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On the Isomorphism Problem for Endomorphisms of Lebesgue Spaces, II

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In part I we studied the Bernoulli property of endomorphisms (i.e. not-invertible measure preserving transformations) of Lebesgue spaces. In this part we will examine the Bernoulli property of special three number-theoretical transformations on $(0, 1]$. The natural extensions of these transformations were proved to be Bernoulli by several authors. But the existence of Bernoulli generators of these transformations themselves has not been considered. We will prove that these transformations are not Bernoulli endomorphisms except the trivial cases.

§6. F-endomorphisms

Let $f(x)$ be a strictly monotone function from $(0, 1]$ into $(0, \infty)$ with absolutely continuous inverse function f^{-1}, and $Tx=\{f(x)\}$ be a transformation on $(0, 1]$ where $\{y\}$ represents the fractional part of $y\in(0, \infty)$. Let $P=\{c_n=f^{-1}(n, n+1]; n=0, 1, 2,\cdots\}$ be the natural partition of $(0, 1]$ associated with T.

For the transformation T and the partition P, we always assume the followings:

(C1) *there exists a T-invariant probability measure* μ *which is absolutely continuous with respect to the ordinary Lebesgue measure* m *on* $(0, 1]$,

(C2) P *is a generator of* T.

We call T on $(0, 1]$ defined by f an *f-endomorphism*.

Remark. Some conditions which assure (C1) and (C2) are known

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(cf. [4] and [5] for example).

In this situation let $g(x)$ denote the density function of μ with respect to m, that is,

$$\mu(A)=\int_A g(x)m(dx)$$

for any measurable set A, then we can easily calculate the value of canonical measure $\mu(c_n|T^{-1}x)$ for a.e. x as follows.[1]

Lemma. *We have*

$$\mu(c_n|T^{-1}x)=\begin{cases}\dfrac{|(f^{-1})'(n+x)|\,g(f^{-1}(n+x))}{g(x)} & \\ \qquad\qquad \textit{if } g(x)>0 \textit{ and } x+n\in f((0,1]), & \\ 0, & \textit{otherwise},\end{cases}$$

for μ-a.e. x.

Using this formula and the result of Part I, we will discuss on the existence or non-existence of Bernoulli generators of the following special three number-theoretical endomorphisms which satisfy the conditions (C1) and (C2).

Firstly let $f_c(x)=\dfrac{1}{x}$ and

$$T_c x=\left\{\frac{1}{x}\right\},$$

which is called *continued-fraction transformation.* In this case

$$g_c(x)=\frac{1}{(1+x)\log 2}>0,$$

and

$$\mu_c(c_n|T_c^{-1}x)=\frac{x+1}{(n+x)(n+x+1)},\ \mu_c\text{-a.e. } x\in(0,1],\ n\geqq 1, \tag{1}$$

where $c_n=\left[\dfrac{1}{n+1},\dfrac{1}{n}\right)$, $n\geqq 1$. Then the probability distributions

1) We denote $T^{-1}x=\{y:\ Ty=x\}$

$\{\mu_c(c_n | T_c^{-1}x);\ n \geqq 1\}$ are not of the same type for μ_c-a.e. $x \in (0, 1]$.

Next let $f_\beta(x) = \beta x$ $(\beta > 1)$ and

$$T_\beta x = \{\beta x\},$$

which is called *β-expansion transformation.*

Lastly let $f_{\beta,\alpha}(x) = \beta x + \alpha$ $(\beta \geqq 2,\ 1 > \alpha \geqq 0)$ and

$$T_{\beta,\alpha} x = \{\beta x + \alpha\},$$

which is called *linear mod* 1 *transformation.*

In the last two cases W. Parry ([4], [5]) proved that the measures μ_β and $\mu_{\beta,\alpha}$ with the following density functions g_β and $g_{\beta,\alpha}$ with respect to m are invariant under T_β and $T_{\beta,\alpha}$ respectively;

$$g_\beta(x) = \frac{1}{F(\beta)} \sum_{k=1}^{\infty} I_{T_\beta^k 1 > x} \frac{1}{\beta^k},$$

$$g_{\beta,\alpha}(x) = \frac{1}{F(\beta,\alpha)} \left(\sum_{k=1}^{\infty} I_{T_{\beta,\alpha}^k 1 > x} \frac{1}{\beta^k} - \sum_{k=1}^{\infty} I_{T_{\beta,\alpha}^k 0 > x} \frac{1}{\beta^k} \right)$$

where $F(\beta)$ and $F(\beta, \alpha)$ are normalizing constants.

In these cases we get

$$\text{(2)} \qquad \mu_\beta(c_n | T_\beta^{-1}x) = \frac{g_\beta\left(\frac{n+x}{\beta}\right)}{\beta \cdot g_\beta(x)} \cdot I_{(0,\, 1 \wedge (\beta - n)]}(x), \quad \mu_\beta\text{-a.e. } x$$

where $c_n = \left(\frac{n}{\beta},\ \frac{n+1}{\beta} \wedge 1\right]$, $0 \leqq n \leqq [\beta] \equiv \beta - \{\beta\}$, and

$$\text{(3)} \qquad \mu_{\beta,\alpha}(c_n | T_{\beta,\alpha}^{-1}x) = \frac{g_{\beta,\alpha}\left(\frac{n+x-\alpha}{\beta}\right)}{\beta \cdot g_{\beta,\alpha}(x)} \cdot I_{(0 \vee (\alpha - n),\, 1 \wedge (\beta + \alpha - n)]}(x),$$

$\mu_{\beta,\alpha}$-a.e. x where $c_n = \left(0 \vee \frac{n-\alpha}{\beta},\ \frac{n+1-\alpha}{\beta} \wedge 1\right]$, $0 \leqq n \leqq [\beta + \alpha]$.

Note that $g_\beta(x)$ and $g_{\beta,\alpha}(x)$ are all positive functions (cf. [6], [11]). Indeed, for example,

$$\frac{1}{F(\beta,\alpha)} \cdot \frac{\beta - 2}{\beta - 1} \leqq g_{\beta,\alpha}(x) \leqq \frac{1}{F(\beta,\alpha)} \cdot \frac{\beta}{\beta - 1}$$

and $g_{2,\alpha}(x)\equiv 1$. Hence the formulas (1)~(3) hold also for m-a.e. x.

Therefore, using (2) and (3), we obtain

$$\text{(4)} \quad \begin{cases} \mu_\beta(c_n \mid T_\beta^{-1}x)>0,\ 0\leqq n\leqq[\beta]-1, & \text{for } m\text{-a.e. } x\in(0,1], \\ \mu_\beta(c_{[\beta]}\mid T_\beta^{-1}x) \begin{cases} =0, & \text{for } m\text{-a.e. } x\in(\{\beta\},1], \\ >0, & \text{otherwise;} \end{cases} \end{cases}$$

$$\text{(5)} \quad \begin{cases} \mu_{\beta,\alpha}(c_n \mid T_{\beta,\alpha}^{-1}x)>0,\ 1\leqq n\leqq[\beta+\alpha]-1, & \text{for } m\text{-a.e. } x\in(0,1], \\ \mu_{\beta,\alpha}(c_{[\beta+\alpha]}\mid T_{\beta,\alpha}^{-1}x) \begin{cases} =0, & \text{for } m\text{-a.e. } x\in(\{\beta+\alpha\},1], \\ >0, & \text{otherwise,} \end{cases} \\ \mu_{\beta,\alpha}(c_0 \mid T_{\beta,\alpha}^{-1}x) \begin{cases} =0, & \text{for } m\text{-a.e. } x\in(0,\alpha], \\ >0, & \text{otherwise.} \end{cases} \end{cases}$$

Then (4) and (5) imply that the probability distributions $\{\mu_\beta(c_n \mid T_\beta^{-1}x);\ 0\leqq n\leqq[\beta]\}$ (and $\{\mu_{\beta,\alpha}(c_n\mid T_{\beta,\alpha}^{-1}x);\ 0\leqq n\leqq[\beta+\alpha]\}$) are not of the same type for μ_β-a.e. x (and $\mu_{\beta,\alpha}$-a.e. x respectively), except the case of integer β.

Recalling the necessary condition for any endomorphism to be Bernoulli (see 2° of §3 in Part I), we have

Theorem 4. *Continued-fraction transformation is not a Bernoulli endomorphism. β-expansion transformations and linear mod* 1 *transformations are not Bernoulli endomorphisms except the case of integer β. If β is an integer, T_β and $T_{\beta,\alpha}$ are both Bernoulli endomorphisms, and are isomorphic under the isomorphism $S_\gamma x=\{x+\gamma\}$ (Weyl automorphism) where $\gamma=\dfrac{\alpha}{\beta-1}$.*

It is easy to see that the Bernoulli generators of T_β and $T_{\beta,\alpha}$ are given by $P=\{c_n;\ 0\leqq n\leqq\beta-1\}$, and $\bar{P}=\{c_0\cup c_\beta,\ c_1,\ c_2,\cdots,\ c_{\beta-1}\}$ respectively for integer β.

We remark that for the above three transformations of special type the Bernoulli property of their natural extensions are proved by several authors ([1], [2], [9], [10], [11], for example), but these transformations themselves are not Bernoulli. Moreover continued-fraction transformation has no Markov generator of countable atoms.

§7. Remarks

In this section we will make some remarks on f-endomorphisms satisfying conditions (C1) and (C2).

1° Firstly assume that the invariant measure μ of T is equivalent to the Lebesgue measure m with density function $g(x)$(i.e. $g(x)>0$ m-a.e. x). Let's consider the following isomorphism θ from $((0, 1], \mu)$ onto $((0, 1], m)$

$$\theta(x)=\int_0^x g(y)m(dy)$$

and put

$$\tilde{f}(x)=\theta(\{f(\theta^{-1}(x))\})+[f(\theta^{-1}(x))],\ x\in(0, 1].$$

Then, denoting T_f and $T_{\tilde{f}}$ the mod 1 transformations induced by f and $\tilde{f}$ respectively, we have

$$T_{\tilde{f}}=\theta\cdot T_f\cdot\theta^{-1},$$

and $T_{\tilde{f}}$ has m as its invariant measure. Therefore we can reduce the study of (T_f, μ) to $(T_{\tilde{f}}, m)$.

2° Noting the above remark 1°, we now consider an f-endomorphism T_f with positive f such that $f(0+)=0$ of which invariant measure is the ordinary Lebesgue measure m on $(0, 1]$ and the natural partition P is a generator. In this situation we can get a necessary and sufficient condition for T_f to be Bernoulli endomorphism.

Assume T_f is Bernoulli with the probability distribution $\lambda=\{\lambda_i;\ i\geqq 0\}$ with distinct positive elements. By lemma of §6 and 2° of §3 in Part I, there exists a permutation $\sigma_x=(\sigma_x(0), \sigma_x(1),\cdots)$ of $\{0, 1, 2,\cdots\}$ for a.e. $x\in(0, 1]$ such that

$$(f^{-1})'(\sigma_x(i)+x)=\lambda_i,\ i\geqq 0.$$

Then, taking the measurable sets $q_\sigma=\{x\in(0, 1];\ \sigma_x=\sigma\}$, we have the following

(B) *there exists a measurable partition* $Q=\{q_\sigma;\ \sigma\in\Sigma\}$ *of* $(0, 1]$ *such that*

$$(f^{-1})'(\sigma(i)+x)=\lambda_i,\ x\in q_\sigma,\ i\geqq 0,$$

where Σ is a family of permutations of $\{0, 1, 2, \cdots\}$.

Conversely, assume that f satisfies the above condition (B) with some probability distribution $\lambda=\{\lambda_i;\ i\geqq 0\}$ with distinct positive elements. Then the Bernoulli partition (the proper partition) for T_f considered in §3 of Part I is

$$R_{T_f}=\{r_i=\{y;\ (f^{-1})'(f(y))=\lambda_i\};\ i\geqq 0\}.$$

Therefore T_f is Bernoulli with the probability distribution λ with distinct positive elements if and only if f satisfies the condition (B) and R_{T_f} is a generator of T_f. However it seems difficult to examine whether R_{T_f} is a generator or not.

Consider the case $f(0+)\neq 0$ (f is increasing). In order that T_f is Bernoulli it is necessary that $f(1)-f(0+)$ is an integer ($\geqq 2$). Hence we can choose γ such that $f(\{\gamma\})=\gamma$. Let

$$\tilde{f}(x)=\begin{cases} f(\{x+\gamma\})-\gamma, & \text{if } x\leqq 1-\{\gamma\}, \\ f(\{x+\gamma\})-\gamma+f(1)-f(0+), & \text{otherwise.}\end{cases}$$

Then we have $\tilde{f}(0+)=0$ and $S_{-\gamma}\cdot T_f\cdot S_\gamma=T_{\tilde{f}}$ where $S_\gamma x=\{x+\gamma\}$. Thus we can reduce the study of T_f to $T_{\tilde{f}}$.

3° Let's take

$$f_{\alpha,\gamma}(x)=f(\{x+\gamma\})+\alpha+f([x+\gamma]+)-f(0+),\qquad 0<\alpha,\ \gamma<1.$$

Then $T_{f_{\alpha,\gamma}}$ is isomorphic to T_f under the Weyl automorphism $S_\alpha x=\{x+\alpha\}$ if $\alpha+\gamma=1$. However in the case $\alpha+\gamma\neq 1$, we cannot conclude that $T_{f_{\alpha,\gamma}}$ is Bernoulli even if T_f is so except the special case of T_β and $T_{\beta,\alpha}$ (see the last statement of Theorem 4). For example, let's consider the following case:

$$f(x)=\begin{cases} \dfrac{x}{p} & x\in(0,\ p], \\[2ex] \dfrac{x-p}{1-p}+1 & x\in(p,\ 1],\ p\neq\dfrac{1}{2},\ 0<p<1.\end{cases}$$

Then $T_{f_{\frac{1}{2},\frac{1}{2}p}}$ is a Markov endomorphism with transition matrix

$$\begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

and hence $T_{f_{\frac{1}{2},\frac{1}{2}p}}$ is not Bernoulli, but T_f is evidently Bernoulli.

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