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On the Isomorphism Problem for Endomorphisms of Lebesgue Spaces, I

By

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§1. Introduction

D. S. Ornstein and others solved the isomorphism problem for Bernoulli automorphisms of Lebesgue spaces and got some conditions for automorphisms to be Bernoulli. The main result is that two Bernoulli automorphisms with the same entropy are isomorphic. Another result of them tells us that mixing Markov automorphisms are Bernoulli. Their results can be rephrased into the terminology of the representation of stationary processes as follows. Let $\{\xi_n; n=0, \pm 1, \pm 2, \cdots\}$ be a strictly stationary process and S be the corresponding shift transformation. If S satisfies one of conditions given by D. S. Ornstein and others, then $\{\xi_n\}$ can be represented in the form

$$\xi_n = f(\cdots, \eta_{-1}, \eta_0, \eta_1, \eta_2, \cdots)$$

where η_0 is a measurable function of $\{\xi_n\}$ and $\eta_n = S^n \eta_0$, $n=0, \pm 1, \pm 2, \cdots$, are independent.

In this connection, there is an important problem called "innovation problem". We will state it in the formulation given by M. Rosenblatt. Let $\mathscr{B}_n$ denote the σ-field generated by ξ_k, $k \leqq n$, and $\mathscr{A}_n$ the σ-field generated by η_n. Can one find a random variable η_0 measurable with respect to $\mathscr{B}_0$, independent of $\mathscr{B}_{-1}$, $\mathscr{B}_0 = \mathscr{B}_{-1} \vee \mathscr{A}_0$ and such that ξ_n is measurable with respect to $\bigvee_{-\infty}^{n} \mathscr{A}_i$?

We now consider an isomorphism problem for automorphisms which is equivalent to the innovation problem mentioned above. Let $\tilde{T}$ be an

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automorphism of a Lebesgue space $(\tilde{X}, \tilde{\mathscr{F}}, \tilde{\mu})$, and $\tilde{P}$ its generator (i.e. $\bigvee_{-\infty}^{\infty} \tilde{T}^n \tilde{P} = \varepsilon$). In this situation, can one find a Bernoulli generator $\tilde{Q}$ of $\tilde{T}$ such that $\bigvee_{-\infty}^{0} \tilde{T}^n \tilde{Q} = \bigvee_{-\infty}^{0} \tilde{T}^n \tilde{P}$? Define the factor endomorphism T of $X = \tilde{X} / \bigvee_{-\infty}^{0} \tilde{T}^n \tilde{P}$ induced by $\tilde{T}$, then the above question is also equivalent to ask as follows. Is the endomorphism T isomorphic to a Bernoulli endomorphism? Concerning with this problem Ya. G. Sinai got an interesting result: Every endomorphism T with positive entropy $h(T) > 0$ has a Bernoulli partition with the same valued entropy as $h(T)$.

In this paper (Part I) we will be concerned with the above mentioned problem. Firstly we study the isomorphism between two Bernoulli endomorphisms. We will see that they are not isomorphic (even if they have the same entropy) except trivial cases. Next we give a condition for a class of Markov endomorphisms to be Bernoulli, which is a generalization of the condition given by M. Rosenblatt.

On the other hand, several authors discussed the Bernoulli property of some special number-theoretical transformations. Those transformations are not automorphisms but endomorphisms, so they proved actually that natural extensions of them have Bernoulli generators. We are interested in the Bernoulli property of those endomorphims themselves not of their natural extensions. We will prove in Part II that the continued-fraction transformation, β-expansion transformations and linear mod 1 transformations are not Bernoulli except the trivial cases.

We are also interested in the isomorphism problem for more general endomorphisms, for example the isomorphism between Markov endomorphisms. We will discuss this problem in Part III.

§2. Preliminaries

Throughout this paper $(X, \mathscr{F}, \mu)$ denotes a non-atomic Lebesgue probability space. A measure-preserving transformation T of X (i.e. $T^{-1}\mathscr{F} \subset \mathscr{F}$ and $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathscr{F}$) is called an *endomorphism* of $(X, \mathscr{F}, \mu)$. In addition if T is invertible (i.e. T is 1-1 and $T\mathscr{F} = \mathscr{F}$), T is called an automorphism. We denote partitions of X by P, Q, $R, \cdots$, which are always assumed to be measurable. Let us denote the cell of partition P containing the point x by $C_P(x)$. As usual ε stands for the partition into the individual points of X and ν stands for the trivial partition consisting of the unique cell X. For each measurable partition R there corresponds the canonical system of measures (the conditional measures) $\{\mu(\cdot|r);\ r\in R\}$. We refer the paper by V. A. Rohlin for these notions.

Let P_i be measurable partitions of X_i, $i=1, 2$, respectively. We denote $\mathrm{dist}(P_1)=\mathrm{dist}(P_2)$ if the distributions of P_1 and P_2 are of the same type, namely if they have no cell with positive measure or if they have the same number of cells with positive measures and there is a 1-1 correspondence between these cells such that the corresponding cells have the same measure. We use also the notation $\mathrm{dist}(\pi^{(1)})=\mathrm{dist}(\pi^{(2)})$ for probability vectors $\pi^{(i)}=(\pi_j^{(i)};\ 1\leqq j\leqq J)$ of which definition is analogous to the above one (so we omit to state it). For measurable partitions P and R, $\mathrm{dist}(P|r)$ stands for the distribution of P with respect to the canonical measure $\mu(\cdot|r)$ for $r\in R$.

Two endomorphisms T_i of $(X_i, \mathscr{F}_i, \mu_i)$, $i=1, 2$, respectively are called isomorphic if there exists an isomorphism (mod 0) ψ from X_1 onto X_2 such that $\psi T_1=T_2\psi$.

A measurable partition P is called a generator of an endomorphism T if $\bigvee_0^\infty T^{-n}P=\varepsilon$. A measurable partition P is a Bernoulli partition (a Markov partition respectively) for T if $\{T^{-n}P;\ n\geqq 0\}$ are independent (Markovian), that is $\mu(A|\bigvee_{n=1}^\infty T^{-n}P)=\mu(A)$ $(\mu(A|\bigvee_{n=1}^\infty T^{-n}P)=\mu(A|T^{-1}P))$ a.e. for all P-measurable A.

An endomorphism is called a *Bernoulli endomorphism* (a *Markov endomorphism* respectively) if it has a Bernoulli generator (a Markov generator).

§3. Isomorphism Theorem for Bernoulli Endomorphisms

In this section we will study the isomorphism between Bernoulli endomorphisms. We remark firstly the followings.

1° *Let P, Q, R be measurable partitions of X. If $P\vee R=Q\vee R$, then* $\mathrm{dist}\,(P|r)=\mathrm{dist}(Q|r)$ *for* a.e. $r\in R$.

2° *Let T be an endomorphism of $(X, \mathscr{F}, \mu)$ with Bernoulli generator Q. If P is a generator of T, then* $\mathrm{dist}(P|r)=\mathrm{dist}(Q)$ *for* a.e. $r\in\bigvee_1^\infty T^{-n}P$. *Especially if P is also a Bernoulli generator, then* $\mathrm{dist}(P)=\mathrm{dist}(Q)$.

Indeed, noticing $\bigvee_0^\infty T^{-n}Q=\bigvee_0^\infty T^{-n}P=\varepsilon$ and $\bigvee_1^\infty T^{-n}Q=\bigvee_1^\infty T^{-n}P=T^{-1}\varepsilon$, apply 1°.

Theorem 1. *Let T_i be Bernoulli endomorphisms of $(X_i, \mathscr{F}_i, \mu_i)$ with Bernoulli generators P_i, $i=1, 2$, respectively. Then they are isomorphic if and only if* $\mathrm{dist}(P_1)=\mathrm{dist}(P_2)$.

Proof. Assume T_1 and T_2 are isomorphic. Then the image P of P_2 by the isomorphism is also a Bernoulli generator of T_1. Hence by 2° we have $\mathrm{dist}(P_1)=\mathrm{dist}(P)=\mathrm{dist}(P_2)$. The converse is evident.

We will now discuss the uniqueness of Bernoulli generator. Let T be an endomorphism of $(X, \mathscr{F}, \mu)$. Put

$$\mu_T(x)=\mu(\{x\}\mid C_{T^{-1}\varepsilon}(x))$$

where $C_{T^{-1}\varepsilon}(x)$ denotes the cell of the partition $T^{-1}\varepsilon$ containing the point x. The function μ_T is obviously measurable and invariant under isomorphisms, and the reciprocal value $1/\mu_T(x)$ is called "Jacobian" by W. Parry. Let R_T be the measurable partition of X generated by μ_T (i.e. the partition into the inverse images of points). We call R_T the *proper partition* for T.

Assume that T has a Bernoulli generator P. Then for a.e. $p\in P$ we have

$$\mu_T(x)=\mu(p\mid C_{T^{-1}\varepsilon}(x))=\mu(p), \qquad \text{a.e. } x\in p$$

by 2°. Therefore it is easy to see $R_T\leqq P$. Thus R_T is a Bernoulli partition for T. Moreover if P is countable and the distribution of P has distinct probabilities, then $R_T=P$.[1)] Thus we have

Theorem 2. *If T has a countable Bernoulli generator with distinct probabilities, then R_T is the unique Bernoulli generator of T.*

§4. A Criterion for Markov Endomorphisms to be Bernoulli

In this section we consider endomorphisms with countable Markov generators and give a criterion for such endomorphisms to be Bernoulli.

1) A countable partition is a measurable partition which has only the cells with positive measures.

Let T be a Markov endomorphism with countable Markov generator $P=\{p_1, p_2, \cdots\}$. Let $\Pi=(\pi_{ij}; i, j=1, 2, \cdots)$ be the transition matrix defined by (T, P) where

$$\pi_{ij}=\mu(p_j \mid T^{-1}p_i), \qquad i, j=1, 2, \cdots.$$

Let $\pi^{(i)}=(\pi_{i1}, \pi_{i2}, \cdots)$ denote the i-th row vector of Π. From 2° of §3 we get

1° *If T has a Bernoulli generator, then*

(U) *there exists a probability vector $\rho=(\rho_1, \rho_2, \cdots)$ with positive ρ_i's such that* $\mathrm{dist}(\pi^{(i)})=\mathrm{dist}(\rho)$ *for all i.*

In this case ρ is the distribution of the Bernoulli generator.

We call a Markov generator P satisfying the above condition (U) a *uniform* Markov generator, and ρ the *common distribution* of P.

2° *If T has a uniform Markov generator P with the common distribution ρ consisting of distinct probabilities ρ_i's, then the proper partition R_T for T defined in §3 is a Bernoulli partition for T and has the distribution ρ.*

Indeed, first we notice that $R_T=\{r_1, r_2, \cdots\}$, $r_i=\{x; \mu_T(x)=\rho_i\}$, $i=1, 2, \cdots$. Given k and i there exists the unique j such that $\pi_{kj}=\rho_i$, so we define

$$\psi(k; i)=j, \text{ if } \pi_{kj}=\rho_i,$$

and

$$\psi(k; i_n, \cdots, i_1, i_0)=\psi(\psi(k; i_n, \cdots, i_1); i_0)$$

inductively. Since $\mu_T(x)=\mu(\{x\} \mid C_{T^{-1}\varepsilon}(x))=\mu(p_j \mid T^{-1}p_k)=\pi_{kj}$ for $x \in T^{-1}p_k \cap p_j$, we have

$$r_i=\bigcup_{\pi_{kj}=\rho_i}(T^{-1}p_k \cap p_j)=\bigcup_k(T^{-1}p_k \cap p_{\psi(k;i)})$$

and hence

$$\mu(r_i)=\sum_k \mu(T^{-1}p_k \cap p_{\psi(k;i)})=\sum_k \mu(T^{-1}p_k)\mu(p_{\psi(k;i)} \mid T^{-1}p_k)=\rho_i.$$

Using the same argument we get

$$T^{-n}r_{i_n}\cap T^{-n+1}r_{i_{n-1}}\cap\cdots T^{-1}r_{i_1}\cap r_{i_0}$$

$$=\bigcup_k(T^{-n-1}p_k\cap T^{-n}p_{\psi(k;i_n)}\cap\cdots\cap T^{-1}p_{\psi(k;i_n,\cdots,i_1)}\cap p_{\psi(k;i_n,\cdots,i_0)})$$

and hence

$$\mu(T^{-n}r_{i_n}\cap T^{-n+1}r_{i_{n-1}}\cap\cdots\cap r_{i_0}) \tag{1}$$

$$=\sum_k\mu(p_k)\rho_{i_n}\rho_{i_{n-1}}\cdots\rho_{i_0}=\rho_{i_n}\rho_{i_{n-1}}\cdots\rho_{i_0}$$

$$=\mu(r_{i_n})\mu(r_{i_{n-1}})\cdots\mu(r_{i_0}).$$

Thus 2° is proved.

Let T be a Markov endomorphism with countable Markov generator $P=\{p_1, p_2,\cdots\}$. Assume that T has a Bernoulli generator (which is necessarily countable by 1°) with distinct probabilities. Then by Theorem 2, the proper partition R_T is the unique Bernoulli generator of T.

Let us calculate the conditional probability $\mu(p_j\mid\bigvee_0^n T^{-i}R_T)$ for later uses. Let

$$C=T^{-n}r_{s_n}\cap T^{-n+1}r_{s_{n-1}}\cap\cdots\cap T^{-1}r_{s_1}\cap r_{s_0}$$

be any cell of $\bigvee_0^n T^{-i}R_T$. We have

$$p_j\cap C=\bigcup_{k\in D(j;s_n,\dots,s_0)}(T^{-n-1}p_k\cap T^{-n}p_{\psi(k;s_n)}\cap\cdots\cap T^{-1}p_{\psi(k;s_n,\dots,s_1)}\cap p_j) \tag{2}$$

where $D(j;s_n,\cdots,s_0)=\{k;\psi(k;s_n,\cdots,s_0)=j\}$, and so

$$\mu(p_j\cap C)=\sum_{k\in D(j;s_n,\dots,s_0)}\mu(p_k)\rho_{s_n}\cdots\rho_{s_0}.$$

Comparing this with (1) we get

$$\mu(p_j\mid\bigvee_0^n T^{-i}R_T)(x)=\sum_{k\in D(j;s_n,\dots,s_0)}\mu(p_k), \tag{3}$$

$$\text{for } x\in T^{-n}r_{s_n}\cap\cdots\cap r_{s_0}.$$

Now we will state

Theorem 3. *Let T be a Markov endomorphism with Markov generator*

$P=\{p_1, p_2, \cdots\}$. *In order that* T *has a countable Bernoulli generator with distinct probabilities, it is necessary and sufficient that* P *is a uniform Markov generator with common distribution consisting of distinct probabilities and satisfies the condition*

(P) *for any* $\delta>0$ *there exist* $j_0, m, i_m, \cdots, i_0$ *such that*

$$\sum_{k\in D(j_0;i_m,\dots,i_0)} \mu(p_k)>1-\delta.$$

Proof. For the proof of the necessity it remains to prove (P). Suppose the condition (P) is not satisfied, then there exists $\delta>0$ such that $\sum_{k\in D(j;i_m,\dots,i_0)} \mu(p_k)\leqq 1-\delta$ for all $j, m, i_m, \cdots, i_0$. Hence by (3) and the fact that R_T is a generator of T we get

$$1_{p_j}(x)=\lim_{n\to\infty}\mu(p_j\,|\bigvee_0^n T^{-i}R_T)(x)\leqq 1-\delta \qquad \text{a.e.}$$

which is a contradiction.

To prove the sufficiency it is enough to show that the proper partition R_T for T is a generator of T. For this we will prove $P\leqq \bigvee_0^\infty T^{-n}R_T$. Given $\delta>0$, we take $j_0, m, i_m, \cdots, i_0$ in (P) and put

$$B=T^{-m}r_{i_m}\cap\cdots\cap T^{-1}r_{i_1}\cap r_{i_0},$$

$$\tau(x)=\begin{cases}\min\{t\geqq 0;\ T^t x\in B\},\\ \infty, \text{ if } \{t\geqq 0;\ T^t x\in B\}=\phi,\end{cases}$$

and

$$E(t)=\{x;\ \tau(x)=t\}.$$

Then we have $E(t)\subset T^{-t}B$ and $E(t)$ is $\bigvee_0^{t+m} T^{-n}R_T$-measurable. Moreover $\mu(\tau(x)<\infty)=1$ i.e. $\mu(\bigcup_0^\infty E(t))=1$ because R_T is a Bernoulli partition. Take a cell F of $\bigvee_0^{t+m} T^{-n}R_T$ such that $F\subset E(t)$. Since $F\subset T^{-t}B$, F should be of the form

$$F=F(s_{t-1},\cdots, s_0)=T^{-t-m}r_{i_m}\cap\cdots\cap T^{-t}r_{i_0}\cap T^{-t+1}r_{s_{t-1}}\cap\cdots\cap r_{s_0},$$

and moreover using the expression (2) we have

$$T^{-t}p_{j_0}\cap p_j\cap F=\bigcup_{k\in D(j_0;i_m,\dots,i_0)}(T^{-t-m-1}p_k\cap\cdots\cap T^{-t}p_{j_0}$$

$$\cap T^{-t+1}p_{\psi(j_0;s_{t-1})}\cap\cdots\cap T^{-1}p_{\psi(j_0;s_{t-1},\dots,s_1)}\cap p_j,$$

if $j=\psi(j_0;s_{t-1},\cdots,s_0)$ and $T^{-t}p_{j_0}\cap p_j\cap F=\phi$ otherwise. This implies

$$\begin{aligned}\mu(p_j|F) &\geqq \mu(T^{-t}p_{j_0}\cap p_j|F)\\ &=\sum_{k\in D(j_0;i_m,\dots,i_0)}\mu(p_k)\delta_{j,\psi(j_0;s_{t-1},\dots,s_0)}\\ &\geqq (1-\delta)\delta_{j,\psi(j_0;s_{t-1},\dots,s_0)}.\end{aligned}$$

Define a partition $P'=\{p_1', p_2',\cdots\}$ by

$$p_j'=\bigcup_t\bigcup_{\substack{s_{t-1},\dots,s_0\\ \psi(j_0;s_{t-1},\dots,s_0)=j}}F(s_{t-1},\cdots,s_0),$$

then we get

$$\begin{aligned}\sum_j\mu(p_j\cap p_j'|F)&=\sum_j\mu(p_j|F)\delta_{j,\psi(j_0;s_{t-1},\dots,s_0)}\\ &\geqq 1-\delta\end{aligned}$$

which implies

$$\sum_j\mu(p_j\Delta p_j')\leqq 2-2\sum_j\mu(p_j\cap p_j')\leqq 2\delta.$$

Hence partition P can be approximated by $\bigvee_0^\infty T^{-n}R_T$-measurable partition (i.e. $P\overset{2\delta}{\subset}\bigvee_0^\infty T^{-n}R_T$ for any $\delta>0$). Thus we get $P\leqq\bigvee_0^\infty T^{-n}R_T$ which completes the proof.

Remark 1. Let's define matrices $M(i)=(m_{kj}(i))$, $i=1, 2,\cdots$, by

$$m_{kj}(i)=\begin{cases}1 & \text{if } \pi_{kj}=\rho_i>0,\\ 0 & \text{otherwise,}\end{cases}$$

and let $m_{kj}(i_m,\cdots,i_0)$ denote the (k, j)-element of the product $M(i_m)M(i_{m-1})\cdots M(i_0)$. Obviously $m_{kj}(i_m,\cdots,i_0)=1$ if and only if $\psi(k;i_m,\cdots,i_0)=j$. The condition (P) is equivalent to

(P′) *for any N there exist $j_0, m, i_m,\cdots, i_0$ such that*

$$m_{kj_0}(i_m, \cdots, i_0) = 1 \qquad \textit{for } 1 \leqq k \leqq N.$$

Notice this condition is a generalization of what is called "point collapsing" by M. Rosenblatt.

Remark 2. We were so far concerned with a uniform Markov generator P with common distribution consisting of distinct probabilities. Even when the common distribution of a uniform Markov generator P of T does not consist of distinct probabilities, we can formulate a sufficient condition for T to be Bernoulli by modifying the definition of matrices $M(i)$'s. Let us construct matrices $M(i) = (m_{kj}(i))$, $i = 1, 2, \cdots$, with the properties (i) each row vector consists of only one 1 and others 0, (ii) if $m_{kj}(i) = 1$ then $\pi_{kj} = \rho_i > 0$ and (iii) $\sum_i m_{kj}(i) \leqq 1$ for all k and j. Notice such constructions of $M(i)$'s are not unique. If there exists a sequence of such matrices $\{M(i); i = 1, 2, \cdots\}$ satisfying the condition (P′), then T is a Bernoulli endomorphism.

§5. Examples

We will now give some concrete examples. Firstly we consider (2×2) and (3×3)-Markov endomorphisms (i.e. endomorphisms with Markov generators of 2 cells and 3 cells).

Example 1. Tow (2×2)-Markov endomorphisms are isomorphic if and only if their transition matrices coincide up to the change of numbering.

Example 2. The classification of uniform (3×3)-Markov endomorphisms with the common distribution (a, b, c).

Case 1. a, b and c are positive and distinct. This case is divided into the following five classes which are mutually not isomorphic:

(i) $\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$, $\begin{pmatrix} a & c & b \\ b & a & c \\ c & b & a \end{pmatrix}$,

(ii) $\begin{pmatrix} b & c & a \\ a & b & c \\ c & a & b \end{pmatrix}$, $\begin{pmatrix} b & a & c \\ c & b & a \\ a & c & b \end{pmatrix}$,

(iii) $\begin{pmatrix} c & a & b \\ b & c & a \\ a & b & c \end{pmatrix}$, $\begin{pmatrix} c & b & a \\ a & c & b \\ b & a & c \end{pmatrix}$,

(iv) $\begin{pmatrix} b & c & a \\ c & a & b \\ a & b & c \end{pmatrix}$, $\begin{pmatrix} c & a & b \\ a & b & c \\ b & c & a \end{pmatrix}$, $\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$,

$\begin{pmatrix} c & b & a \\ b & a & c \\ a & c & b \end{pmatrix}$, $\begin{pmatrix} a & c & b \\ c & b & a \\ b & a & c \end{pmatrix}$, $\begin{pmatrix} b & a & c \\ a & c & b \\ c & b & a \end{pmatrix}$,

(v) *others*.

All endomorphisms of the fifth class satisfy the condition (P′) and hence they are Bernoulli.

Case 2. $a=b$. This case consists of only one Bernoulli class.

Example 3. Uniform (3×3)-Markov endomorphisms with the common distribution (a, b). Notice that Theorem 3 includes also this case, so we can see which endomorphism of this case is Bernoulli using the condition (P′). For example, the following three Markov endomorphisms

(i) $\begin{pmatrix} a & b & 0 \\ a & 0 & b \\ a & b & 0 \end{pmatrix}$, $\begin{pmatrix} a & 0 & b \\ a & b & 0 \\ 0 & a & b \end{pmatrix}$, $\begin{pmatrix} a & b & 0 \\ a & 0 & b \\ 0 & b & a \end{pmatrix}$

are Bernoulli which are of course all isomorphic to Bernoulli endomorphism with the distribution (a, b). On the other hand the following six

(ii) $\begin{pmatrix} 0 & a & b \\ b & 0 & a \\ a & b & 0 \end{pmatrix}$, (iii) $\begin{pmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{pmatrix}$, (iv) $\begin{pmatrix} b & 0 & a \\ a & b & 0 \\ 0 & a & b \end{pmatrix}$,

(v) $\begin{pmatrix} 0 & a & b \\ a & b & 0 \\ b & 0 & a \end{pmatrix}$, (vi) $\begin{pmatrix} 0 & a & b \\ a & 0 & b \\ 0 & b & a \end{pmatrix}$, (vii) $\begin{pmatrix} 0 & a & b \\ 0 & b & a \\ b & a & 0 \end{pmatrix}$

are not Bernoulli, and moreover they are not mutually isomorphic. Any one of this case of Markov endomorphisms is isomorphic to one of the above seven types of Markov endomorphisms. For the details see Part III.

Example 4. Number-theoretical endomorphisms. Continued-fraction transformation defined by

$$Tx=\left\{\frac{1}{x}\right\}, \qquad x\in(0, 1]$$

where $\{y\}$ denotes the fractional part of y, is neither a Bernoulli endomorphism nor a Markov endomorphism with countable generator. β-expansion transformation defined by

$$Tx=\{\beta x\}, \qquad x\in(0, 1]$$

where $\beta>1$, is a Bernoulli endomorphism if and only if β is an integer. Linear mod 1 transformation defined by

$$Tx=\{\beta x+\alpha\}, \qquad x\in(0, 1]$$

where $\beta\geqq 2$, $0\leqq\alpha<1$, is a Bernoulli endomorphism if and only if β is an integer.

We will explain the details of the above examples in Parts II and III.

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