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On Normality of a Family of Holomorphic Functions

By

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0. On normality of a family of continuous functions, the theorem of Ascoli-Arzelà is well known. Especially, if we include the case of compact divergence, for a family of holomorphic functions, normality follows from the condition of equicontinuity at each point. In this paper we consider the normality in the wide sense. For a compact metric space K, we can construct another compact metric space $\ll \mathrm{Comp}(K) \gg$ consisting of all closed subsets of K. Using this space, in a σ-compact, locally compact metric space X, we can define the concept of convergence of a sequence of closed subsets of X and by means of this convergence, every family of closed subsets is always normal. As a consequence of this fact, we can prove that every family of continuous mappings from a connected, σ-compact and locally compact metric space to another σ-compact, locally compact metric space with some additional condition is normal if it is equicontinuous at each point. The method of our proof described in the section 1 seems to be interesting.

On normality of a family of holomorphic functions, many results are obtained by G. Julia [6]. From them, we denote the following interesting theorem:

Let $\mathfrak{F}$ be a family of holomorphic functions in a domain in $\mathbf{C}^2$. If every function $f \in \mathfrak{F}$ does not take two fixed different values, then $\mathfrak{F}$ is normal.

Now, if we consider the normality of a family of holomorphic functions in a domain D in the strict sense, that is, if every sequence of the family

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has a subsequence which converges uniformly to a holomorphic function on every compact set in D, then the domain of normality is a domain of holomorphy. This is known as ≪Conjecture of Julia≫. Using the word graph of holomorphic function, this is described in other words as follows.

Let $\mathfrak{F}$ be a family of holomorphic functions in a domain D *in* $\boldsymbol{C}^n$. *Suppose that the set* $\mathfrak{F}(p)=\{f(p); f\in\mathfrak{F}\}$ *is bounded for every point* $p\in$ D. *If* $\mathrm{D}\times\boldsymbol{C}$ *is a domain of normality of the family* $\mathfrak{G}_{\mathfrak{F}}$ *of graphs of* $\mathfrak{F}$, *then* D *is a domain of holomorphy.*

We shall give a proof of the above in the section 2.

In the section 3, we give an application of the theorem of Julia cited above. If D is a domain in the complex plane, then we get the converse of the theorem of Hurwitz.

1. In this section we consider a σ-compact, locally compact metric space $\mathrm{X}=(\mathrm{X}, d_{\mathrm{X}})$, where d_{X} is a metric in X. Let $\{E_j\}$ be a sequence of non empty closed subsets of X.

Definition 1. The sequence $\{E_j\}$ converges geometrically to a closed set E in X if and only if:

(i) If E is not empty then for any point $p\in E$, there exists a sequence $\{p_j\}$ such that $p_j\in E_j$ and $p_j\rightarrow p$.

(ii) For any compact set K in X with $K\cap E=\phi$, there exists a $j_\circ$ such that $K\cap E_j=\phi$ for all $j\geqq j_\circ$.

From this definition we have immediately:

(1) *Any subsequence of* $\{E_j\}$ *also converges geometrically to* E.

(2) *If a sequence* $\{p_j\}$ *with* $p_j\in E_j$ *has accumulating points, then they belong to* E.

(3) *The condition* (ii) *is equivalent to the following*:

(ii)′ *For any positive number* ε *and for any compact set* K *in* X, *there exists a* $j_\circ$ *such that for all* $j\geqq j_\circ$, *the set* $E_j\cap K$ *is included in the set* $E^{(\varepsilon)}\cap K$, *where* $E^{(\varepsilon)}=\bigcup_{p\in E}\{q\in\mathrm{X}; d_{\mathrm{X}}(q, p)<\varepsilon\}$.

In fact, (1) and (2) are direct consequences of the definition. We shall show (3).

(ii)→(ii)′: We may assume that E is not empty. Suppose that there exists a positive number δ and a compact set K such that the set $\{j;\ E_j \cap K - E^{(\delta)} \neq \phi\}$ of positive integers is an infinite set. For simpliciy, we may assume that $E_j \cap K - E^{(\delta)} \neq \phi$ for all j. Let $\{p_j\}$ be a sequence such that $p_j \in E_j \cap K - E^{\delta}$. Since K is compact, this sequence has accumulating points. Let p be one of them. Then $p \in E$ by the property (2). Thus the set $B_X(p: \delta) = \{q \in X;\ d_X(q, p) < \delta\}$is included in $E^{(\delta)}$ and $p_j \in B_X(p: \delta)$ for all sufficiently large j. This is a contradiction.

(ii)′→(ii): Let K be a compact set with $K \cap E = \phi$. Then there exists a positive number ε such that $K \cap E^{(\varepsilon)} = \phi$. Thus there exists a $j_\circ$ such that $E_j \cap K = \phi$ for all $j \geqq j_\circ$.

Here we explane the necessary fundamentals of the notion of the space Comp (K)[1]. Let K be a compact metric space. For two non empty closed subsets F_1, F_2 of K, we define the metric $\check{d}$ between F_1 and F_2 by

$$\check{d}(F_1, F_2) = \sup_{x \in F_1} \inf_{y \in F_2} d(x, y) + \sup_{t \in F_2} \inf_{s \in F_1} d(s, t),$$

where d is a metric in K.

This metric is called Hausdorff metric. From this metric, Comp (K), the set of all non empty closed subsets of K, becomes a compact metric space[2]. If $\{E_j\}$ converges to E in Comp (K), the following two properties are easily verified:

(i) *For any point $p \in E$, there exists a sequence $\{p_j\}$ such that $p_j \in E_j$* and $p_j \to p$.

(ii) *For any sequence $\{p_j\}$ with $p_j \in E_j$, the accumulating points of $\{p_j\}$ belong to E.*

Now let $\{K_j\}$ be an exhaution of X by compact sets: $K_1 \Subset K_2 \ldots,\ \cup K_j = X$. Take a sequence $\{E_j\}$ of closed subsets of X. For any compact set K_ν, if the set $\{j;\ E_j \cap K_\nu \neq \phi\}$ of positive integers is finite, we define that $\{E_j\}$ converges to a null set. Otherwise, there exists a $j_\circ$ such that $\{j;\ E_j \cap K_{j_\circ} \neq \phi\}$ is an infinite set. For simplicity, we may assume that $E_j \cap K_{j_\circ} \neq \phi$ for all j. From the compactness of the space Comp $(K_{j_\circ})$, we can choose

1) More precisely see [1], [7].
2) See [1] section 5, Chapter 2.

a subsequence $\{E_j^{(j_0)}\}$ of $\{E_j\}$ such that $\{E_j^{(j_0)} \cap K_{j_0}\}$ converges to a closed set F_{j_0} in Comp (K_{j_0}). From the sequence $\{E_j^{(j_0)}\}$ we can also choose a subsequence $\{E_j^{(j_0+1)}\}$ such that $\{E_j^{(j_0+1)} \cap K_{j_0+1}\}$ converges to a closed set F_{j_0+1} in Comp (K_{j_0+1}). We proceed this process. Then it is easily seen that

$$E = \cup F_{j_0+j-1}$$

is closed and that the diagonal sequence $\{E_j^{(j_0+j-1)}\}$ converges geometrically to E. Consequently we have

Lemma 1. *Let* X *be a* σ*-compact, locally compact metric space. Then every family of closed subsets of* X *is normal in the sense of geometric convergence.*

Now let $\mathrm{Y}=(\mathrm{Y}, d_{\mathrm{Y}})$ be a σ-compact metric space which satisfies the following condition:

For any compact set $K \subset \mathrm{Y}$ *and for any positive number* R, *the set*

$$\overline{K^{(R)}} = \overline{\bigcup_{p \in K} \{q \in \mathrm{Y};\ d_{\mathrm{Y}}(q, p) < R\}}$$

is compact.

The space Y which satisfies the above condition is necessarily locally compact. Let $\mathfrak{F}$ be a family of continuous mappings from the connected, locally compact and σ-compact metric space X to Y cited above.

Definition 2. $\mathfrak{F}$ is called equicontinuous at $p \in \mathrm{X}$ if and only if given any positive number ε, there exists a positive number r such that $d_{\mathrm{Y}}(f(p'), f(p)) < \varepsilon$ for all $f \in \mathfrak{F}$ and for all $p' \in B_{\mathrm{X}}(p : r)$.

Let $C(\mathrm{X}, \mathrm{Y})$ be the set of all continuous mappings from X to Y. The topology on $C(\mathrm{X}, \mathrm{Y})$ is the usual compact uniform topology.

Definition 3. A sequence $\{f_j\}$ of $C(\mathrm{X}, \mathrm{Y})$ is called to be compact divergence if and only if given any compact set K in X and compact set K' in Y, there exists a j_0 such that $f_j(K) \cap K' = \phi$ for all $j \geqq j_0$.

Definition 4. $\mathfrak{F}$ is called to be normal if and only if given a sequence of $\mathfrak{F}$, there exists a subsequence which is either compact divergence or compact uniform convergence.

Theorem 1. *If $\mathfrak{F}$ is equicontinuous at each point of* X, *then $\mathfrak{F}$ is normal.*

Proof. Let $\mathfrak{G}_f$ be a graph of $f \in \mathfrak{F}$, i.e., $\mathfrak{G}_f = \{(p, f(p)) \in X \times Y; p \in X\}$. Since f is continuous $\mathfrak{G}_f$ is closed. Let $\mathfrak{G} = \{\mathfrak{G}_f; f \in \mathfrak{F}\}$ be a family of graphs and let d be the metric in $X \times Y$ defined by

$$d((p, q), (p', q')) = \{d_X(p, p')^2 + d_Y(q, q')^2\}^{1/2},$$

then $X \times Y$ becomes a σ-compact, locally compact metric space. Thus by Lemma 1, $\mathfrak{G}$ is normal in $X \times Y$. Take a sequence $\{f_j\}$ from $\mathfrak{F}$. Since $\mathfrak{G}$ is normal we can choose a sequence $\{\mathfrak{G}\nu_j\}$ such that $\{\mathfrak{G}\nu_j\}$ converges geometrically to a closed set S in $X \times Y$, where $\mathfrak{G}\nu_j = \mathfrak{G}_{f_{\nu_j}}$. For convenience, we may assume that $\{\mathfrak{G}_j\}$ itself converges geometrically to S. For a point $p_\circ \in X$, put

$$S(p_\circ) = \{(p_\circ, q) \in X \times Y; q \in Y\} \cap S.$$

Then we shall show

Fact 1. *$S(p_\circ)$ contains at most one point.*

In fact, suppose that $(p_\circ, q)$, $(p_\circ, q')$ be two different points of $S(p_\circ)$. Since $\{\mathfrak{G}_j\}$ converges to S geometrically, there exists two sequences $\{(p_j, q_j)\}$, $\{(p'_j, q'_j)\}$ such that they converge to $(p_\circ, q)$ and $(p_\circ, q')$ respectively. From the equicontinuity of $\mathfrak{F}$ at $p_\circ$, there exists a positive number r such that

$$d_Y(f_j(p), f_j(p_\circ)) < \varepsilon = \frac{1}{8} d_Y(q, q')$$

for all $p \in B_X(p_\circ : r)$ and for all j. Since $p_j \to p_\circ$, $p'_j \to p_\circ$, $q_j \to q$ and $q'_j \to q'$ respectively, there exists a $j_\circ$ such that $d_X(p_j, p_\circ) < r$, $d_X(p'_j, p_\circ) < r$, $d_Y(q_j, q) < \varepsilon$, $d_Y(q'_j, q') < \varepsilon$ for all $j \geqq j_\circ$. Then it holds that

$$d_Y(q, q') \leqq d_Y(q, q_j) + d_Y(q_j, f_j(p_\circ)) + d_Y(f_j(p_0), q'_j)$$

$$+d_Y(q'_j, q')<4\varepsilon=\frac{1}{2}d_Y(q, q').$$

Since $q\neq q'$ this is a contradiction and then Fact 1 is proved.

Put $e=\{p\in X; S(p)=\phi\}$. Then

Fact 2. *e is open.*

To prove this, we have only to show the following

Lemma 2. *$p\in e$ if and only if the sequence $\{f_j(p)\}$ is a divergent sequence, i.e., for any compact set $K\subset Y$, there exists a $j_\circ$ such that $f_j(p)\notin K$ for all $j\geqq j_\circ$.*

In fact, suppose that Lemma 2 is proved. Let $p_\circ\in e$. Take a positive number r such that $d_Y(f_j(p), f_j(p_\circ))<1$ for all $p\in B_X(p_\circ: r)$. Let K be a compact set in Y. Then there exists a $j_\circ$ such that $f_j(p_\circ)\notin\overline{K^{(2)}}$ for all $j\geqq j_\circ$. For every $p\in B_X(p_\circ: r)$, since K is compact there exists a $q_j\in K$ such that $d_Y(f_j(p), K)=d_Y(f_j(p), q_j)$, where of course q_j depends on p. Then we have

$$d_Y(f_j(p), K)=d_Y(f_j(p), q_j)\geqq d_Y(q_j, f_j(p_\circ))-d_Y(f_j(p), f_j(p_\circ))$$

$$\geqq d_Y(f_j(p_\circ), K)-1>1.$$

Therefore $f_j(p)\notin K$ for all $j\geqq j_\circ$. This means that $B_X(p_\circ: r)\subset e$ and Fact 2 is proved.

Proof of Lemma 2. Let $p\in e$. If $\{f_j(p)\}$ is not a divergent sequence, there exists a subsequence $\{f_{\nu_j}(p)\}$ which converges to some point $q\in Y$. Since $\{(p, f_{\nu_j}(p))\}$ converges to (p, q) and since $(p, f_{\nu_j}(p))\in\mathfrak{S}_{\nu_j}$, we have that $(p, q)\in S$. Thus $S(p)\neq\phi$ and $p\notin e$. This is a contradiction.

Conversely, let $\{f_j(p)\}$ be a divergent sequence. If $(p, q)\in S$, then there exists a sequence $\{(p_j, q_j)\}$ such that $(p_j, q_j)\in\mathfrak{S}_j$, $p_j\to p$ and $q_j\to q$. Let r be a positive number such that $d_Y(f_j(p'), f_j(p))<1$ for all $p'\in B_X(p: r)$ and for all j. Take a $j_\circ$ such that $p_j\in B_X(p: r)$ and $d_Y(q_j, q)<1$ for all $j\geqq j_\circ$. Then we have

$$d_Y(f_j(p), q) \leqq d_Y(f_j(p), q_j) + d_Y(q_j, q) = d_Y(f_j(p), f_j(p_j)) + d_Y(q_j, q) < 2$$

for all $j \geqq j_\circ$. This implies that $\{f_j(p)\}$ is not a divergent sequence. This is a contradiction and then $S(p) = \phi$.

Fact 3. $X - e$ *is open.*

In fact, let $p_\circ \in X - e$, then $S(p_\circ)$ consists of exactly one point $(p_\circ, q_\circ)$. Take a sequence $\{(p_j, q_j)\}$ such that $(p_j, q_j) \in \mathfrak{G}_j$, $p_j \to p_\circ$ and $q_j \to q_\circ$. Take a positive number r such that $d_Y(f_j(p), f_j(p_\circ)) < \frac{1}{3}$ for all $p \in B_X(p_\circ : r)$ and for all j. There exists a $j_\circ$ such that $d_Y(q_j, q_\circ) < \frac{1}{3}$, $d_X(p_j, p_\circ) < r$ for all $j \geqq j_\circ$. Then

$$d_Y(f_i(p), q_\circ) \leqq d_Y(f_j(p), f_j(p_\circ)) + d_Y(f_j(p_\circ), q_j) + d_Y(q_j, q_\circ) \leqq 1$$

for all $p \in B_X(p_\circ : r)$ and for all $j \geqq j_\circ$. This means that $\{f_j(p)\}$ contains a convergent sequence for all $p \in B_X(p_\circ : r)$, so that $B_X(p_\circ : r) \subset X - e$. Thus $X - e$ is open.

Since X is connected, as a result of these facts we have that $e = \phi$ or $e = X$.

(1) In the case that $e = X$:

Let K_1, K_2 be two compact sets in X and Y respectively. Since $S = \phi$ there exists a $j_\circ$ such that $\mathfrak{G}_j \cap (K_1 \times K_2) = \phi$ for all $j \geqq j_\circ$. Thus, if $p \in K_1$ then $f_j(p) \notin K_2$ for all $j \geqq j_\circ$. Consequently we have that $f_j(K_1) \cap K_2 = \phi$ for all $j \geqq j_\circ$, so that $\{f_j\}$ is compactly divergent.

(2) In the case that $e = \phi$:

For every point $p \in X$, $S(p) \neq \phi$ and it consists of exactly one point. From this, we can define a mapping η from X to Y such that S is a graph of η. We shall show that the mapping η is continuous in X. Suppose that η is not continuous at $p_\circ$. Then there exists a sequence $\{p_j\}$ of X such that $p_j \to p_\circ$ and $d_Y(\eta(p_j), \eta(p_\circ)) \geqq \delta$ for some positive number δ. Since $(p_\circ, \eta(p_\circ)) \in S$, there exists a sequence $\{(q_j, w_j)\}$ such that $(q_j, w_j) \in \mathfrak{G}_j$, $q_j \to p_\circ$ and $w_j \to \eta(p_\circ)$. Put $\varepsilon = \frac{1}{5}\delta$, then there exists a $j_\circ$ such that

$d_Y(w_j, \eta(p_\circ))<\varepsilon$ and $d_X(q_j, p_\circ)<\varepsilon$ for all $j \geqq j_\circ$. Moreover, since $\mathfrak{F}$ is equicontinuous at $p_\circ$, there exists a positive number r such that $d_X(f_j(p), f_j(p_\circ))<\varepsilon$ for all $p \in B_X(p_\circ : r)$ and for all j. Take a point $p' \in \{p_j\}$ such that $p' \in B_X\left(p_\circ : \frac{r}{2}\right)$. Then

$$d_Y(\eta(p'), \eta(p_\circ)) \geqq \delta,\ d_Y(f_j(p'), f_j(p_\circ))<\varepsilon.$$

Since $(p', \eta(p')) \in S$, we can take a sequence $\{(t_j, s_j)\}$ such that $(t_j, s_j) \in \mathfrak{G}_j$, $t_j \to p'$ and $s_j \to \eta(p')$. Then there exists a k_0 such that $d_X(t_j, p') <\frac{r}{2}$, $d_Y(s_j, \eta(p'))<\varepsilon$ for all $j \geqq k_\circ$. Since $t_j \in B_X(p_\circ : r)$ for all $j \geqq k_\circ$, it holds that

$$d_Y(f_j(t_j), f_j(p_\circ))<\varepsilon,\ d_Y(f_j(p_0), f_j(q_j))<\varepsilon$$

for all j larger than $j_\circ$ and $k_\circ$. Since $w_j=f_j(q_j)$, we have

$$d_Y(\eta(p'), \eta(p_\circ)) \leqq d_Y(\eta(p'), s_j)+d_Y(s_j, f_j(p_\circ))+d_Y(f_j(p_\circ), w_j)$$
$$+d_Y(w_j, \eta(p_\circ))<4\varepsilon=\frac{4}{5}\delta$$

for all j larger than $j_\circ$ and $k_\circ$. This is a contradiction and we have that η is continuous in X.

Now let K_1 be a compact set in X. Take a compact set K_2 in Y and put $K=K_1 \times K_2$. Then

for any positive number ε, *there exists a* $j_\circ$ *such that*

$$\mathfrak{G}_j \cap K \subset \bigcup_{p \in K_1} \{p\} \times B_Y(\eta(p): \varepsilon)$$

for all $j \geqq j_\circ$.

In fact, suppose that there exists a positive number δ such that

$$\{j;\ \mathfrak{G}_j \cap K-\bigcup_{p \in K_1} \{p\} \times B_Y(\eta(p): \delta) \neq \phi\}$$

is infinite. Then there exists a sequence $\{(p_{\nu_j}, q_{\nu_j})\}$ such that

$$(p_{\nu_j}, q_{\nu_j}) \in \mathfrak{G}\nu_j \cap K-\bigcup_{p \in K_1} \{p\} \times B_Y(\eta(p): \delta).$$

We may assume that $p_{\nu_j} \to p \in K_1$, $q_{\nu_j} \to q \in K_2$. Since $p_{\nu_j} \in K_1$, we have that $q_{\nu_j} \notin B_Y(\eta(p_{\nu_j}): \delta)$. That is $d_Y(q_{\nu_j}, \eta(p_{\nu_j})) \geqq \delta$. On the other hand, since η is continuous, there exists a positive number r such that

$$\eta(B_X(p: r)) \subset B_Y\left(\eta(p): \frac{\delta}{4}\right).$$

Take a $j_\circ$ such that $(p_{\nu_j}, q_{\nu_j}) \in B_X(p: r) \times B_Y\left(\eta(p): \frac{\delta}{4}\right)$ for all $j \geqq j_\circ$. Then we have

$$d_Y(q_{\nu_j}, \eta(p_{\nu_j})) \leqq d_Y(q_{\nu_j}, \eta(p)) + d_Y(\eta(p), \eta(p_{\nu_j})) \leqq \frac{\delta}{2}$$

for all $j \geqq j_\circ$. This contradicts the fact that $d_Y(q_{\nu_j}, \eta(p_{\nu_j})) \geqq \delta$, so that

$$\mathfrak{G}_j \cap K \subset \bigcup_{p \in K_1} \{p\} \times B_Y(\eta(p): \varepsilon)$$

for all sufficiently large j. Put

$$K_2 = \overline{\eta(K_1)^{(1)}}$$

then the following statement holds:

There exists a $j_\circ$ such that $\mathfrak{G}_j \cap (K_1 \times Y) = \mathfrak{G}_j \cap (K_1 \times K_2)$ for all $j \geqq j_\circ$.

In fact, suppose that this statement does not hold. Then there exists a sequence $\{(p_{\nu_j}, q_{\nu_j})\}$ such that

$$(p_{\nu_j}, q_{\nu_j}) \in \mathfrak{G}\nu_j \cap (K_1 \times Y) - \mathfrak{G}\nu_j \cap (K_1 \times K_2)\,.$$

Since $p_{\nu_j} \in K_1$, this means that $q_{\nu_j} \notin K_2$, that is $d_Y(q_{\nu_j}, \eta(K_1)) > 1$. For simplicity, we may assume that $p_{\nu_j} \to p_\circ \in K_1$. Since $(p_\circ, \eta(p_\circ)) \in S$ and since $\{\mathfrak{G}\nu_j\}$ converges geometrically to S, there exists a sequence $\{(t_{\nu_j}, s_{\nu_j})\}$ such that $(t_{\nu_j}, s_{\nu_j}) \in \mathfrak{G}\nu_j$, $t_{\nu_j} \to p_\circ$ and $s_{\nu_j} \to \eta(p_\circ)$. Take a positive number r such that $d_Y(f_{\nu_j}(p'), f_{\nu_j}(p_\circ)) < \frac{1}{4}$ for all $p' \in B_X(p_\circ: r)$ and for all j. Then there exists a $j_\circ$ such that $p\nu_j$, $t_{\nu_j} \in B_X(p_\circ: r)$ and $d_Y(s_{\nu_j}, \eta(p_\circ)) < \frac{1}{4}$ for all $j \geqq j_\circ$. It holds that

$$d_Y(q_{\nu_j}, \eta(p_\circ)) \leqq d_Y(q_{\nu_j}, f_{\nu_j}(p_\circ)) + d_Y(f_{\nu_j}(p_\circ), s_{\nu_j}) + d_Y(s_{\nu_j}, \eta(p_\circ)) < \frac{3}{4}$$

for all $j \geqq j_\circ$. This implies that $d_Y(q_{\nu_j}, \eta(K_1)) < \frac{3}{4}$ and this is a contradiction. Summarizing these facts, we have that for any compact set $K_1 \subset X$ and for any positive number ε there exists a $j_\circ$ such that

$$\mathfrak{G}_j \cap (K_1 \times Y) = \mathfrak{G}_j \cap (K_1 \times K_2) \cap \bigcup_{p \in K_1} \{p\} \times B_Y(\eta(p) : \varepsilon)$$

for all $j \geqq j_\circ$. Thus $\{f_j\}$ converges uniformly to η on K_1. This complete the proof of Theorem 1.

2. In this section we consider the case of holomorphic functions. As a consequence of Theorem 1, we have

Corollary of Theorem 1. *Let* D *be a domain in* $\mathbf{C}^n$ *and let* $\mathfrak{F}$ *be a family of holomorphic functions in* D. *If* $\mathfrak{F}$ *is equicontinuous at each point of* D, *then* $\mathfrak{F}$ *is normal in* D.

Now let D be a domain in $\mathbf{C}^n$.

Definition 5. Let $\{A_\nu\}$ be a sequence of principal analytic sets in D. This sequence is called to converge analytically to an analytic set A if and only if given a point $p \in D$, there exists a neighbourhood U of p and holomorphic functions $\{f_\nu\}, f$ in U such that:

(i) f is not identically zero.

(ii) $A_\nu \cap U = \{q \in U; f_\nu(q) = 0\}$, $A \cap U = \{q \in U; f(q) = 0\}$.

(iii) The sequence $\{f_\nu\}$ converges uniformly to f.

From this definition, the following remarks are easily seen.

(1) *If A is not empty, then A is a principal analytic set in* D.

(2) *If a sequence $\{p_\nu\}$ with $p_\nu \in A_\nu$ has accumulating points, then they belong to A.*

(3) *The sequence $\{A_\nu\}$ converges geometrically to A.*

Definition 6. A family $\mathfrak{F}$ of holomorphic functions in D is called to be bounded at $p \in D$ if and only if the set $\mathfrak{F}(p) = \{f(p); f \in \mathfrak{F}\}$ is bounded in the complex plane.

Definition 7. An analytic set A in $\mathrm{D}\times\boldsymbol{C}(w)$ is called to be fine in w at p if and only if the set $A(p)=\{(p, w)\in\boldsymbol{C}^{n+1}\}\cap A$ has no finite accumulating point, where $\boldsymbol{C}(w)$ is a complex plane with coordinate w.

From this definition, as is easily seen, A is not fine in w at p if and only if A includes the complex plane $\{(z, w); z=p\}$. For simplicity, we consider the case of two complex variables x, y.

Lemma 3. *Let $\{f_j\}$ be a sequence of holomorphic functions in* D *satisfying the following conditions*:

(i) *$\{f_j\}$ is bounded at each point of* D.

(ii) *The sequence $\{\mathfrak{G}_j\}$ of graphs of $\{f_j\}$ converges analytically to an analytic set A in* $\mathrm{D}\times\boldsymbol{C}(w)$.

Then $\{f_j\}$ converges uniformly to a holomorphic function on every compact set in D.

Proof[3]. Let $E=\{p\in\mathrm{D}; A \text{ is not fine in } w \text{ at } p\}$. Then

E is a proper analytic set in D.

In fact, since A is a proper analytic set in $\mathrm{D}\times\boldsymbol{C}(w)$, $E\subsetneqq\boldsymbol{C}$. Take a point $p\in\mathrm{D}$ and a polydisc $\Delta\subset\mathrm{D}$ with center at p. Since $\Delta\times\boldsymbol{C}$ is a Cousin II domain, there exists a holomorphic function $\varphi(x, y, w)$ in $\Delta\times\boldsymbol{C}$ such that

$$A\cap(\Delta\times\boldsymbol{C})=\{(x, y, w)\in\Delta\times\boldsymbol{C}; \varphi(x, y, w)=0\}.$$

Then it is easily seen that

$$E\cap\Delta=\{(x, y)\in\Delta; \varphi(x, y, c)=0 \quad \text{for all complex numbers } c\}$$

$$=\bigcap_{c\in\boldsymbol{C}}\{(x, y)\in\Delta; \varphi(x, y, c)=0\}.$$

Thus E is a proper analytic set in D.

Now, we show that the sequence $\{f_j\}$ is uniformly bounded on every

3) This argument was suggested by Prof. H. Fujimoto as an alternative to a more complicated original one of the author. We wish to thank him for his kind advice.

compact set in D. To prove this, we have only to show that for any point $p \in \mathrm{D}$ there exists a neighbourhood U of p such that $\{f_j\}$ is uniformly bounded on U. Let $p_\circ \in \mathrm{D}-E$. Since $\{f_j(p_\circ)\}$ is bounded, there exists a subsequence of $\{f_j(p_\circ)\}$ which converges to a complex number $q_\circ$. Since $(p_\circ, q_\circ) \in A$, there exists a neighbourhood $U=U_1 \times U_2$ of $(p_\circ, q_\circ)$ with $U_1 \subset \mathrm{D}-E$ and holomorphic functions $\{\psi_j\}$, ψ in U such that

$$\mathfrak{G}_j \cap U = \{(x, y, w) \in U; \psi_j(x, y, w)=0\},$$

$$A \cap U = \{(x, y, w) \in U; \psi(x, y, w)=0\}$$

and that the sequence $\{\psi_j\}$ converges uniformly to ψ on U. Since $p_\circ \notin E$, $\psi(p_\circ, w)$ is not identically zero. Therefore there exists a small positive number r such that $\psi(p_\circ, w) \neq 0$ if $|w-q_\circ|=r$. Thus if we choose U_1 sufficiently small, we have $\psi \neq 0$ on $\overline{U}_1 \times \Theta$, where $\Theta=\{w \in \boldsymbol{C}; |w-q_\circ| =r\}$. Put $m=\min\{|\psi(p, w)|; (p, w) \in \overline{U}_1 \times \Theta\}$. Since $\{\psi_j\}$ converges uniformly to ψ there exists a $j_\circ$ such that $|\psi_j-\psi|<\frac{m}{2}$ on $\overline{U}_1 \times \Theta$ for all $j \geqq j_\circ$. Then for any point $p \in U_1$, by the theorem of Hurwitz, the equation $\psi_j(p, w)=0$ has at least one root w_j in the disc $\{w \in \boldsymbol{C}; |w-q_\circ| <r\}$ for all $j \geqq j_\circ$. Thus $\{f_j\}$ is uniformly bounded on U_1. Let $p_\circ \in E$. Since E is a proper analytic set, by a linear change of coordinate if necessary, we can choose a polydisc $U=U_1 \times U_2$ of $p_\circ$ such that $(\partial U_1 \times \partial U_2) \cap E =\phi$. Then $\{f_j\}$ is uniformly bounded on $\partial U_1 \times \partial U_2$ and by the maximal principle $\{f_j\}$ is uniformly bounded on $U_1 \times U_2$.

Now by the theorem of Montel, any sequence of $\{f_j\}$ has a convergent subsequence. To prove that $\{f_j\}$ converges uniformly on every compact set in D, we have only to show that the limit function is independ of the choice of the convergent sequence. Let $\{f_j^{(1)}\}$ and $\{f_j^{(2)}\}$ be two sequences of $\{f_j\}$ which converge to $f^{(1)}$ and $f^{(2)}$ respectively on every compact set in D. Then it is easily seen that $A=$graph of $f^{(1)}=$graph of $f^{(2)}$, so that $f^{(1)}=f^{(2)}$. This complete the proof of Lemma 3.

Definition 8. A family of principal analytic sets in D is called to be analytically normal in D if and only if given a sequence of the family, there exists a subsequence which converges analytically to an analytic set in D.

Definition 9. Let $\mathfrak{S}$ (resp. $\mathfrak{F}$) be a family of principal analytic sets (resp. holomorphic functions) in D. Then D is called a domain of normality of $\mathfrak{S}$ (resp. $\mathfrak{F}$) if and only if $\mathfrak{S}$ (resp. $\mathfrak{F}$) is normal in D and for any schlicht domain $\tilde{\mathrm{D}}$ in $\boldsymbol{C}^2$ such that $\mathrm{D} \subsetneqq \tilde{\mathrm{D}}$, $\mathfrak{S}$ (resp. $\mathfrak{F}$) is no longer normal in $\tilde{\mathrm{D}}$.

For a domain of normality of a family of holomorphic functions, the so-called conjecture of G. Julia is well known[4]. In connection with this conjecture, we have the following

Theorem 2. *Let $\mathfrak{F}$ be a family of holomorphic functions in* D. *Suppose that $\mathfrak{F}(p)$ is bounded at each point p of* D. *If $\mathrm{D} \times \boldsymbol{C}$ is a domain of normality of a family $\mathfrak{G}_{\mathfrak{F}}$ of graphs of $\mathfrak{F}$, then* D *is a domain of holomorphy.*

Proof. Suppose that D is not a domain of holomorphy. Then since D is not holomorphycally convex, there exists a compact set K in D such that

$$\tilde{K} = \{(x, y) \in \mathrm{D};\ |f(x, y)| \leqq \sup_K |f| \text{ for all holomorphic functions in D}\}$$

is not compact in D. Put $\rho = d(K, \partial \mathrm{D})$ and take a point $p_\circ = (x_\circ, y_\circ) \in \tilde{K}$ such that $d(p_\circ, \partial \mathrm{D}) < \frac{\rho}{2}$. It is well known that any holomorphic function in D is also holomorphic in the polydisc $\varDelta = \{(x, y) \in \boldsymbol{C}^2;\ |x - x_\circ| < \rho,\ |y - y_\circ| < \rho\}$. Put $\varDelta' = \left\{(x, y) \in \boldsymbol{C}^2;\ |x - x_\circ| < \frac{2}{3}\rho,\ |y - y_\circ| < \frac{2}{3}\rho\right\}$. Since $d(p_\circ, \partial \mathrm{D}) < \frac{\delta}{2}$, $\varDelta' - \mathrm{D} \neq \phi$. We shall show that

$\mathfrak{G}_{\mathfrak{F}}$ *is normal in* $(\mathrm{D} \cup \varDelta') \times \boldsymbol{C}$.

In fact, take a sequence $\{\mathfrak{G}_\nu\}$ from $\mathfrak{G}_{\mathfrak{F}}$. Since $\mathfrak{G}_{\mathfrak{F}}$ is normal in $\mathrm{D} \times \boldsymbol{C}$, there exists a subsequence $\{\mathfrak{G}_{\nu_j}\}$ which converges analytically to an analytic set A in $\mathrm{D} \times \boldsymbol{C}$. Let f_{ν_j} be the function which represents the graph $\mathfrak{G}_{\nu_j}$. Then by Lemma 3, $\{f_{\nu_j}\}$ converges uniformly to a holomorphic function η on every compact set in D. Expand f_{ν_j} into the convergent Taylor series in $\varDelta'$:

4) See, [5] page 38.

$$f_{\nu_j}(x, y)=\Sigma a^{(\nu_j)}_{k_1,k_2}(x-x_\circ)^{k_1}(y-y_\circ)^{k_2},$$

where

$$a^{(\nu_j)}_{k_1,k_2}=\frac{1}{k_1!k_2!}\ \frac{\partial^{k_1+k_2}f_{\nu_j}(p_\circ)}{\partial x^{k_1}\partial y^{k_2}}$$

Put $K'=\overline{K^{(r)}}$, where $r=\frac{3}{4}\rho$. Since K' is compact, it holds that $\sup_{\nu_j}\sup_{K'}|f_{\nu_j}|=M$. Then by the estimation of Cauchy, we have

$$|a^{(\nu_j)}_{k_1,k_2}|\leqq\frac{1}{k_1!k_2!}\sup_{p\in K}\left|\frac{\partial^{k_1+k_2}f\nu_j(p)}{\partial x^{k_1}\partial y^{k_2}}\right|\leqq\frac{M}{\left(\frac{3}{4}\rho\right)^{k_1+k_2}}.$$

Therefore $\{f_{\nu_j}\}$ is uniformly bounded on Δ'. Thus there exists a subsequence of $\{f_{\nu_j}\}$ which converges uniformly to a holomorphic function in Δ'. For simplicity, we may assume that $\{f_{\nu_j}\}$ converges uniformly to $\tilde{\eta}$ in Δ'. Since $\tilde{\eta}=\eta$ in $\Delta'\cap \mathrm{D}$, the holomorphic function ξ such that $\xi=\eta$ in D and $\xi=\tilde{\eta}$ in Δ' is determined. Then $\{\mathfrak{G}_{\nu_j}\}$ converges analytically to the analytic set $B=\{(p, \xi(p));\ p\in\Delta'\cup\mathrm{D}\}$ in $(\Delta'\cup\mathrm{D})\times\boldsymbol{C}$. Since $\mathrm{D}\times\boldsymbol{C}$ is a domain of holomorphy of $\mathfrak{G}_{\mathfrak{F}}$, this is a contradiction.

Remark of Lemma 3. In Lemma 3, we omit the condition of the boundedness of $\{f_j\}$ at each point and consider under the condition only of an analytic convergence of graphs. Put

$$\Omega=\{p\in\mathrm{D}-E;\ \{f_j(p)\}\text{ is bounded}\}.$$

It is easily seen that the boundedness at p is equivalent to the condition that $A(p)\neq\phi$. Moreover, since A is fine in w at $p\in\mathrm{D}-E$ it is easy to see that $\Omega=\pi(A)-E$, where π is the projection given by $\pi(p, w)=p$. Therefore Ω is open. Thus if Ω is not empty, there exists a holomorphic function η in Ω such that $\{f_j\}$ converges uniformly to η on every compact set in Ω. If $\Omega=\phi$, then $[(\mathrm{D}-E)\times\boldsymbol{C}]\cap A=\phi$. Since $\{\mathfrak{G}_j\}$ converges analytically to A, for any compact set K_1 in $\mathrm{D}-E$ and for any positive number a, there exists a $j_\circ$ such that $\mathfrak{G}_j\cap K=\phi$ for all $j\geqq j_\circ$, where $K=K_1\times\{w\in\boldsymbol{C};\ |w|\leqq a\}$. Thus if $p\in K_1$ then $(p, f_j(p))\notin K$, i.e., $|f_j(p)|>a$ for all $j\geqq j_\circ$. This means that $\{f_j\}$ is compactly divergent in $\mathrm{D}-E$.

Let D be a domain of holomorphy and let $\Omega \neq \phi$, then it is easily seen that *Ω is a holomorphic open set, i.e., any connected component of Ω is a domain of holomorphy.*

3. In this section, we consider an application of the theorem of G. Julia:

Theorem ([6], page 67). *Let* D *be a domain in* $\mathbf{C}^2$ *and let* $\mathfrak{F}$ *be a family of holomorphic functions in* D. *If every function* $f \in \mathfrak{F}$ *does not take two fixed different values, then* $\mathfrak{F}$ *is normal in* D.

Lemma 4. *Let* D *be a domain in* $\mathbf{C}^2$ *and let* $\{f_\nu\}, f$ *be holomorphic functions in* D. *Let* $A=\{a, b, c\}$ *be a set consisting of three different complex numbers. Suppose that for any* $\alpha \in A$, *the sequence* $\{A_{\nu,\alpha}\}$ *of analytic sets in* D *given by*

$$A_{\nu,\alpha}=\{(x, y)\in \mathrm{D}; f_\nu(x, y)=\alpha\},$$

converges geometrically to an analytic set $A_\alpha=\{(x, y)\in \mathrm{D}; f(x, y)=\alpha\}$. *Then* $\{f_\nu\}$ *is normal in* D.

Proof. Take a point $p\in \mathrm{D}$. If $f(p)\notin A$, then there exists a connected open neighbourhood V of p such that $V \Subset \mathrm{D}$ and $f(\overline{V})\cap A=\phi$. Therefore, $A-f(\overline{V})=A$. If $f(p)\in A$, we can choose a connected open neighbourhood U of p such that $U\Subset \mathrm{D}$ and $f(\overline{U})\cap[A-\{f(p)\}]=\phi$. In any case we can choose a connected open neighbourhood U of p such that $U\Subset \mathrm{D}$ and that $A-f(\overline{U})$ contains at least two complex numbers. Let $\{a, b\}\subset A-f(\overline{U})$, then $A_a\cap \overline{U}=\phi$ and $A_b \cap \overline{U}=\phi$. Since $\{A_{\nu,a}\}$, $\{A_{\nu,b}\}$ converge geometrically to A_a, A_b respectively, there exists a ν_0 such that $A_{\nu,a}\cap \overline{U}=\phi, A_{\nu,b}\cap \overline{U}=\phi$ for all $\nu \geqq \nu_0$. That is, the sequence $\{f_\nu\}_{\nu\geqq\nu_0}$ does not take two different values a, b in U. Thus by the theorem of Julia, it is normal in U. Since p is an arbitrary point of D, $\{f_\nu\}$ is normal at each point of D. Then by the diagonal method $\{f_\nu\}$ is normal in D.

Definition 10. A set ω is called a set of uniqueness if and only if any holomorphic function in D which is zero on ω is identically zero.

Theorem 3. *Let* $\{f_\nu\}, f$ *be holomorphic functions in* D *and let* f

be not constant. Let ω *be a set of uniqueness. Suppose that for any* $a \in f(\omega)$ *the sequence* $\{A_{\nu,a}\}$ *of analytic sets in* D *converges geometrically to an analytic set* A_a, *then* $\{f_\nu\}$ *converges uniformly to* f *on every compact set in* D, *where* $A_{\nu,a}$, A_a *are the same as that in Lemma* 4.

Proof. First, the set $f(\omega)$ contains infinitely many complex numbers. In fact, suppose that $f(\omega)$ contains only finitely many complex numbers. Let $f(\omega)=\{a_1, a_2,\dots, a_m\}$. Put $S_i=\{p\in \mathrm{D}; f(p)=a_i\}$. Then the analytic set $S=\cup S_i$ is given by

$$S=\{p\in \mathrm{D};\ g(p)=\prod_{i=1}^{i=m}[f(p)-a_i]=0\}\ .$$

Since f is not constant g is not also constant. But since $S\supset\omega$ and since $g=0$ on S, g is identically zero, this is a contradiction. Thus by Lemma 4 $\{f_\nu\}$ is normal in D. To prove that the sequence $\{f_\nu\}$ converges uniformly to f on every compact set in D, we have only to show that any sequence of $\{f_\nu\}$ contains a subsequence which converges uniformly to f on every compact set in D. Take a sequence $\{f_{\nu_j}\}$ of $\{f_\nu\}$. We may assume that $\{f_{\nu_j}\}$ is compact uniform convergence or compact divergence. Since $\{A_{\nu,a}\}$ converges geometrically to A_a and since A_a is not empty, the case of compact divergence does not occur. Let $\{f_{\nu_j}\}$ converge uniformly to a holomorphic function h on every compact set in D. Let $p_\circ\in\omega$ and let $a=f(p_\circ)$, then since $p_\circ\in A_a$ there exists a sequence $\{p_{\nu_j}\}$ such that $p_{\nu_j}\in A_{\nu_j,a}$ and $p_{\nu_j}\to p_\circ$. Let $K=\{p_\circ, p_{\nu_j};\ j=1, 2,\dots\}$. Since K is compact, for any positive number ε there exists a $j_\circ$ such that

$$\sup_{p\in K}\{|f_{\nu_j}(p)-h(p)|\}<\varepsilon,\ |h(p_{\nu_j})-h(p_\circ)|<\varepsilon$$

for all $j\geqq j_\circ$. Then we have

$$|f_{\nu_j}(p_{\nu_j})-h(p_\circ)|\leqq|f_{\nu_j}(p_{\nu_j})-h(p_{\nu_j})|+|h(p_{\nu_j})-h(p_\circ)|<2\varepsilon.$$

Since $p_{\nu_j}\in A_{\nu_j,a}$ and since $f_{\nu_j}(p_{\nu_j})=a=f(p_\circ)$, we have $f(p_\circ)=h(p_\circ)$. That is, $f=h$ on ω, so that $f=h$ on D.

Remark. In Theorem 3, we can not omit the condition that f is not constant. But since Lemma 4 holds in the case that f is constant, by the same method we can prove the following:

Let U be an open set in the complex plane such that $U \cap f(\mathrm{D}) \neq \phi$. If $\{A_{\nu,a}\}$ converges geometrically to A_a for all $a \in U$, then $\{f_\nu\}$ converges uniformly to f on every compact set in D.

Now we consider the case of one complex variable. If a sequence $\{f_j\}$ of holomorphic functions in a domain $\mathrm{D} \subset \boldsymbol{C}$ converges to a non constant holomorphic function f on every compact set in D, then for any point $p \in \mathrm{D}$ and for any complex number a, there exists a positive number r_0 which satisfies the following property:

For any r with $0<r<r_0$, there exists a j_0 such that the number of a-points of f_j in the disc $\{z \in \boldsymbol{C};\ |z-p|<r\}$ are equal for all $j \geqq j_0$, counted according to multiplicities.

In fact, if $f(p) \neq a$, then there exists a neighbourhood U of p such that $U \Subset \mathrm{D}$ and $\min\{|f(z)-a|\,;\ z \in \overline{U}\} = \delta > 0$. Since $\{f_j\}$ converges uniformly to f on $\overline{U}$ there exists a j_0 such that

$$\max\{|f_j(z)-f(z)|\,;\ z \in \overline{U}\} < \frac{\delta}{2}$$

for all $j \geqq j_0$. Then

$$|f_j(z)-a| \geqq |f(z)-a| - |f_j(z)-f(z)| > \frac{\delta}{2}$$

for all $j \geqq j_0$ and $z \in \overline{U}$. That is $f_j(z) \neq a$ in $\overline{U}$ for all $j \geqq j_0$. If $f(p)=a$, then this is just the theorem of Hurwitz.

Since Theorem 3 holds for a domain in the complex plane, as the converse of the above, we have

Corollary of Theorem 3. *Let $\{f_j\}$, f be holomorphic functions in a domain* D *in* $\boldsymbol{C}$ *and let f be not constant. Suppose that for any point $p \in \mathrm{D}$ and for any complex number a, there exists a positive number r_0 satisfying the following property:*

For any r with $0<r<r_0$, there exists a j_0 such that the number of a-

points of f_j *and* f *are equal in the disc* $\{z\in\boldsymbol{C};\ |z-p|<r\}$ *for all* $j\geqq j_\circ$, *counted according to multiplicities.*

Then $\{f_j\}$ *converges uniformly to* f *on every compact set in* D.

Proof. We have only to show that for every point $p\in\mathrm{D}$, the sequence $\{A_j\}$ of analytic sets given by $A_j=\{z\in\mathrm{D}; f_j(z)=f(p)\}$ converges geometrically to an analytic set $A=\{z\in\mathrm{D}; f(z)=f(p)\}$. The condition (i) of geometric convergence is trivial. Let K be a compact set in D such that $A\cap K=\phi$. For any point $q\in K$, there exists a positive number δ such that $f(z)-f(p)\neq 0$ in the disc $\varDelta=\{z\in\mathrm{D};\ |z-q|<\delta\}$. Evidently we may assume that $\delta<r_\circ$. Then there exists a $j_\circ(q)$ such that $f_j(z)-f(p)\neq 0$ in the disc $\varDelta$ for all $j\geqq j_\circ(q)$. Since K is compact we can choose finitely many such discs $\varDelta_\nu$ with center at q_ν and radius δ_ν such that $K\subset\cup\varDelta_\nu$. Put $j_\circ=\max_\nu j_\circ(q_\nu)$. Then since $f_j(z)-f(p)\neq 0$ in K for all $j\geqq j_\circ$, we have that $A_j\cap K=\phi$ for all $j\geqq j_\circ$. Thus the condition (ii) is proved.

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