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On the Asymptotic Behavior of the Solutions of Some Third and Fourth Order Non-Autonomous Differential Equations

By

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1. Introduction

The purpose of this paper is to investigate the asymptotic behavior of the solutions of non-autonomous differential equations of the form

(1.1) $\dddot{x}+a(t)\ddot{x}+b(t)\dot{x}+c(t)x=p(t),$

(1.2) $\dddot{x}+a(t)\ddot{x}+b(t)\dot{x}+c(t)h(x)=p(t, x, \dot{x}, \ddot{x}),$

(1.3) $\dddot{x}+a(t)f(x, \dot{x})\ddot{x}+b(t)g(x, \dot{x})\dot{x}+c(t)h(x)=p(t, x, \dot{x}, \ddot{x}),$

(1.4) $\ddddot{x}+a(t)f(\dot{x})\dddot{x}+b(t)\phi(\dot{x}, \ddot{x})+c(t)g(\dot{x})+d(t)h(x)=p(t, x, \dot{x}, \ddot{x}, \dddot{x})$

where functions appeared in the equations are real valued. The dots indicate differentiation with respect to t and all solutions considered are assumed to be real.

The problem is to give conditions to ensure that all solutions of (1.1), (1.2), (1.3) and (1.4) tend to zero as $t\to\infty$. This problem has received a considerable amount of attention during the past twenty years, particulary when equations are autonomous. Many of these results are summarized in [14].

In [17] K. E. Swick considered the behavior as $t\to\infty$ of solutions of the differential equations

(1.5) $$\dddot{x}+a\ddot{x}+g(x)\dot{x}+h(x)=e(t),$$

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$$\dddot{x}+p(t)\ddot{x}+q(t)g(\dot{x})+h(x)=e(t) \tag{1.6}$$

where a is a positive constant. In [16] he also considered the asymptotic stability in the large of the trivial solution of the equations

$$\dddot{x}+p(t)\ddot{x}+q(t)g(\dot{x})+r(t)h(x)=0, \tag{1.7}$$

$$\dddot{x}+f(x,\ \dot{x},\ t)\ddot{x}+q(t)g(\dot{x})+r(t)h(x)=0. \tag{1.8}$$

In [6] the author established the conditions under which all solutions of the non-autonomous equations (1.1)$\sim$(1.3) tend to zero as $t\to\infty$.

In this paper we obtain the conditions weaker than that obtained in [6].

Recently the author ([9]) studied the asymptotic behavior of solutions of

$$\dddot{x}+a(t)f(x,\ \dot{x},\ \ddot{x})\ddot{x}+b(t)g(x,\ \dot{x})+c(t)h(x)=p(t,\ x,\ \dot{x},\ \ddot{x}) \tag{1.9}$$

under the condition that

$$\frac{h(x)}{x}\geqq\delta>0 \qquad (x\neq 0).$$

But here we consider the equations (1.3) and (1.4) under the weaker condition that

$$H(x)\equiv\int_0^x h(\xi)\,d\xi\to\infty \qquad \text{as } |x|\to\infty.$$

In [7] the author also investigated the asymptotic behavior of the solutions of the equation

$$\ddddot{x}+f(\ddot{x})\dddot{x}+\phi(\dot{x},\ \ddot{x})+g(\dot{x})+h(x)=p(t,\ x,\ \dot{x},\ \ddot{x},\ \dddot{x}). \tag{1.10}$$

This time we study the non-autonomous equation (1.4). The results obtained here contains the author's result in [7].

The main tools used in this work are Liapunov functions and the generalized Yoshizawa's Theorem ([21; Theorem 14.2]).

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2. Assumptions and Statements of the Results

Theorems 1 and 2 are concerned with the differential equation (1.3). We assume the following assumptions on the functions appeared in (1.3).

Assumptions for Theorems 1 and 2.

(a_1) *$a(t), b(t), c(t)$ are positive and continuously differentiable functions in $I=[0, \infty)$.*

(a_2) *$p(t, x, y, z)$ is continuous in $I\times R^3$.*

(a_3) *$h(x)$ is continuously differentiable for all $x\in R^1$.*

(a_4) *$f(x, y), f_x(x, y), g(x, y)$ and $g_x(x, y)$ are continuous for all $(x, y)\in R^2$.*

Hereafter we use the following notations.

$$a'_+(t)=\max(a'(t), 0), \qquad a'_-(t)=\max(-a'(t), 0)$$

so that $a'(t)=a'_+(t)-a'_-(t)$. Likewise, we denote

$$b'_+(t)=\max(b'(t), 0), \qquad b'_-(t)=\max(-b'(t), 0),$$

$$c'_+(t)=\max(c'(t), 0), \qquad c'_-(t)=\max(-c'(t), 0).$$

Theorem 1. *Suppose that the assumptions $(a_1)\sim(a_4)$ hold and the following conditions are satisfied:*

(i) $A\geqq a(t)\geqq a_0>0, \quad B\geqq b(t)\geqq b_0>0, \quad C\geqq c(t)\geqq c_0>0$ *for* $t\in I$,

(ii) $f_1\geqq f(x, y)\geqq f_0>0, \quad yf_x(x, y)\leqq 0$ *for all* $(x, y)\in R^2$,

(iii) $g_1\geqq g(x, y)\geqq g_0>0, \quad yg_x(x, y)\leqq 0$ *for all* $(x, y)\in R^2$,

(iv) $xh(x)>0 \quad (x\neq 0), \quad H(x)\equiv\int_0^x h(\xi)\,d\xi\to\infty$ *as* $|x|\to\infty$,

(v) $\dfrac{a_0b_0f_0g_0}{C}>h_1\geqq h'(x)$,

(vi) $$\mu\{a'_+(t)f_1-a'_-(t)f_0\}+\{b'_+(t)g_1-b'_-(t)g_0\} -\frac{1}{\mu}c'(t)h_1<\mu b_0g_0-Ch_1$$

where μ is an arbitrarily fixed constant satisfying

$$\frac{Ch_1}{b_0g_0}<\mu<a_0f_0,$$

(vii) $\int_0^\infty c'_+(t)\,dt<\infty, \quad c'(t)\to 0 \qquad as\ t\to\infty,$

(viii) $|p(t, x, y, z)|\leqq p_1(t)+p_2(t)\{H(x)+y^2+z^2\}^{\rho/2}$
$+\Delta(y^2+z^2)^{1/2}$

where ρ, Δ *are constants such that* $0\leqq\rho\leqq 1$, $\Delta\geqq 0$ *and* $p_1(t)$, $p_2(t)$ *are non-negative continuous functions satisfying,*

(ix) $\int_0^\infty p_i(t)\,dt<\infty \qquad (i=1, 2).$

If Δ *is sufficiently small, then every solution* $x(t)$ *of* (1.3) *is uniform-bounded and satisfies*

$$x(t)\to 0, \qquad \dot{x}(t)\to 0, \qquad \ddot{x}(t)\to 0 \qquad as\ t\to\infty.$$

As an immediate consequence of Theorem 1, we have the following result on (1.1).

Corollary 1. *Suppose that the assumption* (a_1) *and the conditions* (i), (vii) *of Theorem* 1 *hold and in addition the following conditions are satisfied*:

(v)' $a_0b_0-C>0,$

(vi)' $\mu a'(t)+b'(t)-\frac{1}{\mu}c'(t)<\mu b_0-C \qquad \left(\frac{C}{b_0}<\mu<a_0\right),$

(ix)' $\int_0^\infty |p(t)|\,dt<\infty.$

Then every solution $x(t)$ *of* (1.1) *is uniform-bounded and satisfies*

$$x(t)\to 0, \qquad \dot{x}(t)\to 0, \qquad \ddot{x}(t)\to 0 \qquad as\ t\to\infty.$$

And also we have the following Corollary 2 concerning the equation (1.2).

Corollary 2. *Suppose that the assumptions* $(a_1)\sim(a_3)$ *and the conditions* (i), (iv), (vii) *of Theorem* 1 *hold and the following conditions are satisfied*:

(v)' $\frac{a_0b_0}{C}>h_1\geqq h'(x),$

(vi)' $\mu a'(t)+b'(t)-\frac{1}{\nu}c'(t)<\mu b_0-Ch_1 \qquad \left(\frac{Ch_1}{b_0}<\mu<a_0, \quad \nu=\frac{\mu}{h_1}\right),$

(viii)' $|p(t, x, y, z)| \leqq p_1(t)$,

(ix)' $\int_0^\infty p_1(t)\,dt < \infty$.

Then every solution $x(t)$ *of* (1.2) *is uniform-bounded and satisfies*

$$x(t) \to 0, \qquad \dot{x}(t) \to 0, \qquad \ddot{x}(t) \to 0 \qquad \text{as } t \to \infty.$$

Remark. Theorem 1 extends the author's earlier results, that is, Corollaries 1 and 2 coincide with Corollary 1 and Theorem 1 in [6] respectively.

In [4], J.O.C. Ezeilo studied the equation

$$\dddot{x} + f_1(x, \dot{x})\ddot{x} + f_2(\dot{x}) + f_3(x) = p(t, x, \dot{x}, \ddot{x}) \tag{2.1}$$

where $p(t, x, y, z)$ satisfies the condition (viii) of our Theorem 1. He required the boundedness and integrability of the functions $p_1(t)$ and $p_2(t)$. Here we only assume the integrability of $p_1(t)$ and $p_2(t)$.

Observe that the condition (v) in Theorem 1 is the usual ≪generalized Routh-Hurwitz conditions≫.

Theorem 2. *Suppose that the assumptions* $(a_1) \sim (a_4)$ *hold and the following conditions are satisfied*:

(i) $A \geqq a(t) \geqq a_0 > 0, \quad B \geqq b(t) \geqq b_0 > 0, \quad C \geqq c(t) \geqq c_0 > 0$ *for* $t \in I$,

(ii) $f(x, y) \geqq f_0 > 0, \quad y f_x(x, y) \leqq 0$ *for all* $(x, y) \in R^2$,

(iii) $g(x, y) \geqq g_0 > 0, \quad y g_x(x, y) \leqq 0$ *for all* $(x, y) \in R^2$,

(iv) $x h(x) > 0 \quad (x \neq 0), \quad H(x) \equiv \int_0^x h(\xi)\,d\xi \to \infty$ *as* $|x| \to \infty$,

(v) $\dfrac{a_0 b_0 f_0 g_0}{C} > h_1 \geqq h'(x)$,

(vi) $\int_0^\infty \{a'_+(t) + b'_+(t) + |c'(t)|\}\,dt < \infty, \quad c'(t) \to 0$ *as* $t \to \infty$,

(vii) $|p(t, x, y, z)| \leqq p_1(t) + p_2(t)\{H(x) + y^2 + z^2\}^{\rho/2} + \Delta(y^2 + z^2)^{1/2}$ *where* ρ, Δ *are constants such that* $0 \leqq \rho \leqq 1$, $\Delta \geqq 0$ *and* $p_1(t)$, $p_2(t)$ *are non-negative continuous functions satisfying*,

(viii) $\int_0^\infty p_i(t)\,dt < \infty \qquad (i = 1, 2)$.

If Δ *is sufficiently small, then every solution* $x(t)$ *of* (1.3) *is uniform-*

bounded and satisfies

$$x(t)\to 0, \quad \dot{x}(t)\to 0, \quad \ddot{x}(t)\to 0 \quad as \ t\to\infty.$$

Remark. In Theorem 2 the functions $f(x, y)$ and $g(x, y)$ are not generally bounded above. Here also we do not need the boundedness of the functions $p_1(t)$ and $p_2(t)$. Theorem 2 is the extension of the author's earlier result ([6; Theorem 2]).

We turn now to the fourth order differential equation (1.4). We make the following assumptions on the functions appeared in (1.4).

Assumptions for Theorem 3.

(A_1) *$a(t)$, $b(t)$, $c(t)$ and $d(t)$ are positive and continuously differentiable functions in $I=[0, \infty)$.*

(A_2) *$f(z)$ is continuously differentiable for all $z\in R^1$.*

(A_3) *$\phi(y, z)$ and $\frac{\partial\phi}{\partial y}(y, z)$ are continuous for all $(y, z)\in R^2$.*

(A_4) *$g(y)$ is continuously differentiable for all $y\in R^1$.*

(A_5) *$h(x)$ is continuously differentiable for all $x\in R^1$.*

(A_6) *$p(t, x, y, z, w)$ is continuous in $I\times R^4$.*

In Theorem 3, the following notations are used:

$$g_1(y)=\frac{g(y)}{y} \qquad (y\neq 0), \qquad g_1(0)=g'(0),$$

$$f_1(z)=\frac{1}{z}\int_0^z f(\xi)\,d\xi \qquad (z\neq 0), \qquad f_1(0)=f(0).$$

Theorem 3. *Suppose that the assumptions $(A_1)\sim(A_6)$ hold and that there exist positive constants such that*

(i) $A\geqq a(t)\geqq a_0>0$, $B\geqq b(t)\geqq b_0>0$, $C\geqq c(t)\geqq c_0>0$,
$D\geqq d(t)\geqq d_0>0$ *for* $t\in I$,

(ii) $f(z)\geqq f_0>0$ *for all* $z\in R^1$,

(iii) $g_1(y)\geqq g_0>0$ *for all* $y\in R^1$, $g(0)=0$,

(iv) $xh(x)>0$ $(x\neq 0)$, $H(x)\equiv\int_0^x h(\xi)\,d\xi\to\infty$ *as* $|x|\to\infty$,

$h_0 - \dfrac{a_0 f_0 \delta_0}{2c_0 g_0 D} \leqq h'(x) \leqq h_0,$

(v) $\phi_y(y, z) \leqq 0, \quad \phi(y, 0) = 0 \qquad$ *in* R^2,

(vi) $0 \leqq \dfrac{\phi(y, z)}{z} - \phi_0 \leqq \dfrac{\varepsilon_0 c_0^3 g_0^3}{BD^2 h_0^2} \qquad (z \neq 0)$

where ε_0 *is a sufficiently small positive constant,*

(vii) $a_0 b_0 c_0 f_0 \phi_0 g_0 - C^2 g_0 g'(y) - A^2 D f_0 h_0 f(z) \geqq \delta_0 > 0$
for all $(y, z) \in R^2$,

(viii) $g'(y) - g_1(y) \leqq \delta < \dfrac{2Dh_0\delta_0}{Ca_0 f_0 c_0^2 g_0^2}$,

(ix) $f_1(z) - f(z) \leqq \dfrac{Cc_0 g_0 \delta}{Aa_0 f_0 D h_0}$,

(x) $\int_0^\infty \{|a'(t)| + b'_+(t) + |c'(t)| + |d'(t)|\} dt < \infty$,
$d'(t) \to 0$ *as* $t \to \infty$, *where* $b'_+(t) = \max(b'(t), 0)$,

(xi) $|p(t, x, y, z, w)| \leqq p_1(t) + p_2(t)\{H(x) + y^2 + z^2 + w^2\}^{\rho/2}$
$+ \Delta(y^2 + z^2 + w^2)^{1/2}$
where ρ, Δ *are constants such that* $0 \leqq \rho \leqq 1$, $\Delta \geqq 0$ *and* $p_1(t)$, $p_2(t)$ *are non-negative continuous functions satisfying,*

(xii) $\int_0^\infty p_i(t) dt < \infty \qquad (i = 1, 2)$.

If Δ *is sufficiently small, then every solution* $x(t)$ *of* (1.4) *is uniform-bounded and satisfies*

$$x(t) \to 0, \quad \dot{x}(t) \to 0, \quad \ddot{x}(t) \to 0, \quad \dddot{x}(t) \to 0 \quad \textit{as } t \to \infty.$$

Remark. Theorem 3 extends the author's result [7] to the non-autonomous equation (1.4). Theorem 3 also contains the results obtained by J. O. C. Ezeilo [4], M. Harrow [11] and M. A. Asmussen [1]. Note that also we do not require the boundedness of $p_1(t)$ and $p_2(t)$ here.

3. Auxiliary Lemmas

Consider a system of differential equations

$$\dot{X} = F(t, X) \tag{3.1}$$

where $X = (x_1, .., x_n)$ and $F(t, X)$ is continuous in $I \times R^n (I = [0, \infty))$.

The following Lemma 1 is well-known ([21]).

Lemma 1. *Suppose that there exists a continuously differentiable function* $V(t, X)$ *defined on* $t\in I$, $\|X\|\geqq R$, *where* R *may be large, which satisfies the following conditions*:

(i) $a(\|X\|)\leqq V(t, X)\leqq b(\|X\|)$, *where* $a(r)\in CI$ (*a family of continuous and increasing functions*), $a(r)\to\infty$ *as* $r\to\infty$ *and* $b(r)\in CI$,

(ii) $\dot{V}_{(3.1)}(t, X)\leqq 0$.

Then the solutions of (3.1) *are uniform bounded.*

Next we consider a system of differential equations

$$\dot{X}=F(t, X)+G(t, X) \tag{3.2}$$

where $F(t, X)$ and $G(t, X)$ are continuous on $I\times Q$ ($I=[0, \infty)$, Q: an open set in R^n). We assume

$$\|G(t, X)\|\leqq G_1(t, X)+G_2(X) \tag{3.3}$$

where $G_1(t, X)$ is non-negative continuous on $I\times Q$ and $\int_0^t G_1(s, X)ds$ is bounded for all t whenever X belongs to any compact subset of Q, and $G_2(X)$ is non-negative continuous in Q.

The following Lemma is a simple extension of the well-known result obtained by T. Yoshizawa [21; Theorem 14.2].

Lemma 2. *Suppose that there exists a non-negative continuously differentiable function* $V(t, X)$ *on* $I\times Q$ *such that* $\dot{V}_{(3.2)}(t, X)\leqq -W(X)$, *where* $W(X)$ *is positive definite with respect to a closed set* Ω *in the space* Q. *Moreover, suppose that* $F(t, X)$ *of the system* (3.1) *is bounded for all* t *when* X *belongs to an arbitrary compact set in* Q *and that* $F(t, X)$ *satisfies the following two conditions with respect to* Ω:

(*a*) $F(t, X)$ *tends to a function* $H(X)$ *for* $X\in\Omega$ *as* $t\to\infty$, *and on any compact set in* Ω *this convergence is uniform.*

(*b*) *Corresponding to each* $\varepsilon>0$ *and each* $Y\in\Omega$, *there exist a* $\delta(\varepsilon, Y)$ *and a* $T(\varepsilon, Y)$ *such that if* $\|X-Y\|<\delta(\varepsilon, Y)$ *and* $t\geqq T(\varepsilon, Y)$, *we have* $\|F(t, X)-F(t, Y)\|<\varepsilon$. *And suppose that* *(c)* $G_2(X)$ *is positive definite with respect to a closed set* Ω *in the space* Q.

Then, every bounded solution of (3.2) *approaches the largest semi-invariant set of the system* $\dot{X}=H(X)$ *contained in* Ω *as* $t\to\infty$.

Proof of Lemma 2. The proof runs analogously as the original proof [21; p. 52~p. 61] using the fact that for any $\lambda>0$

$$\int_t^{t+\lambda} G_2(x(s))\,ds\to 0 \qquad \text{as } t\to\infty$$

whenever $x(t)$ approaches to Ω as $t\to\infty$ e.g.

4. Proof of Theorem 1

In this section it will be assumed that $X=(x, y, z)$ and $\|X\| =\sqrt{x^2+y^2+z^2}$.

We consider, in place of (1.3), the equivalent system

$$(4.1)\quad \begin{cases} \dot{x}=y \\ \dot{y}=z \\ \dot{z}=-a(t)f(x, y)z-b(t)g(x, y)y-c(t)h(x)+p(t, x, y, z). \end{cases}$$

Consider the Liapunov function defined by

$$(4.2)\quad V_0(t, x, y, z)=\mu c(t)H(x)+c(t)h(x)y+b(t)\int_0^y g(x, \eta)\eta\,d\eta$$

$$+\mu a(t)\int_0^y f(x, \eta)\eta\,d\eta+\mu yz+\frac{1}{2}z^2+k$$

where k is a non-negative constant to be determined later in the proof.

Let $\nu=\frac{\mu}{h_1}$, then we have

$$V_0=\frac{1}{2}\mu c(t)\left\{2H(x)+\frac{2}{\mu}h(x)y+\frac{1}{\mu\nu}y^2\right\}$$

$$+\frac{1}{\nu}\int_0^y\{\nu b(t)g(x, \eta)-c(t)\}\eta\,d\eta$$

$$+\mu\int_0^y \{a(t)f(x,\eta)-\mu\}\eta\,d\eta+\frac{1}{2}(z+\mu y)^2+k.$$

Since $h_1 \geqq h'(x)$, we have $2h_1 H(x) \geqq h^2(x)$.

Then it follows

$$|y|\sqrt{2h_1 H(x)} \geqq h(x)y \geqq -|y|\sqrt{2h_1 H(x)}$$

and

$$\left(\sqrt{2H(x)}+\frac{|y|}{\nu\sqrt{h_1}}\right)^2 \geqq \left\{2H(x)+\frac{2}{\nu h_1}h(x)y+\frac{1}{\nu^2 h_1}y^2\right\} \tag{4.3}$$

$$\geqq \left(\sqrt{2H(x)}-\frac{|y|}{\nu\sqrt{h_1}}\right)^2.$$

The left hand side of $(4.3)=2\delta_0 H(x)+\left(\sqrt{2(1-\delta_0)H(x)}+\frac{|y|}{\nu\sqrt{(1-\delta_0)h_1}}\right)^2$

$$-\frac{\delta_0}{(1-\delta_0)\nu^2 h_1}y^2.$$

The right hand side of $(4.3)=2\delta_0 H(x)+\left(\sqrt{2(1-\delta_0)H(x)}-\frac{|y|}{\nu\sqrt{(1-\delta_0)h_1}}\right)^2$

$$-\frac{\delta_0}{(1-\delta_0)\nu^2 h_1}y^2.$$

Hence we have

$$\mu c(t)\delta_0 H(x)+\frac{1}{2}\mu c(t)\left\{\sqrt{2(1-\delta_0)H(x)}+\frac{|y|}{\nu\sqrt{(1-\delta_0)h_1}}\right\}^2$$

$$-\frac{1}{2}c(t)\frac{\delta_0}{(1-\delta_0)\nu}y^2+\frac{1}{\nu}\int_0^y\{\nu b(t)g(x,\eta)-c(t)\}\eta\,d\eta$$

$$+\mu\int_0^y\{a(t)f(x,\eta)-\mu\}\eta\,d\eta+\frac{1}{2}(z+\mu y)^2+k$$

$$\geqq V_0 \geqq \mu c(t)\delta_0 H(x)+\frac{1}{2}\mu c(t)\left\{\sqrt{2(1-\delta_0)H(x)}-\frac{|y|}{\nu\sqrt{(1-\delta_0)h_1}}\right\}^2$$

$$-\frac{1}{2}c(t)\frac{\delta_0}{(1-\delta_0)\nu}y^2+\frac{1}{\nu}\int_0^y\{\nu b(t)g(x,\eta)-c(t)\}\eta\,d\eta$$

$$+\mu\int_0^y\{a(t)f(x,\eta)-\mu\}\eta\,d\eta+\frac{1}{2}(z+\mu y)^2+k.$$

If we take δ_0 as $1-\dfrac{C}{\nu b_0 g_0}>\delta_0>0$, we have

$$V_0\geqq \mu c(t)\delta_0 H(x)+\frac{1}{\nu}\int_0^y\left\{\nu b(t)g(x,\eta)-\frac{c(t)}{1-\delta_0}\right\}\eta\,d\eta$$

$$+\mu\int_0^y\{a(t)f(x,\eta)-\mu\}\eta\,d\eta+\frac{1}{2}(z+\mu y)^2+k$$

$$=\mu c(t)\delta_0 H(x)+\frac{1}{\nu}\int_0^y\left\{\nu b(t)g(x,\eta)-\frac{c(t)}{1-\delta_0}\right\}\eta\,d\eta$$

$$+\mu\int_0^y\{a(t)f(x,\eta)-\mu-\mu\delta_1\}\eta\,d\eta$$

$$+\frac{1}{2}(1+\delta_1)\mu^2\left\{y+\frac{z}{(1+\delta_1)\mu}\right\}^2+\frac{\delta_1}{2(1+\delta_1)}z^2+k.$$

Here we take δ_1 as $\dfrac{a_0 f_0-\mu}{\mu}>\delta_1>0$, then

$$V_0\geqq \mu c(t)\delta_0 H(x)+\frac{1}{\nu}\int_0^y\left\{\nu b(t)g(x,\eta)-\frac{c(t)}{1-\delta_0}\right\}\eta\,d\eta$$

$$+\mu\int_0^y\{a(t)f(x,\eta)-\mu-\mu\delta_1\}\eta\,d\eta+\frac{\delta_1}{2(1+\delta_1)}z^2+k,$$

and we can find a positive number D_1 such that

$$V_0(t,x,y,z)\geqq D_1\{H(x)+y^2+z^2+k\}. \tag{4.4}$$

It is easy to see that there exist two continuous functions $w_1(r)$ and $w_2(r)$ such that

$$w_1(\|X\|)\leqq V_0(t,x,y,z)\leqq w_2(\|X\|) \tag{4.5}$$

for all $X\in R^3$ and $t\in I$ where $w_1(r)\in CIP$ (a family of continuous increasing positive definite functions), $w_1(r)\to\infty$ as $r\to\infty$ and $w_2(r)\in CI$.

Along any solution $(x(t),y(t),z(t))$ of (4.1), we have

$$\dot V_{0(4.1)}=-[\mu b(t)g(x,y)-c(t)h'(x)]y^2-[a(t)f(x,y)-\mu]z^2 \tag{4.6}$$

$$+\frac{1}{2}\mu c'(t)\left\{2H(x)+\frac{2}{\mu}h(x)y+\frac{1}{\mu\nu}y^2\right\}+b(t)y\int_0^y g_x(x,\eta)\eta d\eta$$

$$+\mu a(t)y\int_0^y f_x(x,\eta)\eta d\eta+(\mu y+z)p(t,x,y,z)$$

$$+\int_0^y\left\{\mu a'(t)f(x,\eta)+b'(t)g(x,\eta)-\frac{1}{\nu}c'(t)\right\}\eta d\eta.$$

By the conditions (ii), (iii) and (vi),

$$\dot{V}_{0(4.1)}\leqq-(\mu b_0 g_0-Ch_1)y^2-(a_0 f_0-\mu)z^2+\frac{c'_+(t)}{c(t)}V_0$$

$$+\frac{1}{2}(\mu b_0 g_0-Ch_1)y^2+(1+\mu)(|y|+|z|)|p(t,x,y,z)|$$

$$\leqq-\frac{1}{2}(\mu b_0 g_0-Ch_1)y^2-(a_0 f_0-\mu)z^2+\frac{c'_+(t)}{c(t)}V_0$$

$$+\sqrt{2}(1+\mu)\{p_1(t)+p_2(t)(H(x)+y^2+z^2)^{\rho/2}\}(y^2+z^2)^{1/2}$$

$$+\sqrt{2}\Delta(1+\mu)(y^2+z^2).$$

Note that

$$(H(x)+y^2+z^2)^{\rho/2}\leqq 1+(H(x)+y^2+z^2)^{1/2} \tag{4.7}$$

and if we take $\Delta<\min\left\{\frac{\mu b_0 g_0-Ch_1}{2\sqrt{2}(1+\mu)},\ \frac{a_0 f_0-\mu}{\sqrt{2}(1+\mu)}\right\}$, we can find a positive number D_2 such that

$$\dot{V}_{0(4.1)}\leqq-D_2(y^2+z^2)+\frac{c'_+(t)}{c(t)}V_0 \tag{4.8}$$

$$+\sqrt{2}(1+\mu)\{p_1(t)+p_2(t)\}(y^2+z^2)^{1/2}$$

$$+\sqrt{2}(1+\mu)p_2(t)(H(x)+y^2+z^2).$$

Now we define

$$V(t,x,y,z)=e^{-\int_0^t\gamma(s)ds}\cdot V_0(t,x,y,z) \tag{4.9}$$

where

$$\gamma(s)=\frac{c'_+(s)}{c(s)}+\frac{2\sqrt{2}(1+\mu)}{D_1}\{p_1(s)+p_2(s)\}.$$

Then it is easily verified that there exist two continuous functions $\tilde{w}_1(r)$, $\tilde{w}_2(r)$ satisfying

$$\tilde{w}_1(\|X\|)\leqq V(t,\ x,\ y,\ z)\leqq\tilde{w}_2(\|X\|) \tag{4.10}$$

for all $X\in R^3$ and $t\in I$ where $\tilde{w}_1(r)\in CIP$, $\tilde{w}_1(r)\to\infty$ as $r\to\infty$ and $\tilde{w}_2(r)\in CI$.

Along any solution $(x(t),\ y(t),\ z(t))$ of (4.1) we have

$$\begin{aligned}\dot{V}_{(4.1)}&=e^{-\int_0^t\gamma(s)ds}\cdot[\dot{V}_{0(4.1)}-\gamma(t)V_0]\\&\leqq e^{-\int_0^t\gamma(s)ds}\cdot[-D_2(y^2+z^2)+\sqrt{2}(1+\mu)\{p_1(t)+p_2(t)\}(y^2+z^2)^{1/2}\\&\quad-\sqrt{2}(1+\mu)\{p_1(t)+p_2(t)\}\{H(x)+y^2+z^2+2k\}]\\&\leqq e^{-\int_0^t\gamma(s)ds}\cdot\Big[-D_2(y^2+z^2)\\&\quad-\sqrt{2}(1+\mu)\{p_1(t)+p_2(t)\}\Big\{\Big(\sqrt{y^2+z^2}-\frac{1}{2}\Big)^2-\frac{1}{4}+2k\Big\}\Big].\end{aligned}$$

Setting $k\geqq\frac{1}{8}$, we can find a positive number D_3 such that

$$\dot{V}_{(4.1)}\leqq-D_3(y^2+z^2). \tag{4.11}$$

From the inequalities (4.10), (4.11) and Lemma 1, we see that all the solutions $(x(t),\ y(t),\ z(t))$ of (4.1) are uniform-bounded.

In the system (4.1) we set

$$\begin{aligned}F(t,\ X)&=\begin{pmatrix}x\\ y\\ -a(t)f(x,\ y)z-b(t)g(x,\ y)y-c(t)h(x)\end{pmatrix},\\ G(t,\ X)&=\begin{pmatrix}0\\ 0\\ p(t,\ x,\ y,\ z)\end{pmatrix},\end{aligned} \tag{4.12}$$

then

$$\|G(t, X)\| \leqq p_1(t) + p_2(t)\{H(x) + y^2 + z^2\}^{\rho/2} + \Delta(y^2 + z^2)^{1/2}.$$

Let $G_1(t, X) = p_1(t) + p_2(t)\{H(x) + y^2 + z^2\}^{\rho/2}$ and $G_2(X) = \Delta(y^2 + z^2)^{1/2}$.

It is clear that $F(t, X)$ and $G_1(t, X)$ satisfy the conditions of Lemma 2.

Let $W(X) = D_3(y^2 + z^2)$, then

$$\dot{V}_{(4.1)}(t, x, y, z) \leqq -W(X)$$

and $W(X)$ is positive definite with respect to the closed set $\Omega \equiv \{(x, y, z) \mid x \in R^1, y = 0, z = 0\}$. It follows that on Ω

$$F(t, X) = \begin{pmatrix} 0 \\ 0 \\ -c(t)h(x) \end{pmatrix}.$$

By the condition (i) and (vii), we have $c(t) \to c_\infty$ as $t \to \infty$ where $0 < c_0 \leqq c_\infty \leqq C$. It is also clear that if we take

$$\tilde{H}(X) \equiv \begin{pmatrix} 0 \\ 0 \\ -c_\infty h(x) \end{pmatrix}, \tag{4.13}$$

then the conditions (a) and (b) of Lemma 2 are satisfied.

Moreover $G_2(X)$ is positive definite with respect to the closed set Ω and the condition (c) of Lemma 2 is satisfied.

Since all the solutions of (4.1) are bounded, it follows from Lemma 2 that every solution of (4.1) approaches the largest semi-invariant set of $\dot{X} = \tilde{H}(X)$ contained in Ω as $t \to \infty$.

From (4.13), $\dot{X} = \tilde{H}(X)$ is the system

$$\dot{x} = 0, \quad \dot{y} = 0, \quad \dot{z} = -c_\infty h(x) \tag{4.14}$$

which has the solutions $x = c_1$, $y = c_2$, $z = c_3 - c_\infty h(c_1)(t - t_0)$. To remain in Ω, $c_2 = 0$ and $c_3 - c_\infty h(c_1)(t - t_0) = 0$ for all $t \geqq t_0$ which implies $c_1 = c_3 = 0$.

Therefore the only solution of $\dot{X} = \tilde{H}(X)$ remaining in Ω is $X \equiv 0$, that is, the largest semi-invariant set of $\dot{X} = \tilde{H}(X)$ contained in Ω is the point (0, 0, 0). Then it follows that

$$x(t)\to 0, \quad \dot{x}(t)\to 0, \quad \ddot{x}(t)\to 0 \text{ as } t\to\infty. \qquad \text{Q.E.D.}$$

5. Proof of Theorem 2

Here we consider the system (4.1) and the Liapunov function (4.2), and denote $X=(x, y, z)$ and $\|X\|=\sqrt{x^2+y^2+z^2}$.

By the same arguments as before we obtain the estimates (4.4), (4.5) and (4.6). Then,

$$\begin{aligned}(5.1)\quad \dot{V}_{0(4.1)} \leqq & -(\mu b_0 g_0 - Ch_1)y^2 - (a_0 f_0 - \mu)z^2 + \frac{|c'(t)|}{c(t)} V_0 \\ & + \frac{a'_+(t)}{a(t)} \cdot \mu a(t)\int_0^y f(x, \eta)\eta\, d\eta + \frac{b'_+(t)}{b(t)} \cdot b(t)\int_0^y g(x, \eta)\eta\, d\eta \\ & + \frac{|c'(t)|}{\nu} y^2 + (1+\mu)(|y|+|z|)|p(t, x, y, z)|\end{aligned}$$

where μ is an arbitrarily fixed constant satisfying $\frac{Ch_1}{b_0 g_0} < \mu < a_0 f_0$.

Note that

$$\begin{aligned}\mu a(t)\int_0^y f(x, \eta)\eta\, d\eta &= \mu\int_0^y \{a(t)f(x, \eta) - \mu - \mu\delta_1\}\eta\, d\eta + \frac{1}{2}\mu^2(1+\delta_1)y^2 \\ &\leqq V_0 + \frac{1}{2}\mu^2(1+\delta_1)y^2,\end{aligned}$$

$$\begin{aligned}b(t)\int_0^y g(x, \eta)\eta\, d\eta &= \frac{1}{\nu}\int_0^y \left\{\nu b(t) g(x, \eta) - \frac{c(t)}{1-\delta_0}\right\}\eta\, d\eta + \frac{c(t)}{2\nu(1-\delta_0)} y^2 \\ &\leqq V_0 + \frac{C}{2\nu(1-\delta_0)} y^2\end{aligned}$$

where δ_0, δ_1 are positive constants determined in the Proof of Theorem 1. Then we have

$$\begin{aligned}\dot{V}_{0(4.1)} \leqq & -(\mu b_0 g_0 - Ch_1)y^2 - (a_0 f_0 - \mu)z^2 \\ & + \frac{|c'(t)|}{c(t)} V_0 + \frac{a'_+(t)}{a(t)} V_0 + \frac{a'_+(t)}{2a(t)} \mu^2(1+\delta_1)y^2\end{aligned}$$

$$+\frac{b'_+(t)}{b(t)}V_0+\frac{Cb'_+(t)}{2\nu(1-\delta_0)b(t)}y^2+\frac{|c'(t)|}{\nu}y^2$$

$$+\sqrt{2}(1+\mu)\{p_1(t)+p_2(t)(H(x)+y^2+z^2)^{\rho/2}\}(y^2+z^2)^{1/2}$$

$$+\sqrt{2}\Delta(1+\mu)(y^2+z^2).$$

Using the inequalities (4.4) and (4.7), and taking $\Delta<\min\left\{\frac{\mu b_0 g_0-Ch_1}{\sqrt{2}(1+\mu)},\right.$ $\left.\frac{a_0 f_0-\mu}{\sqrt{2}(1+\mu)}\right\}$, we can find positive numbers D_4 and D_5 such that

$$\dot{V}_{0(4.1)}\leqq -D_4(y^2+z^2)+D_4[a'_+(t)+b'_+(t)+|c'(t)|]V_0 \tag{5.2}$$

$$+\sqrt{2}(1+\mu)\{p_1(t)+p_2(t)\}(y^2+z^2)^{1/2}$$

$$+\sqrt{2}(1+\mu)p_2(t)(H(x)+y^2+z^2).$$

Now we define

$$V(t, x, y, z)=e^{-\int_0^t \gamma(s)ds}\cdot V_0(t, x, y, z) \tag{5.3}$$

where

$$\gamma(s)=D_5(a'_+(s)+b'_+(s)+|c'(s)|)+\frac{2\sqrt{2}(1+\mu)}{D_1}\{p_1(s)+p_2(s)\}.$$

Then there exist two continuous functions $\tilde{w}_1(r)$, $\tilde{w}_2(r)$ such that

$$\tilde{w}_1(\|X\|)\leqq V(t, x, y, z)\leqq \tilde{w}_2((\|X\|) \tag{5.4}$$

for all $X\in R^3$ and $t\in I$ where $\tilde{w}_1(r)\in CIP$, $\tilde{w}_2(r)\to\infty$ as $r\to\infty$ and $\tilde{w}_2(r)$ $\in CI$.

As in the Proof of Theorem 1,

$$\dot{V}_{(4.1)}=e^{-\int_0^t\gamma(s)ds}\cdot[-D_4(y^2+z^2)+\sqrt{2}(1+\mu)\{p_1(t)+p_1(t)\}(y^2+z^2)^{1/2}$$

$$-\sqrt{2}(1+\mu)\{p_1(t)+p_2(t)\}\{H(x)+y^2+z^2+2k\}]$$

$$\leqq e^{-\int_0^t\gamma(s)ds}\cdot\Big[-D_4(y^2+z^2)$$

$$-\sqrt{2}(1+\mu)\{p_1(t)+p_2(t)\}\left(\sqrt{y^2+z^2}-\frac{1}{2}\right)^2-\frac{1}{4}+2k\Big].$$

Setting $k \geqq \frac{1}{8}$, we can find a positive number D_6 such that

$$\dot{V}_{(4.1)} \leqq -D_6(y^2+z^2). \tag{5.5}$$

The remainder of the proof proceeds just as in the Proof of Theorem 1.

Q.E.D.

6. Proof of Theorem 3

In this section it will be assumed that $X=(x, y, z, w)$ and $\|X\|=\sqrt{x^2+y^2+z^2+w^2}$.

The equation (1.4) is equivalent to the system

$$\begin{cases} \dot{x}=y \\ \dot{y}=z \\ \dot{z}=w \\ \dot{w}=-a(t)f(z)w-b(t)\phi(y, z)-c(t)g(y)-d(t)h(x)+p(t, x, y, z, w). \end{cases} \tag{6.1}$$

Our main tool is the function $V_0=V_0(t, x, y, z, w)$ defined by

$$\begin{aligned} 2V_0 = {} & 2\beta\, d(t)\int_0^x h(\xi)\,d\xi+2c(t)\int_0^y g(\eta)\,d\eta+2\alpha b(t)\int_0^z \phi(y, \zeta)\,d\zeta \\ & +2a(t)\int_0^z f(\zeta)\zeta\,d\zeta+2\beta a(t)\,y\int_0^z f(\zeta)\,d\zeta+\{\beta\phi_0 b(t)-\alpha h_0 d(t)\}\,y^2 \\ & -\beta z^2+\alpha w^2+2d(t)h(x)y+2\alpha\, d(t)h(x)z+2\alpha c(t)z\,g(y) \\ & +2\beta\, yw+2zw+k \end{aligned} \tag{6.2}$$

where $\alpha=\frac{1}{a_0 f_0}+\varepsilon$, $\beta=\frac{h_0 D}{c_0 g_0}+\varepsilon$ and ε, k are positive constants to be determined later in the proof. We have

$$2V_0=\frac{a(t)}{f_1(z)}\left\{\frac{w}{a(t)}+f_1(z)z+\beta f_1(z)y\right\}^2+2\varepsilon d(t)\int_0^x h(\xi)\,d\xi \tag{6.3}$$

$$+2d(t)\int_0^x h(\xi)\left[\frac{Dh_0}{c_0 g_0}-\frac{d(t)h'(\xi)}{c(t)g_1(y)}\right]d\xi+c(t)\int_0^y \{g_1(\eta)-g'(\eta)\}\eta\, d\eta$$

$$+2\alpha b(t)\int_0^z \{\phi(y,\zeta)-\phi_0\zeta\}\, d\zeta+\{\beta\phi_0 b(t)-\alpha h_0 d(t)-\beta^2 a(t)f_1(z)\}\, y^2$$

$$+a(t)\int_0^z \{f(\zeta)-f_1(\zeta)\}\zeta\, d\zeta+\{\alpha\phi_0 b(t)-\beta-\alpha^2 c(t)g_1(y)\}\, z^2$$

$$+\left\{\alpha-\frac{1}{a(t)f_1(z)}\right\}w^2+\frac{c(t)}{g_1(y)}\left\{\frac{d(t)}{c(t)}h(x)+yg_1(y)z+\alpha g_1(y)z\right\}^2+k.$$

An elementary computation yields

$$\left[\frac{Dh_0}{c_0 g_0}-\frac{d(t)h'(\xi)}{c(t)g_1(y)}\right]\geqq\frac{Dh_0 c_0 g_0-c_0 g_0 Dh'(\xi)}{c_0 c(t)g_0 g_1(y)}\geqq 0,$$

$$2d(t)\int_0^x h(\xi)\left[\frac{Dh_0}{c_0 g_0}-\frac{d(t)h'(\xi)}{c(t)g_1(y)}\right]d\xi\geqq 0,$$

$$2\alpha b(t)\int_0^z \{\phi(y,\, t)-\phi_0\zeta\}\, d\zeta\geqq 0.$$

From the condition (vii), we have

$$\frac{a_0 b_0 c_0 f_0\phi_0}{C^2}>g'(y),\quad \frac{a_0 b_0 c_0\phi_0 g_0}{A^2 Dh_0}>f(z), \tag{6.4}$$

then

$$\{\beta\phi_0 b(t)-\alpha h_0 d(t)-\beta^2 a(t)f_1(z)\}$$

$$=\beta\{\phi_0 b(t)-\alpha c(t)g_1(y)-\beta a(t)f_1(z)\}+\alpha\{\beta c(t)g_1(y)-h_0 d(t)\}$$

$$=\beta\left\{\phi_0 b(t)-\frac{c(t)}{a_0 f_0}g'(\tilde{y})-\frac{Dh_0}{c_0 g_0}a(t)f(\tilde{z})\right\}-\varepsilon\beta\{c(t)g'(\tilde{y})+a(t)f(\tilde{z})\}$$

$$+\alpha\{\beta c(t)g_1(y)-h_0 d(t)\}$$

$$\geqq\frac{\beta\delta_0}{a_0 c_0 f_0 g_0}-\varepsilon\beta\left\{\frac{a_0 b_0 c_0 f_0\phi_0}{C}+\frac{a_0 b_0 c_0\phi_0 g_0}{ADh_0}\right\}+\alpha\left\{\left(\frac{h_0 D}{c_0 g_0}+\varepsilon\right)c_0 g_0-h_0 D\right\}$$

$$=\frac{1}{ACa_0 f_0 c_0^2 g_0^2}\{ACDh_0\delta_0-\varepsilon a_0^2 b_0 c_0^2 f_0 g_0\phi_0(Af_0 Dh_0+Cg_0)\}$$

$$+\frac{\varepsilon}{ACDh_0a_0c_0f_0g_0}\{ACDh_0\delta_0-\varepsilon a_0^2b_0c_0^2f_0g_0\phi_0(Af_0Dh_0+Cg_0)\}+\varepsilon\alpha c_0g_0.$$

If we take

$$\varepsilon<\frac{ACDh_0\delta_0}{a_0^2b_0c_0^2f_0g_0\phi_0(Af_0Dh_0+Cg_0)}, \tag{6.5}$$

we have

$$\{\beta\phi_0b(t)-\alpha h_0d(t)-\beta^2a(t)f_1(z)\}>\frac{1}{ACa_0f_0c_0^2g_0^2}\{ACDh_0\delta_0$$

$$-\varepsilon a_0^2b_0c_0^2f_0g_0\phi_0(ADf_0h_0+Cg_0)\}.$$

Also using (6.4) we have

$$\{\alpha\phi_0b(t)-\beta-\alpha^2c(t)g_1(y)\}$$

$$\geqq\frac{1}{Ca_0^2Ac_0Df_0^2g_0h_0}\{ACDh_0\delta_0-\varepsilon a_0^2b_0c_0^2f_0g_0\phi_0(Af_0Dh_0+Cg_0)\}$$

$$+\frac{\varepsilon}{a_0ACc_0Df_0g_0h_0}\{ACDh_0\delta_0-\varepsilon a_0^2b_0c_0^2f_0g_0\phi_0(ADf_0h_0+Cg_0)\}+\varepsilon\beta a_0f_0.$$

By (6.5), we have

$$\{\alpha\phi_0b(t)-\beta-\alpha^2c(t)g_1(y)\}$$

$$>\frac{1}{ACa_0^2f_0^2c_0g_0Dh_0}\{ACDh_0\delta_0-\varepsilon a_0^2b_0c_0^2f_0g_0\phi_0(Af_0Dh_0+Cg_0)\}.$$

Further,

$$c(t)\int_0^y\{g_1(\eta)-g'(\eta)\}\eta\,d\eta\geqq-\frac{C\delta}{2}y^2,$$

$$a(t)\int_0^z\{f(\zeta)-f_1(\zeta)\}\zeta\,d\zeta\geqq-\frac{Cc_0g_0\delta}{a_0f_0Dh_0}\cdot\frac{z^2}{2},$$

$$\left\{\alpha-\frac{1}{a(t)f(z)}\right\}w^2\geqq\varepsilon w^2.$$

Then we obtain

$$2V_0 \geqq 2\varepsilon d_0 \int_0^x h(\xi)\, d\xi$$

$$+ \frac{y^2}{2ACa_0 f_0 c_0^2 g_0^2} \{AC(2Dh_0\delta_0 - Ca_0 f_0 c_0^2 g_0^2 \delta)$$

$$-2\varepsilon a_0^2 b_0 c_0^2 f_0 g_0 \phi_0 (Af_0 Dh_0 + Cg_0)\}$$

$$+ \frac{z^2}{2ACa_0^2 f_0^2 c_0 g_0 Dh_0} \{AC(2Dh_0\delta_0 - Ca_0 f_0 c_0^2 g_0^2 \delta)$$

$$-2\varepsilon a_0^2 b_0 c_0^2 f_0 g_0 \phi_0 (Af_0 Dh_0 + Cg_0)\} + \varepsilon w^2 + k.$$

If we take

$$\varepsilon < \frac{AC(2Dh_0\delta_0 - Ca_0 f_0 c_0^2 g_0^2 \delta)}{2a_0^2 b_0 c_0^2 f_0 g_0 \phi_0 (Af_0 Dh_0 + Cg_0)}, \tag{6.6}$$

then there exists a positive number D_1 such that

$$V_0 \geqq D_1\{H(x) + y^2 + z^2 + w^2 + k\}. \tag{6.7}$$

From (6.4) it follows the boundedness of the functions $g_1(y)$ and $f_1(z)$, and we can see easily that there exists a positive number D_2 satisfying

$$V_0 \leqq D_2\{H(x) + y^2 + z^2 + w^2 + k\}. \tag{6.8}$$

Therefore we have

$$D_1(H(x) + y^2 + z^2 + w^2) \leqq V_0 \leqq D_2(H(x) + y^2 + z^2 + w^2 + k). \tag{6.9}$$

Next along any solution $(x(t), y(t), z(t), w(t))$ of (6.1),

$$2\dot{V}_{0(6.1)} = -2\varepsilon c(t) g_1(y) y^2 - 2\left[\frac{h_0 D}{c_0 g_0} c(t) g_1(y) - d(t) h'(x)\right] y^2$$

$$-2[\alpha a(t) f(z) - 1] w^2 - 2[\phi_0 b(t) - \alpha c(t) g'(y) - \beta a(t) f_1(z)] z^2$$

$$-2b(t)\left[\frac{\phi(y, z)}{z} - \phi_0\right]\left(z + \frac{\beta}{2} y\right)^2 + \frac{\beta^2}{2} b(t)\left[\frac{\phi(y, z)}{z} - \phi_0\right] y^2$$

$$+2\alpha b(t) z \int_0^z \phi_y(y, \zeta)\, d\zeta - 2\alpha d(t)[h_0 - h'(x)] yz$$

$$+2(\beta y+z+\alpha w)p(t, x, y, z, w)+2\frac{\partial V_0}{\partial t}$$

$$\leqq -2\varepsilon c_0 g_0 y^2 - 2[\alpha a(t) f(z) - 1]w^2$$

$$-2[\phi_0 b(t) - \alpha c(t) g'(y) - \beta a(t) f_1(z)] z^2 + \frac{\alpha^2}{2} d(t)[h_0 - h'(x)] z^2$$

$$+\frac{\beta^2}{2} b(t)\left[\frac{\phi(y, z)}{z} - \phi_0\right] y^2 + 2(\beta y+z+\alpha w)p(t, x, y, z, w)+2\frac{\partial V_0}{\partial t}.$$

If we take

$$\varepsilon_0 < \varepsilon < \min\left\{\frac{AC(2Dh_0\delta_0 - Ca_0 f_0 c_0^2 g_0^2 \delta)}{2a_0^2 b_0 c_0^2 f_0 g_0 \phi_0 (Af_0 Dh_0 + Cg_0)}, \frac{1}{a_0 f_0}, \frac{Dh_0}{c_0 g_0},\right.$$

$$\left.\frac{ACDh_0\delta_0}{2a_0^2 b_0 c_0^2 f_0 g_0 \phi_0 (Af_0 Dh_0 + Cg_0)}\right\},$$

we can find a positive number D_3 such that

$$\dot{V}_{0(6.1)} \leqq -2D_3(y^2+z^2+w^2) + (\beta y+z+\alpha w)p(t, x, y, z, w) + \frac{\partial V_0}{\partial t}.$$

Let $D_4 = \max(\alpha, \beta, 1)$, then

$$\dot{V}_{0(6.1)} \leqq -2D_3(y^2+z^2+w^2) + \sqrt{3} D_4 (y^2+z^2+w^2)^{1/2} |p(t, x, y, z, w)| + \frac{\partial V_0}{\partial t}$$

$$\leqq -2D_3(y^2+z^2+w^2)$$

$$+\sqrt{3} D_4 (y^2+z^2+w^2)^{1/2} [p_1(t) + p_2(t)\{H(x) + y^2+z^2+w^2\}^{\rho/2}$$

$$+\Delta(y^2+z^2+w^2)^{1/2}] + \frac{\partial V_0}{\partial t}.$$

Taking $\Delta \leqq \frac{D_3}{\sqrt{3} D_4}$, we have

$$\dot{V}_{0(6.1)} \leqq -D_3(y^2+z^2+w^2) \tag{6.10}$$

$$+\sqrt{3} D_4 (y^2+z^2+w^2)^{1/2} [p_1(t) + p_2(t)\{H(x) + y^2+z^2+w^2\}^{\rho/2}]$$

$$+\frac{\partial V_0}{\partial t}.$$

From the assumptions in Theorem 3 and (6.4) we have a positive number D_5 satisfying

$$(6.11)\qquad \frac{\partial V_0}{\partial t} \leqq D_5\{|a'(t)|+b'_+(t)+|c'(t)|+|d'(t)|\}\cdot V_0.$$

Note that

$$(6.12)\qquad (H(x)+y^2+z^2+w^2)^{\rho/2} \leqq 1+(H(x)+y^2+z^2+w^2)^{1/2}.$$

(6.10), (6.11) and (6.12) show that

$$\begin{aligned}(6.13)\qquad \dot{V}_{0(6.1)} \leqq & -D_3(y^2+z^2+w^2)\\ & +D_5\{|a'(t)|+b'_+(t)+|c'(t)|+|d'(t)|\}\cdot V_0\\ & +\sqrt{3}D_4\{p_1(t)+p_2(t)\}(y^2+z^2+w^2)^{1/2}\\ & +\sqrt{3}D_4 p_2(t)(H(x)+y^2+z^2+w^2).\end{aligned}$$

Now we define

$$(6.14)\qquad V(t, x, y, z, w)=e^{-\int_0^t \gamma(s)ds}\cdot V_0(t, x, y, z, w)$$

where

$$\gamma(s)=D_5(|a'(s)|+b'_+(s)+|c'(s)|+|d'(s)|)+\frac{2\sqrt{3}D_4}{D_1}\{p_1(s)+p_2(s)\}.$$

Then it is easy to see that there exist two continuous functions $w_1(r)$, $w_2(r)$ satisfying

$$(6.15)\qquad w_1(\|X\|) \leqq V(t, x, y, z, w) \leqq w_2(\|X\|)$$

for all $X\in R^4$ and $t\in I$ where $w_1(r)\in CIP$, $w_1(r)\to\infty$ as $r\to\infty$ and $w_2(r)\in CI$.

Along any solution $(x(t), y(t), z(t), w(t))$ of (6.1) we have

$$\begin{aligned}\dot{V}_{(6.1)} &= e^{-\int_0^t \gamma(s)ds}\cdot[\dot{V}_{0(6.1)}-\gamma(t)V_0]\\ &\leqq e^{-\int_0^t \gamma(s)ds}\cdot[-D_3(y^2+z^2+w^2)\end{aligned}$$

$$+\sqrt{3}D_4\{p_1(t)+p_2(t)\}\{(y^2+z^2+w^2)^{1/2}$$

$$-\sqrt{3}D_4\{p_1(t)+p_2(t)\}\{H(x)+y^2+z^2+w^2+2k\}]$$

$$\leqq e^{-\int_0^t \gamma(s)ds}\cdot\Big[-D_3(y^2+z^2+w^2)$$

$$-\sqrt{3}D_4\{p_1(t)+p_2(t)\}\Big\{\Big(\sqrt{y^2+z^2+w^2}-\frac{1}{2}\Big)^2-\frac{1}{4}+2k\Big\}\Big].$$

Setting $k\geqq\frac{1}{8}$, we can find a positive number D_6 such that

$$\dot{V}_{(6.1)}\leqq -D_6(y^2+z^2+w^2). \tag{6.16}$$

From the inequalities (6.15) and (6.16), we obtain the uniform boundedness of all the solutions $(x(t), y(t), z(t), w(t))$ of (6.1).

In the system (6.1) we set

$$F(t, X)=\begin{pmatrix} x \\ y \\ z \\ -a(t)f(z)w-b(t)\phi(y, z)-c(t)g(y)-d(t)h(x) \end{pmatrix}, \tag{6.17}$$

$$G(t, X)=\begin{pmatrix} 0 \\ 0 \\ 0 \\ p(t, x, y, z, w) \end{pmatrix},$$

then

$$||G(t, X)||\leqq p_1(t)+p_2(t)\{H(x)+y^2+z^2+w^2\}^{\rho/2}+\Delta(y^2+z^2+w^2)^{1/2}.$$

Let

$$G_1(t, X)=p_1(t)+p_2(t)\{H(x)+y^2+z^2+w^2\}^{\rho/2}$$

and

$$G_2(X)=\Delta(y^2+z^2+w^2)^{1/2}.$$

It is clear that $F(t, X)$ and $G_1(t,X)$ satisfy the conditions of Lemma 2.

Let $W(X)=D_6(y^2+z^2+w^2)$, then

$$\dot{V}_{(6.1)}(t, x, y, z, w) \leqq -W(X)$$

and $W(X)$ is positive definite with respect to the closed set $\Omega \equiv \{(x, y, z, w) \mid x \in R^1, y=0, z=0, w=0\}$.

It follows that on Ω

$$F(t, X)=\begin{pmatrix} 0 \\ 0 \\ 0 \\ -d(t)h(x) \end{pmatrix}$$

By the conditions (i) and (x), we have $d(t) \to d_\infty$ as $t \to \infty$ where $0 < d_0 \leqq d_\infty \leqq D$. It is also clear that if we take

$$\tilde{H}(X) \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ -d_\infty h(x) \end{pmatrix}, \tag{6.18}$$

then the conditions (a) and (b) of Lemma 2 are satisfied.

Moreover $G_2(X)$ is positive definite with respect to the closed set Ω and the condition (c) of Lemma 2 is satisfied.

The remainder of the proof is analogous to that of Theorem 1.

Q.E.D.

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