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A Difference Scheme for Solving the Stefan Problem

By

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It is the aim of this paper to investigate how solution of the Stefan problem can be obtained by solving a new difference scheme in a rectangular lattice and taking the limit of such solution as the mesh size of the lattice tends to zero. For simplicity we shall consider a one-phase Stefan problem of heat equation

$$(1) \qquad a^2\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}=0 \qquad (a\text{: positive constant})$$

in the region $0<t<T$, $0<x<y(t)$, where the boundary conditions

$$(2) \qquad u(0, t)=f(t),$$

$$(3) \qquad u(x(t), t)=0$$

and the initial condition

$$(4) \qquad u(x, 0)=\varphi(x), \quad 0<x<y(0)=l$$

are imposed. The function $x=y(t)$ is the free boundary which is not known and is to be found together with $u(x, t)$ by the Stefan condition

$$(5) \qquad \dot{y}(t)=\kappa\frac{\partial u}{\partial x}(y(t), t) \qquad (\kappa\text{: const.}).$$

With the heat equation we associate an implicit scheme of the form

$$(6) \qquad a^2\frac{u(x_{j+1}, t_n)-2u(x_j, t_n)+u(x_{j-1}, t_n)}{h^2}-\frac{u(x_j, t_n)-u(x_j, t_{n-1})}{k_n}=0.$$

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More precisely we shall introduce a family of rectangular lattices with space mesh h and time steps $k_n (n=1, 2,\dots)$ where h varies in such a way that $\frac{l}{h}=J$ is integer and k_n's are to be found so that the free boundary crosses lattices just at each mesh point (x_{J+n}, t_n), where we put

$$x_j=jh \qquad (j=0, \pm 1, \pm 2,\dots),$$

$$t_n=\sum_{p=1}^{n} k_p \qquad (n=1, 2,\dots).$$

With reference to given positive numbers h and k_n we introduce the divided differences

$$u_x(x_j, t_n)=\frac{1}{h}[u(x_{j+1}, t_n)-u(x_j, t_n)],$$

$$u_{\bar{x}}(x_j, t_n)=\frac{1}{h}[u(x_j, t_n)-u(x_{j-1}, t_n)],$$

$$u_{x\bar{x}}(x_j, t_n)=\frac{1}{h^2}[u(x_{j+1}, t_n)-2u(x_j, t_n)+u(x_{j-1}, t_n)],$$

$$u_{\bar{t}}(x_j, t_n)=\frac{1}{k_n}[u(x_j, t_n)-u(x_j, t_{n-1})].$$

Then $u=u(x_j, t_n)$ shall be the function defined for (x_j, t_n) in the lattice which satisfies the recursion formula

$$a^2 u_{x\bar{x}}(x_j, t_n)-u_{\bar{t}}(x_j, t_n)=0 \tag{7}$$

and the boundary conditions

$$u(0, t_n)=f(t_n), \tag{8}$$

$$u(y_n, t_n)=0, \qquad n=1, 2,\dots \tag{9}$$

where $y_n=x_{J+n}$, and the initial condition

$$u(x_j, 0)=\varphi(x_j), \qquad j=0, 1, 2,\dots, J. \tag{10}$$

Our essential idea is that k_n shall be determined from an analogue to the Stefan condition of the form

$$\frac{h}{k_n} = \kappa u_{\bar{x}}(y_n, t_n) + \beta \frac{k_n}{\sqrt{h}}, \qquad n=1, 2, 3, \ldots \tag{11}$$

with an artificial heat flow term $\beta \frac{k_n}{\sqrt{h}}$ (β is a suitable positive constant).

We can prove that for $h \to 0$, $u(x_j, t_n)$ and y_n approach functions $u(x, t)$ and $y(t)$ under suitable conditions and this pair of functions $\{u(x, t), y(t)\}$ is a solution of (1)–(5).

We have now several works on difference schemes (Douglas and Gallie [1], Vasilev [2] etc). They treat the case in which an inhomogeneous Neumann type boundary condition is imposed at a fixed boundary and it is assumed that the inhomogeneous term is bounded away from zero. In that case a Stefan condition becomes equivalent to an integral relation which is effectively used in the iteration calculation. If the condition, for example,

$$\frac{\partial u}{\partial x}(0, t) = 1 \tag{12}$$

is imposed instead of (2), then the Stefan's condition (5) can be replaced by

$$y(t) = l + \kappa t + \frac{\kappa}{a^2}\left(\int_0^{y(t)} u(x, t)\, dx - \int_0^l \varphi(x)\, dx\right). \tag{5'}$$

And we can consider the system (1), (12), (3), (4) and (5′) instead of the system (1), (12), (3), (4) and (5), while in the case of Dirichlet type boundary condition as here considered by us such replacement cannot be done and in a case of homogeneous Neumann type boundary condition such replacement does not play an effective role. In our new scheme such restriction can be ridden. But essential restriction is that $l > 0$ which may depend only on our method of proof.

Here we consider only the case of Dirichlet type boundary condition. The case of homogeneous Neumann type boundary condition can be treated in the same way. More general case including multi-phase problems also may be treated.

§1. Statement of the Main Result

We shall essentially concerned with the problem (1)–(5): Find $u(x, t)$ and $y(t)>0$ such that

$$a^2\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}=0 \qquad (a\colon \text{const.}) \text{ for } 0<x<y(t),\ t>0, \tag{1.1}$$

$$u(0, t)=f(t) \qquad \text{where } f(t)\leqq 0 \text{ and } t>0, \tag{1.2}$$

$$u(y(t), t)=0 \qquad \text{for } t>0 \text{ and } y(0)=l>0, \tag{1.3}$$

$$u(x, 0)=\varphi(x) \qquad \text{where } \varphi(x)\leqq 0,\ 0\leqq x\leqq l, \text{ and } \varphi(0)=f(0),\ \varphi(l)=0, \tag{1.4}$$

$$\dot{y}(t)=\kappa\frac{\partial u}{\partial x}(y(t), t) \qquad \text{for } t>0. \tag{1.5}$$

The assumptions $f\leqq 0$, $\varphi\leqq 0$ result from the physical background. Existence and uniqueness theorem about the last problem is well known. Furthermore it is known that under the assumption the function $x=y(t)$ is monotone nondecreasing in t (Friedman [3]).

We consider the following difference analogue: Find $\{u(x_j, t_n)\}$ and positive $\{k_n\}$ such that

$$a^2 u_{x\bar{x}}(x_j, t_n)-u_{\bar{t}}(x_j, t_n)=0, \qquad \text{for } 0<x_j<y_n,\ t_n>0, \tag{1.6}$$

$$u(0, t_n)=f(t_n)\ (\leqq 0) \qquad \text{for } t_n>0, \tag{1.7}$$

$$u(y_n, t_n)=0 \qquad \text{for } t_n>0 \text{ and } y_0=l>0, \tag{1.8}$$

$$u(x_j, 0)=\varphi(x_j)\ (\leqq 0) \qquad \text{for } 0\leqq x_j\leqq l \text{ and } \varphi(l)=0, \tag{1.9}$$

$$\frac{h}{k_n}=\kappa v_n+\beta\frac{k_n}{\sqrt{h}} \qquad \text{for } t_n>0, \tag{1.10}$$

or

$$\frac{h}{k_n}=\frac{1}{2}(\kappa v_n+\sqrt{\kappa^2 v_n^2+4\beta\sqrt{h}}), \tag{1.10'}$$

where

$$(1.11) \qquad y_n = x_{J+n} = (J+n)h, \qquad Jh = l$$

$$(1.12) \qquad t_n = \sum_{p=1}^{n} k_p,$$

$$(1.13) \qquad v_n = u_{\bar{x}}(y_n, t_n).$$

Assume that we have already the solutions $u(x_j, t_p)$, k_p for $p \leqq n-1$. Then we shall solve the difference scheme (2.6)–(2.8), (1.10) by the iteration process

$$(1.14) \qquad a^2 u_{x\bar{x}}^{(s)}(x_j, t_n) - \frac{u^{(s)}(x_j, t_n) - u(x_j, t_{n-1})}{k_n^{(s)}} = 0, \; j = 1, 2, \ldots, J+n-1,$$

$$(1.15) \qquad u^{(s)}(0, t_n) = f(t_n),$$

$$(1.16) \qquad u^{(s)}(y_n, t_n) = 0,$$

$$(1.17) \qquad k_n^{(s+1)} = \frac{\sqrt{h}}{2\beta}\left[-\kappa v_n^{(s)} + \sqrt{\kappa^2 v_n^{(s)2} + 4\beta\sqrt{h}}\right], \quad s = 1, 2, 3, \ldots,$$

$$(1.18) \qquad k_n^{(1)} = k_{n-1}.$$

We have

Theorem. *Assume that* $f(t) \in C^1(0 < t < T)$, $\varphi(x) \in C^2(0 < x < l)$. *Then*

i) *At each time step* $t = t_n$, *the iteration process* (2.14)–(2.18) *converges as* $s \to \infty$ *and the limits* $\{u(x_j, t_n)\}$, k_n *satisfy the equations* (2.6)–(2.10).

ii) *The functions* $\{u(x_j, t_n)\}$, $\{y_n\}$ *determined by* (2.6)–(2.10) *converge uniformly to the solution* $u(x, t)$, $y(t)$ *of* (2.1)–(2.5) *respectively as* $h \to \infty$.

We shall prove this theorem in §3 and §4. Before the proof we shall give some preliminaries in the next section. In §5 we shall give some numerical examples.

§2. Preliminaries

Define for $r,\ j \geqq 0$,

$$(2.1)\qquad g(x_r, \xi_j;\ t_n, \tau_{p-1}) = \begin{cases} \dfrac{1}{2\pi h}\displaystyle\int_{-\pi}^{\pi} \prod_{q=p}^{n} \Lambda_q^{-1}[e^{-i(r-j)\omega} - e^{-i(r+j)\omega}]d\omega \\ \qquad\qquad\qquad\qquad \text{for } n \geqq p, \\ \dfrac{1}{h}\delta_{r,j}{}^{*)} \qquad\qquad \text{for } n = p-1, \end{cases}$$

$$(2.2)\qquad G(x_r, \xi_j;\ t_n, \tau_{p-1}) = \begin{cases} \dfrac{1}{2\pi h}\displaystyle\int_{-\pi}^{\pi} \prod_{q=p}^{n} \Lambda_q^{-1}[e^{-i(r-j)\omega} + e^{-i(r+j-1)\omega}]d\omega \\ \qquad\qquad\qquad\qquad \text{for } n \geqq p, \\ \dfrac{1}{h}\delta_{r,j} \qquad\qquad \text{for } n = p-1, \end{cases}$$

where $\Lambda_q = 1 + 4\lambda_q \sin^2\dfrac{\omega}{2}$, $\lambda_q = \dfrac{a^2 k_q}{h^2}$.

Then these functions satisfy the equations

$$(2.3)\qquad \begin{aligned} a^2 g_{x\bar{x}} - g_{\bar{t}} &= 0, \qquad a^2 G_{x\bar{x}} - G_{\bar{t}} = 0, \\ a^2 g_{\xi\bar{\xi}} + g_{\tau} &= 0, \qquad a^2 G_{\xi\bar{\xi}} + G_{\tau} = 0, \end{aligned}$$

and the boundary conditions

$$(2.4)\qquad \begin{aligned} g(0, \xi_j;\ t_n, \tau_{p-1}) &= g(x_r, 0;\ t_n, \tau_{p-1}) = 0, \\ G_x(0, \xi_j;\ t_n, \tau_{p-1}) &= G_{\xi}(x_r, 0;\ t_n, \tau_{p-1}) = 0. \end{aligned}$$

Furthermore we have the conjugate relations

$$(2.5)\qquad \begin{aligned} g_{\bar{x}}(x_r, \xi_j;\ t_n, \tau_{p-1}) &= -G_{\xi}(x_r, \xi_j;\ t_n, \tau_{p-1}) \\ G_x(x_r, \xi_j;\ t_n, \tau_{p-1}) &= -g_{\bar{\xi}}(x_r, \xi_j;\ t_n, \tau_{p-1}). \end{aligned}$$

We call g the Green's function of the first boundary value problem in

*) $\delta_{r,j} = \begin{cases} 1 & (r=j) \\ 0 & (r \neq j) \end{cases}$ is Kronecker's delta.

$x>0$ for the equation (2.6) and G the Green's function of the second boundary value problem.

Lemma 1. *Assume that* $\{k_n\}$ *are given. Then for the solution of the mixed initial-boundary value problem* (1.6)-(1.9) *we have*

$$v_n=[1+a^2k_nG_\xi(y_n, \eta_n; t_n, \tau_{n-1})]^{-1}$$

$$\times[\sum_{j=1}^{J} hG(y_n, \xi_j; t_n, 0)\, \varphi_{\bar{\xi}}(\xi_j)-\sum_{p=1}^{n} k_pG(y_n, 0; t_n, \tau_{p-1})f_\tau(\tau_p) \tag{2.6}$$

$$-a^2\sum_{p=1}^{n-1} k_pG_\xi(y_n, \eta_p; t_n, \tau_{p-1})v_p]$$

where $y_n=x_{J+n}$, $\eta_n=\xi_{J+n}$, $v_n=u_{\bar{x}}(y_n, t_n)$.

And we have also

$$v_n=[1+a^2k_nG_\xi(y_n, \eta_n; t_n, \tau_{n-1})]^{-1}$$

$$\times[-k_nG(y_n, 0; t_n, \tau_{n-1})\, f_\tau(\tau_n) \tag{2.7}$$

$$+\sum_{j=1}^{J+n-1} hG(y_n, \xi_j; t_n, \tau_{n-1})u_{\bar{\xi}}(\xi_j, \tau_{n-1})].$$

Proof. Assume that the functions $\varphi(\xi, \tau)$ and $\psi(\xi, \tau)$ satisfy

$$a^2\varphi_{\xi\bar{\xi}}(\xi_j, \tau_p)-\varphi_\tau(\xi_j, \tau_p)=0$$

and
$$a^2\psi_{\xi\bar{\xi}}(\xi_j, \tau_{p-1})+\psi_\tau(\xi_j, \tau_{p-1})=0.$$

Multiply the former by $hk_p\psi(\xi_j, \tau_p)$ and the latter by $hk_p\varphi(\xi_j, \tau_p)$, add each resulted equation over $j=1, 2,\dots, J+n-1$ and $p=1, 2,\dots, n$ and subtract the latter sum from the former. Then we have

$$a^2\sum_{p=1}^{n} k_p \sum_{j=1}^{J+p-1} h[\psi(\xi_j, \tau_{p-1})\, \varphi_{\xi\bar{\xi}}(\xi_j, \tau_p)-\varphi(\xi_j, \tau_p)\psi_{\xi\bar{\xi}}(\xi_j, \tau_{p-1})]$$

$$-\sum_{p=1}^{n} k_p \sum_{j=1}^{J+p-1} h[\varphi_\tau(\xi_j, \tau_{p-1})\psi(\xi_j, \tau_{p-1})+\varphi(\xi_j, \tau_p)\psi_\tau(\xi_j, \tau_p)]=0.$$

Applying summation by parts,

$$\sum_{j=1}^{J+n} h\psi(\xi_j, \tau_n)\varphi(\xi_j, \tau_n)=\sum_{j=1}^{J} h\psi(\xi_j, 0)\varphi(\xi_j, 0)+$$

$$
\begin{aligned}
&+\sum_{p=1}^{n} h\psi(\eta_p, \tau_p)\varphi(\eta_p, \tau_p)\\
(2.8)\qquad &+a^2\sum_{p=1}^{n} k_p[\psi(\eta_{p-1}, \tau_{p-1})\varphi_\xi(\eta_{p-1}, \tau_p)-\psi_\xi(\eta_{p-1}, \tau_{p-1})\varphi(\eta_{p-1}, \tau_p)]\\
&-a^2\sum_{p=1}^{n} k_p[\psi(0, \tau_{p-1})\varphi_\xi(0, \tau_p)-\psi_\xi(0, \tau_{p-1})\varphi(0, \tau_p)].
\end{aligned}
$$

Now we take

$$\varphi(\xi_j, \tau_p)=u(\xi_j, \tau_p), \qquad \psi(\xi_j, \tau_p)=g(x_r, \xi_j; t_n, \tau_p).$$

Then from (1.7), (1.8), (1.9), (2.4) and the equality

$$u(\eta_{p-1}, \tau_p)=-hv_p \qquad \text{(by (1.8))}$$

we have

$$
\begin{aligned}
u(x_r, t_n)=&\sum_{j=1}^{J} h\,g(x_r, \xi_j; t_n, 0)\varphi(\xi_j)\\
(2.9)\qquad &+a^2\sum_{p=1}^{n} k_p g(x_r, \eta_p; t_n, \tau_{p-1})v_p\\
&+a^2\sum_{p=1}^{n} k_p g_\xi(x_r, 0; t_n, \tau_{p-1})f(\tau_p).
\end{aligned}
$$

Hence by (2.5)

$$
\begin{aligned}
u_{\bar{x}}(x_r, t_n)=&-\sum_{j=1}^{J} hG_\xi(x_r, \xi_j; t_n, 0)\varphi(\xi_j)\\
(2.10)\qquad &-a^2\sum_{p=1}^{n} k_p G_\xi(x_r, \eta_p; t_n, \tau_{p-1})v_p\\
&-a^2\sum_{p=1}^{n} k_p G_{\xi\bar{\xi}}(x_r, \xi_1; t_n, \tau_{p-1})f(\tau_p).
\end{aligned}
$$

Using the equation (2.3) and applying summation by parts to (2.10) we have

$$
\begin{aligned}
u_{\bar{x}}(x_r, t_n)=&\sum_{j=1}^{J} hG(x_r, \xi_j; t_n, 0)\varphi_{\bar{\xi}}(\xi_j)-\sum_{p=1}^{n} k_p G(x_r, 0; t_n, \tau_{p-1})f_\tau(\tau_p)\\
&-a^2\sum_{p=1}^{n} k_p G_\xi(x_r, \eta_p; t_n, \tau_{p-1})v_p\\
&+G(x_r, \xi_1; t_n, \tau_n)f(\tau_n).
\end{aligned}
$$

In the last equation we take $r=J+n$ and solve this equation regarding $u_{\bar{x}}(y_n, t_n)=v_n$ as unknown. Then we get (2.6) since $G(y_n, \xi_1; t_n, \tau_n)=0$.

In the same way we also have (2.7).

Lemma 2. *For small h,*

$$1+a^2G_\xi(y_n, \eta_n; t_n, \tau_{n-1})>\frac{1}{4} \tag{2.11}$$

Proof.

$$\begin{aligned}
&1+a^2G_\xi(y_n, \eta_n; t_n, \tau_{n-1})\\
&=1-\frac{\lambda_n}{\pi}\int_{-\pi}^{\pi} A_n^{-1}\sin^2\frac{\omega}{2}d\omega-\frac{\lambda_n}{\pi}\int_{-\pi}^{\pi} A_n^{-1}\sin\{2(J+n)-1\}\omega\cdot\sin\frac{\omega}{2}d\omega\\
&>\frac{1}{2}\left(1+\frac{1}{\sqrt{1+4\lambda_n}}\right)-\frac{\lambda_n}{\pi\left\{2(J+n)-\frac{1}{2}\right\}}\int_{-\pi}^{\pi}\left(A_n^{-2}+\frac{1}{2}A_n^{-1}\right)d\omega\\
&=\frac{1}{2}\left(1+\frac{1}{\sqrt{1+4\lambda_n}}\right)-\frac{\lambda_n}{\pi\left\{2(J+n)-\frac{1}{2}\right\}}\left[\frac{1}{1+4\lambda_n}+\frac{1}{\sqrt{\lambda_n}}\tan^{-1}\sqrt{4\lambda_n}\right].
\end{aligned}$$

Since

$$\left\{2(J+n)-\frac{1}{2}\right\}h>2Jh=2l \tag{2.12}$$

we have as $h, k_n\to 0$

$$\begin{aligned}
1+a^2G_\xi(y_n, \eta_n; t_n, \tau_{n-1})&>\frac{1}{2}\left(1+\frac{1}{\sqrt{1+4\lambda_n}}\right)-0\,(h)-0(\sqrt{k_n})\\
&>\frac{1}{4}.
\end{aligned}$$

Lemma 3.

$$|G(y_n, 0; t_n, \tau_{p-1})|<\frac{1}{a\sqrt{t_n-\tau_{p-1}}}. \tag{2.13}$$

Proof.

$$|G(y_n, 0; t_n, \tau_{p-1})| = \left|\frac{1}{2\pi h}\int_{-\pi}^{\pi}\prod_{q=p}^{n} \Lambda_q^{-1}(e^{-i(J+n)\omega}+e^{-i(J+n-1)\omega})\, d\omega\right|$$

$$\leqq \frac{1}{\pi h}\int_{-\pi}^{\pi}\left(1+4\sum_{q=p}^{n}\lambda_q \sin^2\frac{\omega}{2}\right)^{-1} d\omega = \frac{2}{h\sqrt{1+4\sum_{q=p}^{n}\lambda_q}}$$

$$< \frac{1}{a\sqrt{t_n-\tau_{p-1}}}.$$

Lemma 4. *For any function* $\psi(\xi_i)$ *we have*

$$|\sum_{j=1}^{J+p} hG(y_n, \xi_j; t_n, \tau_p)\psi(\xi_j)| < \max_{j=1,2,\dots,J+p} |\psi(\xi_j)|. \tag{2.14}$$

Proof. The function

$$w(x_r, t_n) = \sum_{j=1}^{J+p} hG(x_r, \xi_j; t_n, \tau_p)\psi(\xi_j)$$

satisfies the difference equation (7) for $n > p$, $-\infty < j < \infty$ and has the Cauchy data

$$w(x_j, t_p) = \begin{cases} \psi(x_j), & j=1, 2,\dots, J+p \\ \psi(x_{-j+1}), & j=0, 1, 2,\dots, (J+p)+1 \\ 0, & \text{otherwise.} \end{cases}$$

Thus (2.14) follows from the well-known maximum principle.

Lemma 5. *For* $p<n$, *we have*

$$\frac{1}{\pi h^2}\int_{-\pi}^{\pi}\prod_{q=p}^{n}\Lambda_q^{-1}\sin^2\frac{\omega}{2}\, d\omega \leqq \frac{h}{2\sqrt{2}\, a^3(t_n-\tau_{p-1}-\hat{k})^{\frac{3}{2}}} \tag{2.15}$$

where

$$\hat{k} = \hat{k}(p, n) = \max_{q=p,\dots,n} k_q. \tag{2.16}$$

In particular we obtain for $p=n-1$ *that*

$$(2.17) \qquad \frac{1}{\pi h^2}\int_{-\pi}^{\pi} \Lambda_{n-1}^{-1}\, \Lambda_n^{-1} \sin^2\frac{\omega}{2}\, d\omega < \frac{h}{4a^3\sqrt{k_{n-1}k_n}(\sqrt{k_{n-1}}+\sqrt{k_n})}\,.$$

Proof. (2.17) can be obtained by elementary integration. We shall prove (2.15). First we shall show that there are two partial sums $\sum_{\mathrm{I}} k_q$ and $\sum_{\mathrm{II}} k_r$ $(q \neq r)$ such that

$$(2.18) \qquad t_n - \tau_{p-1} = \sum_{\mathrm{I}} k_q + \sum_{\mathrm{II}} k_r$$

and

$$(2.19) \qquad \sum_{\mathrm{I}} k_q \text{ and } \sum_{\mathrm{II}} k_r > \frac{1}{2}(t_n - \tau_{p-1} - \hat{k})$$

hold. In fact it is clear that we can select two partial sums $\sum' k_q$, $\sum'' k_r$ such that

$$\sum_{q=p}^{n} k_q = \sum' k_q + \sum'' k_r + \hat{k} \qquad (q \neq r,\ k_q \neq \hat{k},\ k_r \neq \hat{k}),$$

$$\sum' k_q \leqq \sum'' k_r \leqq \frac{1}{2}\sum_{q=p}^{n} k_q.$$

Hence

$$\sum'' k_r \geqq \frac{1}{2}\Big(\sum_{q=p}^{n} k_q - \hat{k}\Big),$$

$$\sum' k_q + \hat{k} \geqq \frac{1}{2}\sum_{q=p}^{n} k_q.$$

Let

$$\sum_{\mathrm{I}} k_q = \sum' k_q + \hat{k}, \quad \sum_{\mathrm{II}} k_r = \sum'' k_r,$$

which then satisfy (2.18) and (2.19).

It follows from (2.19) that

$$\frac{1}{\pi h^2}\int_{-\pi}^{\pi} \prod_{q=p}^{n} \Lambda_q^{-1} \sin^2\frac{\omega}{2}\, d\omega < \frac{1}{\pi h^2}\int_{-\pi}^{\pi} \frac{\sin^2\frac{\omega}{2}\, d\omega}{\Big(1+4\sum_{\mathrm{I}} \lambda_q \sin^2\frac{\omega}{2}\Big)\Big(1+4\sum_{\mathrm{II}} \lambda_r \sin^2\frac{\omega}{2}\Big)}$$

$$<\frac{1}{\pi h^2}\int_{-\pi}^{\pi}\frac{\sin^2\frac{\omega}{2}\,d\omega}{\left\{1+\frac{2a^2}{h^2}(t_n-\tau_{p-1}-\hat{k})\sin^2\frac{\omega}{2}\right\}^2}$$

and by elementary integration*)

$$=\frac{1}{h^2\left[1+\frac{2a^2}{h^2}(t_n-\tau_{p-1}-\hat{k})\right]^{\frac{3}{2}}}<\frac{h}{2\sqrt{2}\,a^3(t_n-\tau_{p-1}-\hat{k})^{\frac{3}{2}}}.$$

This proves (2.15).

Denote by V_h a bound for $\frac{h}{k_p}$ $(p=1, 2,\ldots, n)$:

$$\frac{h}{k_p}<V_h. \tag{2.20}$$

Then we have

$$(n-p+1)h<(t_n-\tau_{p-1})V_h, \tag{2.21}$$

$$(n-p)h<(t_n-\tau_{p-1}-\hat{k})V_h. \tag{2.22}$$

Lemma 6. *If $\gamma=1$ (K is arbitrary), or $\gamma=\frac{1}{2}$ and K is a half-integer, then*

$$|I_K|\equiv\left|\frac{1}{2\pi h^2}\int_{-\pi}^{\pi}\prod_{q=p}^{n}\Lambda_q^{-1}\sin\gamma\omega\cdot\sin K\omega\,d\omega\right|$$

(2.23)

$$<\frac{(1+2\sqrt{2})\gamma}{2a}\cdot\frac{1}{Kh\sqrt{t_n-\tau_{p-1}-\hat{k}}}+\frac{\sqrt{2}\,\gamma}{a}\cdot\frac{\hat{k}}{Kh(t_n-\tau_{p-1}-\hat{k})^{3/2}}$$

or alternatively

$$|I_K|<\frac{3\gamma}{2a}\frac{\sqrt{t_n-\tau_{p-1}}}{Kh^2}V_h. \tag{2.24}$$

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*) $$\int_{-\pi}^{\pi}\frac{\sin^2\frac{\omega}{2}}{\left(1+\alpha\sin^2\frac{\omega}{2}\right)^2}\,d\omega=\frac{\pi}{(1+\alpha)^{3/2}}\qquad(\alpha\text{: const.})$$

Proof. We put

$$\prod_{q=p}^{n} \Lambda_q^{-1} \sin \gamma\omega = \psi(\omega) . \tag{2.25}$$

Applying integration by parts to I_K we have

$$|I_K| = \frac{1}{2Kh^2\pi} \left| \int_{-\pi}^{\pi} \psi'(\omega) \cos K\omega \, d\omega \right| .$$

Since

$$\psi'(\omega) = \gamma \prod_{q=p}^{n} \Lambda_q^{-1} \left[\cos \gamma\omega - \sum_{q=p}^{n} 8 \Lambda_q^{-1} \lambda_q \sin^2 \frac{\omega}{2} \cos^{2\gamma} \frac{\omega}{2} \right] \tag{2.26}$$

it follows that

$$|I_K| < \frac{\gamma}{2Kh^2\pi} \left[\int_{-\pi}^{\pi} \frac{d\omega}{1 + 4 \sum_{q=p}^{n} \lambda_q \sin^2 \frac{\omega}{2}} {}^{*)} + 8 \sum_{q=p}^{n} \lambda_q \int_{-\pi}^{\pi} \prod_{q=p}^{n} \Lambda_q^{-1} \sin^2 \frac{\omega}{2} \, d\omega \right] \tag{2.27}$$

or alternatively

$$|I_K| < \frac{\left(n - p + \frac{3}{2}\right)\gamma}{Kh^2\pi} \int_{-\pi}^{\pi} \frac{d\omega}{1 + 4 \sum_{q=p}^{n} \lambda_q \sin^2 \frac{\omega}{2}} \tag{2.28}$$

and further that from Lemma 5,

$$\begin{aligned} |I_K| &< \frac{\gamma}{2a} \frac{1}{Kh\sqrt{t_n - \tau_{p-1}}} + \frac{\sqrt{2}\,\gamma}{a^3} \frac{t_n - \tau_{p-1}}{Kh(t_n - \tau_{p-1} - \hat{k})} \\ &= \frac{(1+2\sqrt{2})\gamma}{2a} \frac{1}{Kh\sqrt{t_n - \tau_{p-1}}} + \frac{\sqrt{2}\,\gamma}{a} \frac{\hat{k}}{Kh(t_n - \tau_{p-1} - \hat{k})^{3/2}} , \end{aligned}$$

or alternatively

---

*) $\displaystyle \int_{-\pi}^{\pi} \frac{d\omega}{1 + 4\lambda \sin^2 \frac{\omega}{2}} = \frac{2\pi}{\sqrt{1+4\lambda}} < \frac{\pi}{\sqrt{\lambda}}$

$$|I_K|<\frac{\left(n-p+\frac{3}{2}\right)h\gamma}{Kh^2a\sqrt{t_n-\tau_{p-1}}}<\frac{3\gamma}{2a}\frac{\sqrt{t_n-\tau_{p-1}}}{Kh^2}V_h.$$

Lemma 7.

$$|G_\xi(y_n, \eta_p;\, t_n, \tau_{p-1})|$$

(2.29)
$$<\left[\frac{2+\pi}{2\sqrt{2}\,a^3}V_h+\frac{1+2\sqrt{2}}{2a}\cdot\frac{1}{y_n+\eta_p}\right]\frac{1}{\sqrt{t_n-\tau_{p-1}-\hat{k}}}$$

$$+\frac{\sqrt{2}}{a(y_n+\eta_p)}\cdot\frac{\hat{k}}{\sqrt{t_n-\tau_{p-1}-\hat{k}}}$$

or alternatively

(2.30)
$$|G_\xi(y_n, \eta_p;\, t_n, \tau_{p-1})|<\frac{2+\pi}{2\sqrt{2}\,a^3}\frac{V_h}{\sqrt{t_n-\tau_{p-1}-\hat{k}}}$$

$$+\frac{3\sqrt{t_n-\tau_{p-1}}}{2a(y_n+\eta_p)h}V_h.$$

Proof. It follows from the definition that

$$G_\xi(y_n, \eta_p\colon t_n, \tau_{p-1})$$
$$=\frac{1}{\pi h^2}\int_{-\pi}^{\pi}\prod_{q=p}^{n}\Lambda_q^{-1}\left[-2\sin^2\frac{\omega}{2}\sin(J+n)\omega\sin(J+p)\omega\right.$$
$$\left.+\sin\frac{\omega}{2}\sin(n-p)\omega\cos\frac{\omega}{2}\right]d\omega-I_{2J+n+p}\quad\text{(see (2.23))}.$$

Hence by Lemma 5, (2.22) and 6 *)

$$|G_\xi(y_n, \eta_p;\, t_n, \tau_{p-1})|$$
$$<\frac{[2+(n-p)\pi]}{\pi h^2}\int_{-\pi}^{\pi}\prod_{q=p}^{n}\Lambda_q^{-1}\sin^2\frac{\omega}{2}d\omega+|I_{2J+n+p}|$$
$$<\frac{2+\pi}{2\sqrt{2}\,a^3}\frac{(n-p)h}{(t_n-\tau_{p-1}-\hat{k})^{3/2}}+|I_{2J+n+p}|\ **)$$

*) $|\sin(n-p)\omega|<(n-p)|\omega|<(n-p)\pi\left|\sin\frac{\omega}{2}\right|$, for $|\omega|<\pi$

**) $2+(n-p)\pi<(2+\pi)(n-p)$

$$<\left[\frac{2+\pi}{2\sqrt{2}\,a^3}V_h+\frac{1+2\sqrt{2}}{2a}\frac{1}{y_n+\eta_p}\right]\frac{1}{\sqrt{t_n-\tau_{p-1}-\hat{k}}}$$

$$+\frac{\sqrt{2}}{a}\frac{\hat{k}}{(y_n+\eta_p)(t_n-\tau_{p-1}-\hat{k})^{3/2}}$$

or alternately

$$<\frac{2+\pi}{2\sqrt{2}\,a^3}\cdot\frac{V_h}{\sqrt{t_n-\tau_{p-1}-\hat{k}}}+\frac{3}{2a}\frac{\sqrt{t_n-\tau_{p-1}}}{(y_n+\eta_p)h}V_h.$$

Lemma 8. *Assume that* $k_1\geqq k_2$. *If* h *is small with* $\frac{h}{k_i}<$ const. $(i=1,\ 2)$, *we have*

$$|\Phi_K|\equiv\left|\frac{1}{2\pi h^2}\int_{-\pi}^{\pi}\Lambda_1^{-1}\Lambda_2^{-1}\sin\omega\sin K\omega\;d\omega\right|<\frac{\pi}{4a^3}\frac{Kh}{k_1\sqrt{k_2}} \tag{2.31}$$

or for $Kh>3a\sqrt{k_1}$, *or alternately*

$$<\frac{Kh}{4a^3k_1^{3/2}}e^{-\frac{Kh}{a\sqrt{k_i}}}+\frac{\pi^3}{4a^3}\frac{h}{k_1\sqrt{k_2}}. \tag{2.32}$$

Proof. Since

$$|\Phi_K|<\frac{K}{h^2}\int_{-\pi}^{\pi}\Lambda_1^{-1}\Lambda_2^{-1}\sin^2\frac{\omega}{2}\,d\omega$$

we get (2.31) by (2.17) in Lemma 5.

Next we put

$$\Phi_K=\Phi_{K1}+\Phi_{K2}+\Phi_{K3} \tag{2.33}$$

where

$$\Phi_{K1}=\frac{1}{2\pi h^2}\int_{-\pi}^{\pi}\frac{\omega\sin K\omega}{(1+\lambda_1\omega^2)(1+\lambda_2\omega^2)}d\omega,$$

$$\Phi_{K2}=\frac{1}{2\pi h^2}\int_{-\pi}^{\pi}\frac{(\sin\omega-\omega)\sin K\omega}{\Lambda_1\Lambda_2}d\omega,$$

and

$$\Phi_{K3}=\frac{1}{2\pi h^2}\int_{-\pi}^{\pi}\left[\frac{1}{\Lambda_1\Lambda_2}-\frac{1}{(1+\lambda_1\omega^2)(1+\lambda_2\omega^2)}\right]\omega\sin K\omega\;d\omega.$$

Put $\omega=h\xi$. Then

$$|\Phi_{K1}|<\frac{1}{\pi}\left|\int_0^\infty \frac{\xi\sin Kh\xi}{(1+\lambda_1h^2\xi^2)(1+\lambda_2h^2\xi^2)}\,d\xi\right|^{*)}$$

$$+\frac{1}{\pi}\int_{\frac{\pi}{h}}^\infty \frac{\xi\,d\xi}{(1+\lambda_1h^2\xi^2)(1+\lambda_2h^2\xi^2)}$$

$$=\frac{\exp\left(-\frac{Kh}{a\sqrt{k_1}}\right)-\exp\left(-\frac{Kh}{a\sqrt{k_2}}\right)}{2a^2(k_1-k_2)}+\frac{1}{2\pi a^2(k_1-k_2)}\log\frac{\lambda_2^{-1}+\pi^2}{\lambda_1^{-1}+\pi^2}.$$

Using monotonicity of the function $\frac{Kh}{k^{3/2}}\exp\left(-\frac{Kh}{a\sqrt{k}}\right)$ with respect to k for $Kh>3a\sqrt{k}$ we conclude that

$$|\Phi_{K1}|<\frac{Kh}{4a^3k_1^{3/2}}\exp\left(-\frac{Kh}{a\sqrt{k_1}}\right)+\frac{h^2}{2\pi^3a^2k_2^2}. \tag{2.34}$$

Next by (2.17) in Lemma 5,

$$\begin{aligned}|\Phi_{K2}|&<\frac{\pi^2}{3h^2}\int_0^\pi \Lambda_1^{-1}\Lambda_2^{-1}\sin^2\frac{\omega}{2}\,d\omega<\frac{\pi^3}{12a^3}\frac{h}{\sqrt{k_1k_2}\,(\sqrt{k_1}+\sqrt{k_2})}^{**)}\\&<\frac{\pi^3h}{12a^3\,k_1\sqrt{k_2}}\end{aligned} \tag{2.35}$$

and

$$|\Phi_{K3}|<\frac{1}{\pi h^2}\int_0^\pi \frac{(\lambda_1+\lambda_2)\left(\omega^2-\sin^2\frac{\omega}{2}\right)+\lambda_1\lambda_2\left(\omega^4-16\sin^4\frac{\omega}{2}\right)}{\Lambda_1\Lambda_2(1+\lambda_1\omega^2)(1+\lambda_2\omega^2)}\,d\omega^{***)}$$

---

*) $$\int_0^\infty \frac{\xi\sin m\xi}{(1+a^2\xi^2)(1+b^2\xi^2)}\,d\xi=\begin{cases}\frac{\pi}{2(a^2-b^2)}\left(e^{-\frac{m}{a}}-e^{-\frac{m}{b}}\right) & (a\neq b),\\ \frac{m\pi}{4a^3}e^{-\frac{m}{a}} & (a=b)\end{cases}$$

**) $|\sin\omega-\omega|<\frac{1}{3}|\omega^3|<\frac{\pi^3}{3}\sin^2\frac{\omega}{2}$ for $|\omega|\leqq\pi$

***) $\omega^2-4\sin^2\frac{\omega}{2}<\frac{1}{6}\omega^4$,

$\omega^4-16\sin^4\frac{\omega}{2}<\frac{1}{3}\omega^6$

$$(2.36)\qquad \leqq \frac{(\lambda_1+\lambda_2)}{6\pi h^2\lambda_1\lambda_2}\int_0^\infty \frac{\omega\, d\omega}{\left(1+\frac{4}{\pi^2}\lambda_2\omega^2\right)^2}+\frac{\pi^2}{12h^2\lambda_1}\int_0^\pi \frac{d\omega}{1+4\lambda_2\sin^2\frac{\omega}{2}}$$

$$<\frac{\pi h^2}{16a^4k_2^{\frac{3}{2}}}+\frac{\pi^3 h}{24a^3k_1\sqrt{k_2}}.$$

From (2.33)-(2.36) we get

$$|\Phi_K|<\frac{Kh}{4a^3k_1^{3/2}}\exp\left(-\frac{Kh}{a\sqrt{k_1}}\right)+\left(\frac{1}{2\pi^3a^2}+\frac{\pi}{16a^4}\right)\frac{h^2}{k_2^{\frac{3}{2}}}+\frac{\pi^3}{8a^3}\frac{h}{k_1\sqrt{k_2}}$$

and for small h with $\frac{h}{k_i}<\text{const.}$ $(i=1, 2)$,

$$|\Phi_K|<\frac{Kh}{4a^3k_1^{3/2}}\exp\left(-\frac{Kh}{a\sqrt{k_1}}\right)+\frac{\pi^3}{4a^3}\cdot\frac{h}{k_1\sqrt{k_2}}.$$

This proves the second part of Lemma 8.

§3. Convergence of the Iteration Procedure

We shall prove the first part of Theorem. It is supposed that we know already $u(x_j, t_{n-1})$ $(j=0, 1, 2,\dots, J+n-1)$ and k_{n-1} and have the estimates

$$(3.1)\qquad \begin{aligned}&\max_j |u_{\bar{x}}(x_j, t_{n-1})|<\tilde{M},\\ &\max_j |u_{x\bar{x}}(x_j, t_{n-1})|<\frac{\bar{M}}{a^2}.\end{aligned}$$

In order to prove convergence of the iteration procedure (1.14)-(1.18) it is sufficient to show that there is a constant δ $(0<\delta<1)$ such that

$$(3.2)\qquad |v_n^{(s)}-v_n^{(s-1)}|<\delta|v_n^{(s-1)}-v_n^{(s-2)}| \qquad (s=3, 4,\dots).$$

In fact it follows from (3.2) that $v_n^{(s)}$ converges as $s\to\infty$ and hence $k_n^{(s)}$ also converges to a limit k_n. From maximum principle $u^{(s)}(x_j, t_n)$ are uniformly bounded and hence each subsequence $u^{(s_i)}(x_j, t_n)$ converges to each limit $u(x_j, t_n)(j=0, 1, 2,\dots, J+n)$. It is clear that the limit function satisfies the equation

$$a^2 u_{x\bar{x}}(x_j, t_n) - \frac{u(x_j, t_n) - u(x_j, t_{n-1})}{k_n} = 0$$

and the conditions $u(0, t_n) = f(t_n)$, $u(y_n, t_n) = 0$. Since such function is uniquely determined, the sequence $u^{(s)}(x_j, t_n)$ itself converges to the same limits $u(x_j, t_n)$, $(j = 0, 1, \dots, J+n)$.

Now we shall show that (3.2) is valid under some conditions. Applying the formula (2.7) in Lemma 1 to the solution of (1.14)–(1.16) we have

$$\begin{aligned} v_n^{(s)} &= [1 + a^2 k_n^{(s)} G_\xi(y_n, \eta_n; t_{n-1} + k_n^{(s)}, \tau_{n-1})]^{-1} \\ &\times [\sum_{j=1}^{J+n-1} hG(y_n, \xi_j; t_{n-1} + k_n^{(s)}, \tau_{n-1}) u_\xi(\xi_j, \tau_{n-1}) \\ &- G(y_n, 0; t_{n-1} + k_n^{(s)}, \tau_{n-1}) \{f(\tau_{n-1} + k_n^{(s)}) - f(\tau_{n-1})\}]. \end{aligned} \tag{3.3}$$

First we get from Lemmas 2, 3, 4

$$|v_n^{(s)}| < 4\left(\tilde{M} + \frac{\sqrt{k_n^{(s)}}}{a} M_1\right).$$

Since $k_n^{(s)}$ is at most $\frac{h^{3/4}}{\sqrt{\beta}}$ from (1.17), we have for small h

$$|v_n^{(s)}| < 4\tilde{M} + 1 = M, \qquad s = 1, 2, 3, \dots. \tag{3.4}$$

Hence it follows from (1.17) that

$$\begin{aligned} \frac{h}{k_n^{(s)}} &= \frac{1}{2}(\sqrt{\kappa^2 v_n^{(s-1)2} + 4\beta\sqrt{h}} + \kappa v_n^{(s-1)}) \\ &< \kappa M + \frac{\beta}{\kappa M}\sqrt{h} \\ &< 2\kappa M \end{aligned} \tag{3.5}$$

for small h.

We consider the difference $v_n^{(s)} - v_n^{(s-1)}$: using the notation

$$D(\cdot(k_n^{(s)})) = \cdot(k_n^{(s)}) - \cdot(k_n^{(s-1)})$$

we have

$$
(3.6)\quad
\begin{aligned}
|v_n^{(s)}-v_n^{(s-1)}| &< |D(\Gamma_1^{-1}(k_n^{(s)}))|\cdot|\Gamma_2(k_n^{(s)})| \\
&+|\Gamma_1^{-1}(k_n^{(s-1)})|\cdot[\,|\sum_{j=1}^{J+n-1} hD(G(y_n,\xi_j;t_{n-1}+k_n^{(s)},\tau_{n-1})u_\xi(\xi_j,\tau_{n-1})| \\
&+|G(y_n,0;t_{n-1}+k_n^{(s)},\tau_{n-1})D(f(\tau_{n-1}+k_n^{(s)}))| \\
&+k_n^{(s-1)}|f_\tau(\tau_n)|\cdot|D(G(y_n,0;t_{n-1}+k_n^{(s)},\tau_{n-1})|\,]
\end{aligned}
$$

where

$$
(3.7)\quad
\begin{aligned}
\Gamma_1(k_n^{(s)}) &= 1+a^2k_n^{(s)}G_\xi(y_n,\xi_n;t_{n-1}+k_n^{(s)},\tau_{n-1}), \\
\Gamma_2(k_n^{(s)}) &= \sum_{j=1}^{J+n-1} hG(y_n,\xi_j;t_{n-1}+k_n^{(s)},\tau_{n-1})u_\xi(\xi_j,\tau_{n-1}) \\
&\quad -k_n^{(s)}G(y_n,0;t_{n-1}+k_n^{(s)},\tau_{n-1})\cdot f_\tau(\tau_n).
\end{aligned}
$$

It is easy to see that from Lemmas 2, 3, 4, and (3.5)

$$(3.8)\quad |\Gamma_1^{-1}(k_n^{(s-1)})|<4$$

$$(3.9)\quad |\Gamma_2(k_n^{(s)})|<2\tilde{M} \qquad \text{(for small } h)$$

$$(3.10)\quad |G(y_n,0;t_{n-1}+k_n^{(s)},\tau_{n-1})D(f(\tau_{n-1}+k_n^{(s)}))|\,C_1\frac{|D(k_n^{(s)})|}{\sqrt{h}} \qquad \left(C_1=\frac{\sqrt{2\kappa M}\,M_1}{a}\right),$$

and from Lemma 5,

$$
(3.11)\quad
\begin{aligned}
&|k_n^{(s-1)}f_\tau(\tau_n)D(G(y_n,0;t_{n-1}+k_n^{(s)},\tau_{n-1}))| \\
&< \frac{4a^2k_n^{(s-1)}M_1}{\pi h^3}\int_{-\pi}^{\pi}(\Lambda^{(s)}\Lambda^{(s-1)})^{-1}\sin^2\frac{\omega}{2}d\omega\,|D(k_n^{(s)})| \\
&< \frac{a^2k_n^{(s-1)}M_1|D(k_n^{(s)})|}{h^3\sqrt{\lambda_n^{(s)}\lambda_n^{(s-1)}}(\sqrt{\lambda_n^{(s)}}+\sqrt{\lambda_n^{(s-1)}})} \\
&< \frac{C_1}{\sqrt{h}}D(k_n^{(s)}),
\end{aligned}
$$

where

$$\Lambda^{(s)}=1+4\lambda_n^{(s)}\sin^2\frac{\omega}{2}, \qquad \Lambda^{(s-1)}=1+4\lambda_n^{(s-1)}\sin^2\frac{\omega}{2},$$

(3.12)

$$\lambda_n^{(s)}=\frac{a^2k_n^{(s)}}{h^2}, \qquad \lambda_n^{(s-1)}=\frac{a^2k_n^{(s-1)}}{h^2}.$$

By the mean value theorem

$$|D(\Gamma_1^{-1}(k_n^{(s)}))|<16|[a^2G_\xi(y_n, \eta_n; t_{n-1}+\bar{k}, \tau_{n-1})$$
$$+a^2\bar{k}\frac{dG}{d\bar{k}}(y_n, \eta_n; t_{n-1}+\bar{k}, \tau_{n-1})]|\cdot|D(k_n^{(s)})|$$
$$<\frac{16a^2}{\pi h^2}\left[\int_{-\pi}^{\pi}\frac{\sin^2\frac{\omega}{2}\,d\omega}{\left(1+4\bar{\lambda}\sin^2\frac{\omega}{2}\right)^2}\right.$$
$$\left.+\left|\int_{-\pi}^{\pi}\frac{\sin\frac{\omega}{2}\sin\left\{2(J+n)-\frac{1}{2}\right\}\omega\,d\omega}{\left(1+4\bar{\lambda}\sin^2\frac{\omega}{2}\right)^2}\right|\right]\cdot|D(k_n^{(s)})|$$

where $\bar{k}$ is a value between $k_n^{(s)}$ and $k_n^{(s-1)}$, $\bar{\lambda}=\frac{a^2\bar{k}}{h^2}$. By (2.18) in Lemma 5 and (2.24) in Lemma 6,

$$|D(\Gamma_1^{-1}(k_n^{(s)}))|<\left[\frac{2}{a}\frac{h}{\bar{k}^{3/2}}+8(1+4\sqrt{2})a\frac{1}{\left\{2(J+n)-\frac{1}{2}\right\}h\sqrt{\bar{k}}}\right]\cdot|D(k_n^{(s)})|$$

(3.13)

$$<C_2\frac{|D(k_n^{(s)})|}{\sqrt{h}}, \quad C_2=\frac{4\sqrt{2}(\kappa M)^{3/2}}{a}+\frac{4(8+\sqrt{2})a\sqrt{\kappa M}}{l}.$$

(by (3.5))

Finally we consider the sum

$$B\equiv\sum_{j=1}^{J+n-1}hD(G(y_n, \xi_j; t_{n-1}+k_n^{(s)}, \tau_{n-1}))u_\xi(\xi_j, \tau_{n-1}). \tag{3.14}$$

Here

$$D(G(y_n, \xi_j; t_{n-1}+k_n^{(s)}, \tau_{n-1}))$$
$$=-\frac{1}{2\pi h}\int_{-\pi}^{\pi}\frac{4a^2}{h^2}\sin^2\frac{\omega}{2}(\Lambda^{(s)}\Lambda^{(s-1)})^{-1}[e^{-i(J+n-j)\omega}$$
$$+e^{-i(J+n+j-1)\omega}]d\omega\cdot D(k_n^{(s)})$$
$$=a^2\Phi_{\xi\xi}(\xi_j)D(k_n^{(s)}),$$

where

$$\Phi(\xi_j)=\frac{1}{2\pi h}\int_{-\pi}^{\pi}(\Lambda^{(s)}\Lambda^{(s-1)})^{-1}[e^{-i(J+n-j)\omega}+e^{-i(J+n+j-1)\omega}]d\omega.$$

Hence

$$B=a^2D(k^{(s)})\sum_{j=1}^{J+n-1}h\Phi_{\xi\bar{\xi}}(\xi_j)u_{\bar{\xi}}(\xi_j,\tau_{n-1})$$

$$=a^2D(k^{(s)})[\Phi_{\xi}(\eta_{n-1})u(\eta_{n-1},\tau_{n-1})-\Phi_{\xi}(0)u_{\bar{\xi}}(0,\tau_{n-1})$$

$$-\sum_{j=0}^{J+n-2}h\Phi_{\xi}(\xi_j)u_{\xi\bar{\xi}}(\xi_j,\tau_{n-1})].$$

Thus

(3.15)
$$|B|<a^2[\tilde{M}|\Phi_{\xi}(\eta_{n-1})|+\tilde{M}|\Phi_{\xi}(0)|+\frac{\bar{M}}{a^2}\sum_{j=0}^{J+n-2}h|\Phi_{\xi}(\xi_j)|]\cdot|D(k_n^{(s)})|.$$

Here

$$\Phi_{\xi}(\xi_j)=-\frac{2}{\pi h^2}\int_{-\pi}^{\pi}\{\Lambda^{(s)}\Lambda^{(s-1)}\}^{-1}\sin^2\frac{\omega}{2}\sin(J+n)\omega\sin j\omega\,d\omega$$

$$+\frac{1}{2\pi h^2}\int_{-\pi}^{\pi}\{\Lambda^{(s)}\Lambda^{(s-1)}\}^{-1}\sin\omega\sin(J+n-j)\omega\,d\omega$$

$$-\frac{1}{2\pi h^2}\int_{-\pi}^{\pi}\{\Lambda^{(s)}\Lambda^{(s-1)}\}^{-1}\sin\omega\sin(J+n+j)\omega\,d\omega$$

and from Lemma 5 and Lemma 8

$$|\Phi_{\xi}(\xi_j)|<\frac{h}{2a^3\hat{k}\sqrt{\check{k}}}+\frac{y_n+x_j}{4a^3\hat{k}^{3/2}}\exp\left(-\frac{y_n+x_j}{a\sqrt{\hat{k}}}\right)+\frac{C_3h}{\hat{k}\sqrt{\check{k}}}$$

$$+\begin{cases}\dfrac{y_n-x_j}{4a^3\hat{k}^{3/2}}\exp\left(-\dfrac{y_n-x_j}{a\sqrt{\hat{k}}}\right)+\dfrac{C_3h}{\hat{k}\sqrt{\check{k}}} & (\text{for } y_n-x_j>3a\sqrt{\hat{k}})\\ \text{or alternatively} & \\ \dfrac{C_3(y_n-x_j)}{\hat{k}\sqrt{\check{k}}} & \end{cases}$$

where $\hat{k}=\max(k_n^{(s)}, k_n^{(s-1)})$, $\check{k}=\min(k_n^{(s)}, k_n^{(s-1)})$ and $C_3=\frac{\pi^3}{4a^3}$.

Using (3.4) we have $\frac{h}{\hat{k}\sqrt{\check{k}}}<\frac{(2\kappa M)^{3/2}}{\sqrt{h}}$. Hence

$$|\Phi_\xi(\xi_j)|<\frac{y_n+\xi_j}{4a^3\hat{k}^{3/2}}\exp\left(-\frac{y_n+\xi_j}{a\sqrt{\hat{k}}}\right)+\frac{C_4}{\sqrt{h}}$$

$$(3.16)\qquad +\begin{cases}\dfrac{y_n-\xi_j}{4a^3\hat{k}^{3/2}}\exp\left(-\dfrac{y_n-\xi_j}{a\sqrt{\hat{k}}}\right) & (\text{for } y_n-\xi_j>3a\sqrt{\hat{k}})\\ \text{or alternatively} & \\ \dfrac{C_3(y_n-\xi_j)}{\hat{k}\sqrt{\check{k}}} & \end{cases}$$

$$\left(C_4=\frac{\sqrt{2}(\kappa M)^{3/2}}{a^3}+2C_3(\kappa M)^{3/2}\right).$$

In particular

$$(3.17)\qquad |\Phi_\xi(\eta_{n-1})|<\frac{y_n+\eta_{n-1}}{4a^3\hat{k}^{3/2}}\exp\left(-\frac{y_n+\eta_{n-1}}{a\sqrt{\hat{k}}}\right)+\frac{C_5}{\sqrt{h}}$$

and

$$(3.18)\qquad |\Phi_\xi(0)|<\frac{y_n}{2a^3\hat{k}^{3/2}}\exp\left(-\frac{y_n}{a\sqrt{\hat{k}}}\right)+\frac{C_5}{\sqrt{h}}$$

$$(C_5=C_4+(2\kappa M)^{3/2}C_3).$$

From (3.13)–(3.16) we have

$$|B|<a^2\tilde{M}\left[\frac{y_n+\eta_{n-1}}{4a^3\hat{k}^{3/2}}\exp\left(-\frac{y_n+\eta_{n-1}}{a\sqrt{\hat{k}}}\right)+\frac{y_n}{2a^3\hat{k}^{3/2}}\exp\left(-\frac{y_n}{a\sqrt{\hat{k}}}\right)\right.$$

$$\left.+\frac{2C_5}{\sqrt{h}}\right]\cdot|D(k_n^{(s)})|+\bar{M}\left[\sum_{j=0}^{J+n-2}\frac{y_n+\xi_j}{4a^3\hat{k}^{3/2}}\exp\left(-\frac{y_n+\xi_j}{a\sqrt{\hat{k}}}\right)\right.$$

(3.19)

$$+\frac{C_4\eta_{n-1}}{\sqrt{h}}+\sum_{j=0}^{J_1}h\frac{y_n-\xi_j}{4a^3\hat{k}^{3/2}}\exp\left(-\frac{y_n-\xi_j}{a\sqrt{\hat{k}}}\right)+$$

$$+C_3\sum_{j=J_1+1}^{J+n-2} h\frac{y_n-\xi_j}{\hat{k}\sqrt{\check{k}}}\Bigg]\cdot|D(k_n^{(s)})|,$$

where $J_1=\max\{j;\ y_n-\xi_j>3a\sqrt{\hat{k}}\}$. Here we note that

$$2l<y_n+\eta_{n-1}<2l+2Mt_n,$$

$$l<y_n<l+Mt_n.$$

It follows from (3.19) that

$$|B|<a^2\tilde{M}\Bigg[\frac{l+Mt_n}{2a^3\hat{k}^{3/2}}\Bigg\{\Bigg(\exp\Big(-\frac{2l}{a\sqrt{\hat{k}}}\Big)+\exp\Big(-\frac{l}{a\sqrt{\hat{k}}}\Big)\Bigg\}+\frac{2C_5}{\sqrt{h}}\Bigg]\cdot|D(k_n^{(s)})|$$

$$+\bar{M}\Bigg[\frac{1}{4a^3\hat{k}^{3/2}}\int_0^{y_n}\Big\{(y_n-x)\exp\Big(-\frac{y_n-x}{a\sqrt{\hat{k}}}\Big)+(y_n+x)\exp\Big(-\frac{y_n+x}{a\sqrt{\hat{k}}}\Big)\Big\}dx$$

$$+\frac{9a^2C_2}{2\sqrt{\check{k}}}+\frac{C_4(l+Mt_n)}{\sqrt{h}}\Bigg]\cdot|D(k_n^{(s)})|$$

and further for small $h(\hat{k})$

$$|B|<\Bigg[\frac{4a^2\tilde{M}}{\sqrt{h}}+\frac{\bar{M}}{2a\sqrt{\hat{k}}}\int_0^\infty\sigma e^{-\sigma}d\sigma+\frac{9a^2C_2\bar{M}}{2\sqrt{\check{k}}}$$

$$+\frac{C_4(l+Mt_n)\bar{M}}{\sqrt{h}}\Bigg]\cdot|D(k_n^{(s)})|<\frac{C_6}{\sqrt{h}}|D(k_n^{(s)})| \tag{3.20}$$

$$\Bigg(C_6=4a^2\tilde{M}+\frac{\bar{M}}{\sqrt{2}}\Big(\frac{1}{a}+9a^2C_3\Big)\sqrt{\kappa M}+C_4(l+MT)\bar{M}\Bigg).$$

Consequently we get from (3.6), (3.8)-(3.11), (3.13)-(3.14) and (3.20)

$$|v_n^{(s)}-v_n^{(s-1)}|<\frac{C_7}{\sqrt{h}}|D(k_n^{(s)})|, \tag{3.21}$$

$$C_7=2\tilde{M}C_2+4(2C_1+C_6).$$

Here we have from (1.17)

$$|D(k_n^{(s)})|<\frac{\kappa\sqrt{h}}{\beta}|v_n^{(s-1)}-v_n^{(s-2)}|. \tag{3.22}$$

It follow from (3.21) and (3.22) that

$$|v_n^{(s)}-v_n^{(s-1)}|<\frac{C_7\kappa}{\beta}|v_n^{(s-1)}-v_n^{(s-2)}|.$$

If β is chosen so large that

(3.23) $$\delta\equiv\frac{C_7\kappa}{\beta}<1$$

then (3.2) holds with δ smaller than 1. Thus we have proved convergence of our iteration procedure.

§4. Convergence of the Scheme as $h\to 0$

We assume that

(4.1) $$\max_{0\leqq t\leqq T}|f(t)|,\ \max_{0\leqq x\leqq l}|\varphi(x)|<M_0$$

(4.2) $$\max_{0<t<T}|\dot{f}(t)|,\ \max_{0<x<l}|\dot{\varphi}(x)|<M_1$$

and

(4.3) $$\max_{0<x<l}|\ddot{\varphi}(x)|<M_2.$$

By the maximum principle we have from (4.1)

(4.4) $$\max_{\substack{0\leqq x_j\leqq y_n\\ 0<t_n\leqq T}}|u(x_j,t_n)|<M_0$$

and also from the assumption $\varphi\leqq 0$, $f\leqq 0$,

(4.5) $$u(x_j,t_n)\leqq 0.$$

We shall see that it is sufficient for convergence proof to show a priori estimate

(4.6) $$\max_{t_n<T}|v_n|<M.$$

In fact we have then for small h

(4.7) $$|u_{\bar{l}}(y_{n-1},t_n)|<2\kappa M^2$$

because

$$0 \leqq u_{\bar{t}}(y_{n-1}, t_n) = -\frac{h}{k_n} v_n = -\frac{v_n}{2}(\sqrt{\kappa^2 v_n^2 + 4\beta\sqrt{h}} + \kappa v_n).$$

The system which $z = u_{\bar{t}}$ satisfies is

$$(4.8) \quad \begin{cases} a^2 z_{x\bar{x}}(x_j, t_n) - z_{\bar{t}}(x_j, t_n) = 0, & 0 < x_j < y_{n-1}, \quad 0 < t_n < T, \\ z(0, t_n) = f_{\bar{t}}(t_n), & 0 < t_n < T, \\ z(y_{n-1}, t_n) = u_{\bar{t}}(y_{n-1}, t_n), & 0 < t_n < T, \\ z(x_j, 0) = a^2 \varphi_{x\bar{x}}(x_j), & 0 < x_j < l. \end{cases}$$

By the maximum principle we get from (4.2), (4.3) and (4.7)

$$(4.9) \quad \max_{\substack{0 < x_j < y_n \\ t_n < T}} |u_{\bar{t}}(x_j, t_n)| < \bar{M} = \max\{M_1, a^2 M_2, 2\kappa M^2\}$$

and

$$(4.10) \quad \max_{\substack{0 < x_j < y_n \\ t_n < T}} |u_{x\bar{x}}(x_j, t_n)| < \frac{\bar{M}}{a^2}.$$

Using the identity

$$u_x(x_j, t_n) = v_n - \sum_{r=j+1}^{J+n-1} u_{x\bar{x}}(x_r, t_n) h$$

and (4.6), (4.10) we obtain

$$(4.11) \quad \max_{\substack{0 < x_j < y_n \\ t_n < T}} |u_x(x_j, t_n)| < \tilde{M} = M + \frac{\bar{M}}{a^2}(l + 2\kappa M T).$$

We shall show convergence from a priori estimates (4.4), (4.6), (4.9)–(4.11). Let h_α tend to zero as $\alpha \to \infty$. From (1.10) corresponding $k_{n\alpha}$ tends to zero as $\alpha \to \infty$. Denote by $y_\alpha(t)$ the broken line crossing each right-end-mesh-point (y_n, t_n). Then we have from (1.10)′, (4.6)

$$(4.12) \quad l \leqq y_\alpha(t) \leqq l + 2\kappa M T, \qquad 0 < t < T$$

and also

$$(4.13) \quad 0 < y_\alpha(t^2) - y_\alpha(t^1) < \kappa M(t^2 - t^1), \qquad 0 < t^1 < t^2 < T.$$

Since last inequalities means that the sequence of functions $\{y_\alpha(t)\}$ is uniformly bounded and equi-continuous, it follows that there is a subsequence (which we denote again by $\{y_\alpha(t)\}$ which converges to a continuous function $y(t)$ uniformly in $0 \leqq t \leqq T$. The limit function satisfies by (4.12) and (4.13)

$$(4.14) \qquad l \leqq y(t) \leqq l + 2MT, \qquad 0 < t < T$$

$$(4.15) \qquad 0 < y(t^2) - y(t^1) < \kappa M(t^2 - t^1), \qquad 0 < t^1 < t^2 < T.$$

Let u_α be the solution of system (1.6)–(1.10) corresponding to h_α. It is shown from (4.4), (4.6), (4.9)–(4.11) and (4.15) that a subsequence of $\{u_\alpha\}$ converges to the solution u of (1.1)–(1.4) with the boundary $x = y(t)$ above defined uniformly in $0 < x < y(t)$, $0 < t < T$. (see Petrowsky [4])

We shall show that the pair of functions $(y(t), u(x, t))$ satisfies also the Stefan's condition (1.5) (hence all the system (1.1)–(1.5)). We can define $\{u_\alpha(x, t)\}$ for all (x, t) extended from $\{u_\alpha(x_j, t_n)\}$ appropriately. Then we have from (4.10)

$$(4.16) \qquad |u_{\alpha\bar{x}}(x, t) - u_{\alpha\bar{x}}(x', t)| < \frac{\bar{M}}{a^2}|x - x'|, \qquad 0 < t < T$$

and

$$(4.17) \qquad \left|\frac{\partial u(x, t)}{\partial x} - \frac{\partial u(x', t)}{\partial x}\right| < \frac{\bar{M}}{a^2}|x - x'|, \qquad 0 < t < T.$$

Hence the limits

$$(4.18) \qquad \lim_{x \to y_\alpha(t)} u_{\alpha\bar{x}}(x, t) = v_\alpha(t) \qquad \text{(uniformly in } t\text{)}$$

and

$$(4.19) \qquad \lim_{x \to y(t)} \frac{\partial u}{\partial x}(x, t) = v(t) \qquad \text{(uniformly in } t\text{)}$$

exist. Consequently it follows from (4.18) and (4.19) that

$$(4.20) \qquad \lim_{\alpha \to \infty} v_\alpha(t) = v(t) \qquad \text{(uniformly in } t\text{)}.$$

By (1.10)′

$$y_\alpha(t)=l+\int_0^t \dot{y}_\alpha(t)\,dt=l+\sum k_n\frac{h}{k_n}$$

$$=l+\frac{1}{2}\sum k_n(\sqrt{\kappa^2 v_{\alpha n}^2+4\beta\sqrt{h}}+\kappa v_{\alpha n}).$$

Here take $\alpha\to\infty$. Then we get by (4.20)

$$y(t)=l+\kappa\int_0^t v(t)\,dt.$$

This means that $y(t)$ is differentiable and

$$\dot{y}(t)=\kappa v(t)$$

which is not but the Stefan's condition.

Since the solution of the system (1.1)-(1.5) is unique (Friedman [3]), it follows that not only a subsequence but also the full sequence $\{y_\alpha(t)\}$, $\{u_\alpha(x, t)\}$ themselves converge to $y(t)$ and $u(x, t)$ respectively.

It remains to show (4.6). Applying the formula (2.6) to the solution of (1.6)-(1.10) we have

$$v_n=A_1^{-1}[A_2+A_3+A_4], \tag{4.21}$$

where

$$A_1=1+a^2k_nG_\xi(y_n, \eta_n; t_n, \tau_{n-1}),$$

$$A_2=\sum_{j=1}^{J} hG(y_n, \xi_j; t_n, 0)\varphi_\xi(\xi_j), \tag{4.22}$$

$$A_3=-\sum_{P=1}^{n} k_pG(y_n, 0; t_n, \tau_{p-1})f_\tau(\tau_p)$$

and

$$A_4=-a^2\sum_{P=1}^{n} k_pG_\xi(y_n, \eta_p; t_n, \tau_{p-1})v_p.$$

Directly from Lemmas 2 and 4 we get

$$|A_1|<4, \tag{4.23}$$

$$|A_2|<M_1. \tag{4.24}$$

By Lemma 3,

$$(4.25) \qquad |A_3| < \frac{M_1}{a} \int_0^{t_n} \frac{d\tau}{\sqrt{t_n - \tau}} = \frac{2M_1}{a} \sqrt{t_n} \, .$$

Now we consider A_4:

$$A_4 = A_{41} + A_{42},$$

$$(4.26) \qquad A_{41} = -a^2 \sum_{t_n - \tau_{p-1} > \frac{2h^{\frac{4}{3}}}{\sqrt{\beta}}} k_p G_\xi(y_n, \eta_p; t_n, \tau_{p-1}) v_p,$$

$$A_{42} = -a^2 \sum_{t_n - \tau_{p-1} < \frac{2h^{\frac{3}{4}}}{\sqrt{\beta}}} k_p G_\xi (y_n, \eta_p; t_n, \tau_{p-1}) v_p.$$

First we have from (2.30)

$$|A_{41}| < \Bigg\{ \sum_{t_n - \tau_{p-1} > \frac{2h^{\frac{3}{4}}}{\sqrt{\beta}}} \frac{k_p}{\sqrt{t_n - \tau_{p-1} - \hat{k}(n, p)}} [\gamma_1 V_h + \gamma_2]$$

$$+ \sum_{t_n - \tau_{p-1} > \frac{2h^{\frac{3}{4}}}{\sqrt{\beta}}} \frac{\gamma_3 \hat{k}(n, p)}{(t_n - \tau_{p-1} - \hat{k}_p(n, p))^{\frac{3}{2}}} \Bigg\} ||v||$$

where

$$||v|| = \max_{p=1, \dots, n-1} |v_p|$$

$$\gamma_1 = \frac{2 + \pi}{2\sqrt{2a}}, \qquad \gamma_2 = \frac{(1 + 2\sqrt{2})a}{4l}, \qquad \gamma_3 = \frac{a}{\sqrt{2}\, l}$$

and V_h is a bound for $\frac{h}{k_p}$ $(p = 1, \dots, n-1)$ as in §2. Since $\hat{k}(n, p) < \frac{h^{\frac{3}{4}}}{\sqrt{\beta}}$,

$$|A_{41}| < \left[\sqrt{2} (\gamma_1 V_h + \gamma_2) \int_0^{t_{n-2}} \frac{d\tau}{\sqrt{t_n - \tau}} + \frac{2\sqrt{2} \gamma_3 h^{\frac{3}{4}}}{\sqrt{\beta}} \int_0^{t_n - \frac{2h^{\frac{3}{4}}}{\sqrt{\beta}}} \frac{d\tau}{(t_n - \tau)^{\frac{3}{2}}} \right] ||v||$$

$$(4.27) \qquad < [(\gamma_1' V_h + \gamma_2') \sqrt{t_n} + \gamma_3' h^{\frac{3}{8}}] \, ||v||$$

$$\left(\gamma_1' = 2\sqrt{2} \gamma_1, \qquad \gamma_2' = 2\sqrt{2} \gamma_2, \qquad \gamma_3' = \frac{4\gamma_3}{\beta^{\frac{1}{4}}} \right).$$

Using (2.31) we have

$$|A_{42}|<\Big[\sum_{t_n-\tau_{p-1}<\frac{2h^{\frac{3}{4}}}{\sqrt{\beta}}}\Big\{\frac{\gamma_1 V_h}{\sqrt{t_n-\tau_{p-1}-\hat{k}(n,\,p)}}+\frac{\gamma_4 V_h}{h}\sqrt{t_n-\tau_{p-1}}\Big\}k_p\Big]\,||v||$$

$$\left(\gamma_4=\frac{3a}{4l}\right).$$

Since $k_q>\dfrac{h}{k||v||}$ from (1.10′) we have

$$\sqrt{t_n-\tau_{p-1}-\hat{k}(n,\,p)}>\sqrt{\frac{h}{\kappa||v||}}\;.$$

Hence

$$|A_{42}|<\Big[2\gamma_1\sqrt{\frac{\kappa||v||}{\beta}}V_h\,h^{\frac{1}{4}}+\frac{\gamma_4 V_h}{h}\int_{t_n-\frac{2h^{\frac{3}{4}}}{\sqrt{\beta}}}^{t_n}\sqrt{t_n-\tau}\,d\tau\Big]$$

(4.28) $$=[\gamma_1'' h^{\frac{1}{4}}V_h||v||^{\frac{1}{2}}+\gamma_4' h^{\frac{1}{8}}V_h]\,||v||$$

$$\left(\gamma_1''=2\gamma_1\sqrt{\frac{\kappa}{\beta}},\qquad \gamma_4'=\frac{2\sqrt{2}\,\gamma_4}{3\beta^{\frac{3}{4}}}\right).$$

We obtain from (4.26)–(4.28)

$$|A_4|<[(\gamma_1'V_h+\gamma_2')\sqrt{t_n}+\gamma_3'h^{\frac{3}{8}}+\gamma_1''h^{\frac{1}{4}}V_h||v||^{\frac{1}{2}}$$

$$+\gamma_4'h^{\frac{1}{8}}V_h]\,||v||.$$

Here

$$V_h<\kappa||v||+\sqrt{\beta}\,h^{\frac{1}{4}}$$

from (1.10′). Put

$$|A_4|<[\gamma_3'h^{\frac{3}{8}}+\gamma_4'\sqrt{\beta}\,h^{\frac{3}{8}}+\gamma_1''\sqrt{\beta}\,h^{\frac{1}{2}}||v||^{\frac{1}{2}}$$

(4.29) $$+\gamma_4'\kappa h^{\frac{1}{8}}||v||+\gamma_1''\kappa h^{\frac{1}{4}}||v||^{\frac{3}{2}}+(\gamma_2'+\gamma_1'\sqrt{\beta}\,h^{\frac{1}{4}}$$

$$+\gamma_1'\kappa||v||)\sqrt{t_n}\,]\,||v||\equiv L_1(||v||,\,t_n,\,h)\,.$$

It follows from (4.23)–(4.25) and (4.29) that

$$|v_n|<4\left[M_1+\frac{2M_1}{a}\sqrt{t_n}+L_1(\|v\|,\ t_n,\ h)\right]. \tag{4.30}$$

Put

$$\hat{M}=4M_1+1 \tag{4.31}$$

and take h so small that

$$\begin{gathered}\gamma_2'+\gamma_1'\sqrt{\beta}\,h^{\frac{1}{4}}+\gamma_1'\kappa\hat{M}<2(\gamma_2'+\gamma_1'\kappa\hat{M}),\quad\text{and}\\ (\gamma_3'h^{\frac{3}{8}}+\gamma_4'\sqrt{\beta}\,h^{\frac{3}{8}}+\gamma_1''\sqrt{\beta}\,\hat{M}^{\frac{1}{2}}h^{\frac{1}{2}}+\gamma_4'\kappa\hat{M}h^{\frac{1}{8}}\\ +\gamma_1''\kappa\hat{M}^{\frac{3}{2}}h^{\frac{1}{4}})\hat{M}<\frac{1}{8}\end{gathered} \tag{4.32}$$

and take σ so small that

$$\left\{\frac{M_1}{a}+(\gamma_2'+\gamma_1'\kappa\hat{M})\hat{M}\right\}\sqrt{\sigma}<\frac{1}{16}. \tag{4.33}$$

Then if we assume that

$$\|v\|=\max_{p=1,\dots,n-1}|v_p|<\hat{M}$$

$$t_n<\sigma$$

we have from (4.30) and (4.31)

$$|v_n|<\hat{M}.$$

Thus we get the local a priori estimate

$$|v^n|<\hat{M}\qquad\text{for } t_n<\sigma \text{ and sufficiently small } h. \tag{4.34}$$

Hence convergence follows from the last estimate for $0<t<\sigma$ as we noted above and existence of a local solution of the differential problem (1.1)–(1.5) is established as a by-product. It is well known that global existence in our problem follows from local existence (see Friedman [3]). Therefore we have a priori estimate

$$\max_{\substack{0 \leq x \leq y(t) \\ 0 < t \leq T}} \left| \frac{\partial u}{\partial x}(x, t) \right| < \hat{M}_1 \tag{4.35}$$

where $\hat{M}_1$ is a constant. From convergence properties as noted above we have for sufficiently small h

$$\max_{\substack{0 < x_j \leq y_n \\ t_n < \sigma}} |u_{\bar{x}}(x_j, t_n)| < 2\hat{\hat{M}}_1 \equiv M_1. \tag{4.36}$$

Put

$$M = 4\hat{\hat{M}}_1 + 1 \tag{4.37}$$

and take σ_1 so small that

$$\left\{ \frac{\hat{\hat{M}}_1}{a} + (\gamma'_2 + \gamma'_1 \kappa M) M \right\} \sqrt{\sigma_1} < \frac{1}{16} \tag{4.38}$$

(see (4.33)). Then we get

$$|v(t_n)| < M \qquad \text{for } 0 < t_n < \sigma < \sigma_1 \tag{4.39}$$

as above. Here σ_1 is depending only on M_1, not on t_n. In the same way, we also have

$$|v(t_n)| < M \qquad \text{for } 0 < t_n < \sigma + 2\sigma_1$$

and so on. Thus we get a priori estimate

$$|v(t_n)| < M \qquad \text{for } 0 < t_n < T,$$

which was desired.

§5. Numerical Experiment

We show some results of our numerical experiment using our difference scheme. We take, for example, the following data in the problem (1.1)–(1.5):

$$a = 1.0, \qquad l = 1.0, \qquad \kappa = 1.0,$$

$$f(t) = \begin{cases} -\cos \frac{\pi}{4} t, & 0 < t < 2 \\ 0 \quad , & t > 2 \end{cases}$$

and

$$\varphi(x)=x-1, \qquad 0<x<1.$$

We solved this problem by the scheme (1.6)–(1.10) in the three cases:
i) $\beta=0.05$, $h=0.1$ ii) $\beta=0.05$, $h=0.01$ iii) $\beta=0$, $h=0.1$.
Here by $\beta=0$ we mean the case without the artificial heat flow term in (1.10), that is, we use the formula

$$k_n^{(s+1)}=\frac{h}{\kappa v_n^{(s)}}$$

instead of (1.17) in the iteration procedure. Fig. 1 shows the position of the free boundary in each case. In the third case the calculation could not be continued because the iteration determining k_7 did not converge.

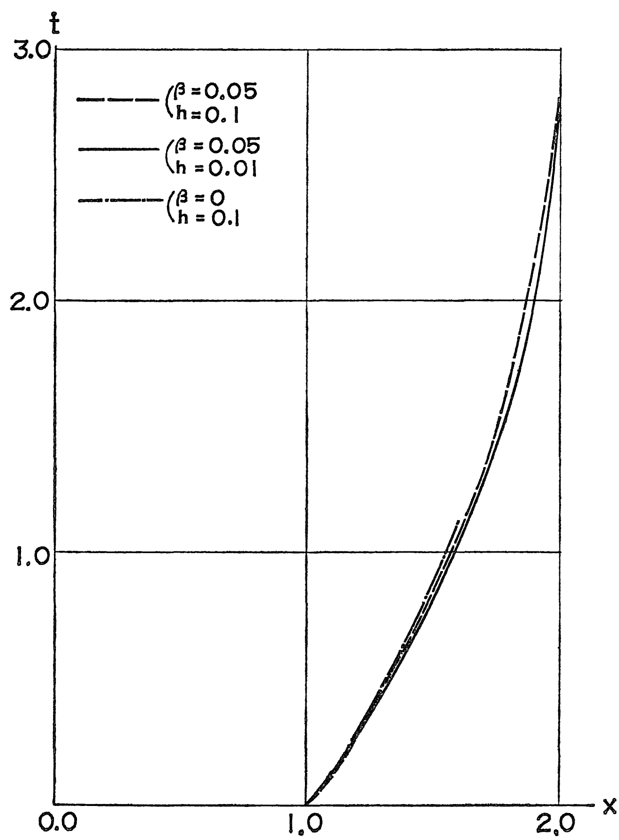

Fig. 1

In the first two cases with artificial heat flow term the calculations could be done as far as we desired and the iteration at each time step converged within check bound $|k_n^{(s)}-k_n^{(s-1)}|<0.0001$ by $5\sim 8$ times (the first case) or $3\sim 4$ times (the second case). From comparison of the first two cases we know that even the first calculation rough mesh size $h=0.1$ shows sufficiently convergent feature.

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