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On the Isomorphism Problem for Endomorphisms of Lebesgue Spaces, III

By

I. KUBO*, H. MURATA** and H. TOTOKI

This is the sequel to the preceding Parts I and II. In Part I we have studied the Bernoulli case, and then we have seen in Part II that some number-theoretical transformations are not Bernoulli. We will now try to study the isomorphism problem for Markov endomorphisms. Firstly we will give a general isomorphism theorem for Markov endomorphisms with countable generators in §8. We are also concerned with Markov endomorphisms having no uniform Markov generator. In §9 we will study the mixing property of a kind of skew product transformation in preparation for further investigation of isomorphism problem for Markov endomorphisms. In §10 we will give an isomorphism theorem for a typical class of Markov endomorphisms (which are uniform but not "point collapsing"). In the last section we will classify (2×2) and (3×3)-Markov endomorphisms as examples of applications of our theorems given in the preceding sections.

Throughout this part, T denotes a Markov endomorphism of Lebesgue space $(X, \mathscr{F}, \mu)$ with a countable Markov generator $P=\{p_j; j=0, 1, 2,\cdots\}$. Its transition matrix will be denoted by $\Pi=(\pi_{ij}; i, j=0, 1, 2,\cdots)$. We define a measurable function $\xi(x)$ by

$$\xi(x)=\xi(x; P)=j, \qquad \text{for } x\in p_j. \tag{1}$$

We also use the notations given in Part I.

We will appeal to the following two invariants. The first one is $\mu_T(x)=\mu(\{x\}\,|\,C_{T^{-1}\varepsilon}(x))$ which was introduced in Part I, and we have

$$\mu_T(x)=\mu(p_{\xi(x)}\,|\,T^{-1}p_{\xi(Tx)})=\pi_{\xi(Tx),\xi(x)} \qquad \text{a.e.} \tag{2}$$

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* Department of Mathematics, Nagoya University, Nagoya.
** Department of Mathematics, Hiroshima University, Hiroshima.

for the Markov endomorphism T. Another one is introduced as follows. Let $D(T)$ denote the family of all distributions ρ such that $\rho=\text{dist}(\varepsilon | C_{T^{-1}\varepsilon}(x))$ on a set of positive measure. $D(T)$ is evidently an invariant. If T has a Markov generator $P=\{p_i\}$, then

$$D(T)=\{\text{dist}(P | T^{-1}p_i)=\text{dist}(\pi^{(i)});\ i=0, 1, 2, \cdots\}, \tag{3}$$

where $\pi^{(i)}=(\pi_{i0}, \pi_{i1}, \cdots)$ denotes the i-th row vector of Π as in §4. Put $\tilde{r}(\rho)=\{x; \text{dist}(\varepsilon | C_{T^{-1}\varepsilon}(x))=\rho\}$ for $\rho\in D(T)$ and define a measurable partition[1)]

$$\tilde{R}_T=\{\tilde{r}(\rho);\ \rho\in D(T)\}. \tag{4}$$

Then it is easy to see that $\tilde{R}_T$ is independent of the choice of Markov generators and $\tilde{R}_T\leqq T^{-1}\varepsilon$.

§8. Isomorphism Theorems for Markov Endomorphisms

We will firstly consider general Markov endomorphisms with countable generators.

Theorem 5. *Let T and S be ergodic Markov endomorphisms with transition matrices $\Pi=(\pi_{ij})$ and $\Gamma=(\gamma_{ij})$ respectively. Then T and S are isomorphic if and only if there exists a measurable integer-valued function $\eta(x)$ such that*

(i) $\gamma_{\eta(Tx), \eta(x)}=\pi_{\xi(Tx), \xi(x)}$ a.e.

(ii) $Q=\{q_j=\{x; \eta(x)=j\}; j=0, 1, 2, \cdots\}$ *is a Markov generator of T.*

Proof. "Only if" part. If T is isomorphic to S, there obviously exists a Markov generator $Q=\{q_j\}$ of T such that $\mu(q_j | T^{-1}q_i)=\gamma_{ij}$. Hence defining $\eta(x)=j$ for $x\in q_j$, we have

$$\pi_{\xi(Tx), \xi(x)}=\mu_T(x)=\gamma_{\eta(Tx), \eta(x)} \qquad \text{a.e.}$$

"If" part. Since

$$\mu(q_{\eta(x)} | T^{-1}q_{\eta(Tx)})=\mu_T(x)=\pi_{\xi(Tx), \xi(x)}=\gamma_{\eta(Tx), \eta(x)} \qquad \text{a.e.}$$

1) Here T is assumed to have a countable Markov generator.

we have $\mu(q_j|T^{-1}q_i)=\gamma_{ij}$ if $\mu(q_j\cap T^{-1}q_i)>0$. Therefore we have $\mu(q_j\cap T^{-1}q_i)=\mu(q_i)\gamma_{ij}$ for all i and j, and so

$$\sum_i \mu(q_i)\gamma_{ij}=\sum_i \mu(q_j\cap T^{-1}q_i)=\mu(q_j)$$

for all j, which implies $\{\mu(q_i); i=0, 1,\cdots\}$ is a stationary distribution of Γ. Since S is ergodic, the stationary distribution of Γ is unique, and hence denoting the Markov generator of S with transition matrix Γ by $Q'=\{q'_i; i=0, 1, 2,\cdots\}$ we have $\mu(q_i)=\nu(q'_i)>0$[1] and $\mu(q_j|T^{-1}q_i)=\gamma_{ij}=\nu(q'_j|S^{-1}q'_i)$ for all i and j. Then the natural mapping induced by (T, Q) and (S, Q') is an isomorphism between T and S.

Let us now consider the class of Markov endomorphisms each of which has a countable Markov generator satisfying

(D) $\mathrm{dist}(\pi^{(i)})$, $i=0, 1, 2,\cdots$, *are all different.*

1° *If there exists a Markov generator satisfying the condition* (D) *for an endomorphism, then it is unique.*

Indeed, if T has a Markov generator P which satisfies the condition (D), then it is easy to see $\tilde{r}(\mathrm{dist}(\pi^{(i)}))=T^{-1}p_i$ i.e. $p_i=T\tilde{r}(\mathrm{dist}(\pi^{(i)}))$. Hence we have $P=T\tilde{R}_T$, where $\tilde{R}_T$ is defined by (4).

The following is a direct conclusion of 1°.

Theorem 6. *Two ergodic Markov endomorphisms with Markov generators satisfying the condition* (D) *are isomorphic, if and only if their transition matrices are the same except the numbering of the cells of Markov generators.*

Let T and S be ergodic Markov endomorphisms with Markov generators satisfying the condition (D), of which transition matrices are denoted by $\Pi=(\pi_{ij})$ and $\Gamma=(\gamma_{ij})$ respectively. Then Theorem 6 implies that T and S are isomorphic if and only if there exists a permutation (a one to one onto mapping) σ such that $\pi_{ij}=\gamma_{\sigma i,\sigma j}$.

§9. Mixing Property of a Skew Product Transformation

In this section we are concerned with a special kind of skew product

1) ν denotes the invariant probability measure of S.

transformation in the sense of H. Anzai [1], and we will study its mixing property in preparation for the following section.

Let T be a Markov endomorphism with a Markov generator $P=\{p_0, p_1,\dots, p_{N-1}\}$, of which transition matrix is $\Pi=(\pi_{ij}; 0\leqq i, j\leqq N-1)$. Let $N_0\neq 1$ be a divisor of N and $Y=\mathbf{Z}/N_0\mathbf{Z}$ where $\mathbf{Z}$ denotes the additive group of all integers. Define $\tilde{X}=X\times Y$ and $\tilde{\mu}=\mu\times\nu$ where ν is the normalized Haar measure on Y.

Let us consider the skew product transformation $\tilde{T}$ defined by

$$\tilde{T}(x, y)=(Tx, \beta y+\tau\xi(x)), \qquad (x, y)\in\tilde{X},$$

where β and τ are integers such that $(\beta, N)=1$.[1)] Evidently $\tilde{T}$ is an endomorphism of $(\tilde{X}, \tilde{\mu})$.

Theorem 7. (i) *$\tilde{T}$ is a Markov endomorphism with Markov generator $\tilde{P}=P\times\varepsilon_Y$ and its transition matrix $\tilde{\Pi}$ is given by*

$$\tilde{\pi}_{(a',b'),(a,b)}=\delta_{b',\tau a+\beta b}\pi_{a',a}.$$

(ii) *$\tilde{T}$ is mixing if and only if T is mixing, $(\tau, N_0)=1$ and there exists an integer M satisfying that for any $0\leqq q\leqq N_0-1$ there is a sequence $\{a(j; q); 0\leqq j\leqq M\}$ such that $a(0; q)=a(M; q)=0$,*

$$\pi_{a(j+1,q),a(j,q)}>0, \qquad 0\leqq j\leqq M-1,$$

and

$$\sum_{j=0}^{M-1}\beta^{M-j-1}a(j, q)=q, \qquad \bmod N_0.$$

Proof. (i) Let us denote $g(x, y)=(\xi(x), y)$. Then we have

$$\tilde{\mu}(g(\tilde{T}^k(x, y))=(a(k), b(k)), 0\leqq k\leqq n)$$

$$=\tilde{\mu}(\xi(T^kx)=a(k), \beta^k y+\tau\sum_{i=0}^{k-1}\beta^{k-1-i}a(i)=b(k), 0\leqq k\leqq n)$$

$$=\mu(\xi(T^kx)=a(k), 0\leqq k\leqq n)\nu(\beta^n y+\tau\sum_{i=0}^{n-1}\beta^{n-1-i}a(i)=b(n))$$

1) (β, N) denotes the greatest common measure of β and N, and $\xi(x)$ is defined by (1).

$$\times \prod_{k=1}^{n} \delta_{b(k),\tau a(k-1)+\beta b(k-1)}$$

$$= \tilde{\mu}(g(\tilde{T}^n(x, y)) = (a(n), b(n))) \prod_{k=1}^{n} \delta_{b(k),\tau a(k-1)+\beta b(k-1)} \pi_{a(k),a(k-1)}.$$

On the other hand it is easy to see

$$\tilde{\mu}(g(\tilde{T}^{k-1}(x, y)) = (a, b) \mid g(\tilde{T}^k(x, y)) = (a', b'))$$
$$= \delta_{b',\tau a+\beta b} \pi_{a',a}.$$

Thus we get the assertion (i).

(ii) "Only if" part. Since $\tilde{T}$ is mixing, there is M such that all components $\tilde{\pi}^{(M)}_{(a',b'),(a,b)}$ of matrix $\tilde{\Pi}^M$ are positive. Hence T is also mixing. Especially $\tilde{\pi}^{(M)}_{(a,b),(0,0)} > 0$ implies that there exists a sequence $a(0)=0, a(1), \cdots, a(M-1), a(M)=a$ such that

$$\pi_{a(j+1),a(j)} > 0, \qquad 0 \leqq j \leqq M-1$$

and

$$\tau \sum_{j=0}^{M-1} \beta^{M-1-j} a(j) = b, \qquad \mod N_0.$$

Furthermore putting $b=1$ we have $(\tau, N_0)=1$. Applying the above argument to $a=0$ and $b=\tau q$, we get the required sequence $\{a(j; q); 0 \leqq j \leqq M\}$.

"If" part. Since T is mixing, there is n_0 such that $\pi^{(n_0)}_{a',a} > 0$ for all a and a'. We will prove $\tilde{\pi}^{(n+m+M)}_{(a',b'),(a,b)} > 0$ for all (a, b), (a', b') and for all $n, m \geqq n_0$. Firstly we have sequences $\alpha(0)=a, \alpha(1), \cdots, \alpha(n-1), \alpha(n)=0$ and $a(0)=0, a(1), \cdots, a(m-1), a(m)=a'$ such that $\pi_{\alpha(j+1),\alpha(j)} > 0, 0 \leqq j \leqq n-1$, and $\pi_{a(j+1),a(j)} > 0, 0 \leqq j \leqq m-1$. Put

$$\alpha(n+M+j) = a(j), \ 0 \leqq j \leqq m.$$

and choose $q(0 \leqq q \leqq N_0 - 1)$ as follows:

$$q = \tau^*(b' - \beta^{n+M+m} b - \tau \sum_{\substack{0 \leqq j \leqq n-1 \\ n+M \leqq j \leqq n+M+m-1}} \beta^{n+M+m-j-1} \alpha(j)) \beta^{*m}, \qquad \mod N_0,$$

where $\tau^* \tau = \beta^* \beta = 1$, mod N_0, and define

$$\alpha(n+j) = a(j; q), \qquad 0 \leqq j \leqq M.$$

On account of

$$\tau \sum_{\substack{0 \leqq j \leqq n-1 \\ n+M \leqq j \leqq n+M+m-1}} \beta^{n+M+m-j-1}\alpha(j) = -\beta^m \tau q + b' - \beta^{n+M+m} b, \qquad \text{mod } N_0,$$

and

$$\tau \sum_{n \leqq j \leqq n+M-1} \beta^{n+M+m-j-1}\alpha(j) = \beta^m \tau q, \qquad \text{mod } N_0,$$

we have

$$\beta^{n+M+m} b + \tau \sum_{j=0}^{n+M+m-1} \beta^{n+M+m-j-1}\alpha(j) = b', \qquad \text{mod } N_0,$$

$$\pi_{\alpha(j+1),\alpha(j)} > 0, \ 0 \leqq j \leqq n+M+m-1,$$

and $\alpha(0)=a$, $\alpha(n+M+m)=a'$. This means $\tilde{\pi}^{(n+M+m)}_{(a',b'),(a,b)}>0$. Thus $\tilde{T}$ is mixing.

The following is a direct consequence of the theorem.

Corollary. *If* $\pi_{ij}>0$ *for all* $0 \leqq i, j \leqq N-1$ *and* $(\tau, N_0)=1$, *then* $\tilde{T}$ *is mixing.*

Example. Even if T is mixing, $\tilde{T}$ is neither necessarily mixing nor ergodic. For example, let T be the Markov endomorphism with transition matrix

$$\begin{pmatrix} 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and $\beta=\tau=1$, then $\tilde{T}$ is not ergodic. Indeed $\tilde{T}$ has two irreducible components $\{(0,0), (1,0), (2,0), (0,2), (1,2), (2,2), (3,1), (3,3)\}$ and its complement.

§10. Uniform Markov Endomorphisms

In this section we will study the isomorphism problem for a special kind of uniform Markov endomorphism using the result of the preceding section. Markov endomorphisms studied here are uniform Markov but not "point collapsing". Namely we are concerned with Markov endomorphisms satisfying the condition

(G) *there exists a finite uniform Markov generator of which common distribution* $\rho=(\rho_i; 0\leqq i\leqq N-1)$ *consists of distinct positive* ρ_i*'s, and its transition matrix* $\Pi=(\pi_{ij}; 0\leqq i, j\leqq N-1)$ *satisfies*

$$\pi_{ij}=\rho_{i+tj+t_0}, \qquad 0\leqq i, j\leqq N-1,$$

for some $0\leqq t, t_0\leqq N-1$ *such that* $(N, t)=1$, *where the addition is taken to be* mod N.

Theorem 8. *Let T and S be Markov endomorphisms with transition matrices* $\Pi=(\pi_{ij}; 0\leqq i, j\leqq N-1)$ *and* $\Gamma=(\gamma_{ij}; 0\leqq i, j\leqq N-1)$ *respectively, which satisfy the condition* (G) *with the same common distribution* $\rho=(\rho_i; 0\leqq i\leqq N-1)$, *i.e.* $\pi_{ij}=\rho_{i+tj+t_0}$ *and* $\gamma_{ij}=\rho_{i+sj+s_0}$ *for all* $0\leqq i, j\leqq N-1$, *and* $(N, t)=(N, s)=1$. *Then T and S are isomorphic if and only if* $t=s$ *and* s_0-t_0 *is a multiple of* $(t+1, N)$.

Proof. "Only if" part. Assume T and S are isomorphic. Then Theorem 5 implies that there is a measurable function $\eta(x)$ such that

$$\pi_{\xi(Tx),\xi(x)}=\gamma_{\eta(Tx),\eta(x)}, \qquad \text{a.e.}$$

where $\xi(x)$ is defined by (1) for the Markov generator of T. Since ρ_i's are distinct we have

$$\xi(Tx)-\eta(Tx)=-s(\xi(x)-\eta(x))-(t-s)\xi(x)+s_0-t_0, \qquad \text{a.e.} \tag{5}$$

Let us firstly suppose $t\neq s$. Put $c=(t-s, N)$, $N_0=N/c$, $\tau=(t-s)/c$ and $\beta=-s$. Note $(N, \beta)=(N_0, \tau)=1$. Let us consider the skew product transformation $\tilde{T}(x, y)=(Tx, \beta y+\tau\xi(x))$ on $\tilde{X}=X\times Y$ where $Y=\mathbb{Z}/N_0\mathbb{Z}$. Corollary to Theorem 7 in §9 implies $\tilde{T}$ is mixing. Now define a measurable function

$$h(x, y)=\omega(cy+\xi(x)-\eta(x))$$

where $\omega(a)=\exp(2\pi ia/N)$. Then we have

$$h(\tilde{T}(x,\ y))=\omega(c(\beta y+\tau\xi(x))+\xi(Tx)-\eta(Tx))$$

$$=\omega(c\beta y+\beta(\xi(x)-\eta(x))+s_0-t_0)$$

$$=(h(x,\ y))^{\beta}\omega(s_0-t_0).$$

Since $(N, \beta)=1$, there is an integer M such that $\beta^M=1$ and $\sum_{j=0}^{M-1}\beta^j=0$ (mod N). Hence we have $h(\tilde{T}^M(x,\ y))=h(x,\ y)$, which contradicts the mixing property of $\tilde{T}$.

Assume now $t=s$, and consider the function

$$h(x)=\omega(\xi(x)-\eta(x)).$$

Taking the same M as above, we have $h(T^M x)=h(x)$. Since T is mixing, $h(x)$ is a constant function i.e. there is $0\leqq d\leqq N-1$ such that $\xi(x)-\eta(x)=d$, mod N, a.e. This and (5) imply

$$d=\xi(Tx)-\eta(Tx)=\beta(\xi(x)-\eta(x))+s_0-t_0$$

$$=\beta d+s_0-t_0, \qquad \text{mod}\ N, \qquad \text{a.e.}$$

and so $(1+t)d=(1-\beta)d=s_0-t_0$ (mod N).

"If" part. Assume $t=s$ and s_0-t_0 is a multiple of $(t+1, N)$. Then there is an integer d such that $s_0-t_0=d(t+1)$ (mod N). Defining

$$\eta(x)=\xi(x)-d, \qquad \text{mod}\ N,$$

we have

$$\eta(Tx)+s\eta(x)+s_0=\xi(Tx)+t\xi(x)+t_0, \qquad \text{mod}\ N,$$

and so $\pi_{\xi(Tx),\xi(x)}=\gamma_{\eta(Tx),\eta(x)}$. Since $\eta(x)$ define the same partition as $\xi(x)$, which is of course a Markov generator of T, Theorem 5 in §8 implies that T and S are isomorphic.

§11. (2×2) and (3×3)-Markov Endomorphisms

As examples of applications of our theorems, we will classify (2×2) and (3×3)-Markov endomorphisms (i.e. endomorphisms with Markov generators of 2 and 3 cells respectively) by means of the terminology of their transition matrices. Since we always neglect events of measure zero, we do not consider matrices having transient states.

A. *Classification of* (2×2)*-Markov endomorphisms.*

Let us denote the transition matrices by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Case 1. dist $(a, b)\neq$dist(c, d). Theorem 6 implies that two Markov endomorphisms of this case are isomorphic if and only if their transition matrices coincide up to the change of numbering.

Case 2. dist$(a, b)=$dist(c, d) and $a\neq b$. This case is divided into three classes (i) Bernoulli class $\begin{pmatrix} a & b \\ a & b \end{pmatrix}$, $\begin{pmatrix} b & a \\ b & a \end{pmatrix}$, (ii) $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ and (iii) $\begin{pmatrix} b & a \\ a & b \end{pmatrix}$. These three classes are not mutually isomorphic. Indeed Theorem 3 implies that (ii) and (iii) are not isomorphic to (i). We can see that (ii) and (iii) are not isomorphic applying Theorem 8.

Case 3. $a=b=c=d=1/2$.

B. *Classification of* (3×3)*-Markov endomorphisms.*

In order to classify (3×3)-Markov endomorphisms we appeal further to the following lemma.

1° *Consider two ergodic Markov endomorphisms T and S with transition matrices*

$$\Pi=\begin{pmatrix} a' & b' & c \\ a'' & b'' & c \\ d' & e' & f \end{pmatrix}, \qquad \Gamma=\begin{pmatrix} a & b & c \\ a & b & c \\ d & e & f \end{pmatrix}$$

respectively. If $c>0$, $(a', b')\cong(a'', b'')\cong(a, b)$ and $(d', e')\cong(d, e)$[1]*, then T and S are isomorphic.*

Indeed, let $P=\{p_0, p_1, p_2\}$ be a Markov generator of T with transition matrix Π, and define $\alpha(i, j)$, $0\leqq i\leqq 2$, $0\leqq j\leqq 1$, as follows: $\pi_{0,\alpha(0,0)}=\pi_{1,\alpha(1,0)}=a$, $\pi_{0,\alpha(0,1)}=\pi_{1,\alpha(1,1)}=b$, $\pi_{2,\alpha(2,0)}=d$, $\pi_{2,\alpha(2,1)}=e$. Then putting

1) $(a', b')\cong(a, b)$ means $(a', b')=(a, b)$ or $(a', b')=(b, a)$.

$$q_i = \bigcup_{j=0}^{2} (T^{-1} p_j \cap p_{\alpha(j,i)}), \quad i=0, 1,$$

$$q_2 = p_2,$$

we get a Markov generator $Q=\{q_0, q_1, q_2\}$ of T with transition matrix Γ. To see the Markov property of Q, it is enough to note that $\mu(q_j \mid C_{T^{-1}\varepsilon}(x))$ is constant a.e. on each $T^{-1}q_i$, $0 \leqq i,\ j \leqq 2$. We can prove that Q is a generator of T by means of an analogous idea to the proof of Theorem 3 (in Part I) noting that $T^n x$, $n \geqq 0$, visits certainly $q_2 = p_2$ for a.e. x.

Now we are in a position to state the classification of (3×3)-Markov endomorphisms. We will naturally restrict ourselves to ergodic ones. Let $\Pi = (\pi_{ij};\ 0 \leqq i,\ j \leqq 2)$ denote the transition matrices.

Case 1. $\mathrm{dist}(\pi^{(i)})$, $0 \leqq i \leqq 2$, are all different. Theorem 6 gives us the classification of this case. Namely two Markov endomorphisms of this case are isomorphic if and only if their transition matrices coincide up to the change of numbering.

Case 2. Two of $\mathrm{dist}(\pi^{(i)})$, $0 \leqq i \leqq 2$, are the same and the rest is different, that is $D(T) = \{\rho_1, \rho_2\}$, $\rho_1 \neq \rho_2$. Consider two (3×3)-Markov endomorphisms T and S with transition matrices Π and Γ of this case respectively. Suppose firstly that T and S are isomorphic. Then we have $D(T) = D(S) = \{\rho_1, \rho_2\}$. Choosing a suitable numbering we can assume

$$\text{(6)} \qquad \begin{cases} \mathrm{dist}(\pi^{(0)}) = \mathrm{dist}(\pi^{(1)}) = \mathrm{dist}(\gamma^{(0)}) = \rho_1 \\ \mathrm{dist}(\pi^{(2)}) = \mathrm{dist}(\gamma^{(2)}) = \rho_2 \end{cases}$$

without loss of generality.

Case 2.1. $\mathrm{dist}(\gamma^{(1)}) = \rho_1$. Let $P = \{p_0, p_1, p_2\}$ be a Markov generator of T with Π. Then the partition defined by (4) has the following two elements

$$\tilde{r}_1 = \tilde{r}(\rho_1) = T^{-1} p_0 \cup T^{-1} p_1, \quad \tilde{r}_2 = \tilde{r}(\rho_2) = T^{-1} p_2.$$

By our assumption T has also a Markov generator $Q = \{q_0, q_1, q_2\}$ with Γ, and then we have $\tilde{r}_2 = T^{-1} q_2$ and so $q_2 = p_2$ and $\gamma_{22} = \pi_{22}$. Moreover we have

$$\mu(T\tilde{r}_2 \mid C_{T^{-1}\varepsilon}(x)) = \begin{cases} \pi_{i2}, & \text{for } x \in T^{-1}p_i, \\ \gamma_{i2}, & \text{for } x \in T^{-1}q_i, \end{cases} \quad 0 \leqq i \leqq 2. \tag{7}$$

Since the left hand side of (7) is independent of the choice of Markov generators, we have $(\pi_{02}, \pi_{12}) \cong (\gamma_{02}, \gamma_{12})$.

First we will prove that if $\pi_{02} \neq \pi_{12}$ then $P=Q$ and so $\Pi=\Gamma$ up to the change of numbering. Indeed, noting that the left hand side of (7) takes two different values on $\tilde{r}_1$, decompose $\tilde{r}_1$ into $\hat{r}_0$ and $\hat{r}_1$ according to its values. Then it is easy to see $\hat{R}=\{\hat{r}_0, \hat{r}_1, \hat{r}_2=\tilde{r}_2\}=T^{-1}P$. By the same reason we have also $\hat{R}=T^{-1}Q$, and so $P=Q$.

Next, let us assume (6), $\mathrm{dist}(\gamma^{(1)})=\rho_1$ and $\pi_{02}=\pi_{12}$. Then 1° implies that T and S are isomorphic. Thus we obtain the following classification: Suppose $\mathrm{dist}(\pi^{(0)})=\mathrm{dist}(\pi^{(1)})=\mathrm{dist}(\gamma^{(0)})=\mathrm{dist}(\gamma^{(1)})\neq\mathrm{dist}(\pi^{(2)})=\mathrm{dist}(\gamma^{(2)})$. Then the condition

$$(\pi_{02}, \pi_{12}) \cong (\gamma_{02}, \gamma_{12}), \quad \pi_{22}=\gamma_{22} \tag{8}$$

is necessary for T and S being isomorphic. Conversely, under the above condition (8), (a) when $\pi_{02}\neq\pi_{12}$ T and S are isomorphic if and only if $\Pi=\Gamma$ up to the change of numbering, and (b) when $\pi_{02}=\pi_{12}$ T and S are always isomorphic.

Case 2.2. $\mathrm{dist}(\gamma^{(1)})=\rho_2$. If ρ_1 or ρ_2 consists of three positive elements, then considering the invariant $\mu_T(x)$, T and S are not isomorphic. Otherwise T and S are reduced to (2×2)-Markov endomorphisms using 1°, hence there are cases such that T and S are isomorphic.

Case 3. Uniform case. Let us denote the common distribution by $\rho=(a, b, c)$, where we assume a and b are positive and c is non-negative.

Case 3.1. $a=b$ and $c\geqq 0$. Remark 2 of §4 (Part I) implies that all Markov endomorphisms of this case are isomorphic to a Bernoulli endomorphism with the distribution (a, a, c).

Case 3.2. a, b and c are positive and distinct. In this case we have the following five classes:

(i) $\begin{pmatrix} a & c & b \\ b & a & c \\ c & b & a \end{pmatrix}$, $\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$,

(ii) $\begin{pmatrix} b & a & c \\ c & b & a \\ a & c & b \end{pmatrix}$, $\begin{pmatrix} b & c & a \\ a & b & c \\ c & a & b \end{pmatrix}$,

(iii) $\begin{pmatrix} c & b & a \\ a & c & b \\ b & a & c \end{pmatrix}$, $\begin{pmatrix} c & a & b \\ b & c & a \\ a & b & c \end{pmatrix}$,

(iv) $\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$, $\begin{pmatrix} c & a & b \\ a & b & c \\ b & c & a \end{pmatrix}$, $\begin{pmatrix} b & c & a \\ c & a & b \\ a & b & c \end{pmatrix}$,

$\begin{pmatrix} b & a & c \\ a & c & b \\ c & b & a \end{pmatrix}$, $\begin{pmatrix} a & c & b \\ c & b & a \\ b & a & c \end{pmatrix}$, $\begin{pmatrix} c & b & a \\ b & a & c \\ a & c & b \end{pmatrix}$,

(v) *others* (*Bernoulli class*).

Indeed, all endomorphisms of the fifth class are isomorphic to a Bernoulli endomorphism with (a, b, c) as they were already discussed in §5. Since all endomorphisms of classes (i)~(iv) do not satisfy the condition (P′) in §4, they are isomorphic to no endomorphism of class (v). It is obvious that all matrices belonging to each class (i)~(iv) coincide with each other by the change of numbering. Finally the first one of each class satisfies the condition (G) in §10, for which (i) $t=2$, $t_0=0$, (ii) $t=2$, $t_0=1$, (iii) $t=2$, $t_0=2$ and (iv) $t=1$, $t_0=0$ by taking $\rho=(\rho_0, \rho_1, \rho_2)=(a, b, c)$. Therefore they are not isomorphic to each other by Theorem 8.

Case 3.3. $a\neq b$ and $c=0$. The arguments about classes (i)~(iv) of case 3.2 are still valid in this case (notice that Theorem 8 holds even if one element of the common distribution is zero). Therefore we have the classes (i)$\sim$(iv) just putting $c=0$ in case 3.2. The fifth class of case 3.2 is divided into three classes: (va) the ones isomorphic to Bernoulli endomorphism with (a, b), which satisfy the condition (P′) in §4; (vb) the ones which are reduced to the (2×2)-Markov endomorphism with transition matrix $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ using 1°; (vc) the ones which are reduced to the (2×2)-Markov endomorphism with transition matrix $\begin{pmatrix} b & a \\ a & b \end{pmatrix}$ using 1°. Examples of Markov endomorphisms of these three classes are as follows:

$$\text{(va)} \begin{pmatrix} a & b & 0 \\ a & 0 & b \\ a & b & 0 \end{pmatrix} \qquad \text{(vb)} \begin{pmatrix} 0 & a & b \\ a & 0 & b \\ 0 & b & a \end{pmatrix} \qquad \text{(vc)} \begin{pmatrix} 0 & a & b \\ 0 & b & a \\ b & a & 0 \end{pmatrix}.$$

Except the class (va), no endomorphism of this case is Bernoulli. We discussed already in A that classes (vb) and (vc) are not mutually isomorphic. Thus it remains to verify that classes (vb), (vc) and (i)$\sim$(iv) are not isomorphic. Suppose that (vb) and (i) are isomorphic, for example. Then we have two Markov generators $P=\{p_0, p_1, p_2\}$ and $Q=\{q_0, q_1\}$ with transition matrices belonging to class (i) and $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ respectively. Let $R=R_T$ be the proper partition for T defined in §3 (Part I). It is easy to see $R\vee P=P\vee T^{-1}P$ and hence $P\vee \bigvee_{k=0}^{\infty} T^{-k}R=\varepsilon$, and $Q\vee \bigvee_{k=0}^{\infty} T^{-k}R=\varepsilon$ analogously. Therefore $\mathrm{dist}(P|r)=\mathrm{dist}(\varepsilon|r)=\mathrm{dist}(Q|r)$ for a.e. $r\in \bigvee_{k=0}^{\infty} T^{-k}R$, but it is easy to see that $\mathrm{dist}(P|r)=(1/3, 1/3, 1/3)$ and $\mathrm{dist}(Q|r)=(1/2, 1/2)$ for a.e. r which is a contradiction. Thus we have the classification (i), (ii), (iii), (iv), (va), (vb) and (vc).

References

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