Publ. RIMS, Kyoto Univ.
9 (1974), 397–428

The Principle of Limiting Absorption for the Non-selfadjoint Schrödinger Operator in $\mathbb{R}^N$ ($N \neq 2$)

By

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§0. Introduction

Let us consider the Schrödinger operator

$$L = -\sum_{j=1}^{N} D_j D_j + Q \qquad \left(D_j = \frac{\partial}{\partial x_j} + i b_j(x)\right) \tag{0.1}$$

in the whole N-dimensional euclidean space $\mathbb{R}^N$. Here $b_j(x)$, $j=1, 2,\ldots, N$, are real-valued functions on $\mathbb{R}^N$ and Q denotes the multiplication operator by a complex-valued function $Q(x)$ on $\mathbb{R}^N$. Let $u_{\lambda\pm i\mu}$ be solutions of the equations

$$(L-(\lambda \pm i\mu))u = f \qquad (\lambda \in \mathbb{R},\ \mu > 0). \tag{0.2}$$

If the limits

$$\lim_{\mu \downarrow 0} u_{\lambda\pm i\mu} = u_{\pm} \tag{0.3}$$

exist and $u_{\pm}$ solve the equation

$$(L-\lambda)u = f, \tag{0.4}$$

then it is said that the limiting absorption principle holds for L. The meaning of the limit is to be determined suitably[1)].

Ikebe-Saitō [7], which will be refered to as I-S, deals with the case that $Q(x)$ is a real-valued function, i.e., the case that L is a formally self-adjoint Schrödinger operator. In I-S $Q(x)$ is assumed to be decomposed

Communicated by S. Matsuura, February 1, 1973.
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1) For the literature of the limiting absorption principle see, e.g., Èidus [3].

as $Q(x)=V_0(x)+V(x)$ such that $V_0(x)$ and $V(x)$ are real-valued functions and

$$(0.5)\qquad \begin{cases} V_0(x)=O(|x|^{-\delta}), & \dfrac{\partial V_0}{\partial |x|}=O(|x|^{-1-\delta}), \\ V(x)=O(|x|^{-1-\delta}) & \end{cases}$$

at infinity with $\delta>0$. Some asymptotic conditions at infinity are imposed on $b_j(x)$, too[2]. Then it is shown in I–S that the limiting absorption principle holds for any $\lambda\neq 0$.

In the case that L is non-selfadjoint, however, it is possible that for some $\lambda\neq 0$ there exists a non-trivial solution of the equation (0.4) with $f=0$, where the limiting absorption principle does not hold for such a singular point λ, and it is important to study the set σ of all singular points of L.

The limiting absorption method for non-selfadjoint Schrödinger operators is closely related to the spectral and scattering theory for them. Mochizuki [11] considered the operator (0.1) with $b_j(x)\equiv 0$ and a complex-valued function $Q(x)=O(|x|^{-2-\delta})$ in $\mathbb{R}^3$. He showed, among others, that the set $\sigma\cup\{0\}$ is a compact set of $\mathbb{R}$, and that for any open interval e which does not contain any singular point the "spectral measure"

$$(0.6)\qquad E(e)=\lim_{\mu\downarrow 0}\frac{1}{2\pi i}\int_e \{(L-(\lambda+i\mu))^{-1}-(L-(\lambda-i\mu))^{-1}\}\,d\lambda$$

is well-defined. Further, he developed the spectral and scattering theory for L. Goldstein [4] obtained the similar results for the operator (0.1) in $\mathbb{R}^N$ ($N=1, 2,\ldots$) with $b_j(x)\equiv 0$ and $Q(x)=(1+|x|)^{-\alpha}g(x)$, where $\alpha>N/2$ and $g(x)\in L_2(\mathbb{R}^N)\cap L_\infty(\mathbb{R}^N)$[3]. Pavlov [12] showed that $\sigma\cup\{0\}$ is a compact set with the Lebesgue measure 0 for the operator (0.1) in $\mathbb{R}^N$ ($N=1, 3$) with $b_j(x)\equiv 0$ and $x^2Q(x)\in L_1(\mathbb{R})$ ($N=1$) or $Q(x)$, $\sum_{j=1}^{3}\left|\frac{\partial}{\partial x_j}Q(x)\right|=O(|x|^{-3-\delta})$ ($N=3$). In [12] some asymptotic relations with respect to the set σ are also given. Recently Ikebe [5] has shown

2) For more precise conditions imposed on $Q(x)$ and $b_j(x)$, see Assumption 1.1 of I–S.

3) To be more precise, for $N=3$ we may assume that $Q(x)=O(|x|^{-2-\gamma})$ and for $N=1$ we may assume that $Q(x)\in L_2(\mathbb{R}^1)$.

that the singular points form a discrete set for the reduced wave equation with a complex refractive index $n(x)$

$$\Delta u+\kappa^2 n(x)^2 u=0 \tag{0.7}$$

in $\mathbb{R}^3$, where $\mathrm{Re}\, n(x)\geqq 1$, $\mathrm{Im}\, n(x)\geqq 0$, and $n(x)-1$ has a compact support. A spectral theory for the equation (0.7) is also developed in [5].

In the present paper we shall assume that $N\neq 2$ and $Q(x)$ can be decomposed as $Q(x)=V_0(x)+V(x)$ such that $V_0(x)$ and $V(x)$ are real-valued and complex-valued functions on $\mathbb{R}^N$, respectively, and they satisfy (0.5)[4]. In addition $V_0(x)$ and $V(x)$ are assumed to be bounded on $\mathbb{R}^N$. Let us set

$$\mathbb{C}_+=\{\kappa=\kappa_1+i\kappa_2/\kappa_2\geqq 0\}-\{0\}, \tag{0.8}$$

and for a real β let us define a Hilbert space $L_{2,\beta}$ by

$$L_{2,\beta}=\{f(x)/\int_{\mathbb{R}^N}(1+|x|)^{2\beta}|f(x)|^2 dx<\infty\} \tag{0.9}$$

with its inner product

$$(f, g)_\beta=\int_{\mathbb{R}^N}(1+|x|)^{2\beta}f(x)\cdot\overline{g(x)}\,dx \tag{0.10}$$

and norm

$$\|f\|_\beta=[(f,f)_\beta]^{1/2}. \tag{0.11}$$

Σ denotes the set of all $\kappa\in\mathbb{C}_+$ for which there exists a non-trivial solution $u\in L_{2,-\frac{1+\varepsilon}{2}}$ of the equation

$$(L-\kappa^2)u=0 \tag{0.12}$$

with the "radiation condition"

$$\int_{|x|\geqq 1}(1+|x|)^{-1+\varepsilon}|\mathscr{D}u|^2 dx<\infty \qquad (0<\varepsilon\leqq\min(1, \delta/2)), \tag{0.13}$$

where

4) For the case $N=2$, see Saitō [14].

(0.14) $$|\mathscr{D}u|^2=\sum_{j=1}^{N}|\mathscr{D}_j u|^2, \qquad \mathscr{D}_j u=D_j u+\frac{N-1}{2|x|}\cdot\frac{x_j}{|x|}u-i\kappa\frac{x_j}{|x|}u.$$

Then it will be shown that for any pair $(\kappa, f)\in(\boldsymbol{C}_+-\Sigma)\times L_{2,\frac{1+\varepsilon}{2}}$ the equation $(L-\kappa^2)u=f$ with the radiation condition (0.13) has a unique solution $u=u(\kappa, f)\in L_{2,-\frac{1+\varepsilon}{2}}$, which is an $L_{2,-\frac{1+\varepsilon}{2}}$-valued continuous function on $(\boldsymbol{C}_+-\Sigma)\times L_{2,\frac{1+\varepsilon}{2}}$, and the estimate

(0.15) $$\|u(\kappa, f)\|_{-\frac{1+\varepsilon}{2}}\leqq\frac{C}{|\kappa|}\|f\|_{\frac{1+\varepsilon}{2}} \qquad ((\kappa, f)\in K\times L_{2,\frac{1+\varepsilon}{2}})$$

holds with a positive constant $C=C(K, L, \varepsilon)$ depending only on K, L, and ε, where K is a (possibly unbounded) closed subset of $\boldsymbol{C}_+$ such that $K\subset\{\kappa\in\boldsymbol{C}_+/|\kappa|\geqq a\}$ with some $a>0$ and $K\cap\Sigma=\phi$. As for the properties of Σ we shall show that Σ is a bounded set of $\boldsymbol{C}_+$ and that $(\Sigma\cap\boldsymbol{R})\cup\{0\}$ is a compact null set.

Let us outline the contents of the present paper. We shall state the main results of this paper in §1. §2 is devoted to giving some a priori estimates for L. In §3 we shall show the limiting absorption principle for L. The properties of the singular points of L will be studied in §4.

Finally let us give the list of the notation which will be used in the following sections without further reference.

$\boldsymbol{R}$: real numbers.

$\boldsymbol{C}$: complex numbers.

$\boldsymbol{C}_+=\{\kappa=\kappa_1+i\kappa_2\in\boldsymbol{C}/\kappa_2\geqq 0\}-\{0\}$.

$M_a=\{\kappa=\kappa_1+i\kappa_2\in\boldsymbol{C}/|\kappa|>a,\ \kappa_2>0\}$ $\qquad (a>0)$.

$D_j=\partial_j+ib_j(x)$ $\qquad \left(\partial_j=\dfrac{\partial}{\partial x_j}\right)$.

$\mathscr{D}_j=\mathscr{D}_j^{(\kappa)}=D_j+\dfrac{N-1}{2|x|}\tilde{x}_j-i\kappa\tilde{x}_j$ $\qquad (\tilde{x}_j=x_j/|x|)$.

$Du=(D_1u, D_2u,\dots, D_Nu)$.

$\mathscr{D}u=(\mathscr{D}_1u, \mathscr{D}_2u,\dots, \mathscr{D}_Nu)$.

$D_ru=\sum_{j=1}^{N}D_ju\cdot\tilde{x}_j$ $\qquad (r=|x|)$.

$\mathscr{D}_ru=\sum_{j=1}^{N}\mathscr{D}_ju\cdot\tilde{x}_j$ $\qquad (r=|x|)$.

$B_r = \{x \in \mathbb{R}^N / |x| \leqq r\} \qquad (r > 0).$

$B_{rs} = \{x \in R^N / r \leqq |x| \leqq s\} \qquad (0 < r < s).$

$E_r = \{x \in \mathbb{R}_N / |x| \geqq r\} \qquad (r > 0).$

$S_r = \{x \in \mathbb{R}^N / |x| = r\} \qquad (r > 0).$

$\left[\int_{S_T} - \int_{S_t}\right] f \, dS = \int_{S_T} f \, dS - \int_{S_t} f \, dS.$

$L_{2,\beta}(G)$ ($\beta \in \mathbb{R}$) denotes the class of all functions f on G such that $(1+|x|)^\beta f$ is square integrable over G. Here G is a measurable set of $\mathbb{R}^N$. The norm $\| \ \|_{\beta,G}$ and inner product $(\ , \ )_{\beta,G}$ of $L_{2,\beta}(G)$ are defined by

$$\|f\|_{\beta,G} = \left[\int_G (1+|x|)^{2\beta} |f(x)|^2 dx\right]^{1/2}$$

and

$$(f, g)_{\beta,G} = \int_G (1+|x|)^{2\beta} f(x) \overline{g(x)} \, dx,$$

respectively. We set $L_{2,\beta}(\mathbb{R}^N) = L_{2,\beta}$, $\| \ \|_{\beta,\mathbf{R}^N} = \| \ \|_\beta$ and $(\ , \ )_{\beta,\mathbf{R}^N} = (\ , \ )_\beta$. When $\beta = 0$, we shall omit the subscript 0 as in $L_2(G)$, $(\ , \ )_G$ etc.

H_m is all L_2 functions with L_2 distribution derivatives up to the m-th order, inclusive.

C^m is all m-times continuously differentiable functions on $\mathbb{R}^N$.

C_0^∞ denotes the set of all infinitely continuously differentiable functions with compact support in $\mathbb{R}^N$.

M_{loc} is the class of all locally M functions.

$\mathbb{B}(X, Y)$ denotes the set of all bounded linear operators on X into Y, X and Y being Banach spaces. We set $\mathbb{B}(X, X) = \mathbb{B}(X)$.

$\mathbb{C}(X, Y)$ is all compact operators on X into Y, where X, Y are Banach spaces. We set $\mathbb{C}(X, X) = \mathbb{C}(X)$.

§1. Outline of Results

We consider the second order, elliptic partial differential operator

$$Lu = -\sum_{j=1}^N D_j D_j u + Q(x) u \tag{1.1}$$

in $\mathbb{R}^N$, where N is a positive integer and the complex-valued function $Q(x)$ on $\mathbb{R}^N$ can be decomposed as

(1.2) $$Q(x)=V_0(x)+V(x).$$

Throughout this paper we assume the following

Assumption 1.1. *Let N be a positive integer such that $N\neq 2$.*

(V_0) *$V_0(x)$ is a real-valued, measurable function on $\mathbb{R}^N$ such that the radial derivative $\frac{\partial V_0}{\partial|x|}$ exists and the estimates*

$$|V_0(x)|\leqq C(1+|x|)^{-\delta}, \qquad \frac{\partial V_0}{\partial|x|}\leqq C(1+|x|)^{-1-\delta}$$

hold on the whole space $\mathbb{R}^N$ with positive constants C and δ.

(V) *$V(x)$ is a complex-valued, measurable function on $\mathbb{R}^N$ which satisfies*

$$|V(x)|\leqq C(1+|x|)^{-1-\delta}$$

for all $x\in\mathbb{R}^N$ with the same C and δ as in (V_0). We set $V_1(x)=\mathrm{Re}\,V(x)$ and $V_2(x)=\mathrm{Im}\,V(x)$[5]*, i.e., $V(x)=V_1(x)+iV_2(x)$.*

(B) *$b_j(x)$ is a real-valued C^1 function on $\mathbb{R}^N$ satisfying*

$$|B_{jk}(x)|\leqq C(1+|x|)^{-1-\delta}$$

for $x\in\mathbb{R}^N$, j, $k=1, 2,\ldots, N$, where C and δ are as above, and $B_{jk}(x)=\partial_j b_k(x)-\partial_k b_j(x)$.

From Assumption 1.1 we see that for all $u\in H_{2,loc}$ Lu is well-defined in the sense of distributions and we have $Lu\in L_{2,loc}$. In the sequel we set

(1.3) $$D(L)=H_{2,loc}.$$[6]

Definition 1.2. Let ε be a positive number satisfying $0<\varepsilon\leqq 1$ and $0<\varepsilon\leqq\delta/2$. We call a value $\kappa\in\mathbb{C}_+$ a singular point of L, if the equation

(1.4) $$\begin{cases}(L-\kappa^2)u=0,\\ \|\mathscr{D}u\|_{\frac{-1+\varepsilon}{2},E_1}<\infty,\ u\in L_{2,-\frac{1+\varepsilon}{2}}\cap H_{2,loc}\end{cases}$$

has a non-trivial solution u. We denote by $\Sigma=\Sigma(L)=\Sigma(L,\varepsilon)$ the set of all singular points of L. The set $\Sigma_{\mathbb{R}}$ is defined by

5) Re α and Im α mean the real and imaginary parts respectively.

6) $D(T)$ is the domain of T.

$$\Sigma_{\mathbb{R}} = \Sigma \cap \mathbb{R}. \tag{1.5}$$

Let us note that Σ does not contain the origin 0. As has been shown in I-S (Lemma 1.9), if $V(x)$ is a real-valued function, then $\Sigma \subset \{\kappa = ib/b > 0\}$, and in particular $\Sigma_{\mathbb{R}} = \phi$[7]. But we can easily construct a complex-valued function $Q(x)$ satisfying Assumption 1.1 for which we have $\Sigma_{\mathbb{R}} \neq \phi$.

In order to state the main results of this paper we consider a restriction of the differential operator L in L_2. It follows from the well-known results of Ikebe-Kato [6] that the restriction of the differential operator $-\sum_{j=1}^{N} D_j D_j$ to C_0^∞

$$C_0^\infty \ni u \longmapsto -\sum_{j=1}^{N} D_j D_j u \in L_2 \tag{1.6}$$

has a unique self-adjoint extension A in L_2. Denote the domain of A by D_0. Then $u \in L_2$ belongs to D_0 if and only if $u \in L_2 \cap H_{2,loc}$ and $-\sum_{j=1}^{N} D_j D_j u \in L_2$. Since the operator defined by (1.6) is symmetric and non-negative, it has the Friedrichs extension which should be equal to A. Hence we have $\sum_{j=1}^{N} \|D_j u\|^2 < \infty$ for any $u \in D_0$. Thus we have

$$D_0 = \{u \in L_2 / u \in H_{2,loc},\ -\sum_{j=1}^{N} D_j D_j u \in L_2,\quad D_j u \in L_2,\ j = 1, 2, \dots, N\}. \tag{1.7}$$

Define an operator H by

$$\begin{cases} Hu = Au + Qu \\ D(H) = D_0. \end{cases} \tag{1.8}$$

Q is a bounded linear operator on L_2, and hence H is a densely defined, closed linear operator. When $z \in \mathbb{C}$ belongs to the resolvent set $\rho(H)$ of H, the bounded liniear operator

$$R(z) = (H - z)^{-1} \tag{1.9}$$

7) In I-S we assumed the unique continuation property for L. In this paper, however, the boundedness of $Q(x)$ is assumed and hence the unique continuation property for L holds good (see, e.g., Aronszajn [1]).

on L_2 is defined. $R(z)$ transforms L_2 onto D_0.

The following theorems are the main results of the present paper. First we shall state a theorem on the properties of Σ.

Theorem 1.3. *Let Assumption* 1.1 *be fulfilled.*

(i) *Then Σ is a bounded set of $\mathbb{C}_+$, i.e., there exists a positive number μ_0 such that*

$$\Sigma \subset \{\kappa \in \mathbb{C}_+ / |\kappa| \leqq \mu_0\}. \tag{1.10}$$

Σ_R is a bounded set of $\mathbb{R}$ with the Lebesgue measure 0.

(ii) *For any $a>0$ $\Sigma \cap \bar{M}_a$ is a compact set of $\mathbb{C}_+$, $\bar{M}_a$, being the closure of M_a, i.e.,*

$$\bar{M}_a = \{\kappa = \kappa_1 + i\kappa_2 / |\kappa| \geqq a,\ \kappa_2 \geqq 0\}. \tag{1.11}$$

Further, $\Sigma - \Sigma_R$ is an isolated, bounded set having no limit point in $\{\kappa \in \mathbb{C}_+ / \mathrm{Im}\,\kappa > 0\}$.

(iii) *Let $\kappa \in \mathbb{C}_+$, $\mathrm{Im}\,\kappa > 0$. Then $\kappa \in \Sigma$ if and only if κ^2 belongs to the point spectrum of H. We have $\kappa^2 \in \rho(H)$ for $\kappa \in \mathbb{C}_+ - \Sigma$ with $\mathrm{Im}\,\kappa > 0$.*

Theorem 1.4 *(limiting absorption principle). Let Assumption* 1.1 *be fulfilled, and let $\kappa \in \mathbb{R} - \Sigma_R \cup \{0\}$. Then for any $f \in L_{2,\frac{1+\varepsilon}{2}}$ and any sequence $\{\kappa_n\}$ which satisfies*

$$\kappa_n \longrightarrow \kappa,\ \kappa_n \in \mathbb{C}_+ - \Sigma \text{ and } \mathrm{Im}\,\kappa_n > 0, \tag{1.12}$$

there exists

$$\lim_{n\to\infty} R(\kappa_n^2) f = u(\kappa, f) \tag{1.13}$$

in $L_{2,-\frac{1+\varepsilon}{2}}$. The limit $u(\kappa, f)$ thus obtained is independent of the choice of the sequence $\{\kappa_n\}$ and is a unique solution of the equation

$$\begin{cases} (L-\kappa^2)u = f, \\ \|\mathscr{D}u\|_{\frac{-1+\varepsilon}{2}, E_1} < \infty,\ u \in H_{2,loc} \cap L_{2,-\frac{1+\varepsilon}{2}}. \end{cases} \tag{1.14}$$

Theorem 1.4 can be obtained immediately from the following more general theorem.

Theorem 1.5. *Let Assumption* 1.1 *be fulfilled. Assume that* M *is an open set of* $\mathbb{C}$ *such that* $\bar{M}\cap\Sigma=\phi$ *and* $M\subset M_a$ *with some* $a>0$, $\bar{M}$ *being the closure of* M.

(i) *Then there exists a unique solution* $u=u(\kappa,f)$ *of the equation* (1.14) *for any pair* $(\kappa,f)\in\bar{M}\times L_{2,\frac{1+\varepsilon}{2}}$.

(ii) *The solution* $u=u(\kappa,f)$, $(\kappa,f)\in\bar{M}\times L_{2,\frac{1+\varepsilon}{2}}$, *satisfies the estimates*

$$\|u\|_{-\frac{1+\varepsilon}{2}}\leqq\frac{C}{|\kappa|}\|f\|_{\frac{1+\varepsilon}{2}}, \tag{1.15}$$

$$\|\mathscr{D}u\|_{\frac{-1+\varepsilon}{2}}+\gamma(N)\left\|\frac{u}{r}\right\|_{B_1}\leqq C\|f\|_{\frac{1+\varepsilon}{2}} \tag{1.16}$$

and

$$\|u\|_{-\frac{1+\varepsilon}{2},E_\rho}\leqq\frac{C}{|\kappa|}(1+\rho)^{-\varepsilon/2}\|f\|_{\frac{1+\varepsilon}{2}}\qquad(\rho\geqq1), \tag{1.17}$$

where $\kappa=\kappa_1+i\kappa_2$, $C=C(M,L,\varepsilon)$ *is a positive constant*[8], *and* $\gamma(N)$ *is a non-negative constant such that* $\gamma(1)=\gamma(3)=0$ *and* $\gamma(N)>0$ *for* $N\geqq4$.

(iii) *If we define an operator* $(L-\kappa^2)^{-1}$ *by*

$$(L-\kappa^2)^{-1}f=u(\kappa,f)\qquad(f\in L_{2,\frac{1+\varepsilon}{2}}) \tag{1.18}$$

for $\kappa\in\bar{M}$, *then the operator* $(L-\kappa^2)^{-1}$ *is a* $\mathbb{B}(L_{2,\frac{1+\varepsilon}{2}},L_{2,-\frac{1+\varepsilon}{2}})$*-valued, continuous function on* $\bar{M}$, *and we have*

$$\|(L-\kappa^2)^{-1}\|\leqq\frac{C}{|\kappa|}\qquad(\kappa\in\bar{M}), \tag{1.19}$$

where $C=C(M,L,\varepsilon)$ *is a positive constant and* $\|(L-\kappa^2)^{-1}\|$ *means the operator norm of* $(L-\kappa^2)^{-1}$ *in* $\mathbb{B}(L_{2,\frac{1+\varepsilon}{2}},L_{2,-\frac{1+\varepsilon}{2}})$.

(iv) $(L-\kappa^2)^{-1}\in\mathbb{C}(L_{2,\frac{1+\varepsilon}{2}},L_{2,-\frac{1+\varepsilon}{2}})$ *for all* $\kappa\in\bar{M}$,

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8) Here and in the sequel we mean by $C=C(A,B,\cdots)$ that C is a positive constant depending only on A, $B,\cdots$.

(v) $(L-\kappa^2)^{-1}$ *is a* $\mathbb{B}(L_{2,\frac{1+\varepsilon}{2}})$*-valued, analytic function on* M.

Remark 1.6. (iv) of the above theorem can be generalized as follows: Let $\{f_n\}$ be any bounded sequence of $L_{2,\frac{1+\varepsilon}{2}}$, and let $\{\kappa_n\}$ be any sequence contained in $\bar{M}$. Then the sequence $\{(L-\kappa_n^2)^{-1}f_n\}$ is relatively compact in $L_{2,-\frac{1+\varepsilon}{2}}$.

We shall show some properties of the spectrum $\sigma(H)$ of H. Let T be a linear closed operator in a Banach space X. Its point spectrum, continuous spectrum and residual spectrum are denoted by $\sigma_p(T)$, $\rho_c(T)$ and $\sigma_r(T)$, respectively. Further, we define the essential spectrum $\sigma_e(T)$ as the set of all $z\in\mathbb{C}$ satisfying the following condition:

$$(1.20)\qquad \begin{cases} \text{There exists a sequence } \{u_n\}\subset D(T) \text{ such that } \{u_n\} \text{ is} \\ \text{a normalized sequence in } X, \text{ and we have } (T-z)u_n\to 0 \\ \text{strongly in } X \text{ and } u_n\to 0 \text{ weakly in } X \text{ as } n\to\infty. \end{cases}$$

Theorem 1.7. *Let* H *be as defined in* (1.7).

(i) *Then* $\sigma_e(H)=\sigma_e(A)\subset[0,\infty)$.

(ii) $\sigma(H)\cap(\mathbb{C}-[0,\infty))\subset\sigma_p(H)$ *and* $\sigma_p(H)\cap(0,\infty)=\phi$. *The eigenvalues in* $\mathbb{C}-[0,\infty)$, *if they exist, are of finite multiplicity and they form an isolated, bounded set having no limit point in* $\mathbb{C}-[0,\infty)$.

(iii) $\sigma(H)\cap(0,\infty)=\sigma_e(H)\cap(0,\infty)=\sigma_c(H)\cap(0,\infty)$.

(iv) $\sigma_r(H)$ *consists of at most only one point* $z=0$.

Remark 1.8. If it is assumed that $b_j(x)=0$, then, as has been mentioned in Mochizuki [11], we obtain $\sigma_r(H)=\phi$.

The rest of this paper will be concerned with the proofs of these theorems. Theorem 1.5 and Remark 1.6 will be proved in §3. We shall show Theorems 1.3 and 1.7 in §4.

§2. A Priori Estimates for L

This section is devoted to showing some a priori estimates for the formally non-selfadjoint Schrödinger operator L. These results will be useful in the following sections to justify the limiting absorption principle for L.

Let us introduce a function space $D_\kappa = D_{\kappa,\varepsilon}$ by

$$D_\kappa = \{u \in H_{2,loc}/u \in L_{2,-\frac{1+\varepsilon}{2}},\ (L-\kappa^2)u \in L_{2,\frac{1+\varepsilon}{2}}\}, \tag{2.1}$$

where $\kappa \in C_+$ with $\mathrm{Im}\kappa > 0$ and ε is as in Definition 1.2. We shall first investigate the properties of $u \in D_\kappa$. To this end we prepare

Lemma 2.1. (i) *Let $\alpha \in \mathbb{R}$ and $\kappa \in \mathbb{C}$. Let $u \in H_{2,loc} \cap L_{2,\alpha}$ and $(L-\kappa^2)u \in L_{2,\alpha}$. Then we have $D_j u \in L_{2,\alpha}$, $j = 1, 2, \ldots, N$.*

(ii) *Let $\alpha, \beta \in \mathbb{R}, \alpha \leqq \beta$, and $\kappa \in C_+$ with $\mathrm{Im}\,\kappa > 0$. Let $u \in H_{2,loc} \cap H_{2,\alpha}$ such that $D_j u \in L_{2,\alpha}$ $(j = 1, 2, \ldots, N)$ and $(L-\kappa^2)u \in L_{2,\beta}$. Then we have $u \in L_{2,\gamma}$, where $r = \min(\alpha + 1/2, \beta)$.*

Proof. The proof of (i) and (ii) with $\mathrm{Re}\,\kappa \neq 0$ is quite similar to the one of Lemma 2.4 of I–S, and hence we shall omit it. Let us prove (ii) with $\mathrm{Re}\,\kappa = 0$. We set $\kappa = ib$, $b > 0$, and $(L-\kappa^2)u = (L+b^2)u = f$. Take the real part of $((L+b^2)u, \phi u)_{B_r} = (f, \phi u)_{B_r}$, $\phi(r) = (1+r)^{2\gamma}$. Then we obtain, using partial integration,

$$\begin{aligned}\int_{B_r} \phi |Du|^2 dx + b^2 \int_{B_r} \phi |u|^2 dx + \int_{B_r} \phi(V_0 + V_1)|u|^2 dx \\ = \mathrm{Re}\int_{S_r} \phi(D_r \bar{u})u\ dS + \mathrm{Re}\int_{B_r} \frac{d\phi}{dr}(D_r u)\bar{u}\ dx \\ + \mathrm{Re}\int_{B_r} \phi f \bar{u}\ dx.\end{aligned} \tag{2.2}$$

Since $V_0(x) + V_1(x) = O(|x|^{-\delta})$ at infinity, there exists $R > 0$ such that for any $r > R$

$$\int_{B_{Rr}} \phi |V_0 + V_1|\ |u|^2 dx \leqq \frac{b^2}{2} \int_{B_{Rr}} \phi |u|^2 dx \leqq \frac{b^2}{2} \int_{B_r} \phi |u|^2 dx. \tag{2.3}$$

Hence, noting that

$$\varliminf_{r\to\infty} \int_{S_r} \phi |(D_r u)\bar{u}|\, dS = 0, \tag{2.4}$$

we have $\|\phi^{1/2}u\| < \infty$, i.e., $u \in L_{2,\gamma}$, which completes the proof. Q.E.D.

Lemma 2.2. *Let $N \geqq 3$ and let $u \in H_1(\mathbb{R}^N)_{loc}$.*

Then $u/|x| \in L_2(\boldsymbol{R}^N)_{loc}$.

Proof. In the case $N=3$ the proof is given in Courant-Hilbert [2], p. 446–p. 447. When $N \geqq 4$, we can prove this lemma in quite a similar way as in the case $N=3$. Q.E.D.

Using these lemmas we obtain

Proposition 2.3. (i) *Let* $\kappa \in \boldsymbol{C}_+$ *with* $\mathrm{Im}\,\kappa > 0$ *and let* $u \in D_\kappa$. *Then* u, $D_j u \in L_{2, \frac{1+\varepsilon}{2}}$ $(j=1, 2,\dots, N)$.

(ii) *Let* $\kappa \in \boldsymbol{C}_+$ *with* $\mathrm{Im}\,\kappa > 0$ *and let* $u \in D_\kappa$. *Then we have the estimate*

$$\|u\|_{\frac{1+\varepsilon}{2}} \leqq C\{\|u\|_{-\frac{1+\varepsilon}{2}} + \|(L-\kappa^2)u\|_{\frac{1+\varepsilon}{2}}\} \tag{2.5}$$

with a positive constant $C = C(\kappa, L, \varepsilon)$. *As a function of* κ, C *is bounded when* κ *moves in any compact set contained in* $\{\kappa = \kappa_1 + i\kappa_2 / \kappa_2 > 0\}$.

(iii) *For any* $\kappa \in M_a$ $(a>0)$ *and any* $u \in D_\kappa$ *we have*

$$\kappa_2 \|u\|_{\frac{1-\varepsilon}{2}} \leqq C\{\|u\|_{-\frac{1+\varepsilon}{2}} + \|\mathscr{D} u\|_{\frac{-1+\varepsilon}{2}} + \|(L-\kappa^2)u\|_{\frac{1+\varepsilon}{2}}\} \tag{2.6}$$

with a positive constant $C = C(a, L, \varepsilon)$, *where we set* $\kappa_2 = \mathrm{Im}\,\kappa$.

Proof. (i) follows directly from Lemma 2.1. Let us show (ii). Taking the real and imaginary part of $((L-\kappa^2)u, \phi u) = (f, \phi u)$, $\phi(r) = (1+r)^{2\alpha}$ $\left(\alpha \leqq \frac{1+\varepsilon}{2}\right)$, we obtain

$$\begin{aligned} \int_{\boldsymbol{R}^N} \phi |Du|^2 dx = &-\mathrm{Re}\int_{\boldsymbol{R}^N} \frac{d\phi}{dr}(D_r u)\bar{u}\, dx \\ &+\mathrm{Re}\int_{\boldsymbol{R}^N} \phi f \bar{u}\, dx \\ &-\int_{\boldsymbol{R}^N} \phi(V_0 + V_1 - \kappa_1^2 + \kappa_2^2)|u|^2 dx \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} 2\kappa_1\kappa_2 \int_{\boldsymbol{R}^N} \phi |u|^2\, dx = &\mathrm{Im}\int_{\boldsymbol{R}^N} \frac{d\phi}{dr}(D_r u)\bar{u}\, dx \\ &+\int_{\boldsymbol{R}^N} \phi V_2 |u|^2\, dx - \mathrm{Im}\int_{\boldsymbol{R}^N} \phi f \bar{u}\, dx, \end{aligned} \tag{2.8}$$

respectively, where we have used partial integration. Since $\frac{d\phi}{dr} = \alpha(1+r)^{-1}\phi$ and $|D_r u| \leqq |Du|$, it follows from (2.7) that

$$\|Du\|_\alpha^2 \leqq |\alpha| \|Du\|_{\alpha-1}\|u\|_{\alpha-1} + \|f\|_\alpha \|u\|_\alpha + (2C+\kappa_1^2)\|u\|_\alpha^2, \tag{2.9}$$

where C is the constant given in Assumption 1.1, which implies that

$$\|Du\|_\alpha \leqq C_1\|u\|_\alpha + \|f\|_\alpha \qquad (C_1 = C_1(\kappa_1, L, \alpha)). \tag{2.10}$$

If $2\kappa_1^2 \geqq \kappa_2^2$, then we obtain from (2.8)

$$\|u\|_\alpha^2 \leqq \frac{1}{\sqrt{2}\kappa_2^2}\{|\alpha| \|Du\|_{\alpha-1}\|u\|_{\alpha-1} + C\|u\|_\alpha \|u\|_{\alpha-1-2\varepsilon} + \|f\|_\alpha \|u\|_\alpha\}, \tag{2.11}$$

where we should note that $|V(x)| \leqq C(1+|x|)^{-1-2\varepsilon}$. If $2\kappa_1^2 < \kappa_2^2$, then we obtain from (2.7)

$$\int_{\boldsymbol{R}^N} \phi(\kappa_2^2 - \kappa_1^2 + V_0 + V_1)|u|^2\, dx \leqq |\alpha| \|Du\|_{\alpha-1}\|u\|_{\alpha-1} + \|f\|_\alpha \|u\|_\alpha. \tag{2.12}$$

The left-hand side of (2.12) estmated as follows: Since $V_0(x)+V_1(x) = O(|x|^{-\delta})$ at infinity, we can find $R = R(\kappa_2) > 0$ such that $|V_0(x)+V_1(x)| \leqq \kappa_2^2/4$ for $x \in E_R$. Thus we have, noting that $\kappa_2^2 - \kappa_1^2 \geqq \kappa_2^2/2$,

$$\begin{aligned} \int_{\boldsymbol{R}^N} \phi(\kappa_2^2 - \kappa_1^2 + V_0 + V_1)|u|^2\, dx &\geqq \int_{\boldsymbol{R}^N} \phi\left(\frac{\kappa_2^2}{2} + V_0 + V_1\right)|u|^2\, dx \\ &\geqq \frac{\kappa_2^2}{4}\|u\|_\alpha^2 - \int_{B_R} \phi|V_0 + V_1|\ |u|^2\, dx. \end{aligned} \tag{2.13}$$

It follows from (2.12) and (2.13) that

$$\frac{\kappa_2^2}{4}\|u\|_\alpha^2 \leqq |\alpha| \|Du\|_{\alpha-1}\|u\|_{\alpha-1} + \|f\|_\alpha \|u\|_\alpha + 2C(1+R)^2\|u\|_{\alpha-1}^2, \tag{2.14}$$

where we have made use of the estimate

$$(2.15)\qquad \int_{B_R}\phi|V_0+V_1||u|^2dx\leqq 2C(1+R)^2\int_{B_R}\phi(1+|x|)^{-2}|u|^2\,dx$$
$$\leqq 2C(1+R)^2\|u\|_{\alpha-1}^2.$$

Thus we obtain from (2.11) and (2.14)

$$(2.16)\qquad \|u\|_\alpha\leqq C_2\{\|Du\|_{\alpha-1}+\|u\|_{\alpha-1}+\|f\|_\alpha\}$$
$$(C_2=C_2(\kappa_2,\,L,\,\varepsilon)).$$

From (2.16) with $\alpha=(1+\varepsilon)/2$ and (2.10) with $\alpha=(-1+\varepsilon)/2$ we have

$$(2.17)\qquad \|u\|_{\frac{1+\varepsilon}{2}}\leqq C_3\{\|u\|_{\frac{-1+\varepsilon}{2}}+\|f\|_{\frac{1+\varepsilon}{2}}\}$$
$$(C_3=C_3(\kappa,\,L,\,\varepsilon)).$$

Similarly, setting $\alpha=(-1+\varepsilon)/2$ in (2.16) and $\alpha=(-3+\varepsilon)/2$ in (2.10), we obtain

$$(2.18)\qquad \|u\|_{\frac{-1+\varepsilon}{2}}\leqq C_4\{\|u\|_{\frac{-3+\varepsilon}{2}}+\|f\|_{\frac{1+\varepsilon}{2}}\}$$
$$(C_4=C_4(\kappa,\,L,\,\varepsilon)).$$

By noting that $0<\varepsilon\leqq 1$ and C_3 and C_4 are bounded when κ moves in any compact set in $\{\kappa\in\mathbb{C}/\kappa_2>0\}$, (ii) follows (2.17) and (2.18). Finally we shall show (iii). From the relations $D_ru=\mathscr{D}_ru-\frac{N-1}{2r}u+i\kappa u$ and (2.8) it follows that

$$(2.19)\qquad \kappa_2\|u\|_{\frac{1-\varepsilon}{2}}^2=\frac{1}{2\kappa_1}\Big\{\mathrm{Im}\int_{\mathbb{R}^N}\frac{d\phi}{dr}(\mathscr{D}_ru)\bar{u}\,dx+\int_{\mathbb{R}^N}\phi V_2|u|^2\,dx$$
$$+\kappa_1\int_{\mathbb{R}^N}\frac{d\phi}{dr}|u|^2\,dx-\mathrm{Im}\int_{\mathbb{R}^N}\phi f\bar{u}\,dx.$$

Here we have made use of the fact that $\mathscr{D}_ru\in L_{2,\frac{1+\varepsilon}{2}}$ which is obtained from (i) and Lemma 2.2. (2.6) with $\kappa\in M_a\cap\{\kappa/|\kappa_1|\geqq a/\sqrt{3}\}=J_1$ follows easily from (2.19). Let us now assume that $\kappa\in M_a\cap\{\kappa/|\kappa_1|<a/\sqrt{3}\}=J_2$. Since $2\kappa_1^2<\kappa_2^2$ and $\kappa_2>(a\sqrt{2})/\sqrt{3}$ for $\kappa\in J_2$, (2.6) with $\kappa\in J_2$ follows from (2.14) with $\alpha=(1-\varepsilon)/2$ and (2.10) with $\alpha=-(1+\varepsilon)/2$. Thus we have proved Proposition 2.3 completely. Q.E.D.

In the following two Propositions 2.4 and 2.6 a priori estimates for functions belonging to D_κ will be given.

Proposition 2.4[9]. *Let $a>0$. Then there exists a positive constant $C=C(a, L, \varepsilon)$ such that for $\kappa\in M_a$ and $u\in D_\kappa$*

$$(2.20)\qquad ||\mathscr{D}u||_{\frac{-1+\varepsilon}{2}}+\gamma(N)\left\|\frac{u}{r}\right\|_{B_1}\leqq C\{||u||_{-\frac{1+\varepsilon}{2}}+||f||_{\frac{1+\varepsilon}{2}}\},$$

where $f=(L-\kappa^2)u$ and $\gamma(N)$ is a non-negative constant which satisfies $\gamma(1)=\gamma(3)=0$ and $\gamma(N)>0$ for $N\geqq 4$.

This proposition will be proved after showing the next lemma.

Lemma 2.5. *Let $\phi(r)$ be a real-valued continuous function on $(0, \infty)$ such that $\phi(r)$ is piecewise continuously differentiable. Then we have for any $u\in H_{2,loc}$*

$$(2.21)\qquad \int_{B_{tT}}\left(\kappa_2\phi+\frac{1}{2}\cdot\frac{\partial\phi}{\partial r}\right)|\mathscr{D}u|^2\,dx$$

$$+\int_{B_{tT}}\left(\frac{\phi}{r}-\frac{\partial\phi}{\partial r}\right)(|\mathscr{D}u|^2-|\mathscr{D}_r u|^2)\,dx$$

$$+\int_{B_{tT}}c_N\left\{\kappa_2\frac{\phi}{r^2}-\frac{1}{2}\frac{\partial}{\partial r}\left(\frac{\phi}{r^2}\right)\right\}|u|^2\,dx$$

$$=\int_{B_{tT}}\left\{\frac{1}{2}\left(\frac{\partial\phi}{\partial r}V_0+\phi\frac{\partial V_0}{\partial r}\right)-\kappa_2\phi V_0\right\}|u|^2dx$$

$$-\mathrm{Re}\int_{B_{tT}}\phi Vu\cdot\overline{\mathscr{D}_r u}\,dx+\mathrm{Re}\int_{B_{tT}}\phi f\overline{\mathscr{D}_r u}\,dx$$

$$-\mathrm{Im}\int_{B_{tT}}\sum_{j,k=1}^{N}\phi B_{jk}(\mathscr{D}_j u)\tilde{x}_k\cdot\bar{u}\,dx$$

$$-\frac{1}{2}\left[\int_{S_T}-\int_{S_t}\right]\phi\left\{|\mathscr{D}u|^2-2|\mathscr{D}_r u|^2\right.$$

$$\left.+\left(V_0+\frac{c_N}{r^2}\right)\right\}|u|^2\,dS,$$

9) Cf. Lemma 1.7 of I-S.

where $f=(L-\kappa^2)u$, $0<t<T<\infty$, $\kappa=\kappa_1+i\kappa_2$ *and*

(2.22) $$c_N=\frac{1}{4}(N-1)(N-3).$$

Proof. Integrate $\phi(L-\kappa^2)u\cdot\bar{u}=\phi f\cdot\bar{u}$ over B_{tT} and take the real part. Then we can proceed as in the proof of Lemma 2.2 of I-S to obtain (2.21).

Q.E.D.

Proof of Proposition 2.4. Set in (2.21)

(2.23) $$\phi(r)=\begin{cases} r & (0\leqq r\leqq 1) \\ \frac{1}{2^\varepsilon}(1+r)^\varepsilon & (r>1), \end{cases}$$

and $0<t<1<T<\infty$. Since it is easy to see that

(2.24) $$\begin{cases} \frac{1}{2}\cdot\frac{\partial\phi}{\partial r}=\begin{cases} \frac{1}{2} & (0\leqq r\leqq 1) \\ \frac{\varepsilon}{2^{1+\varepsilon}}(1+r)^{-1+\varepsilon} & (r>1), \end{cases} \\ \frac{\phi}{r}-\frac{\partial\phi}{\partial r}\geqq 0, \qquad |\mathscr{D}u|-|\mathscr{D}_r u|\geqq 0, \\ -\frac{1}{2}c_N\cdot\frac{\partial}{\partial r}\left(\frac{\phi}{r^2}\right)\begin{cases} =\frac{1}{2}c_N\frac{1}{r^2} & (0\leqq r\leqq 1) \\ \geqq -\frac{\varepsilon}{2^{1+\varepsilon}}c_N(1+r)^{-1-\varepsilon} & (r>1), \end{cases} \end{cases}$$

we obtain

(2.25) the left-hand side of (2.21)

$$\geqq \frac{\varepsilon}{2^{1+\varepsilon}}\|\mathscr{D}u\|^2_{\frac{-1+\varepsilon}{2},B_{tT}}+\frac{c_N}{2}\left\|\frac{u}{r}\right\|^2_{B_1}$$

$$-\frac{\varepsilon}{2^{1+\varepsilon}}c_N\|u\|^2_{-\frac{1+\varepsilon}{2},B_{1T}}.$$

We shall now estimate the right-hand side of (2.21).

(2.26) the right-hand side of (2.21)

$$\leqq C_1\|u\|^2_{-\frac{1+\varepsilon}{2}}+C_2\kappa_2\|u\|^2_{-\frac{\varepsilon}{2}}$$

$$+C_3\int_{\boldsymbol{R}^N}(1+r)^{-1-\varepsilon}|u||\mathscr{D}u|\,dx$$

$$+C_4\int_{\boldsymbol{R}^N}(1+r)^{\varepsilon}|f||\mathscr{D}u|\,dx$$

$$+\frac{1}{2}\int_{S_T}\phi(2|\mathscr{D}_r u|^2+|V_0||u|^2)\,dS$$

$$+\frac{1}{2}\int_{S^t}\phi\Big(|\mathscr{D}u|^2+|V_0||u|^2+c_N\Big|\frac{u}{r}\Big|^2\Big)dS$$

$$=I_1+I_2+I_3+I_4+I_5(T)+I_6(t)$$

with positive constants $C_j=C_j(L, \varepsilon)$, $j=1, 2, 3, 4$, where we have made use of Assumption 1.1 and the fact that $c_N\geqq 0$. Noting that

$$\lim_{T\to\infty} I_5(T)=0, \qquad \lim_{t\to 0} I_6(t)=0, \tag{2.27}$$

which follow from the fact that $\mathscr{D}_j u$, $u\in L_2$ and $u/r\in L_{2,loc}$ (Lemma 2.2 and Proposition 2.3, (i)), we obtain from (2.25) and (2.26) by using Schwarz' inequality

$$\frac{\varepsilon}{2^{1+\varepsilon}}\|\mathscr{D}u\|^2_{\frac{-1+\varepsilon}{2}}+\frac{c_N}{2}\Big\|\frac{u}{r}\Big\|^2_{B_1} \tag{2.28}$$

$$\leqq C\{\|u\|^2_{-\frac{1+\varepsilon}{2}}+\kappa_2\|u\|_{\frac{1-\varepsilon}{2}}\|u\|_{-\frac{1+\varepsilon}{2}}$$

$$+\|\mathscr{D}u\|_{\frac{-1+\varepsilon}{2}}(\|u\|_{-\frac{1+\varepsilon}{2}}+\|f\|_{\frac{1+\varepsilon}{2}})\}$$

with $C=C(L, \varepsilon)$, which, together with (iii) of Proposition 2.3, yields (2.20).

Q.E.D.

The above proposition will be used to show the next

Proposition 2.6 [10]. *Let $a>0$ and let $\kappa\in M_a$, $u\in D_\kappa$. Then we have*

10) Cf. Lemma 1.8 of I-S.

(2.29) $$\|u\|^2_{-\frac{1+\varepsilon}{2},E_\rho}$$

$$\leqq C(1+\rho)^{-\varepsilon}\left(\frac{1}{|\kappa|}\|u\|^2_{-\frac{1+\varepsilon}{2}}+\frac{1}{|\kappa|^2}\|f\|^2_{\frac{1+\varepsilon}{2}}+\frac{1}{|\kappa|}\|u\|_{-\frac{1+\varepsilon}{2}}\|f\|_{\frac{1+\varepsilon}{2}}\right)$$

for any $\rho\geqq 0$ *with* $C=C(a, L, \varepsilon)$, *where* $f=(L-\kappa^2)u$.

Proof. We can find positive numbers b and c such that $M_a\subset J_1\cup J_2$, where

(2.30) $$\begin{cases} J_1=\{\kappa=\kappa_1+i\kappa_2\in M_a/\kappa_1^2-\frac{1}{2}\kappa_2^2\geqq b^2\}, \\ J_2=\{\kappa=\kappa_1+i\kappa_2\in M_a/\kappa_2^2-\kappa_1^2\geqq c^2\}. \end{cases}$$

Assume that $\kappa\in J_1$. We have from the definition of $\mathscr{D}_r u$

(2.31) $$|\mathscr{D}_r u(x)|^2=\left|D_r u(x)+\frac{N-1}{2r}u(x)+\kappa_2 u(x)\right|^2$$

$$+\kappa_1^2|u(x)|^2-2\kappa_1\,\mathrm{Im}\,\{(D_r u)\bar{u}\}$$

$$\geqq\kappa_1^2|u(x)|^2-2\kappa_1\,\mathrm{Im}\,\{(D_r u)\bar{u}\},$$

and hence we have the inequality

(2.32) $$\kappa_1^2\int_{S_r}|u|^2\,dx\leqq\int_{S_r}|\mathscr{D}_r u|^2 dS+2\kappa_1\,\mathrm{Im}\int_{S_r}D_r u\cdot\bar{u}\,dS.$$

Combining (2.32) with

(2.33) $$-\mathrm{Im}\int_{S_r}D_r u\cdot\bar{u}\,dS+\int_{B_r}V_2|u|^2dx-2\kappa_1\kappa_2\int_{B_r}|u|^2\,dx$$

$$=\mathrm{Im}\int_{B_r}f\cdot\bar{u}\,dx,$$

which is obtained by integrating $(L-\kappa^2)u\cdot\bar{u}=f\cdot\bar{u}$ over B_r and taking the imaginary part, we have

(2.34) $$\kappa_1^2\int_{S_r}|u|^2dS\leqq\int_{S_r}|\mathscr{D}_r u|^2dS+2|\kappa_1|\int_{B_r}|V_2||u|^2dx$$

$$+2|\kappa_1|\int_{Br}|f||u|\,dx$$

$$\leqq \int_{Sr} |\mathscr{D}u|^2 dS + 2C|\kappa_1| \|u\|^2_{-\frac{1+\varepsilon}{2}}$$

$$+2|\kappa_1| \|f\|_{\frac{1+\varepsilon}{2}} \|u\|_{-\frac{1+\varepsilon}{2}},$$

C being given as in Assumption 1.1. Multiply both sides of (2.34) by $(1+r)^{-1-\varepsilon}$ and integrate from ρ to ∞. Then we arrive at

$$\|u\|^2_{-\frac{1+\varepsilon}{2}, E_\rho} \leqq \frac{1}{\kappa_1^2} \|\mathscr{D}u\|^2_{-\frac{1+\varepsilon}{2}, E_\rho} \tag{2.35}$$

$$+\frac{1}{\varepsilon|\kappa_1|}(1+\rho)^{-\varepsilon}\{2C\|u\|^2_{-\frac{1+\varepsilon}{2}}$$

$$+2\|f\|_{\frac{1+\varepsilon}{2}}\|u\|_{-\frac{1+\varepsilon}{2}}\}$$

$$\leqq (1+\rho)^{-\varepsilon}\left\{\frac{1}{\kappa_1^2}\|\mathscr{D}u\|_{-\frac{1+\varepsilon}{2}} + C_0\left(\frac{1}{|\kappa_1|}\|u\|^2_{-\frac{1+\varepsilon}{2}} + \frac{1}{|\kappa_1|}\|u\|_{-\frac{1+\varepsilon}{2}}\|f\|_{\frac{1+\varepsilon}{2}}\right)\right\}$$

with a positive constant $C_0 = C_0(a, L, \varepsilon)$. From (2.35) and Proposition 2.4 we get (2.29) with κ replaced by κ_1. Since $\kappa \in J_1$, it follows that

$$|\kappa| \leqq |\kappa_1| + \kappa_2 \leqq (1+\sqrt{2})|\kappa_1|. \tag{2.36}$$

Thus we arrive at (2.29) with $\kappa \in J_1$. Next we shall assume that $\kappa \in J_2$. From the relation

$$|\mathscr{D}_r u|^2 = |D_r u + \frac{N-1}{2r}u - i\kappa_1 u|^2 + \kappa_2^2 |u|^2 \tag{2.37}$$

$$+2\cdot\mathrm{Re}\left\{\left(D_r u + \frac{N-1}{2r}u - i\kappa_1 u\right)(\kappa_2 \bar{u})\right\}$$

we obtain

$$\kappa_2^2 \int_{S_r} |u|^2 dS \leqq \int_{S_r} |\mathscr{D}_r u|^2 \, dS - 2\kappa_2 \cdot \mathrm{Re} \int_{S_r} (D_r u)\bar{u} \, dS. \tag{2.38}$$

In order to estimate the second term of the right-hand side of (2.38) we integrate $(L-\kappa^2)u \cdot \bar{u} = f \cdot \bar{u}$ over B_r and take the real part, whence follows

$$(2.39)\qquad -\mathrm{Re}\int_{S_r}(D_r u)\bar{u}\,dS = -\int_{B_r}|Du|^2 dx - \int_{B_r}(V_0+V)|u|^2 dx$$

$$-(\kappa_2^2-\kappa_1^2)\int_{B_r}|u|^2\,dx + \mathrm{Re}\int_{B_r} f\bar{u}\,dx$$

$$\leqq \int_{B_r}(2C(1+|x|)^{-\delta}-c^2)|u|^2\,dx + \int_{B_r}|f|\,|u|\,dx$$

$$\leqq 2C\|u\|_{B_R}^2 + \|f\|_{\frac{1+\varepsilon}{2}}\|u\|_{-\frac{1+\varepsilon}{2}}$$

$$\leqq 2C(1+R)^{1+\varepsilon}\|u\|_{-\frac{1+\varepsilon}{2}}^2 + \|f\|_{\frac{1+\varepsilon}{2}}\|u\|_{-\frac{1+\varepsilon}{2}}$$

with $R=R(c)$, where we used the technique quite similar to the one used in obtaining (2.13). Then we can proceed as in the case $\kappa\in J_1$ to obtain from (2.38) and (2.39) the estimate (2.29) with κ replaced by κ_2. Since $|\kappa|\leqq 2|\kappa_2|$ for $\kappa\in J_2$, (2.29) holds good. Therefore we have (2.29) for all $\kappa\in M_a$. Q.E.D.

To establish the following theorem is our main purpose in this section.

Theorem 2.7. *Let Assumption* 1.1 *be satisfied and let* Σ *be the set of the singular points of* L *defined in Definition* 1.2. *Assume that* M *is an open set of* $\mathbf{C}$ *such that* $M\subset M_a$ *with some* $a>0$ *and* $\bar{M}\cap\Sigma=\phi$, *where* $\bar{M}$ *is the closure of* M *in* $\mathbf{C}$. *Let* $\kappa\in M$ *and let* $u\in D_\kappa$. *Then there exists a positive constant* $C=C(M, L, \varepsilon)$ *such that we have the estimates*

$$(2.40)\qquad \|\mathscr{D}u\|_{\frac{-1+\varepsilon}{2}} + \gamma(N)\left\|\frac{u}{r}\right\|_{B_1} \leqq C\|f\|_{\frac{1+\varepsilon}{2}},$$

$$(2.41)\qquad \|u\|_{-\frac{1+\varepsilon}{2},E_\rho} \leqq \frac{C}{|\kappa|}(1+\rho)^{-\varepsilon/2}\|f\|_{\frac{1+\varepsilon}{2}}\qquad (\rho\geqq 0),$$

where $f=(L-\kappa^2)u$ *and* $\gamma(N)$ *is as in Proposition* 2.4. *In particular, setting* $\rho=0$ *in* (2,41), *we have*

$$(2.42)\qquad \|u\|_{-\frac{1+\varepsilon}{2}} \leqq \frac{C}{|\kappa|}\|f\|_{\frac{1+\varepsilon}{2}}.$$

Proof. Since (2.40) and (2.41) follow from (2.42), Propositions 2.4 and 2.6, it suffices to show (2.42). Let us now suppose that (2.42) is false. Then we can find sequences $\{\kappa_n\} \subset M$ and $\{u_n\} \subset D_{\kappa n}$, $n=1, 2,\dots$, such that

$$(2.43) \qquad \|u_n\|_{-\frac{1+\varepsilon}{2}}=1, \qquad \frac{1}{|\kappa_n|}\|f_n\|_{\frac{1+\varepsilon}{2}} \leqq \frac{1}{n}, \qquad f_n=(L-\kappa_n^2)u_n.$$

The sequence $\{\kappa_n\}$ satisfies one of the following conditions (1) and (2):

(1) $\{\kappa_n\}$ is a bounded sequence. In this case we may assume with no loss of generality that $\kappa_n \to \kappa$, $n\to\infty$, with some $\kappa \in \bar{M}$.

(2) $\{\kappa_n\}$ is an unbounded sequence. Then it may be assumed that $|\kappa_n| \to \infty$, $n\to\infty$.

Let us consider the condition (1). We recall the usual interior estimate

$$(2.44) \qquad \|Du\|_{B_s} \leqq C(\|u\|_{B_r}+\|f\|_{B_r})$$

$$(0<s<r,\ u\in H_{2,loc},\ f=(L-\kappa^2)u,\ \kappa\in K)$$

with $C=C(r-s, L, K)>0$, K being a bounded set of $\mathbf{C}$. Since the boundedness of the sequences $\{\|u_n\|_{B_r}\}$ and $\{\|f_n\|_{B_r}\}$ follows from (2.43) and the condition (1), we can apply (2.44) to show the boundedness of $\{\|Du_n\|_{B_s}\}$ for each $s>0$, whence we obtain the boundedness of $\{\|\partial_j u_n\|_{B_s}\}$ for any $s>0$ and any $j=1, 2,\dots, N$. Therefore $\{u_n\}$ is relatively compact in $L_{2,loc}$, and hence we may assume that

$$(2.45) \qquad u_n \longrightarrow u \quad \text{in } L_{2,loc}$$

with $u\in L_{2,loc}$. Then the interior estimate (2.44) can be applied again to see that

$$(2.46) \qquad u_n \longrightarrow u \quad \text{in } H_{1,loc}.$$

From (2.43) we obtain

$$(2.47) \qquad \|f_n\|_{\frac{1+\varepsilon}{2}} \leqq \frac{|\kappa_n|}{n} \longrightarrow 0 \qquad (n \longrightarrow 0),$$

which, together with (2.45), implies that u is a weak solution of the equation $(L-\kappa^2)u=0$. Hence, as has been shown in Ikebe-Kato [6], $u\in H_{2,loc}$ and u is seen to be a strong solution of $(L-\kappa^2)u=0$. We can now apply Proposition 2.6 to obtain

(2.48) $$\|u_n\|^2_{-\frac{1+\varepsilon}{2}, E_\rho} = O(\rho^{-\varepsilon}) \qquad (\rho \geqq 0)$$

uniformly for n, and hence $u_n \to u$ in $L_{2,-\frac{1+\varepsilon}{2}}$ and $\|u\|_{-\frac{1+\varepsilon}{2}} = 1$. On the other hand, since the sequence $\{\|\mathscr{D} u_n\|_{\frac{-1+\varepsilon}{2}, E_1}\}$ is bounded (Proposition 2.4) and $u_n \to u$ in $H_{1,loc}$, we have $\|\mathscr{D} u\|_{\frac{-1+\varepsilon}{2}, E_1} < \infty$. Thus u is a solution of the equation $(L - \kappa^2)u = 0$, $\|\mathscr{D} u\|_{\frac{-1+\varepsilon}{2}, E_1} < \infty$, whence follows $u = 0$ since $\kappa \in \bar{M}$ does not belong to the set of the singular points. This contradicts $\|u\|_{-\frac{1+\varepsilon}{2}} = 1$. Finally let us consider the case that $\{\kappa_n\}$ satisfies the condition (2). In this case by the use of Proposition 2.6 with $\rho = 0$ and (2.43) we have

(2.49) $$1 = \|u_n\|^2_{-\frac{1+\varepsilon}{2}} \leqq C\left\{\frac{1}{|\kappa_n|} + \frac{1}{|\kappa_n|^2}\|f_n\|^2_{\frac{1+\varepsilon}{2}} + \frac{1}{|\kappa_n|}\|f_n\|_{\frac{1+\varepsilon}{2}}\right\}$$

$$\leqq C\left(\frac{1}{|\kappa_n|} + \frac{1}{n^2} + \frac{1}{n}\right) \to 0$$

as $n \to \infty$, which is a contradiction. Q.E.D.

§3. The Limiting Absorption Principle

The results obtained in the preceding section can be applied to justify the principle of limiting absorption for the non-selfadjoint Schrödinger operator L. We shall now prove Theorem 1.5. Let M be an open set of $\boldsymbol{C}$. In Theorem 1.5 M is assumed to satisfy the following condition

(I) $M \subset M_a$ with some $a > 0$ and $\bar{M} \cap \Sigma = \phi$.

In this section we shall assume that, in addition to (I), M satisfies the following

(II) $\kappa^2 \in \rho(H)$ for any $\kappa \in M$.

It will be shown in §4 that (I) implies (II). And then Theorem 1.5 will be proved completely (see Remark 4.8 in §4). The proof of Theorem 1.5 for M satisfying (I) and (II) will be divided into several steps.

Proof of (i) *and* (ii) *of Theorem* 1.5 (*existence, uniqueness and estimates*). Let $(\kappa, f) \in M \times L_{2,\frac{1+\varepsilon}{2}}$, and set

(3.1) $$u=u(\kappa, f)=R(\kappa^2)f, \qquad R(\kappa^2)=(H-\kappa^2)^{-1}.$$

Since $u\in D_0$ and $f\in L_{2,\frac{1+\varepsilon}{2}}$, u is a unique solution of

(3.2) $$(L-\kappa^2)u=f, \qquad \|\mathscr{D}u\|_{\frac{-1+\varepsilon}{2},E_1}<\infty, \qquad u\in H_{2,loc}\cap L_{2,-\frac{1+\varepsilon}{2}},$$

and $u\in D_\kappa$. Theorem 2.7 can be applied to show that the estimates (2.40), (2.41) and (2.42) are valid for $u=u(\kappa, f)$. Now let $(\kappa, f)\in \bar{M}\times L_{2,\frac{1+\varepsilon}{2}}$, and let $\{\kappa_n\}$ be a sequence in M converging to κ. We put $u_n=u(\kappa_n, f)$. From (2.42) we see that $\{u_n\}$ is a bounded sequence in $L_{2,-\frac{1+\varepsilon}{2}}$, and hence $\{u_n\}$ is a bounded sequence in $L_2(B_t)$ for any $t>0$, too. By using the interior estimate (2.44) we can find a subsequence $\{u_n'\}$ of $\{u_n\}$ such that

(3.3) $$u_n'\longrightarrow u \qquad \text{in } H_{1,loc}$$

with $u\in H_{1,loc}$. u can be easily seen to be a weak solution of the equation $(L-\kappa^2)u=f$, whence follows that $u\in H_{2,loc}$ and u is a strong solution as in the proof of Theorem 2.7. The estimates (2.40) ~ (2.42) enable us to show that u_n' coverges to u in $L_{2,-\frac{1+\varepsilon}{2}}$ and $\|\mathscr{D}u\|_{\frac{-1+\varepsilon}{2},E_1}<\infty$. Thus u is a solution of the equation (3.2) and the estimates (1.15), (1.16) and (1.17) for u can be easily shown, since each u_n' satisfies the estimates (2.40) ~ (2.42). The uniqueness of u follows from the fact that $\kappa\notin\Sigma$.

Q.E.D.

Now that (i) and (ii) of Theorem 1.5 have been established, the operator $(L-\kappa^2)^{-1}$ can be defined by (1.18)

(3.4) $$(L-\kappa^2)^{-1}f=u(\kappa, f) \qquad (f\in L_{2,\frac{1+\varepsilon}{2}}).$$

Proof of Theorem 1.5, (iii), (iv) *and Remark* 1.6. It follows from (1.15) that $(L-\kappa^2)^{-1}$ is a bounded linear operator from $L_{2,\frac{1+\varepsilon}{2}}$ into $L_{2,-\frac{1+\varepsilon}{2}}$ and (1.19) holds good. Now let us turn to the proof of Remark 1.6. If the sequence $\{u(\kappa_n, f_n)\}$ contains a subsequence $\{u(\kappa_n', f_n')\}$ such that $|\kappa_n'|\to\infty$, then by (1.15) and the boundedness of $\{f_n\}$ in $L_{2,\frac{1+\varepsilon}{2}}$ we have

(3.5) $$\|u(\kappa_n', f_n')\|_{-\frac{1+\varepsilon}{2}}\leqq\frac{C}{|\kappa_n'|}\|f_n'\|_{\frac{1+\varepsilon}{2}}\to 0,$$

i.e., there exists a subsequence of $\{u(\kappa_n, f_n)\}$ which converges to 0 in $L_{2,-\frac{1+\varepsilon}{2}}$. If $\{\kappa_n\}$ is a bounded sequence, then we may assume with no loss of generality that

$$(3.6)\qquad \begin{cases} \kappa_n \longrightarrow \kappa \\ f_n \longrightarrow f \quad \text{weakly in } L_{2,\frac{1+\varepsilon}{2}} \qquad (n \longrightarrow \infty) \end{cases}$$

with $\kappa \in \bar{M}$ and $f \in L_{2,\frac{1+\varepsilon}{2}}$. Using the interior estimate (2.44) and the estimates given in (ii) of Theorem 1.5, we can proceed as in the proof of (i) of Theorem 1.5 to show that we can find a subsequence of $\{u(\kappa_n, f_n)\}$ converging to $u(\kappa, f)$, the solution of (3.2), in $L_{2,-\frac{1+\varepsilon}{2}}$. This completes the proof of Remark 1.6. (iv) of Theorem 1.5 follows from Remark 1.6. Finally let us prove the continuity in κ of $(L-\kappa^2)^{-1}$ in $\mathbf{B}(L_{2,\frac{1+\varepsilon}{2}}, L_{2,-\frac{1+\varepsilon}{2}})$. Suppose that $(L-\kappa^2)^{-1}$ is not continuous at $\kappa \in \bar{M}$. Then we obtain sequences $\{\kappa_n\} \subset \bar{M}$ and $\{f_n\} \subset L_{2,\frac{2+\varepsilon}{2}}$ satisfying

$$(3.7)\qquad \begin{cases} \kappa_n \longrightarrow \kappa, \qquad \|f_n\|_{\frac{1+\varepsilon}{2}} = 1, \\ \|(L-\kappa^2)^{-1} f_n - (L-\kappa_n^2)^{-1} f_n\|_{-\frac{1+\varepsilon}{2}} \geqq \gamma \end{cases}$$

with $\kappa \in \bar{K}$ and $\gamma > 0$. We put $u_n = (L-\kappa^2)^{-1} f_n$ and $v_n = (L-\kappa_n^2)^{-1} f_n$. $\{f_n\}$ is assumed to converge weakly to some $f \in L_{2,\frac{1+\varepsilon}{2}}$ with no loss of generality. By almost the same argument as used in the proof of Remark 1.6 it can be shown that there is a sequence $\{n'\}$ of positive integers such that

$$(3.8)\qquad u_{n'} \longrightarrow u(\kappa, f), \qquad v_{n'} \longrightarrow u(\kappa, f)$$

in $L_{2,-\frac{1+\varepsilon}{2}}$ as $n' \longrightarrow \infty$. We therefore have

$$(3.9)\qquad \|u_{n'} - v_{n'}\|_{-\frac{1+\varepsilon}{2}} \longrightarrow 0 \qquad (n' \longrightarrow \infty),$$

which contradicts (3.7). Q.E.D.

Proof of Theorem 1.5, (v) (*analyticity of* $(L-\kappa^2)^{-1}$). Let $\kappa \in M$.

Then it follows from (ii) of Proposition 2.3 and (1.15) that

$$\|u\|_{\frac{1+\varepsilon}{2}} \leqq C\|f\|_{\frac{1+\varepsilon}{2}} \qquad (f \in L_{2,\frac{1+\varepsilon}{2}},\ u=(L-\kappa^2)^{-1}f) \tag{3.10}$$

with $C=C(\kappa)$ which is locally bounded on M. Hence for $\kappa \in M$ the operator $(L-\kappa^2)^{-1}$ can be regarded as a bounded linear operator on $L_{2,\frac{1+\varepsilon}{2}}$ and its operator norm is locally bounded on M. Since $(L-\kappa^2)^{-1}f=R(\kappa^2)f$ for $(\kappa, f) \in M \times L_{2,\frac{1+\varepsilon}{2}}$, the analyticity of $(L-\kappa^2)^{-1}$ on M is clear from the resolvent equation

$$R(\kappa^2)-R(\mu^2)=(\mu^2-\kappa^2)R(\kappa^2)R(\mu^2), \tag{3.11}$$

and the proof is complete. Q.E.D.

§4. The Properties of Σ

We shall now prove Theorems 1.3 and 1.7 which are concerned with the set of the singular points of L and the spectrum of H. At the same time the gap in the proof of Theorem 1.5 given in §3 will be filled.

First we shall show some properties of the essential spectrum $\sigma_e(T)$ of a closed operator T in a Hilbert space. Since the proof is not difficult, we shall omit it.

Lemma 4.1. (i) $\rho(T) \cap \sigma_e(T)=\phi$.

(ii) *$\sigma_e(T) \supset \sigma_c(T)$ and $\sigma_e(T) \supset \sigma_p^\infty(T)$, where $\sigma_p^\infty(T)$ denotes the point spectrum of T with infinite multiplicity.*

(iii) *$\sigma_e(T)$ contains all accumulation points of $\sigma_c(T)$ and $\sigma_p(T)$.*

T^* denotes the adjoint operator of T and we set $(T^*)^*=T^{**}$ if they are well-defined. The next lemma which shows the invariance of $\sigma_e(T)$ by T-compact perturbation is well-known.

Lemma 4.2. *Let T be a densely defined, closed linear operator in a Hilbert space with the densely defined T^* and $T^{**}=T$. Assume, further, that B is a T-compact linear operator with $B^{**}=B$. Then we have*

$$\sigma_e(T)=\sigma_e(T+B). \tag{4.1}$$

For the definition of T-compactness see, e.g., Kato [9], p. 194.

Since $H=A+Q$, where A is a self-adjoint operator and Q is a bounded linear operator, we have

$$\begin{cases} H^*=A+Q^*, \qquad Q^*u=\overline{Q(x)}u \\ D(H^*)=D_0 \end{cases} \tag{4.2}$$

and $H^{**}=H$. Another self-adjoint operator H_0 in L_2 is defined by

$$H_0=A+V_0, \tag{4.3}$$

which is the restriction in L_2 of a differential operator

$$L_0=-\sum_{j=1}^{N} D_jD_j+V_0=L-V. \tag{4.4}$$

Using the interior estimate (2.44) and the fact that $Q(x)$, $V_0(x)=O(|x|^{-\delta})$ at infinity, we can see that Q and V_0 are A-compact operators. Thus we have from Lemm 4.2

Proposition 4.3. $\sigma_e(H)=\sigma_e(H_0)=\sigma_e(A)\subset[0, \infty)$.

Obviously L_0 satisfies Assumption 1.1 with $V=0$. Therfore the result given in §2 and §3 are still valid for L_0. The set of the singular point of L_0 is denoted by Σ_0. The following proposition is due to Ikebe-Uchiyama [8] (see also Remark on the proof of Lemma 2.5. of I-S).

Proposition 4.4. (i) $\sigma_p(A)\cap(0, \infty)=\sigma_p(H_0)\cap(0, \infty)=\sigma_p(H)\cap(0, \infty)=\sigma_p(H^*)\cap(0, \infty)=\phi$.

(ii) $\Sigma_0\cap \boldsymbol{R}=\phi$.

Set

$$\boldsymbol{C}'_+=\{\kappa\in\boldsymbol{C}_+/\mathrm{Im}\,\kappa>0\} \tag{4.5}$$

and

$$\Sigma'=\Sigma\cap\boldsymbol{C}'_+, \qquad \Sigma'_0=\Sigma_0\cap\boldsymbol{C}'_+. \tag{4.6}$$

We shall show some properties of Σ' and Σ'_0.

Proposition 4.5. (i) $\kappa\in\Sigma'$ $[\kappa\in\Sigma'_0]$ *if and only if* $\kappa\in\boldsymbol{C}'_+$ *and*

$\kappa^2 \in \sigma_p(H)$ [$\kappa \in \mathbb{C}'_+$ *and* $\kappa^2 \in \sigma_p(H_0)$].

(ii) Σ' [Σ'_0] *forms a discrete set (having no limit point in* $\mathbb{C}'_+$).

(iii) $\Sigma_0 = \Sigma'_0$ *is a bounded set of* $\{\kappa = ib/b > 0\}$. *And we have* $\kappa^2 \in \rho(H_0)$ *for any* $\kappa \in \mathbb{C}'_+ - \Sigma'_0$.

Proof. Let $\kappa \in \Sigma'$ and let u be a non-trivial solution of the equation (1.4). Then by (i) of Proposition 2.3 $u \in D_0$, and hence u is an eigenfunction of H associated with the eigenvalue κ^2. Conversely, if $u \in D_0$ is an eigenfunction of H associated with the eigenvalue κ^2, $\kappa \in \mathbb{C}'_+$, then u is a non-trivial solution of (1.4). Thus we have proved (i). (ii) follows from (i), Lemma 4.1 and Proposition 4.3. By (ii) of Proposition 4.4 we have $\Sigma_0 = \Sigma'_0$. Since H_0 is a self-adjoint operator which is bounded below, $\sigma(H_0)$ is contained in the real line and is bounded below. This implies, together with (i), that $\Sigma_0 = \Sigma'_0$ is a bounded set of $\{\kappa = ib/b > 0\}$. Let $\kappa \in \mathbb{C}'_+ - \Sigma'_0$. Then $\mathrm{Im}\,\kappa^2 \neq 0$, $\kappa^2 \notin \sigma_p(H_0)$ by (i) and $\kappa^2 \notin \sigma_c(H_0)$ by Lemma 4.1 and Proposition 4.3, and hence $\kappa^2 \in \rho(H_0)$. Thus we have proved (iii), and the proof is complete. Q.E.D.

Let M_0 be an open set of $\mathbb{C}$ such that $\bar{M}_0 \cap \Sigma_0 = \phi$ and $M_0 \subset M_a$ with $a > 0$. It follows from (iii) of Proposition 4.5 that M_0 satisfies the conditions (I) and (II) with L replaced by L_0 which are given in §3. Therefore Theorem 1.5 holds good with L and M replaced by L_0 and M_0. For $\kappa \in \mathbb{C}_+ - \Sigma_0$ we can thus define an operator $T(\kappa)$ by

$$T(\kappa) = -(L_0 - \kappa^2)^{-1} V. \tag{4.7}$$

Since $-V \in \mathbb{B}(L_{2,-\frac{1+\varepsilon}{2}}, L_{2,\frac{1+\varepsilon}{2}})$, $T(\kappa)$ is a $\mathbb{B}(L_{2,-\frac{1+\varepsilon}{2}})$-valued continuous fuctction on $\mathbb{C}_+ - \Sigma_0$ with the estimate for its operator norm

$$\| T(\kappa) \| \leqq \frac{C}{|\kappa|} \qquad (\kappa \in M_0,\ C = C(M_0, L, \varepsilon)) \tag{4.8}$$

(Themorem 1.5, (iii)). Moreover we have $T(\kappa) \in \mathbb{C}(L_{2,-\frac{1+\varepsilon}{2}})$ for $\kappa \in \mathbb{C}_+ - \Sigma_0$ (Theorem 1.5, (iv)), and hence the spectrum $\sigma(T(\kappa))$ is discrete in $\mathbb{C} - \{0\}$ and each non-zero eigenvalue of $T(\kappa)$ is of finite multiplicity. From these results a characterization of the set Σ can be obtained as follows.

Proposition 4.6. *Let* $\kappa \in \mathbb{C}_+ - \Sigma_0$. *Then* $\kappa \in \Sigma$ *if and only if*

$1 \in \sigma(T(\kappa))$, *i.e., there exists a non-trivial solution* $u \in L_{2,-\frac{1+\varepsilon}{2}}$ *of the eqation* $T(\kappa)u = u$.

Proof. Assume that $\kappa \in \Sigma$, $\kappa \notin \Sigma_0$. Then there exists a non-trivial solution u of the equation (1.4). Since (1.4) can be rewritten in the form

$$\text{(4.9)} \qquad \begin{cases} (L_0 - \kappa^2)u = -Vu \\ \|\mathscr{D}u\|_{\frac{-1+\varepsilon}{2}, E_1} < \infty, \qquad u \in H_{2,loc} \cap L_{2,-\frac{1+\varepsilon}{2}}. \end{cases}$$

we obtain $u = -(L_0 - \kappa^2)^{-1} Vu = T(\kappa)u$. Conversely let us assume the existence of a non-trivial $u \in L_{2,-\frac{1+\varepsilon}{2}}$ satisfying $u = T(\kappa)u$, i.e., $u = -(L_0 - \kappa^2)^{-1} Vu$. Thus u satisfies (4.9), which implies that u solves (1.4).

Q. E. D.

The next proposition will fill the gap in the proof of Theorem 1.5 given in §3.

Proposition 4.7. $\kappa \in \boldsymbol{C}'_+ - \Sigma'$ *if and only if* $\kappa \in \boldsymbol{C}'_+$ *and* $\kappa^2 \in \rho(H)$.

Proof. The "if" part directly follows from the "only if" part of (i) of Proposition 4.5. Now we shall show the "only if" part. Let $\kappa \in \boldsymbol{C}'_+ - \Sigma'$. Since Σ' and Σ'_0 are discrete in $\boldsymbol{C}'_+$ ((ii) of Proposition 4.5), we can find a sequence $\{\kappa_n\} \subset \boldsymbol{C}'_+$ such that $\kappa_n \neq \kappa$, $\kappa_n \notin \Sigma' \cup \Sigma'_0$ and $\kappa_n \to \kappa$ as $n \to \infty$. Let us show that $(H - \kappa^2)D_0 \supset L_{2,\frac{1+\varepsilon}{2}}$. In fact let $f \in L_{2,\frac{1+\varepsilon}{2}}$. Since $(L_0 - \kappa_n^2)^{-1}$ is well-defined and $I - T(\kappa_n)$ is invertible by Propotition 4.6, we can define $u_n \in L_{2,-\frac{1+\varepsilon}{2}} \cap H_{2,loc}$ by

$$\text{(4.10)} \qquad u_n = (I - T(\kappa_n))^{-1}(L_0 - \kappa_n^2)^{-1} f,$$

which implies that $(I + (L_0 - \kappa_n^2)^{-1} V)u_n = (L_0 - \kappa_n^2)^{-1} f$, i.e., $(L - \kappa_n^2)u_n = f$. We have $u_n \in D_{\kappa_n}$. Theorem 2.7 can be applied to show that there exists a subsequence $\{u'_n\}$ of $\{u_n\}$ such that $\{u'_n\}$ converges in $L_{2,-\frac{1+\varepsilon}{2}}$ to the solution u of the equation $(L - \kappa^2)u = f$. Noting that $u \in D_\kappa$, we obtain $u \in D_0$ by Proposition 2.3, (i), and hence $f = (H - \kappa^2)u \in (H - \kappa^2)D_0$. Thus we have shown the relation $(H - \kappa^2)D_0 \supset L_{2,\frac{1+\varepsilon}{2}}$, whence follows, together with the denseness of $L_{2,\frac{1+\varepsilon}{2}}$ in L_2, that $\kappa^2 \notin \sigma_r(H)$. On the other hand $\kappa^2 \notin \sigma_c(H)$ (Proposition 4.3) and $\kappa^2 \notin \sigma_p(H)$ ((i) of Proposition 4.5). Therefore we can coclude that $\kappa^2 \in \rho(H)$. Q.E.D.

Remark 4.8. From the above proposition we see that the condition (II) in §3 is implied by the condition (I). Thus we have proved Theorem 1.5 and Remark 1.6 completely.

By the use of Proposition 4.6 that the rest of Theorem 1.3 will be shown.

Proposition 4.9. *Σ is a bounded set of $\mathbb{C}_+$.*

Proof. Suppose that there exists a sequence $\{\kappa_n\} \subset \Sigma$ satisfying $|\kappa_n| \to \infty$ as $n \to \infty$. For each n we obtain $u_n \in L_{2,-\frac{1+\varepsilon}{2}}$ such that

$$(4.11) \qquad u_n = T(\kappa_n) u_n, \qquad \|u_n\|_{-\frac{1+\varepsilon}{2}} = 1.$$

Because of the boundedness of Σ_0 we may assume that $\kappa_n \notin \Sigma_0$ for all n. Then from (4.8) and (4.11) it follows that

$$(4.12) \qquad 1 = \|u_n\|_{-\frac{1+\varepsilon}{2}} \leqq \|T(\kappa_n) u_n\|_{-\frac{1+\varepsilon}{2}} \leqq \frac{C}{|\kappa_n|},$$

where $C = C(L, \varepsilon)$ is independent of n. As n tends to ∞, the right-hand side of (4.12) converges to 0, which is a cntradiction. Q.E.D.

Proposition 4.10. *Let $a > 0$. Then $\Sigma \cap \bar{M}_a$ is a closed set of $\mathbb{C}_+$, $\bar{M}_a$ being the closure of M_a.*

Proof. Let $\kappa_n \in \Sigma \cap \bar{M}_a$ such that

$$(4.13) \qquad \kappa_n \longrightarrow \kappa_0 \qquad (n \longrightarrow \infty)$$

with $\kappa_0 \in \bar{M}_a$. u_n denotes the solution of the equation $(L - \kappa_n^2) u_n = 0$ with $u_n \in H_{2,loc} \cap L_{2,-\frac{1+\varepsilon}{2}}$, $\|\mathscr{D} u_n\|_{\frac{-1+\varepsilon}{2}, E_1} < \infty$, and $\|u_n\|_{-\frac{1+\varepsilon}{2}} = 1$. From (ii) of Proposition 4.5 it follows that κ_0 is real and $|\kappa_0| \geqq a$. Thus, setting $M_0 = \{\kappa = \kappa_1 + i\kappa_2 / |\kappa_1| > a/2,\ \kappa_2 > 0\}$, we may assume that κ_n, $\kappa_0 \in \bar{M}_0$. Since $|\kappa_1| > a/2$ for $\kappa \in M_0$, $\bar{M}_0 \cap \Sigma_0 = \phi$ by (iii) of Proposition 4.5. Thus Proposition 4.6 can be applied to show that

(4.14) $$u_n = T(\kappa_n)u_n, \qquad \|u_n\|_{-\frac{1+\varepsilon}{2}} = 1.$$

According to Remark 1.6 with L and f_n replaced by L_0 and $-Vu_n$, respectively, the sequence $\{T(\kappa_n)u_n\}$ is relatively compact. Hence it follows from (4.14) that there exists a subsequence $\{u_n'\}$ of $\{u_n\}$ which satisfies

(4.15) $$u_n' \longrightarrow u_0 \qquad \text{in } L_{2,-\frac{1+\varepsilon}{2}}$$

with $u_0 \in L_{2,-\frac{1+\varepsilon}{2}}$. Taking account of the continuity of $T(\kappa)$ in $\mathbb{B}(L_{2,-\frac{1+\varepsilon}{2}})$, we obtain from (4.14) and (4.15)

(4.16) $$u_0 = T(\kappa_0)u_0, \qquad \|u_0\|_{-\frac{1+\varepsilon}{2}} = 1,$$

which implies $\kappa_0 \in \Sigma$. Q.E.D.

Proposition 4.11. *$\Sigma_{\mathbb{R}}$ is a set with the Lebesgue measure 0 in $\mathbb{R}$.*

Proof. It suffieces to show that $e_a = \bar{M}_a \cap \Sigma_{\mathbb{R}}$ is a null set for any $a > 0$. We can prove this in almost the same way as the one used in proving Lemm 6.2 of Kuroda [10]. For an arbitrary $\kappa_0 \in e_a$, noting that $T(\kappa_0)$ is compact and $1 \in \sigma(T(\kappa_0))$, we can find a circle γ in $\mathbb{C}$ with center 1 and its radius less than 1 such that $\gamma \subset \rho(T(\kappa_0))$. Then there exists a positive number μ such that $\gamma \subset \rho(T(\kappa))$ for any $\kappa \in \{\kappa \in \mathbb{C}_+ / |\kappa - \kappa_0| < \mu\}$. In fact, let us suppose that $\kappa_n \to \kappa_0$, $\kappa_n \in \mathbb{C}_+$ and $\gamma \cap \sigma(T(\kappa_n)) \neq \phi$ for any $n = 1, 2, \ldots$. Then, proceeding as in the proof of Proposition 4.10, we can easily show that $\gamma \cap \sigma(T(\kappa_0)) \neq \phi$, which is a contradiction. Since e_a is compact by Propositions 4.9 and 4.10, e_a is covered by a finite number of intervals of the type $I = (\kappa - \mu, \kappa + \mu)$. Therefore it is sufficient to prove that $I_{\kappa_0} \cap e_a$ is a null set. We set

(4.17) $$P(\kappa) = \frac{1}{2\pi i}\int_\gamma (T(\kappa) - \xi)^{-1} d\xi.$$

Then, as is well-known, $P(\kappa)$ is a finite dimensional (oblique) projection and $f(\kappa) = \det(1 - T(\kappa)P(\kappa)) = 0$ for $\kappa \in \{\kappa \in \mathbb{C}_+ / |\kappa - \kappa_0| < \mu\}$ if and only if $1 \in \sigma(T(\kappa))$. It follows from the analyticity of $T(\kappa)$ on $\mathbb{C}_+'$ that $f(\kappa)$ is analytic on $\{\kappa \in \mathbb{C}_+' / |\kappa - \kappa_0| < \mu\}$ and is continuous on $\{\kappa \in \mathbb{C}_+ / |\kappa - \kappa_0| < \mu\}$.

Hence $f(\kappa)$ cannot vanish on a subset of I_{κ_0} whose Lebesgue measure is positive. Thus we have shown that $I_{\kappa_0} \cap e_a$ is a null set. Q.E.D.

Proof of Theorem 1.3. (i) follows from Propositions 4.9 and 4.11. (ii) follows from Proposition 4.10 and (ii) of Proposion 4.5. We obtain (iii) from (i) of Proposition 4.5 and Proposition 4.7. Q.E.D.

Proof of Theorem 1.7. (i) is clear from Proposition 4.3. Since $\mathbf{C}-[0, \infty)=\{\kappa^2/\kappa\in\mathbf{C}'_+\}$, it follows from Proposition 4.7 and (i) of Proposition 4.5 that $\mathbf{C}-[0, \infty)=\rho(H)\cup\sigma_p(H)$, which implies that $\sigma_p(H)\supset\sigma(H)\cap(\mathbf{C}-[0, \infty))$. The relation $\sigma_p(H)\cap(0, \infty)=\phi$ is obtained in (i) of Proposition 4.4. The rest of (ii) can be obtained from Proposition 4.3 and Lemma 4.1, (ii) and (iii). Let us show (iv). Since we have proved $\sigma_p(H)\cap(0, \infty)=\sigma_p(H^*)\cap(0, \infty)=\phi$ in (i) of Proposition 4.4, $\sigma_r(H)\cap(0, \infty)$ can be easily seen to be an empty set. On the other hand we have $\sigma_r(H)\cap(\mathbf{C}-[0, \infty))=\phi$ by the first assertion of (ii). Hence (iv) follows. Finally (iii) follows directly from the relation $\sigma_p(H)\cap(0, \infty)=\sigma_r(H)\cap(0, \infty)=\phi$.

Q.E.D.

Note Added (July 1, 1973): After this paper was written we have informed by Prof. S.T. Kuroda that the relation $\sigma_e(A)=[0, \infty)$ holds under our assumption. For the proof see Kuroda [13].

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