm{Re}\, z\in(0, 1/2)$ and is continuous and uniformly bounded for $\mathrm{Re}\, z\in[0, 1/2]$, we have

$$e^{z^2}\mathrm{A}_n(z)=(2\pi)^{-1}\int_{-\infty}^{\infty}(z-it)^{-1}e^{-t^2}\mathrm{A}_n(it)\mathrm{d}t \tag{6.14}$$
$$-(2\pi)^{-1}\int_{-\infty}^{\infty}(z-it-1/2)^{-1}e^{-(t-i/2)^2}\mathrm{A}_n(it+1/2)\mathrm{d}t$$

for $\mathrm{Re}\, z\in(0, 1/2)$. Hence

$$(\mathrm{d}/\mathrm{d}z)e^{z^2}\mathrm{A}_n(z) \tag{6.15}$$
$$=(2\pi)^{-1}\int_{-\infty}^{\infty}(z-it-1/2)^{-2}e^{-(t-i/2)^2}\mathrm{A}_n(it+1/2)\mathrm{d}t$$
$$-(2\pi)^{-1}\int_{-\infty}^{\infty}(z-it)^{-2}e^{-t^2}\mathrm{A}_n(it)\mathrm{d}t.$$

Since $\|\mathrm{A}_n(it)\|$ and $\|\mathrm{A}_n(it+1/2)\|$ are uniformly bounded in t and n, the integral converges uniformly in n. Hence (6.15) is strongly convergent as $n\to\infty$, uniformly in z over any compact subset of $\{z;\ \mathrm{Re}\, z\in(0, 1/2)\}$.

By $l_1^{1/2}\Psi\geqq\Psi(h_n)\geqq l_2^{1/2}\Psi$ and $\Phi=\lim\Psi(h_n)$, we have $l_1^{1/2}\Psi\geqq\Phi\geqq l_2^{1/2}\Psi$. By Theorem 3(8) of [2], there exists $A\in\mathfrak{M}$ such that $A\Psi=\Phi$, $\|A\|\leqq l_1^{1/2}$ and $\|A^{-1}\|\leqq l_2^{-1/2}$. Then $A=\lim_n \mathrm{A}_n(1/2)$.

For $Q'\in\mathfrak{M}'$ we have

$$\|(\sigma_t^{\varphi_n}\{\mathrm{A}_n(1/2)^{-1}\}-\sigma_t^{\varphi}\{A^{-1}\})Q'\Phi\|$$
$$=\|Q'(\sigma_t^{\varphi_n}\{\mathrm{A}_n(1/2)^{-1}\}\Psi(h_n)-\sigma_t^{\varphi}\{A^{-1}\}\Phi)$$
$$-Q'\sigma_t^{\varphi_n}\{\mathrm{A}_n(1/2)^{-1}\}(\Psi(h_n)-\Phi)\|$$
$$\leqq\|Q'\|\|(\Delta_{\Psi(h_n)}^{it}-\Delta_{\Phi}^{it})\Psi\|+\|Q'\|l_2^{-1/2}\|\Psi(h_n)-\Phi\|$$

which converges to 0 as $n\to\infty$ uniformly in t over a compact set due to $\lim\Psi(h_n)=\Phi$ and Theorem 10 of [2], where $\varphi=\omega_\Phi$. Hence

$$\lim_n \sigma_t^{\varphi_n}\{\mathrm{A}_n(1/2)^{-1}\}=\sigma_t^{\varphi}\{A^{-1}\}$$

uniformly in t over a compact set.

By (6.11),

$$u_t^{\psi\varphi_n} = \Delta_\Psi^{it}(\Delta_{\chi_n})_{21}^{-it}\{\lambda/(1-\lambda)\}^{-it}$$

is strongly convergent, uniformly over a compact set of t.

Since $\|A_n(z)^{-1}\|$ is uniformly bounded, the analyticity and the continuity of $A_n(z)$ imply the same properties for $A_n(z)^{-1}$. Hence

$$\text{(6.16)} \qquad A_n(s+it)^{-1} = \sigma_t^\psi(A_n(s)^{-1})u_t^{\psi\varphi_n} = u_t^{\psi\varphi_n}\sigma_t^{\varphi_n}(A_n(s)^{-1})$$

is strongly convergent as $n\to\infty$, for $s\in[0, 1/2]$, which is proved in exactly the same way as before by use of the Cauchy integral formula of the form (6.14).

Combining the convergences of (6.15) and (6.16), we have the strong convergence of

$$A_n(z)^{-1}(d/dz)\{e^{z^2}A_n(z)\} \equiv F_n(z),$$

as $n\to\infty$ uniformly in z over any compact subset of $\{z;\ \mathrm{Re}\, z\in(0, 1/2)\}$.

By Theorem 1 of [3] and (4.6), we have

$$(d/dt)u_t^{\varphi_n\psi} = iu_t^{\varphi_n\psi}\sigma_t^\psi(h_n).$$

Hence

$$\text{(6.17)} \qquad e^{-z^2}F_n(z) - 2z = \sigma_{-iz}^\psi(h_n) \in \mathfrak{M}$$

for $z = it$. Hence

$$(e^{-z^2}F_n(z)-2z)Q'\Psi = Q'\Delta_\Psi^z h_n\Psi,\ Q'\in\mathfrak{M}'$$

which holds for all z satisfying $\mathrm{Re}\, z\in[0, 1/2]$. Since w-lim $h_n = h$ and $F_n(z)$ has a strong limit, we have

$$\lim_n (e^{-z^2}F_n(z)-2z)Q'\Psi = Q'\Delta_\Psi^z h\Psi$$

for $Q'\in\mathfrak{A}_{\Psi 2}$ (see §3 of [2] for the definition of $\mathfrak{A}_{\Psi 2}$) and $\mathrm{Re}\, z\in(0, 1/2)$. Since $\lim F_n(z)\in\mathfrak{M}$, this implies the existence of an analytic continuation $\sigma_z^\psi(h)$ of $\sigma_t^\psi(h)$ to z in $\{z;\ \mathrm{Re}\, z\in(0, 1/2)\}$.

By (6.15) and the uniform boundedness of $A_n(z)$, we obtain the uniform boundedness of $\sigma_z^\psi(h_n)$ over any compact subset of $\{z;\ \mathrm{Re}\, z\in(0,$

$1/2)\}$ and over all n. Hence $\lim_n \sigma_z^\psi(h_n)=\sigma_z^\psi(h)$ and

$$\Delta_\Psi^{t_m} h_n \Delta_\Psi^{t_{m-1}} h_n \dots \Delta_\Psi^{t_1} h_n \Psi$$

$$=\sigma_{-it_m}^\psi(h_n)\dots\sigma_{-i(t_m+\dots+t_1)}^\psi(h_n)\Psi$$

is strongly convergent as $n\to\infty$ for $t_j>0$, $t_1+\dots+t_m<1/2$.

Since $\|h_n\|$ is uniformly bounded by $\max(|\log l_1|, |\log l_2|)$, we have

$$\lim_n \Psi(h_n)=\Psi(h)$$

by Lebesgue dominated convergence Theorem (for inner product with other vectors and for its norm). Hence $\Phi=\Psi(h)$. Q.E.D.

Theorem 6.3. *If* $l_1\geqq l_2>0$ *and* $l_1^{1/2}\Psi\geqq\Phi\geqq l_2^{1/2}\Psi$ *for cyclic and separating* Ψ *and* Φ, *then there exists* $h\in\mathfrak{M}$ *such that* $h^*=h$, $\Phi=\Psi(h)$ *and* $\log l_1\geqq h\geqq\log l_2$.

Remark. If φ and ψ are normal faithful states satisfying $l_1\psi\geqq\varphi\geqq l_2\psi$, then the unique representative ξ_φ and ξ_ψ in a fixed canonical cone V_Ψ satisfies $l_1^{1/2}\xi_\psi\geqq\xi_\varphi\geqq l_2^{1/2}\xi_\psi$ by Theorem 3(8) and (9) of [2] and hence there exists $h\in\mathfrak{M}$ such that $h^*=h$, $\Phi=\Psi(h)$ and $\log l_1\geqq h\geqq\log l_2$ due to this theorem.

Proof. By Theorem 3(8) of [2] and the assumption $l_1^{1/2}\Psi\geqq\Phi\geqq l_2^{1/2}\Psi$, there exists $Q\in\mathfrak{M}$ such that $\Phi=Q\Psi$. By Theorem 3(7) of [2] and $l_1^{1/2}\Psi\geqq Q\Psi\geqq l_2^{1/2}\Psi$, $(\Delta_\Psi^{iz}Q\Delta_\Psi^{-iz})^-\in\mathfrak{M}$ for $\mathrm{Im}\, z\in[0, 1/2]$ and

$$l_1^{1/2}\geqq(\Delta_\Psi^{-1/4}Q\Delta_\Psi^{1/4})^-\geqq l_2^{1/2}$$

where the bar indicates the closure. Let

$$k\equiv\log(\Delta_\Psi^{-1/4}Q\Delta_\Psi^{1/4})^-.$$

Then $k\in\mathfrak{M}$ and $\log l_1\geqq 2k\geqq\log l_2$. We have

$$\Phi=Q\Psi=\Delta_\Psi^{1/4}(\Delta_\Psi^{-1/4}Q\Delta_\Psi^{1/4})\Psi=\Delta_\Psi^{1/4}e^k\Psi.$$

Let $k_\beta\equiv\int\sigma_t^\psi(k)f_\beta^G(t)dt$ where f_β^G is given by (3.11) of [2]. Then

$k_\beta^* = k_\beta$, $\|k_\beta\| \leqq \|k\|$ and $\lim_{\beta\to 0} k_\beta = k$. Hence

$$\|\Delta_\Psi^{1/4}(e^{k_\beta} - e^k)\Psi\|^2 \leqq 2\|(e^{k_\beta} - e^k)\Psi\|^2 \to 0$$

by (3.13) of [2] and $\Delta_\Psi^{1/2}(e^{k_\beta} - e^k)\Psi = J_\Psi(e^{k_\beta} - e^k)\Psi$. Let

$$\Phi_\beta = \Delta_\Psi^{1/4} e^{k_\beta}\Psi.$$

We have $\Phi_\beta \in V_\Psi$ and $\lim \Phi_\beta = \Phi$.

Since $\int f_\beta^G(t)\mathrm{d}t = 1$ and $f_\beta^G(t) \geqq 0$, we have

$$\log l_1 \geqq 2k_\beta \geqq \log l_2,\ \ l_1^{1/2} \geqq e^{k_\beta} \geqq l_2^{1/2}.$$

By Theorem 3(7) of [2], we have

$$l_1^{1/2}\Psi \geqq \Phi_\beta \geqq l_2^{1/2}\Psi.$$

Since

$$\sigma_t^\phi(\exp k_\beta) = \exp \sigma_t^\phi(k_\beta)$$

and $\sigma_t^\phi(k_\beta)$ has an analytic continuation to an entire function

$$\sigma_z^\phi(k_\beta) = \int \sigma_t^\phi(k) f_\beta^G(t-z)\mathrm{d}t \in \mathfrak{M},$$

$(\Delta_\Psi^{it} e^{k_\beta} \Delta_\Psi^{-it})$ has an analytic continuation to an entire function $\exp \sigma_z^\phi(k_\beta)$. Hence

$$\Phi_\beta = (\Delta_\Psi^{1/4} e^{k_\beta} \Delta_\Psi^{-1/4})\Psi = \exp(\sigma_{-i/4}^\phi(k_\beta))\Psi.$$

Since $Q_1 \equiv \sigma_{-i/4}^\phi(k_\beta)$ satisfies $J_\Psi Q_1 \Psi (= \Delta_\Psi^{1/2} Q_1^* \Psi) = Q_1 \Psi$ and has the property that $\sigma_t^\phi(Q_1) = \sigma_{t-i/4}^\phi(k_\beta)$ has an analytic continuation to an entire function $\sigma_z^\phi(Q_1) = \sigma_{z-i/4}^\phi(k_\beta)$, Proposition 5.4 is applicable and there exists $h_\beta \in \mathfrak{M}$ such that $\Phi_\beta = \Psi(h_\beta)$, $h_\beta^* = h_\beta$. Lemma 6.2 then implies Theorem.

Q.E.D.

Remark. The above proof implies that if $\Phi = \Delta_\Psi^{1/4} e^k \Psi$, $k = k^* \in \mathfrak{M}$, and $\log l_2 \leqq 2k \leqq \log l_1$, then there exists $h \in \mathfrak{M}$, $h^* = h$ such that $\Phi = \Psi(h)$, and $\log l_2 \leqq h \leqq \log l_1$.

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Note added in proof: Theorem 3(8) of [2] has been misused in proofs of Propositions 4.13, 4.14, Lemma 6.2 and Theorem 6.3. Hence we need the assumption $l_1^{1/2}\omega_\Psi \geqq \omega_\Phi \geqq l_2^{1/2}\omega_\Psi$ in Propositions 4.13 and 4.14. However, Lemma 6.2 and Theorem 6.3 hold without modification.

For Lemma 6.2, we modify its proof after (6.13) as follows: Since $h-h_n$ is weakly convergent to 0 and $(-H_\Psi+i+h)^{-1}$ is strongly convergent,

$$(-H_\Psi+i+h_n)^{-1}-(-H_\Psi+i+h)^{-1}$$
$$=(-H_\Psi+i+h)^{-1}(h-h_n)(-H_\Psi+i+h_n)^{-1}$$

is weakly convergent to 0. By (6.12′), $(\Delta_X)_{21}=\exp(-H_\Psi+h)$, which implies that Φ and $\Psi(h)$ have the same modular automorphisms. Hence $\Phi=e^\alpha\Psi(h)$ for a selfadjoint α affiliated with the center. Then $(\Delta_X)_{21}$ is calculated to be $\exp(-H_\Psi+h+\alpha)$. Hence $\alpha=0$ and $\Phi=\Psi(h)$.

For Theorem 6.3, $l_1^{1/2}\Psi \geqq \Phi \geqq l_2^{1/2}\Psi$ directly implies $\Phi=\Delta_\Psi^{1/4}e^k\Psi$ for a $k\in\mathfrak{M}$ and $\log l_1 \geqq 2k \geqq \log l_2$.