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# On Canonical Forms of Singularities of $C^\infty$ Function Germs of Higher Codimension

By

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## §1. Introduction

Let $f$ be a germ at 0 in $\mathbb{R}^n$ of a $C^\infty$-function with $f(0)=0$, and assume that 0 is a singular point of $f$. Then we call it the *codimension* of $f$ the codimension of the ideal generated by all the first partial derivatives of $f$ in the ring of germs of functions $C^\infty: \mathbb{R}^n \to \mathbb{R}$ which vanish at 0.

It is trivial that if $n=1$, function germs of codimension $r$ have only the canonical forms $\pm x^{2+r}$.

In [1] Cerf showed the canonical forms of germs of codimension 1 or 2. When $n=2$, we find in Thom [5] all possible canonical forms of codimension not exceeding four. The main purpose of this paper is to study codimensions of $C^\infty$-function germs and to extend the results above to the cases codimension $\leqq 6$ and $\leqq 8$ when $n=2$.

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## §2. Main Results

Let $\mathscr{E}_n$ denote the ring of germs at 0 in $\mathbb{R}^n$ of $C^\infty$ functions, and $\mathfrak{m}_n$ denote the maximal ideal of $\mathscr{E}_n$ which consists of elements vanishing at 0.

The following generalizes Morse theorem.

**Theorem 1.** *Let $f$ be an element in $\mathfrak{m}_n^2$, and let the rank of the hessian of $f$ be $i$. Then $f$ is equivalent (i.e. be transformed through a change of coordinates) to an element of the form*

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$$\sum_{j=1}^{i} \pm x_j^2 + g(x_{i+1},\dots, x_n)$$

*where g is contained in* $\mathfrak{m}_{n-i}^3$.

Here the codimension of $f$ is equal to the one of $g$ in $\mathfrak{m}_{n-i}$. Thus it is enough to treat elements in $\mathfrak{m}_n^3$. Codimensions of elements in $\mathfrak{m}_n^3$ equal or exceed $2^n-1$(§4). Hence, if we treat elements in $\mathfrak{m}_n^2$ with codimension $\leqq 6$, then we may assume $n=2$. We write simply codim$f$ instead of the codimension of $f$.

When $n=2$, elements in $\mathfrak{m}_2^3$ with codim $\leqq 8$ are equivalent to ones of the forms in the following table (§5).

| Codim | Canonical forms |
|---|---|
| 3 | $x^3 \pm xy^2$. |
| 4 | $x^2y \pm y^4$. |
| 5 | $x^2y \pm y^5$, $x^3 \pm y^4$. |
| 6 | $x^2y \pm y^6$, $x^3 + xy^3$. |
| 7 | $x^2y \pm y^7$, $x^3 + y^5$. |
| 8 | $x^2y \pm y^8$;<br>$x^4 - 2x^2y^2 + txy^3 + y^4$, $t$: a parameter, $t>0$;<br>$x^4 + tx^2y^2 + y^4$, $t<2$, $t \neq -2$;<br>$x^4 + tx^2y^2 - y^4$, $t \geq 2\times 3^{1/2}$;<br>$x^4 + (-2\times 3^{1/2} + s_0 \sin t)x^2y^2 + (8\times 3^{-3/4} + s_0\cos t)xy^3 - y^4$,<br>$s_0$: a sufficiently small positive constant, $t \neq t_1, t_2$. |

When $n \geq 3$ and codim $=2^n-1$, we need $2^n - n^2/2 - n/2 - 1$ parameters (§4).

Diff$_n$ denotes the group of local diffeomorphisms of $C^\infty$-class around 0 in $\boldsymbol{R}^n$. Naturally Diff$_n$ is a transformation group of $\mathscr{E}_n$.

**Theorem 2.** *Let $f$ be an element in $\mathfrak{m}_n^2$. Then we have* codim$f$ = *the codimension* of the orbit of* Diff$_n$ *passing $f$ in* $\mathfrak{m}_n^2$.

* We give a natural definition of the codimensions of the orbits in §4.

## §3. Proof of Theorem 1

If an element $f$ is contained in $\mathfrak{m}_n^k$ and not in $\mathfrak{m}_n^{k+1}$, then, using some linear transformation we may asume that $f$ is *regular* in $x_n$ of order $k$, i.e. $f(0,\dots, 0, x_n)$ has zero of order exactly $k$ at $x_n=0$. Hence Theorem 1 follows from the next lemma.

**Lemma 3.** *Let $f$ be an element in $\mathfrak{m}_n$. Suppose that $f$ is regular in $x_n$ of order $k$. Then $f$ is equivalent to an element of the form*

$$\pm x_n^k + g_1(x_1,\dots, x_{n-1})x_n^{k-2} + g_2(x_1,\dots, x_{n-1})x_n^{k-3} +$$

$$\cdots + g_{k-1}(x_1,\dots, x_{n-1})$$

*where $g_i$ are contained in $\mathfrak{m}_{n-1}$, and the local diffeomorphism which is used here takes the form*

$$\tau=(\tau_1(x),\dots, \tau_n(x))=(x_1,\dots, x_{n-1}, \tau_n(x)).$$

*Proof.* Levinson [2] treats the analytic case. It is shown already in [4] that there exist a polynomial in $x_n$ with coefficients in $\mathscr{E}_{n-1}$ $\sum_{i=0}^{2k} r_i(x_1,\dots, x_{n-1})x_n^i$ and an element $\tau$ in $\mathrm{Diff}_n$ such that

$$f\circ\tau=\sum_{i=0}^{2k} r_i x_n^i$$

$$\tau=(\tau_1(x),\dots, \tau_n(x))=(x_1,\dots, x_{n-1}, \tau_n(x)).$$

Let $F$ be an analytic function in $x_n$, $u_0,\dots, u_{2k}$ variables defined by

$$F=\sum_{i=0}^{2k} (r_i(0)+u_i)x_n^i.$$

$\sum_{i=0}^{2k} r_i x_n^i$ is regular in $x_n$ of order $k$, and so is $F$. Then the corresponding result of Levinson [2] shows that there exist a polynomial in $x_n$ with coefficients in $\mathscr{E}_{2k+1}$ of the form

$$\pm x_n^k + g_1(u_0,\dots, u_{2k})x_n^{k-2} + \cdots + g_{k-1}(u_0,\dots, u_{2k})$$

and an element $y_n(x_n, u_0,\ldots, u_{2k})$ in $\mathfrak{m}_{2k+2}$ regular in $x_n$ of order 1 such that

$$\sum_{i=0}^{2k}(r_i(0)+u_i)\,y_n^i=\pm x_n^k+g_1x_n^{k-2}+\cdots+g_{k-1}.$$

Let

$$u(x_1,\ldots, x_{n-1})=(u_0(x),\ldots, u_{2k}(x))$$
$$=(r_0(x)-r_0(0),\ldots, r_{2k}(x)-r_{2k}(0)).$$

We have

$$\sum_{i=0}^{2k}r_i(z_n)^i=\pm x_n^k+(g_1\circ u)x_n^{k-2}+\cdots+g_{k-1}\circ u$$

where $z_n(x_1,\ldots, x_n)$ is some element in $\mathfrak{m}_n$ and regular in $x_n$ of order 1. Let

$$\tau'=(x_1,\ldots, x_{n-1}, z_n(x)).$$

Then we see

$$f\circ\tau\circ\tau'=\pm x_n^k+(g_1\circ u)x_n^{k-2}+\cdots+g_{k-1}\circ u.$$

This proves Lemma 3.

## §4. Codimension of Elements in $\mathfrak{m}_n^3$

Let $f$ be an element in $\mathfrak{m}_n^2$. Then codim $f$ is equal to the supremum of the codimensions in $\mathfrak{m}_n/\mathfrak{m}_n^k$ $(k=1, 2,\ldots)$ of the image of the sublinear space spanned by $\dfrac{\partial f}{\partial x_i}x_1^{\alpha}\cdots x_n^{\beta}$, $0\leqq\alpha, \beta$, $1\leqq i\leqq n$, and is also equal to the sum of the codimension in $\mathfrak{m}_n^k/\mathfrak{m}_n^{k+1}$ of the intersection of $\mathfrak{m}_n^k/\mathfrak{m}_n^{k+1}$ and the sublinear space above mod $\mathfrak{m}_n^{k+1}$ when $k=1, 2,\ldots$. If a relation "the sublinear space above mod $\mathfrak{m}_n^{k+1}\supset\mathfrak{m}_n^k/\mathfrak{m}_n^{k+1}$" is satisfied, then we have the corresponding relation of the above where $k$ is replaced by $k+m(m\geqq 0)$. From this, when we see whether codim $f$ is larger than $k$ or not, we only have to look over the partial differential coefficients of order $\leqq k+2$. Let $S_k$ be the subset of $\mathfrak{m}_n^2$ which consists of elements with codim $\geqq k$.

Then we have

$$S_k+\mathfrak{m}_n^{k+2}=S_k.$$

We can easily prove that the set $S_k/\mathfrak{m}_n^p$, $p\geqq k+2$ is an algebraic set in $\mathfrak{m}_n^2/\mathfrak{m}_n^p$.

We remark that

$$\operatorname{codim} x_1^3+\cdots+x_n^3=2^n-1.$$

Let us compute the codimensions of elements in $\mathfrak{m}_n^3$.

**Lemma 4.** *Let $f$ be an element in $\mathfrak{m}_n^3$. Then we have*

$$\operatorname{codim} f\geqq 2^n-1.$$

*Proof.* Let $f_1,\dots,f_n$ be homogeneous polynomials of degree 2. Then we have an inequality

(1) the codimension of the ideal in $\mathfrak{m}_n$ which is generated by $f_i$

$$\geqq 2^n-1,$$

the reason is the following. Let for each $i\geqq 1$, $g_{i1}$, $g_{i2},\dots$ be homogeneous polynomials of degree $i$ such that the natural image of the set $\{g_{ij}\}_j$ into the quotient space $\mathfrak{m}_n^i/\mathfrak{m}_n^{i+1}+\sum_{j=1}^{n} f_j\mathfrak{m}_n^{i-2}$ (if $i>1$, $\mathfrak{m}_n/\mathfrak{m}_n^2$ if $i=1$) is a basis. Then the intersection of $\mathfrak{m}_n^k/\mathfrak{m}_n^{k+1}$ and the sublinear space of $\mathfrak{m}_n/\mathfrak{m}_n^{k+1}$ which is spanned by the elements $f_i x_1^\alpha\cdots x_n^\beta$ is generated by the elements $g_{ij}f_\alpha\cdots f_\beta$ where $\alpha\leqq\cdots\leqq\beta$ $i=k-2$, $k-4,\dots$, and $i+2\times$the number of the set $\{\alpha,\dots,\beta\}=k$. If $f_i=x_i^2$, and if $g_{ij}$ are elements of the form $x_\alpha\cdots x_\beta$, $1\leqq\alpha<\cdots<\beta\leqq n$, then the elements $g_{ij}f_\alpha\cdots f_\beta$ are linearly independent in $\mathfrak{m}_n^k/\mathfrak{m}_n^{k+1}$. Hence the codimension of the sublinear space in $\mathfrak{m}_n$ spanned by $f_i x_1^\alpha\cdots x_n^\beta$ is equal to or larger than $\operatorname{codim} x_1^3+\cdots+x_n^3$. This proves the inequality (1).

$S_{p+2}/\mathfrak{m}_n^p$ is an algebraic set in $\mathfrak{m}_n^2/\mathfrak{m}_n^p$, $p=2^n+1$. Hence it is enough to prove that if $f$ is an element in $\mathfrak{m}_n^3$ whose partial differential coefficients at 0 of order 3 are near to the ones of $x_1^3+\cdots+x_n^3$, then we have $\operatorname{codim} f=2^n-1$. Let $g_{ij}$ be defined as above when $f_i=x_i^2$. Then the set $\left\{g_{ij}\dfrac{\partial f}{\partial x_\alpha}\cdots\dfrac{\partial f}{\partial x_\beta}\right\}$, for $\alpha\leqq\cdots\leqq\beta$, $i=k$, $k-2,\dots$, $i+2\times\#\{\alpha,\dots,\beta\}=k$, is a basis of $\mathfrak{m}_n^k/\mathfrak{m}_n^{k+1}$. From this we only have to prove that for each $k$, the above $g_{ij}\frac{\partial f}{\partial x_\alpha}\cdots\frac{\partial f}{\partial x_\beta}$ with $i\neq k$ is a system of generators of the intersection of $\mathfrak{m}_n^k/\mathfrak{m}_n^{k+1}$ which is spanned by $\frac{\partial f}{\partial x_i}x_1^\alpha\cdots x_n^\beta \bmod \mathfrak{m}_n^{k+1}$. Let $g$ be an element in $\mathfrak{m}_n^k$ which is equal to some element mod $\mathfrak{m}_n^{k+1}$ of the form $\sum_{i=1}^n \varphi_i\frac{\partial f}{\partial x_i}$ where $\varphi_i\in\mathscr{E}_n$. If the elements $\varphi_i$ are containd in $\mathfrak{m}_n^p$ and at least one element $\varphi_s$ is not in $\mathfrak{m}_n^{p+1}$, and if $p<k-2$. Let $\bar{\varphi}_i$ be linear combinations of $g_{ij}\frac{\partial f}{\partial x_\alpha}\cdots\frac{\partial f}{\partial x_\beta}$ with $i+2\times\#\{\alpha,\dots,\beta\}=p$ such that

$$\varphi_i=\bar{\varphi}_i \qquad \bmod \mathfrak{m}_n^{p+1}.$$

Then $\sum_{i=1}^n \varphi_i\frac{\partial f}{\partial x_i}$ is zero mod $\mathfrak{m}_n^{p+3}$, and so is $\sum_{i=1}^n \bar{\varphi}_i\frac{\partial f}{\partial x_i}$. From this we see that

$$\sum_{i=1}^n \bar{\varphi}_i\frac{\partial f}{\partial x_i}=0 \qquad \text{in } \mathfrak{m}_n.$$

And we have

$$g=\sum_{i=1}^n \varphi_i\frac{\partial f}{\partial x_i}=\sum_{i=1}^n (\varphi_i-\bar{\varphi}_i)\frac{\partial f}{\partial x_i} \qquad \bmod \mathfrak{m}_n^{k+1}.$$

Repeating this operation, we can take the elements $\varphi_i$ in $\mathfrak{m}_n^{k-2}$. The elements $\bar{\varphi}_i$ which are defined as above satisfy

$$g=\sum_{i=1}^n \bar{\varphi}_i\frac{\partial f}{\partial x_i} \qquad \bmod \mathfrak{m}_n^{k+1}.$$

This completes the proof.

Let $k$ be a positive integer. We introduce an equivalence relation in $\mathrm{Diff}_n$ as follows. Elements in $\mathrm{Diff}_n$ are equivalent if they have the same partial differential coefficients of order $\leqq k$. And $\mathrm{Diff}_n^k$ denotes the quotient space of $\mathrm{Diff}_n$ by the above equivalence relation. The space $\mathrm{Diff}_n^k$ is a Lie group. $\mathrm{Diff}_n$ being a transformation group of $\mathscr{E}_n$, $\mathrm{Diff}_n^k$ acts on $\mathscr{E}_n/\mathfrak{m}_n^{k+1}$. Hence any orbit of $\mathrm{Diff}_n^k$ in $\mathscr{E}_n/\mathfrak{m}_n^{k+1}$ is a submanifold in $\mathscr{E}_n/\mathfrak{m}_n^{k+1}$.

Let $f$ be an element in $\mathfrak{m}_n^2$. Then we call the upper bound of the set {the codimensions of the orbits of $\mathrm{Diff}_n^k$ passing $f$ in $\mathfrak{m}_n^2/\mathfrak{m}_n^{k+1}$} the *codimension* of the orbit of $\mathrm{Diff}_n$ passing $f$ in $\mathfrak{m}_n^2$.

*Proof of Theorem* 2. We can regard elements in $\mathrm{Diff}_n^k$ as taking the following form

$$\Big(\sum_{|\alpha|=1}^{k} a_{1\alpha}x^\alpha,\dots,\sum_{|\alpha|=1}^{k} a_{n\alpha}x^\alpha\Big)$$

where $\alpha$ are $n$-integers and where $\alpha=(\alpha_1,\dots,\alpha_n)$, $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $|\alpha|=\alpha_1+\cdots+\alpha_n$. Let $I$ be the identity of the group $\mathrm{Diff}_n^k$ Let $f$ be an element in $\mathfrak{m}_n^2/\mathfrak{m}_n^{k+1}$. Let $\varphi$ be the map from $\mathrm{Diff}_n^k$ to $\mathfrak{m}_n^2/\mathfrak{m}_n^{k+1}$ defined by

$$\varphi(\dots,a_{i\alpha},\dots)=f\circ\Big(\sum_{|\alpha|=1}^{k} a_{1\alpha}x^\alpha,\dots,\sum_{|\alpha|=1}^{k} a_{n\alpha}x^\alpha\Big).$$

Then the dimension of the orbit of $\mathrm{Diff}_n^k$ passing $f$ is the rank of the map $\varphi$ at $I$. We easily see that

$$\frac{\partial\varphi}{\partial a_{i\alpha}}(I)=\frac{\partial f}{\partial x_i}x^\alpha.$$

Hence the rank of the map $\varphi$ at $I$ is the dimension of the sublinear space spanned by $\frac{\partial f}{\partial x_i}x^\alpha$, $|\alpha|\geqq 1$. By Nakayama's lemma the codim in $\mathfrak{m}_n^2$ of the space which is spanned by $\frac{\partial f}{\partial x_i}x^\alpha$, $|\alpha|\geqq 1$ is the one in $\mathfrak{m}_n$ of the space spanned by $\frac{\partial f}{\partial x_i}x^\alpha$, $|\alpha|\geqq 0$. Q.E.D.

**Corollary 5.** *Let $f$, $g$ be elements in $\mathfrak{m}_n^2$ with* $\mathrm{codim}\, ft+g(1-t)=a$ *finite constant, and let for each $0\leqq t_0\leqq 1$, $f-g$ be contained in the ideal generated by all the first partial derivatives of the element $ft_0+g(1-t_0)$. Then $f$ and $g$ are equivalent.*

The proof is easy from the fact in [3] that a germ is *finitely determined* if and only if the codimension is finite.

The dimension of the space $\mathfrak{m}_n^2/\mathfrak{m}_n^3$ is $n^2/2+n/2$. From this we have the next corollary.

**Corollary 6.** *Let $f$ be an element in $\mathfrak{m}_n^3$. Then the orbit of* $\mathrm{Diff}_n$ *passing $f$ in $\mathfrak{m}_n^3$ is of codimension $\geqq 2^n-n^2/2-n/2-1$.*

## §5. The Case in Two Variables

In virtue of Lemma 3, any element in $\mathfrak{m}_2^3$ and not in $\mathfrak{m}_2^4$ is equivalent to one of the form

$$F=x^3+3ay^2x+2by^3+f(y)x+g(y)$$

where $f\in\mathfrak{m}_1^3$, $g\in\mathfrak{m}_1^4$.

There are three cases.

$$b^2+a^3\neq 0; \tag{1}$$

$$b^2+a^3=0,\ b\neq 0; \tag{2}$$

$$a=b=0. \tag{3}$$

*The case* (1). We can see easily that codim $F=3$ and that the ideal generated by all the first partial derivatives of $F$ contains $\mathfrak{m}_2^3$. Applying Corollary 5, we see that $F$ is equivalent to an element of the forms

$$x^3-xy^2, \quad \text{if} \quad a<-b^{2/3};$$

$$x^3\pm y^3, \quad \text{if} \quad a>-b^{2/3}.$$

*The case* (2). Assume that codim $F$ is finite. Through some linear transformation, $F$ takes the form

$$x^2y+xf(y)+g(y)+h(x,\ y)x^2$$

where $f\in\mathfrak{m}_1^3$, $g\in\mathfrak{m}_1^4$, $h\in\mathfrak{m}_2^2$. Let $\tau$ be an element in $\mathrm{Diff}_2$ defined by

$$\tau=(x,\ y-h(x,\ y)).$$

Then

$$F\circ\tau=x^2y+x(f\circ\tau)+g\circ\tau$$

takes the form

$$x^2y+x\bar{f}(y)+\bar{g}(y)+x^2\bar{h}(x,\ y)$$

where $\bar{f}\in\mathfrak{m}_1^3$, $\bar{g}\in\mathfrak{m}_1^4$, $\bar{h}\in\mathfrak{m}_2^3$. Repeating this operation, we may assume that the element above $h$ is cotaind in $\mathfrak{m}_2^k$ where $k$ is taken large enough. Then we show that $F$ is equivalent to an element of the form

$$x^2y+xf(y)+g(y).$$

Moreover, transforming by some element in $\mathrm{Diff}_2$ of the form

$$\tau=(x-\bar{f}(y)/2,\ \varphi(y)),$$

we see that $F$ is equivalent to an element of the form

$$x^2y\pm y^n.$$

*The case* (3). Put

$$F=x^3+f(y)x+g(y)$$

where $f\in\mathfrak{m}_1^3$, $g\in\mathfrak{m}_1^4$, and let $n$, $m$ be the upper bounds of integers which satisfy respectively $\mathfrak{m}_1^n\ni f$, $\mathfrak{m}_1^m\ni g$. Then we can prove in the same way as in the above that $F$ is equivalent to

$$x^3\pm y^m, \quad \text{if} \quad m\leqq n+1;$$

$$x^3\pm y^n x, \quad \text{if} \quad 2n\leqq m+1.$$

## §6. Appendix. Universal Unfolding

In this section we give proofs to some statements in Thom [5] and [6].

Let $V$ be an element in $\mathfrak{m}_n^2$ with codim $V=k$, and let $g_1,\dots, g_k$ be ones in $\mathfrak{m}_n$ such that the natural image of the set $\{g_i\}$ is a basis of the quotient ring of $\mathfrak{m}_n$ by the ideal generated by the first partial derivatives of $V$. Then the expression

$$V(x)+\sum_{i=1}^{k}u_i g_i(x)$$

is called the *universal unfolding* of $V$. Let $G(x, u)$ be the element in $\mathscr{E}_{n+k+1}$ defined by

$$G(x, u) = u_0 + \text{ the universal unfolding.}$$

The following shows the universality.

**Theorem 7.** *Let $F(x_1, \ldots, x_n, v_1, \ldots, v_m)$ be an element in $\mathscr{E}_{n+m}$ such that*

$$F(x, 0) = V.$$

*Then there exist a $k+1$-tuple $u = (u_0, \ldots, u_k)$ of elements in $\mathfrak{m}_m$ and an elemet $\tau$ in $\mathrm{Diff}_{n+m}$ of the form*

$$(\tau_1(x, v), \ldots, \tau_n(x, v), v_1, \ldots, v_m) \tag{1}$$

*such that we have*

$$G(x, u(v)) \circ \tau = F.$$

*Proof.* As $V$ is finitely determined, we may assume that $V$ is a polynomial. Let $\mathfrak{p}$ and $\mathfrak{q}$ be the ideals in $\mathscr{E}_n$ and $\mathscr{E}_{n+m}$ generated by the derivatives $\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}$ and $\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}$ respectively. Let $f_i$, $i = 1, \ldots, s$ be elements in $\mathfrak{p}$ satisfying that the natural image of the set $\{f_i\}$ is a basis of the quotient ring of $\mathfrak{p}$ by the ideal $\mathfrak{p}^2\mathfrak{m}_n^2$. Then $\{g_i, f_j\}$ is a basis of the quotient ring of $\mathfrak{m}_{n+m}$ by the ideal $\mathfrak{q}^2\mathfrak{m}_{n+m}^2 + \mathfrak{m}_m\mathscr{E}_{n+m}$. Malgrange's preparation theorem shows that there exist elements $h_i(v)$, $i = 0, \ldots, k+s$ in $\mathfrak{m}_m$ such that the element

$$F - V - h_0 - \sum_{i=1}^{k} h_i g_i - \sum_{i=1}^{s} h_{i+k} f_i$$

is contained in the ideal $\mathfrak{q}^2\mathfrak{m}_{n+m}^2$. In [4] it is shown that if $H$ is an element in $\mathscr{E}_{n+m}$ such that the element $F$–$H$ is contained in $\mathfrak{q}^2\mathfrak{m}_{n+m}^2$, then $F$ and $H$ are equivalent through a local diffeomorphism of the form (1). From these fact, we only have to prove in the case

$$F = V + h_0 + \sum_{i=1}^{k} h_i g_i + \sum_{i=1}^{s} h_{k+i} f_i.$$

Let $K$, $L$ be the elements in $\mathscr{E}_{n+k+s+1}$ defined by

$$K(x, t) = V + t_0 + \sum_{i=1}^{k} t_i g_i + \sum_{i=1}^{s} t_{k+i} f_i,$$

$$L(x, t)=V+t_0+\sum_{i=1}^{k} t_i g_i.$$

Let $\overline{\mathfrak{m}}_{n+k+s+1}$ denote the ideal in $\mathscr{E}_{n+k+s+1}$ generated by $k+s+1$ elements $t_i$. By the hypothesis, there exist $n$ elements $a_i$ in $\overline{\mathfrak{m}}_{n+k+s+1}$ such that we have

$$\sum_{i=1}^{s} t_{k+i} f_i=\sum_{i=1}^{n} a_i \frac{\partial V}{\partial x_i}.$$

Let $\pi$ be the element in $\mathrm{Diff}_{n+k+s+1}$ defined by

$$(x_1+a_1,\dots, x_n+a_n, t_0,\dots, t_{k+s}).$$

Then

$$L(x, t)\circ\pi-K\in\overline{\mathfrak{m}}^2_{n+k+s+1}.$$

Hence there exist $k+s+1$ elements $b_i(t)$ in $\mathfrak{m}^2_{k+s+1}$ such that $L\circ\pi$ and $K+b_0+\sum_{i=1}^{k} b_i g_i+\sum_{i=1}^{s} b_{k+1} f_i$ are equivalent through a local diffeomorphism of the corresponding form of (1) where $v$ is replaced by $t=(t_0,\dots, t_{k+s})$. Put

$$\chi=(x_1,\dots, x_n, t_0+b_0,\dots, t_{k+s}+b_{k+s}).$$

Then we have

$$\chi\in\mathrm{Diff}_{n+k+s+1},$$

$$K\circ\chi=K+b_0+\sum_{i=1}^{k} b_i g_i+\sum_{i=1}^{s} b_{k+i} f_i.$$

Therefore $L\circ\pi\circ\chi^{-1}$ and $K$ are equivalent through a (1)-type local diffeomorphism. This shows that $K$ is equivalent through a (1)-type local diffeomorphism to an element $\bar{L}$ in $\mathscr{E}_{n+k+s+1}$ of the form

$$\bar{L}=V+c_0+\sum_{i=1}^{k} c_i g_i$$

where $c_i(t)$ are contained in $\mathfrak{m}_{k+s+1}$.
Put particularly

$$t_i=h_i,\ i=0,\dots, k+s.$$

Then Theorem 7 follows.

**Remark 8.** In the theorem above, we consider the case $m=k+1$ and $v_{i+1}=u_i$ for $0\leqq i\leqq k$. Let $F$ be sufficiently near to $G$ as an element in the finitely dimensional vector space $\mathscr{E}/\mathfrak{m}^s$ for some $s$. Then $u(v)$ is a local diffeomorphism. This means the unfolding.

**Remark 9.** Even if we treat the topological equivalence, we can not drop $u_0$ from $G(x, u)$ in Theorem 7 and Remark 8. For example, let $V(x)=x^4$, $g_1(x)=x$ and $g_2(x)=x^2$. The figures 1 and 2 correspond $x^4+\varepsilon u_2x^3+u_2x^2$ and $x^4+u_2x^2+u_1x$ respectively. These show that $x^4+\varepsilon u_2x^3+u_2x^2$ with $\varepsilon$, $u_2<0$ can not be topologically equivalent to an element of the form $(x+a)^4+(x+a)^2b+(x+a)c$ for any real numbers $a$, $b$, $c$. Hence $G(x, u)=x^4+u_2x^2+u_1x+u_0$ needs $u_0$.

From Remark 8 we can deduce immediately

**Corollary 10.** ([6], p. 52.) *Let $V$ be an element in $\mathfrak{m}_n^2$ with* codim $V=k$, *and let $G(x,u)$ be the universal unfolding of $V$. Suppose that an element $F$ in $\mathscr{E}_{n+k}$ is sufficiently near to $G$ and satisfies an equation $F(x, 0)=V$. Let $\tilde{G}=(G(x, u), u)$ and $\tilde{F}=(F(x, u), u)$. Then there exist elements $\tau$ and $\chi$ in* $\mathrm{Diff}_{n+k}$ *and* $\mathrm{Diff}_{k+1}$ *respectively of the forms*

$$\tau=(\tau_1(x, u),\dots, \tau_n(x, u), \tau_{n+1}(u),\dots, \tau_{n+k}(u))$$

$$\chi=(\chi_1(y, u), \chi_2(u),\dots, \chi_{k+1}(u))$$

*such that the square*

$$\begin{array}{ccc} (x, u) & \xrightarrow{\tilde{G}} & (y, u) \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\chi} \\ (x, u) & \xrightarrow{\tilde{F}} & (y, u) \end{array}$$

*commutes.*

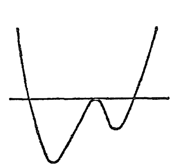

Figure 1. $\varepsilon$, $u_2<0$

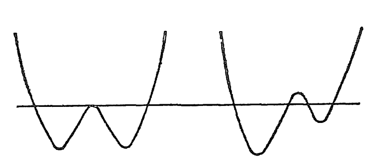

Figure 2. $u_2<0$

**Note Added after Submission:** The author found that some of his results were obtained also by V. Arnol'd (Normal forms for functions near degenerate critical points, the Weyl groups of $A_k$, $D_k$, $E_k$ and Lagrangian singularities, Functional Analysis and its Applications, 6, No. 4, 254–272, (1973)), and that J. Mather's lectures on "Right equivalence" given at Warwick University (1973) have also some connection with our work.

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