Publ. RIMS, Kyoto Univ.
9 (1974), 675-682

On the Growth of Solutions of Nonlinear Diffusion Equation $u_t = \triangle u + F(u)$

By

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1. Introduction

In connection with the model, proposed by R. Fisher [2], of the natural selection in biology, A. Kolmogoroff-I. Petrowsky-N. Piscounoff [6] discussed the asymptotic behavior for $t \to \infty$ of solution of nonlinear diffusion equation

$$(1) \qquad \frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2} + (1-p)^2 p, \qquad (-\infty < x < \infty),$$

$$p(x, 0) = f(x).$$

Here and in what follows $f(x)$ is assumed to be a continuous function with $0 \leqq f \leqq 1$. Recently, N. Ikeda-Y. Kametaka [7] considered the nonlinear diffusion equation of the form

$$(2) \qquad u_t = \triangle u + G(u) \qquad (x \in R^N,\ t > 0)$$

where $G(s)$ is a C^∞-function on [0, 1] with $G(0) = G(1) = 0$ such that $G''(s) < 0$ $(0 \leqq s \leqq 1)$, and showed that the zero solution is unstable, while the constant function 1 is stable in the sense that any solution of (2) converges to 1 as $t \to \infty$ uniformly on any compact set if the nonnegative initial data is not identically zero however "small" it may be. There are many interesting equations of the form (2) such as $u_t = \triangle u + (1-u)u$ (Logistic equation with dissipative effect in the population growth); $u_t =$

Communicated by S. Matsuura, July 4, 1973.
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$\triangle u+(1-u^2)u$ (a special case of E. Abrahams-T. Tsuneto equation in the superconductivity [1]). However the proof due to Ikeda-Kametaka, based on a reduction to the case of bounded domain, is not applicable at least directly to Eq. (1) discussed by A. Kolmogoroff-I. Petrowsky-N. Piscounoff, since the condition $G''(s)<0$ does not hold. The purpose of the present note is to study *quantitatively* the asymptotic behavior for $t\to\infty$ of solutions of not only Eq. (2), but also Eq. (1). More precisely, Let us consider the equation:

$$u_t=\triangle u+a(x,t,u) \qquad (x\in R^N,\ t>0) \tag{3}$$

($u_t=\partial u/\partial t$: R^N; N-dimensional Euclidean space: $\triangle=\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_N^2}$) with the initial data $u(x,0)=f(x)$, where $a(x,t,s)$ is assumed to be a continuous function of x, t, $s(x\in R^N,\ t>0,\ 0\leqq s\leqq 1)$ satisfying the following condition.

Condition on $a(x,t,s)$; there exist positive constants c_0, c_0', real m and m' with $m\geqq 1$, $m'\geqq 1$, such that

$$0<c_0(1-s)^m s\leqq a(x,t,s)\leqq c_0'(1-s)^{m'}s$$

for $x\in R^N$, $t\geqq 0$ and $0<s<1$.

Our result is the following.

Theorem. *Under the above condition on a, let $f(x)$ be a continuous function on R^N such that $0\leqq f(x)\leqq 1$; $f(x)\not\equiv 0$; $f(x)\not\equiv 1$ $(x\in R^N)$. Let u be a solution of Eq.* (3) *with the initial value $f(x)$. Then we have:*

(I) *(the lower bound for u); For any compact set K in R^N there exist constants M and $t_0>0$ such that $u(x,t)$ is estimated from below for $t\to\infty$:*

$$1-Mt^{-1/m}\leqq u(x,t)\leqq 1$$

for all $x\in K$ and $t>t_0$.

(II) *the upper bound for u); If for some $R>0$*

$$\sup_x f(x)<1 \qquad (|x|>R)$$

then $u(x,t)$ is estimated from above for $t\to\infty$:

$$0\leqq u(x,\ t)\leqq 1-M't^{-1/(m'-1)} \qquad (m'\neq 1)$$

$$0\leqq u(x,\ t)\leqq 1-M'e^{-kt} \qquad (m'=1)$$

for all $x\in R^N$ *and* $t>t_0$ *where* t_0, k, M' *are positive constants depending on* f.

Applying the above theorem to Eq. (1) and Eq. (2), we see that any solution u_1, u_2 of Eq. (1) (Eq. (2)) with a nonnegative initial value $f(x)$ $(0\leqq f\leqq 1,\ f\not\equiv 0, f\not\equiv 1)$ such that $f(x)$ has compact support, converges to 1 in the following manner: For any compact set K,

$$1-M_1t^{-1/2}\leqq u_1(x,\ t)\leqq 1-M_1't^{-1}$$

$$(1-M_2t^{-1}\leqq u_2(x,\ t)\leqq 1-M_2'e^{-kt})$$

for $x\in K$ and $t>t_0$ respectively where M_1, M_1', M_2, M_2', k, t_0 are positive constants depending on f and K.

The proof of the theorem is based on the well-known comparison theorem for parabolic equations (Westphal-Prodi Theorem; e.g., see S. Kaplan [4], J. Szarski [10], Protter-Weinberger [9]). If the initial function $f(x)$ is uniformly positive, i.e.,

$$\inf_x f(x)\ \ (\equiv\gamma)>0 \qquad (x\in R^N)$$

Then the part (I) of our theorem is a direct consequence of the comparison theorem (For probabilistic approach to this case, see M. Freidlin [3]): it suffices to notice that $u(x,\ t)$ is estimated from below by the solution $v(t)$ of the ordinary differential equation $v_t=c_0(1-v)^m v$ $(v_t=dv/dt)$, $v(0)=\gamma$. In the interesting case that $f(x)$ has compact support, we need more sophisticated treatments.

2. Proof of Theorem

We first show that $u(x,\ t)$ has the following estimates from below and above:

(4) $$v_*(x,\ t)\leqq u(x,\ t)$$

(5) $$v^*(x,\ t)\geqq u(x,\ t) \qquad (x\in R^N,\ t>0)$$

where

$$v_*(x,\ t)=\left(\frac{1}{4\pi t}\right)^{\frac{N}{2}}\int_{R^N} e^{-(x-y)^2/4t}f(y)\,dy$$

and

$$v^*(x,\ t)=1-e^{-c_0't}\left(\frac{1}{4\pi t}\right)^{\frac{N}{2}}\int_{R^N} e^{-(x-y)^2/4t}(1-f(y))\,dy.$$

Since $0\leqq a(x,\ t,\ u)\leqq c_0'(1-u)$ by the assumption, u satisfies

$$u_t\leqq \triangle u+c_0'(1-u)$$

and

$$u_t\geqq \triangle u$$

with the initial value $u(x,\ 0)=f(x)$, while the v_* and v^* satisfy

$$\partial v_*/\partial t=\triangle v_*;\ \ v_*(x,\ 0)=f(x)$$

and

$$\partial v^*/\partial t=\triangle v^*+c_0'(1-v^*);\ \ v^*(x,\ 0)=f(x).$$

Hence, applying the comparison theorem, we have the desired bound (4) and (5).

Using these estimates, we shall construct comparison functions to derive more refined bounds for u.

After choosing s_0 so that $s_0>2^{m+1}NC_0^{-1}$, we put

$$M=1-\frac{1}{2}\left(\frac{1}{4\pi s_0}\right)^{\frac{N}{2}}\inf_{x_0}\int_{R^N} e^{-(x_0-y)^2/2s_0}f(y)\,dy \qquad (x_0\in K) \tag{6}$$

and

$$t_0=2^{m+1}C_0^{-1}Mm^{-1}(1-M)^{-1}+2s_0; \tag{7}$$

note that $0<M<1$ since $f(y)>0$ for some y. Then the function

$$v(x,\ t)=\left(1-M\left(\frac{t_0}{t+t_0}\right)^{1/m}\right)\exp\left(-(x-x_0)^2/(4t+2s_0)\right)$$

has the following properties:

(i) $\qquad v(x,\ 0)\leqq u(x,\ s_0)$

(ii) $$v_t \leqq \triangle v + a(x, t+s_0, v), \qquad (x \in R^N,\ t>0)$$

Indeed, by (6) we have

$$v(x, 0) = (1-M)\exp(-(x-x_0)^2/(2s_0))$$
$$\leqq \left(\frac{1}{4\pi s_0}\right)^{\frac{N}{2}} \int_{R^N} \exp\left(-\frac{(x-x_0)^2}{2s_0} - \frac{(x_0-y)^2}{2s_0}\right) dy$$
$$\leqq \left(\frac{1}{4\pi s_0}\right)^{\frac{N}{2}} \int_{R^N} \exp\left(-\frac{(x-y)^2}{4s_0}\right) f(y)\, dy$$
$$\leqq u(x, s_0), \qquad [\text{by } (4)]$$

showing (i). To see (ii), we set

$$h(t) = 1 - M\left(\frac{t_0}{t+t_0}\right)^{1/m}$$

and

$$e(x, t) = \exp\left(-\frac{(x-x_0)^2}{4t+2s_0}\right).$$

Then, after noting that $v = h(t)e(x, t)$, a straightforward calculations show

$$v_t - \triangle v - c_0(1-h)^m v$$
$$= \left\{\frac{M}{m}\left(\frac{t_0}{t+t_0}\right)^{1/m} \frac{1}{t+t_0} + h(t)\frac{|x-x_0|^2}{(2t+s_0)^2}\right\} e(x, t)$$
$$- \left\{-\frac{N}{2t+s_0} + \frac{|x-x_0|^2}{(2t+s_0)^2}\right\} h(t)e(x, t)$$
$$- c_0 M^m \frac{t_0}{t+t_0} h(t)e(x, t)$$
$$= \left\{\frac{M}{m}\left(\frac{t_0}{t+t_0}\right)^{1/m} \frac{1}{h(t)} + N\frac{t+t_0}{2t+s_0} - c_0 M^m t_0\right\} \frac{h(t)e(x, t)}{t+t_0}.$$

We shall show that $\{\cdots\}$ is non-positive. Since $h(t) \geqq 1-M$ and $M \geqq 1/2$ by $0 \leqq f \leqq 1$, we have

$$\{\cdots\} \leqq Mm^{-1}(1-M)^{-1} + Nt_0/s_0 - c_0 2^{-m} t_0$$

$$\leqq -2^{-m-1}c_0(t_0-2^{m+1}Mm^{-1}(1-M)^{-1}c_0^{-1})$$

$$-t_0 s_0^{-1}2^{-m-1}c_0(s_0-2^{m+1}Nc_0^{-1})$$

$$\leqq 0, \qquad \text{[by (7)]}$$

showing that $\{\cdots\}\leqq 0$. Hence,

$$v_t \leqq \triangle v + c_0(1-h(t))^m v\,.$$

Since by the assumption

$$c_0(1-h)^m v \leqq c_0(1-h(t)e(x, t))^m v$$

$$= c_0(1-v)^m v$$

$$\leqq a(x, t+s_0, v)\,,$$

we have (ii).

Since $u(x, t+s_0)$ is a solution of $u_t=\triangle u+a(x, t+s_0, u)$, we can apply the comparison theorem to $u(x, t+s_0)$ and $v(x, t)$, and obtain the inequality $u(x, t+s_0)\geqq v(x, t)$ $(x\in R^N, t\geqq 0)$. In particular, at $x=x_0$ we have

$$u(x_0, t+s_0)\geqq v(x_0, t)$$

$$=1-M\left(\frac{t_0}{t+t_0}\right)^{1/m}$$

for $x_0\in K$ and $t\geqq 0$, from which the estimate in the part (1) of Theorem follows. Next we turn to the proof of the part (11). By the assumption

$$u_t \leqq \triangle u + c_0' u.$$

Since the function $v(x, t)$ defined by

$$v(x, t)=e^{c_0' t}\left(\frac{1}{4\pi t}\right)^{\frac{N}{2}}\int_{R^N} e^{-(x-y)^2/4t} f(y)\,dy \tag{8}$$

is a solution of $v_t=\triangle v+c_0' v$ with the initial data $f(x)$, it follows from the comparison theorem that

$$u(x, t)\leqq v(x, t)\,.$$

If we set $\gamma_1 = \sup_x f(x)$ ($|x| \geqq R$), and if we take t_1, so small that $\gamma_1 e^{c_0' t_1} \leqq \gamma_1 + \frac{1}{3}(1-\gamma_1)$ (note $\gamma_1 < 1$), then we have

$$(9) \qquad e^{c_0' t_1}\left(\frac{1}{4\pi t_1}\right)^{\frac{N}{2}} \int_{|y|>R} \exp\left(-\frac{(x-y)^2}{4t_1}\right) f(y)\, dy < \gamma_1 e^{c_0' t_1} < \gamma_1 + \frac{1}{3}(1-\gamma_1)$$

for all $x \in R^N$. On the other hand, then exists an R_0 such that

$$(10) \qquad e^{c_0' t}\left(\frac{1}{4\pi t_1}\right)^{\frac{N}{2}} \int_{|y|<R} \exp\left(-\frac{(x-y)^2}{4t_1}\right) f(y)\, dy < \frac{1}{3}(1-\gamma_1)$$

for $|x| > R_0$. Combining (10) with (9), we have $v(x, t) < \gamma_2$ for $|x| > R_0$, where $\gamma_2 = \gamma_1 + 2(1-\gamma_1)/3$ (<1). Hence,

$$u(x, t_1) < \gamma_2 \qquad (|x| > R_0)$$

Since, by (5) $u(x, t_1) < 1$ for all x in $|x| \leqq R_0$ (note $f \not\equiv 1$), and since $\gamma_2 < 1$, there exists a constant $\gamma_3 < 1$ such that

$$u(x, t_1) < \gamma_3$$

for all $x \in R^N$. Using the solution $v(t)$ of the ordinary differential equation $v_t = c_0'(1-v)^m$, $v(0) = \gamma_3$ which has the following properties;

$$\text{(i}'\text{)} \quad v(0) > u(x, t_1) \qquad\qquad \text{(ii)} \quad v_t \geqq \Delta v + a(x, t+t_1, v)$$

we can see that $u(x, t+t_1) < v(t)$ ($x \in R^N$, $t > 0$). Since the $v(t)$ is explicitly given by

$$v(t) = 1 - ((m'-1)t/c_0' + (1-\gamma_3)^{m'-1})^{-1/(m'-1)} \qquad (m' \neq 1)$$

$$v(t) = 1 - (1-\gamma_3)e^{-c_0' t} \qquad (m' = 1)$$

the desired upper bound for u easily follows. Theorem is thus proved.

The main result of the present paper has been reported at the symposium on the Navier-Stokes equations and the related topics (November 1971), held at the Research Institute of Mathematical Sciences (Kôkyûroku of the

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