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Strongly Hyperbolic Systems with Variable Coefficients

By

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§1. Introduction

We consider the first order partial differential equation in $R^p\times[0, T]$, $T>0$,

$$L[u]=\frac{\partial}{\partial t}u-\sum_{j=1}^{p}A_j(x, t)\frac{\partial}{\partial x_j}u-B(x, t)u=f(x, t), \tag{1.1}$$

where $A_j(x, t)$ and $B(x, t)$ are matrices of order m, infinitely differentiable with respect to t and $x=(x_1,\dots, x_p)$, and u and f are vector-valued functions with m components. We consider the Cauchy problem for this equation with intial values given at $t=t_0>0$. We say that the Cauchy problem for (1.1) is uniformly well posed, if for any $f(x, t)$ infinitely differentiable and for any intial value $u(x, t_0)$ infinitely differentiable, there exists uniquely the infinitely differentiable solution $u(x, t)$ of (1.1) in $\Omega(x_0, t_0, \varepsilon)=\{(x, t); |x-x_0|\leqq\lambda_0(t_0+\varepsilon-t),\ t_0\leqq t\leqq t_0+\varepsilon\}$ for any ε, $0<\varepsilon\leqq\varepsilon_0$, where (x_0, t_0) is an arbitrary point in $R^p\times[0, T]$ and λ_0 and ε_0 are positive constants. We denote by L_0 the principal part of L, that is, $L_0=\frac{\partial}{\partial t}-\sum A_j(x, t)\frac{\partial}{\partial x_j}$. We say that L_0 is strongly hyperbolic, if the Cauchy problem for (1.1) is uniformly well posed for any lower order $B(x, t)$ infinitely differentiable.

We suppose that the multiplicity of the characteristic roots is independent of x, t and ξ. More precisely

$$\det(\lambda-\sum A_j(x, t)\xi_j)=\prod_{j=1}^{l}(\lambda-\lambda_j(x, t:\xi))^{\nu_j} \tag{1.2}$$

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where $\nu_j(j=1, 2,\dots, l)$ are positive integers independent of (x, t, ξ) and

$$\lambda_i(x, t: \xi)\neq\lambda_j(x, t: \xi) \qquad (i\neq j). \tag{1.3}$$

Then we have

Theorem 1.1. *Suppose that* (1.2) *and* (1.3) *are valid. Then L_0 is strongly hyperbolic, if and only if $\sum A_j(x, t)\xi_j$ is diagonalizable.*[1)]

Remark 1. When the coefficients of L are constant matrices, K. Kasahara and M. Yamaguti proved in [7] that L_0 is strongly hyperbolic if and only if $\sum A_j\xi_j$ is uniformly diagonalizable.

Remark 2. In the case of variable coefficients, it is known that $\sum A_j(x, t)\xi_j$ is necessarily diagonalizable for L^2-well posedness of (1.1) (cf. [4], [6], [14]).

We note that the statements in Remark 1 and Remark 2 hold without the hypothesis that the multiplicity of characteristic roots is constant. But our Theorem 1.1 is not valid, if we do not assume that the multiplicity of characteristic roots is constant. For example,

$$L_0=\frac{\partial}{\partial t}-\begin{pmatrix} t & 1 \\ 0 & -t \end{pmatrix}\frac{\partial}{\partial x}$$

is strongly hyperbolic, that is

$$\begin{cases} (L_0+B(x, t))u(x, t)=f(x, t) \\ u(x, 0)=u_0(x) \end{cases} \tag{1.4}$$

is well posed for any lower order 2×2 matrix $B(x, t)$, inifinitely differentiable. We can see from Remark 2 that (1.4) is not L^2-well posed. But we have a positive integer s_0 such that there holds for any non-negative integer k,

$$||u(t)||_k\leqq C\left\{||u_0||_{k+s_0}+\int_0^t||f(s)||_{k+s_0}ds\right\}, \quad ^{2)}$$

1) We say that $\sum A_j(x, t)\,\xi_j$ is diagonalizable, if for any fixed point (x, t, ξ) there exists a non-singular matrix N so that $N(\sum A_j(x, t)\xi_j)N^{-1}$ is a diagonal matrix.
2) This estimate is obtained by use of the result which O.A. Oleinik derived in [1].

where

$$\|u(t)\|_k^2=\sum_{|\alpha|+j\leqq k}\int\left|\left(\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u(x, t)\right|^2 dx.$$

In this example L_0 is not diagonalizable at $t=0$.

We say that the Cauchy problem for (1.1) with intial plane $t_0=0$ is L^2-well posed, if for any $u_0(x)$ in $L^2(R^p)$ given at $t=0$ and for any $f(x, t)$ in $\mathscr{E}_t^0(L^2(R^p))$, there exists the solution $u(x, t)$ in $\mathscr{E}_t^0(L^2(R^p))$ satisfying

$$\|u(t)\|\leqq C\left\{\|u_0\|+\int_0^t\|f(s)\|ds\right\}, \quad \text{for} \quad t>0$$

where $\|\cdot\|$ is the norm of $L^2(R^p)$.

Now we assume instead of (1.3),

$$(1.3)' \qquad \inf_{\substack{(x,t)\in R^p\times[0,T]\\ |\xi|=1,\, i\neq j}} |\lambda_i(x, t: \xi)-\lambda_j(x, t: \xi)|\geqq\delta>0.$$

Then it follows from Theorem 1.1 that,

Corollary. *We suppose that* (1.2) *and* (1.3)′ *are valid. Then the following statements are equivalent*:

(i) L_0 *is strongly hyperbolic.*

(ii) $\sum A_j(x, t)\xi_j$ *is diagonalizable.*

(iii) *The Cauchy problem* (1.1) *is* L^2*-well posed.*

Theorem 1.1 implies (i)⇒(ii). (iii)⇒(i) is trivial. We note that T. Kano proved directly (iii)⇒(ii) in [5].

Though S. Mizohata has proved already in [10] and [11] the sufficiency of Theorem 1.1 and (ii)⇒(iii) in Corollary, we shall explain these facts. Namely, under the assumptions (1.2), (1.3) and the diagonalizability of $\sum A_j(x, t)\xi_j$, we can construct locally the symmetrizer of $\sum A_j(x, t)\xi_j$ and can derive the finite propagation speed of the solution of (1.1) and the existence of the solution of (1.1) for any u_0 and $f(x, t)$, in $L^2(R^p)$ and in $\mathscr{E}_t^0(L^2(R^p))$, with compact supports, respectively. We put as the symmetrizer of $A(x, t: \xi)=\sum A_j(x, t)\xi_j$,

$$(1.5) \qquad S(x, t: \xi)=\frac{1}{2\pi i}\oint_\Gamma(\lambda-A(x, t: \xi)^*)^{-1}(\lambda-A(x, t: \xi))^{-1}$$
$$\times P_\lambda(x, t: \lambda, \xi)^{-1}P(x, t: \lambda, \xi)\,d\lambda,$$

where $A(x, t: \xi)^*$ is the adjoint matrix of $A(x, t: \xi)$, $P(x, t: \lambda, \xi)$ is the determinant of $(\lambda - A(x, t: \xi))$, $P_\lambda(x, t: \lambda, \xi) = \frac{\partial}{\partial \lambda} P(x, t: \lambda, \xi)$, and Γ is a Jordan's curve which contains the characteristic roots of $A(x, t: \xi)$ and does not contain the poles of $P P_\lambda^{-1}(x, t: \lambda, \xi)$. Then $S(x, t: \xi)$ has the following properties: if $A(x, t: \xi)$ is diagonalizable and (1.2), (1.3) are valid,

$$(1.6) \qquad < S(x, t: \xi)h, h> \quad \geqq \delta(x, t: \xi)|h|^2,$$

where, for any compact set K in R^p,

$$(1.7) \qquad \inf_{\substack{(x,t)\in K\times[0,T] \\ |\xi|=1}} \delta(x, t: \xi) = d(K) > 0,$$

and

$$(1.8) \qquad (S(x, t: \xi)A(x, t: \xi))^* = S(x, t: \xi)A(x, t: \xi).$$

Moreover under the assumption (1.3)′, we have

$$(1.7)' \qquad \inf_{\substack{(x,t)\in R^p\times[0,T] \\ |\xi|=1}} \delta(x, t: \xi) = d_0 > 0.$$

The proof of these properties is due to K. O. Friedrichs [3]. The sufficiency of Theorem 1.1 can be proved by use of the properties (1.6), (1.7) and (1.8) and (ii)⇒(iii) in Corollary by use of (1.7)′ (See S. Mizohata [10], [11]).

In the next section we shall prove the necessity of Theorem 1.1 by use of the modified method which P.D. Lax introduced in [8].

Remark 3. For the single higher order equation $P\left(x, t: \frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)$ of homogeneous order m, it is known that the characteristic roots of $P(x, t; \lambda, \xi)$ are real distinct, if and only if $P\left(x, t; \frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)$ is strongly hyperbolic. This fact follows from the results which S. Mizohata and Y. Ohya in [12] and [13], and H. Flaschka and G. Strang in [2] proved.

§2. Proof of Necessity of Theorem 1.1

Here we shall prove that $\sum A_j(x, t)\xi_i$ is diagonalizable if L_0 is strongly hyperbolic. We need an inequality derived by the closed graph theorem. If the Cauchy problem (1.1) is uniformly well posed, it follows from the closed graph theorem that there exists a positive constant $C(\Omega)$ and a positive integer s_0 such that

$$|u|_{0,\Omega}\leqq C(\Omega)\{|Lu|_{s_0,\Omega}+|u|_{s_0,D}\}, \tag{2.1}$$

where $|\cdot|_{s,\Omega}$ is a supremum norm of $C^s(\Omega)$, and $\Omega=\{(x, t);\ |x-x_0|\leqq \lambda_0(t'-t),\ t_0\leqq t\leqq t'\}$ and $D=\Omega\cap\{t=t_0\}$. We note that a constant $C(\Omega)$ may be generally dependent on Ω. We put $\Omega_\mu=\left\{(x, t);\ |x-x_0|\leqq\lambda_0\left(t_0+\frac{1}{\mu}-t\right),\ t_0\leqq t\leqq t_0+\mu^{-1}\right\}$ and $D_\mu=\Omega_\mu\cap\{t=t_0\}$. Then we have

Lemma 2.1. *Suppose that the Cauchy problem of* (1.1) *is uniformly well posed. Then there exist positive integers s_0, s_1 and a constant C_0 such that it holds for any $\mu\geqq 1$ and for any $u(x, t)$ in $C^\infty(\Omega_1)$,*

$$|u|_{0,\Omega_\mu}\leqq C_0\mu^{s_1}\{|Lu|_{s_0,\Omega_\mu}+|u|_{s_0,D_\mu}\}, \tag{2.2}$$

where s_0, s_1 and C_0 are independent of μ.

Proof. Suppose that there exist functions $u_\mu(x, t)$ for any μ such that

$$|u_\mu|_{0,\Omega_\mu}\geqq \mu^{s(\mu)}\{|Lu_\mu|_{s_0,\Omega_\mu}+|u_\mu|_{s_0,D_\mu}\}, \tag{2.3}$$

where $s(\mu)\to\infty$ for $\mu\to\infty$. Without loss of generality we may assume $|u_\mu|_{0,\Omega_\mu}=1$. We put $Lu_\mu=f_\mu$, and $u_\mu(x, t_0)=g_\mu$. We can extend $f_\mu(x, t)$ and $g_\mu(x)$ to the domain $\Omega_1=\{|x-x_0|\leqq\lambda_0(t_0+1-t),\ t_0\leqq t\leqq t_0+1\}$ such that

$$\tilde{f}_\mu(x, t)=f_\mu(x, t) \qquad \text{in } \Omega_\mu$$
$$\tilde{g}_\mu(x)=g_\mu(x) \qquad \text{in } D_\mu,$$

and

$$\begin{cases} |\tilde{f}_\mu|_{s_0,\Omega_1}\leqq M^{2\log_2\mu}|f_\mu|_{s_0,\Omega_\mu}, \\ |\tilde{g}_\mu|_{s_0,D_1}\leqq M^{\log_2\mu}\ |g_\mu|_{s_0,D_\mu}, \end{cases} \tag{2.4}$$

where M is independent of μ.[3] Let $\tilde{u}_\mu(x, t)$ be the solution of (1.1) for $\tilde{f}_\mu(x, t)$ and $\tilde{g}_\mu(x)$. Then the uniqueness of the solution implies

3) These inequalities will be proved in appendix.

$$\tilde{u}_\mu(x, t)=u_\mu(x, t) \qquad \text{in} \quad \Omega_\mu.$$

Hence we have

$$|\tilde{u}_\mu|_{0,\Omega_1} \geqq 1. \tag{2.5}$$

On the other hand, by virtue of (2.1), we have

$$|\tilde{u}_\mu|_{0,\Omega_1} \leqq C(\Omega_1)(|\tilde{f}_\mu|_{s_0,\Omega_1}+|\tilde{g}_\mu|_{s_0,D_1}),$$

from which we get, combining with (2.3) and (2.4), the following inequality

$$|\tilde{u}_\mu|_{0,\Omega_1} \leqq C(\Omega_1)(M^{2\log_2\mu})\mu^{-s(\mu)}.$$

This and (2.5) cannot be compatible if μ is sufficiently large.

Now, we shall prove our theorem by contradiction. Assume that $\sum A_j(x, t)\xi_j$ is not diagonalizable. Then for some $B(x, t)$, the Cauchy problem for differential operator $L_0+B(x, t)$ can not be uniformly well posed.

Lemma 2.2. *Assume that* (1.2) *and* (1.3) *are valid. Then there exist an open set* U_0 *in* $R^p \times [0, T] \times (R^p-\{0\})$ *and a non-singular matrix* $N(x, t; \xi)$, *of which elements are as smooth as those of* $A(x, t; \xi)$ *in* U_0, *such that,*

$$N(x, t: \xi)A(x, t: \xi)=\begin{pmatrix} D_1 & & & 0 \\ & D_2 & & \\ & & \ddots & \\ 0 & & & D_k \end{pmatrix} N(x, t: \xi) \tag{2.6}$$

where D_i are $m_i \times m_i$ *Jordan's forms, that is,*

$$D_i=\begin{pmatrix} \lambda_i(x, t: \xi) & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i(x, t: \xi) \end{pmatrix}, \quad (i=1, 2,\dots, k), \tag{2.7}$$

$$\sum m_i=m.$$

We shall prove this lemma in the appendix.

Remark. It is known that characteristic roots $\lambda_i(x, t: \xi)$ of $A(x, t: \xi)$

are real, if the Cauchy problem of (1.1) is uniformly well posed (cf. S. Mizohata [9]).

Now we assume that $A(x, t: \xi)$ is not diagonalizable, that is, for some i_0 we have $m_{i_0} \geqq 2$. We may assume $i_0=1$ without loss of generality. Moreover we may assume that an open set U_0 in Lemma 2.2 is a neighbourhood of $(0, 0, \xi_0)$. We note that characteristic roots $\lambda_i(x, t: \xi)$ are analytic with respect to ξ. We write $\lambda_1(x, t: \xi)=a(x, t: \xi)$. Then we can find an analytic function $\lambda^0(x, t)$ (real valued) satisfying

$$\lambda_t^0(x, t)=\sum_{j+|\nu|\leqq r_0}(j!\nu!)^{-1}(t^j x^\nu)\left(\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\nu a(0, 0; \lambda_x^0)$$

$$\lambda^0(x, 0)=x\cdot\xi_0$$

Then we have

$$\begin{cases} \lambda_t^0=a(x, t: \lambda_x^0)+O(x^{\nu_0}t^{j_0}), \ \nu_0+j_0=r_0+1 \\ \lambda^0(x, 0)=x\cdot\xi_0 \end{cases} \tag{2.8}$$

We put $N_0(x, t)=N(x, t: \lambda_x^0(x, t))$. Then we note that

$$N_0(x, t)A(x, t: \lambda_x^0)N_0^{-1}(x, t)=\begin{pmatrix} D_1 & 0 \\ 0 & \tilde{D} \end{pmatrix}+O(x^{\nu_0}t^{j_0}) \tag{2.9}$$

where

$$D_1=\begin{pmatrix} a(x, t: \lambda_x^0) & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & a(x, t: \lambda_x^0) \end{pmatrix}, \quad m_1\times m_1 \text{ matrix.}$$

We transform L by $N_0(x, t)$. We put $v=N_0u$. Then we have

$$\tilde{L}[v]=\frac{\partial}{\partial t}v-\sum_j \tilde{A}_j(x, t)\frac{\partial}{\partial x_j}v+\sum_j N_0A_jN_{0x_j}^{-1}v$$
$$+N_0N_{0t}^{-1}v-N_0BN_0^{-1}v,$$

where

$$\tilde{A}_j(x, t)=N_0A_j(x, t)N_0^{-1}.$$

Now we choose $B(x, t)$ such that

$$N_0 A_j N_{0x_j}^{-1} - N_0 N_{0t}^{-1} - N_0 B N_0^{-1} = B_0,$$

where $(m_1, 1)$ element of B_0 is -1 and the other elements are zero. We consider the Cauchy problem for $\tilde{L}$, that is,

$$(2.10) \quad \begin{cases} \tilde{L}[v] = \left(\dfrac{\partial}{\partial t} - \sum \tilde{A}_j(x, t)\dfrac{\partial}{\partial x_j} + B_0\right) v(x, t) = f(x, t) \\ v(x, 0) = v_0(x) \end{cases}$$

Then the Cauchy problem for (2.10) is uniformly well posed, if the Cauchy problem for L is uniformly well posed. Hence the inequality (2.2) with $(x_0, t_0) = (0, 0)$ holds for $\tilde{L}$, that is, for any $\mu \geqq 1$,

$$(2.11) \quad |v|_{0,\Omega_\mu} \leqq C_0 \mu^{s_1} \{ |\tilde{L}v|_{s_0,\Omega_\mu} + |v|_{s_0,D_\mu} \}.$$

Our purpose is to construct the asymptotic solution $\{v(x, t)\}$ of (2.10) with $\tilde{f} = 0$ and $v_0 = 0$, which violates the inequality (2.11).

We construct the asymptotic solution of the following form,

$$(2.12) \quad v(x, t: n) = \sum_{j \geqq 0} e^{in\lambda(x,t:n)} h_j n^{-j/m_1},$$

where h_j are constant vectors of forms ${}^t(h_j^{(1)}, \ldots, h_j^{(m_1)}, 0, \ldots, 0)$, $h_j^{(i)}$ $(i = 1 \ldots m_1)$ being constant numbers and $\lambda(x, t: n)$ is of form as

$$(2.13) \quad \lambda(x, t: n) = \sum_{j \geqq 0} (\lambda^0(x, t) + \sigma_j t) n^{-j/m_1}, \ (\sigma_j \text{ constant.})$$

We write,

$$h(n) = \sum_{j \geqq 0} h_j n^{-j/m_1},$$

$$\sigma(n) = \sum_{j \geqq 0} \sigma_j n^{-j/m_1},$$

$$e(n) = \sum_{j \geqq 0} n^{-j/m_1},$$

and

$$h^0(n) = \sum_{j \geqq 0} {}^t(h_j^{(1)}, \ldots, h_j^{(m_1)}) n^{-j/m_1}.$$

Applying $\tilde{L}$ to v defined in (2.12) we have

$$\tilde{L}[v]=in\Big(\lambda_t(x,\ t:\ n)-\sum_{k=1}^{p}\tilde{A}_k\lambda_{x_k}(x,\ t:\ n)-\frac{i}{n}B_0\Big)h(n)$$

(2.14)

$$=in\begin{pmatrix} \Lambda(n)h^0(n) \\ 0 \end{pmatrix}+O(nx^{\nu_1}t^{j_1}),\qquad |\nu_1|+j_1=r_0$$

where $i=\sqrt{-1}$, and

$$\Lambda(n)=\begin{pmatrix} \sigma(n) & -e(n) & & 0 \\ 0 & \ddots & \ddots & -e(n) \\ \frac{i}{n} & 0 & \ddots & \sigma(n) \end{pmatrix},\ (m_1\times m_1 \text{ matrix}) .$$

We determine $\sigma(n)$ such that det $\Lambda(n)=0$, that is,

$$\det \Lambda(n)=\sigma(n)^{m_1}+\frac{i}{n}e(n)^{m_1-1}$$

(2.15)

$$=(\sum\sigma_j n^{-j/m_1})^{m_1}+\frac{i}{n}(\sum n^{-j/m_1})^{m_1-1}$$

$$=\sum_{j\geqq 0}F_j n^{-j/m_1},$$

where

$$F_j=\sum_{\substack{\nu_0+\cdots+\nu_j=m_1\\ \nu_1+2\nu_2+\cdots+j\nu_j=j}}\binom{m_1}{\nu_0}\binom{m_1-\nu_0}{\nu_1}\cdots\binom{m_1-\nu_0-\nu_1\cdots-\nu_{j-2}}{\nu_{j-1}}\sigma_0^{\nu_0}\sigma_0^{\nu_1}\cdots\sigma_j^{\nu_j}$$

(2.16)

$$+i\sum_{\substack{\nu_0+\cdots+\nu_j=m_1-1\\ \nu_1+2\nu_2+\cdots+(j-m_1)\nu_{j-m_1}=j-m_1}}\binom{m_1-1}{\nu_0}\cdots\binom{m_1-1-\nu_0-\cdots-\nu_{j-2}}{\nu_{j-1}}.$$

From this formula, we have $F_0=\sigma_0^{m_1}$. Hence we have

$$\sigma_0=0$$

which implies $F_j=0$ $(j<m_1)$. Next, we obtain

$$F_{m_1}=\sigma_1^{m_1}+i=0$$

Hence we can choose σ_1 such that

(2.17) $$\operatorname{Im}\sigma_1<0.$$

As σ_0 is zero, we have generally from (2.16)

$$F_j = m_1 \sigma_1^{m_1-1} \sigma_{j-m_1+1} + P_j(\sigma_1, \dots, \sigma_{j-m_1}) = 0,\ j > m_1, \tag{2.18}$$

where P_j is polynomials of $(\sigma_1, \dots, \sigma_{j-m_1})$. Hence we can determine σ_j so that (2.18) holds.

Next we determine $h^0(n)$ such that $\Lambda(n)h^0(n)=0$. As we can write

$$\Lambda(n)h^0(n) = \begin{pmatrix} \sigma(n) & -e(n) & & 0 \\ & \ddots & \ddots & \\ 0 & & \ddots & -e(n) \\ \frac{i}{n} & & & \sigma(n) \end{pmatrix} \begin{pmatrix} h_1(n) \\ \vdots \\ \vdots \\ h_{m_1}(n) \end{pmatrix} = 0$$

we obtain the relations,

$$h_k(n) = \left(\frac{e(n)}{\sigma(n)}\right)^{m_1-k} h_{m_1}(n),\ k=1, \dots, m_1-1.$$

We choose $h_{m_1}(n) = \sigma(n)^{m_1-1}$. Then we have $h_k(n) = e(n)^{m_1-k}\sigma(n)^{k-1}$. Hence we can expand $h^0(n)$ as

$$h^0(n) = \sum_{j\geqq 0} h_j^0 n^{-j/m_1}, \tag{2.19}$$

where h_j^0 are constant vectors. Here we note

$$h_0^0 = {}^t(1, 0, \dots, 0) \neq 0. \tag{2.20}$$

Now, we define $v_N(x, t: n)$ as

$$v_N(x, t: n) = \sum_{j=0}^{N} e^{in\lambda_N(x,t:n)} h_j n^{-j/m_1},$$

where

$$\lambda_N(x, t: n) = \sum_{j=0}^{N} (\lambda^0(x, t) + \sigma_j t) n^{-j/m_1}$$

and $h_j = {}^t(h_j^0, 0, \dots, 0)$. Then we have

$$L[v_N] = e^{in\lambda_N}(O(n^{-N_1}) + O(n x^{\nu_1} t^{j_1})),\ |\nu_1| + j_1 = r_0 \tag{2.21}$$

where N_1 becomes larger and larger if we choose N larger and larger. We put

$$\rho(n, \mu)=\max_{(x,t)\in\bar{\Omega}_\mu}\{-\mathrm{Im}\, n\lambda_N(x, t: n)\}.$$

Then by virtue of (2.17), we obtain

$$\rho(n, \mu)\geqq cn^{1-1/m_1}\mu^{-1}$$

where c is independent of n and μ. Here we choose $\mu=n^{(m_1-1)/2m_1}$. Then we have

$$\rho(n, \mu)>cn^{(m_1-1)/2m_1}. \tag{2.22}$$

From (2.21), we obtain

$$|L[v_N]|_{s_0,\Omega_\mu}\leqq \text{const. } e^{\rho(n,\mu)}(n^{-N_1+s_0}+n^{s_0+1}\mu^{-r_0+s_0}) \tag{2.23}$$

where const. is independent of n and μ and $r_0=\nu_1+j_1$. Noting that $\mathrm{Im}\,\lambda_N(x, 0; n)=0$, we have

$$|v_N|_{s_0,D_\mu}\leqq \text{const. } n^{s_0}. \tag{2.24}$$

On the other hand we obtain by virtue of (2.20),

$$|v_N|_{0,\Omega_\mu}\geqq c_1 e^{\rho(n,\mu)}. \tag{2.25}$$

Hence it follows from (2.11), (2.23), (2.24) and (2.25) that

$$c_1 e^{\rho(n,\mu)}\leqq \text{const. }\{e^{\rho(n,\mu)}(n^{s_0-N_1}+n^{s_0+1-(r_0-s_0)(m_1-1)/2m_1}) + n^{s_0}\}n^{s_1(m_1-1)/2m_1}. \tag{2.26}$$

Here we choose N_1 and r_0 such that $N_1=s_0+1+s_1(m_1-1)/2m_1$ and $s_0+1-(r_0-s_0)(m_1-1)/2m_1=-1-s_1(m_1-1)/2m_1$. Then (2.26) and (2.22) can not be compatible, if n is sufficiently large. This proves the necessity of our Theorem 1.1.

Appendix

Lemma A.1. *Let $g(x)$ be in $C^{s_0}(D_\mu)$, where $D_\mu=\{x\in R^p, |x|\leqq\mu^{-1}\}$, $\mu\geqq 1$. Then there exists $\tilde{g}(x)$ in $C^{s_0}(D_1)$ such that*

$$\tilde{g}(x)=g(x) \qquad \textit{in } D_\mu, \tag{A.1}$$

and

(A.2) $$|\tilde{g}|_{s_0,D_1} \leqq M^{\log_2\mu} |g|_{s_0,D_\mu},$$

where M is independent of μ.

Proof. Under the change of variables, $|x|=r$, the domain D_μ is transformed into the domain $\tilde{D}_\mu=\{(r, \omega),\ 0\leqq r\leqq \mu^{-1},\ \omega\in S^{p-1}\}$, where S^{p-1} is the unit sphere. And $g(x)$ is transformed into $g(r, \omega)$ in $C^{s_0}(\tilde{D}_\mu)$. We extend $g(r, \omega)$ to the domain $D_\mu^{(1)}=\{(r, \omega),\ 0\leqq r\leqq 2\mu^{-1},\ \omega\in S^{p-1}\}$,

$$g^{(1)}(r, \omega)=g(r, \omega),\ 0\leqq r\leqq \mu^{-1},\ \omega\in S^{p-1}$$

$$=\sum_{i=1}^{s_0+1} a_i g\left(\mu^{-1}-\frac{i}{s_0+1}(r-\mu^{-1}),\ \omega\right),\ \mu^{-1}<r\leqq 2\mu^{-1},$$

where $\sum_{i=1}^{s_0+1} a_i\left(\frac{i}{s_0+1}\right)^j=1,\ j=0, 1,\dots, s_0$. Then $g^{(1)}(r, \omega)$ belongs to $C^{s_0}(D_\mu^{(1)})$ and satisfies

$$|g^{(1)}|_{s_0,D_\mu^{(1)}}\leqq M|g|_{s_0,D_\mu}.$$

Next, we extend $g^{(1)}(r, \omega)$ to a domain $D_\mu^{(2)}=\{(r, \omega),\ 0\leqq r\leqq 4\mu^{-1},\ \omega\in S^{p-1}\}$,

$$g^{(2)}(r, \omega)=g^{(1)}(r, \omega),\ 0\leqq r\leqq 2\mu^{-1},$$

$$=\sum_{i=1}^{s_0+1} a_i g^{(1)}(2\mu^{-1}-\frac{i}{s_0+1}(r-2\mu^{-1}), \omega),$$

$$2\mu^{-1}<r<4\mu^{-1}.$$

Generally, for any positive integer m we define $g^{(m+1)}(r, \omega)$ as

$$g^{(m+1)}(r, \omega)=g^{(m)}(r, \omega),\ 0\leqq r\leqq 2^m\mu^{-1}$$

$$=\sum_{i=1}^{s_0+1} a_i g^{(m)}\left(2^m\mu^{-1}-\frac{i}{s_0+1}(r-2^m\mu^{-1}),\ \omega\right)$$

$$2^m\mu^{-1}<r\leqq 2^{m+1}\mu^{-1}.$$

Then $g^{(m+1)}(r, \omega)$ belongs to $C^{s_0}(D_\mu^{(m+1)})$, where $D^{(m+1)}=\{(r, \omega),\ 0\leqq r\leqq 2^{m+1}\mu^{-1},\ \omega\in S^{p-1}\}$, and satisfies

$$|g^{(m+1)}|_{s_0,D_\mu^{(m+1)}}\leqq M|g^{(m)}|_{s_0,D_\mu^{(m)}}$$

$$\leqq M^{m+1}|g|_{s_0,D_\mu}.$$

Hence, if we put $m+1=\log_2\mu$, we have $D^{(m+1)}=D_1$ and $\tilde{g}(x)=g^{(m+1)}(r, \omega)$ satifying (A.1) and (A.2).

Q.E.D.

Corollary. *Let $f(x, t)$ be in $C^s(\Omega_\mu)$, where $\Omega_\mu=\{(x, t), |x|<\lambda_0(\mu^{-1}-t), 0\leqq t\leqq\mu^{-1}\}$, $\mu\geqq 1$. Then there exists $\tilde{f}(x, t)$ in $C^{s_0}(\Omega_1)$ such that*

$$\tilde{f}(x, t)=f(x, t) \text{ in } \Omega_\mu,$$

and

$$|\tilde{f}|_{s_0,\Omega_1}<M^{2\log_2\mu}|f|_{s_0,\Omega_\mu}.$$

We can prove this corollary by the same method as in Lemma A.1, if we change variables (x, t) into (r, ω, τ) as $r=|x|$, $\tau=\mu^{-1}-t-r$.

Lemma A.2. *Let $A(z)$ be a $m\times m$ matrix of which elements are infinitely differentiable functions of z in an open set B_1 in R^n and of which eigenvalues are all zero. Then there exists a non-singular matrix $N(z)$ of which elements are infinitely differentiable in an open set $B\subset B_1$, such that*

$$N(z)^{-1}A(z)N(z)=\begin{pmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_l \end{pmatrix}, \quad \text{for } z \text{ in } B \tag{A.3}$$

where

$$J_i=\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}.$$

Proof. Since the eigenvalues of $A(z)$ are all zero, there exists an integer $\nu(z)$ for any z such that $A(z)^\nu=0$ and $A(z)^{\nu-1}\neq 0$. We put

$$\nu_0=\max_{z\in\bar{B}_1}\nu(z)=\nu(z_0),$$

Then we have a neighbourhood B_2 of z_0 in B_1 such that $A(z)^{\nu_0}=0$ and $A(z)^{\nu_0-1}\neq 0$ for any z in B_2. Moreover we have an open set B_3 in B_2 such that the rank of $A(z)^i$ is constant for any $i=1, 2,\ldots, \nu_0-1$ and for any z in B_3. We define $W_i(z)$ as

$$W_i(z)=\{h\in C^m,\ A(z)^i h=0\}.$$

Then we note that dim $W_i(z)=m_i$ is constant for z in B_3 and

$$C^m=W_{\nu_0}(z)\supset W_{\nu_0-1}(z)\supset\cdots\supset W_1(z).$$

We put $r_i=m_i-m_{i-1}$ $(i=2,\dots,\nu_0)$ and $r_1=m_1$. Then there exist constant vectors $\{h_1,\dots,h_{r_{\nu_0}}\}$ such that

$$W_{\nu_0}(z)=\{h_1,\dots,h_{r_{\nu_0}}\}+W_{\nu_0-1}(z)\qquad\text{(direct sum)}$$

for any z in $B_4\subset B_3$, where B_4 is some open set in B_3. Then we have $A(z)h_j$ in $W_{\nu_0-1}(z)$ and $\{A(z)h_1,\dots,A(z)h_{r_{\nu_0}}\}\cap W_{\nu_0-2}(z)=\{0\}$ for any $z\in B_4$. Hence we have $r_{\nu_0}\leqq r_{\nu_0-1}$. Hence there exsit constant vectors $\{h_{r_{\nu_0}+1},\dots,h_{r_{\nu_0-1}}\}$ such that

$$W_{\nu_0-1}(z)=\{h_1,\dots,Ah_{r_{\nu_0}},h_{r_{\nu_0}+1},\dots,h_{r_{\nu_0-1}}\}$$
$$+W_{\nu_0-2}(z)\qquad\text{(direct sum)}$$

for any z in $B_5\subset B_4$, where B_5 is some open set in B_4. Then we have $A(z)^2h_j(j=1,\dots,r_{\nu_0})$ and $A(z)h_j(j=r_{\nu_0}+1,\dots,r_{\nu_0-1})$ in $W_{\nu_0-2}(z)$ and $\{A(z)^2h_1,\dots,A(z)^2h_{r_{\nu_0}},A(z)h_{r_{\nu_0}+1},\dots,A(z)h_{r_{\nu_0-1}}\}\cap W_{\nu_0-3}(z)=\{0\}$ for z in B_5. Hence we have $r_{\nu_0-1}\leqq r_{\nu_0-2}$. Similarly, we have $r_{\nu_0}\leqq r_{\nu_0-1}\leqq\cdots\leqq r_1$. Therefore we can choose constant vectors $h_1,\dots,h_{r_1}$ such that, $i=1,\dots,\nu_0$

$$W_i(z)=\{A(z)^{\nu_0-i}h_1,\dots,A(z)^{\nu_0-i}h_{r_{\nu_0}},A(z)^{\nu_0-i-1}h_{r_{\nu_0}+1},\dots,$$
$$A(z)^{\nu_0-i-1}h_{r_{\nu_0-1}},\dots,h_{r_{i+1}+1},\dots,h_{r_i}\}$$
$$+W_{i-1}(z)\qquad\text{(direct sum)}$$

for z in $B_{4+i}\subset B_{4+i-1}$, where B_{4+i} is some open set in B_{4+i-1}. Then $A(z)^k h_{r_{i+1}+1},\dots,A(z)^k h_{r_i}$, $(i=1,2,\dots,\nu_0;\ k=0,1,\dots,i-1)$ become the basis of $W_{\nu_0}(z)=C^m$ for z in $B_{3+\nu_0}$

$$N_j^{(i)}(z)=(h_j,A(z)h_j,\dots,A(z)^{i-1}h_j),\ j=r_{i+1},\dots,r_i.$$

Then we obtain

$$A(z)N_j^{(i)}(z)=N_j^{(i)}(z)\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$$

We put

$$N(z)=(N_1^{(\nu_0)},\dots N_{r_{\nu_0}}^{(\nu_0)}, N_{r_{\nu_0}+1}^{(\nu_0-1)},\dots, N_{r_{\nu_0-1}}^{(\nu_0-1)},\dots, N_{r_2+1}^{(0)},\dots, N_{r_1}^{(0)}).$$

Then $N(z)$ satisfies (A.3) for z in $B=B_{\nu_0+3}$.

Proof of Lemma 2.2. For the fixed point $z_0=(x_0, t_0, \xi_0)$ we denote by $\{h_1^{(i)},\dots, h_{m_i}^{(i)}\}$ the basis of the generalized eigenspace of $A(z_0)$ corresponding to the eigen value $\lambda_i(z_0)$ of $A(z_0)$. We put

$$h_j^{(i)}(z)=\frac{1}{2\pi\sqrt{-1}}\oint_{|\lambda-\lambda_i(z_0)|=r_0}(\lambda-A(z))^{-1}h_j^{(i)}\,d\lambda, \quad (j=1,\dots, m_i).$$

Then $\{h_1^{(i)}(z),\dots, h_{m_1}^{(i)}(z)\}$ constructs the basis of the generalized root space of $A(z)$ corresponding to $\lambda_i(z)$ the eigenvalue of $A(z)$, for z in a neighbourhood of z_0. We put $N_1(z)=(h_1^{(1)}(z),\dots, h_{m_1}^{(1)}(z)\dots, h_1^{(l)}(z),\dots, h_m^{(l)}(z))$. Then we have

$$N_1(z)A(z)N_1(z)^{-1}=\begin{pmatrix} A_1(z) & & 0 \\ & A_2(z) & \\ & & \ddots \\ 0 & & A_l(z)\end{pmatrix},$$

where $A_i(z)$ has only the eigenvalue $\lambda_i(z)$. Hence we can apply Lemma A.2 to $(\lambda_i(z)-A_i(z))$ for z in B_0.

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