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On Cohomology Theories of Infinite CW-complexes, IV

By

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Let E and X be CW-spectra and $\{X^\lambda\}$ the set of all finite subspectra of X. The cohomology group $E^*(X)$ is topologized by taking the subgroups $F^\lambda E^*(X)=\mathrm{Ker}\{E^*(X)\to E^*(X^\lambda)\}$ as neighborhoods of zero in $E^*(X)$. In general $E^*(X)$ is not Hausdorff. In the previous papers [II] and [III] with the same title we studied conditions on CW-spectra E and X under which $E^*(X)$ is Hausdorff. We are going to continue the investigation.

There arises a natural *question whether* $E^*(X_\wedge Y)$ *is Hausdorff when* $E^*(X)$ *and* $E^*(Y)$ *are Hausdorff*. In the present paper we treat of this question and give the following answer for well-known cohomology theories $E^*=$(reduced) ordinary cohomology H^*, complex K-cohomology K^* and complex cobordism MU^*.

Theorem. *Let* E^* *denote* H^*, K^* *or* MU^* *and* X *and* Y *be* CW-*spectra. Assume that* Y *has finite skeletons. If both* $E^*(X)$ *and* $E^*(Y)$ *are Hausdorff, then* $E^*(X_\wedge Y)$ *is so, too.*

First we attack our question in a few special cases when $E^*(X)$ is Hausdorff. If $H_*(X)\otimes R$ is a free R-module, then we get a desirable answer under certain restrictions on $\pi_*(E)$ where R is a subring of the rational numbers Q (Theorem 1). Applying it to $E^*=MU^*$ we perform the proof of Theorem in the case $E^*=MU^*$ (Theorem 3).

Next we restrict ourselves to the cohomology theory $E^*=H^*$ or K^*. The universal coefficient theorem gives a necessary and sufficient condition on $E_*(X)$ under which $ER^*(X)$ is Hausdorff (Theorem 4). With the

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aid of this new criterion we discuss conditions that $ER^*(X_\wedge Y)$ is Hausdorff (Theorem 5). As a corollary we obtain Theorem in the cases $E^*=H^*$ and K^*.

We adopt all notations and notions used in [III].

1. Ring-spectrum E with Coherent Ring $\pi_*(E)$

1.1. As typical examples of $E^*(X)$ which are Hausdorff we have the following:

I) *X is a finite CW-spectrum,*

II) *$\pi_n(E)$ is a finite abelian group for each degree n, or $\pi_*(E)$ is a Q-module* [III, Propositions 3, 4],

III) *$\pi_n(E)$ is a finitely generated R-module for each degree n and* $\pi_*(X)\otimes Q=0$ [III, Theorem 2],

IV) *$\pi_n(E)$ is a finitely generated free R-module for each degree n and X is a connective CW-spectrum such that $H_*(X)\otimes R$ is a free R-module* [III, Theorem 5],

where R is a subring of the rational numbers Q.

First of all, we are going to make an attack on our question in the above special cases I)–IV). In the case II) we have nothing to do because $E^*(X)$ is always Hausdorff. On the other hand, it is evident that $\pi_*(X_\wedge Y)\otimes Q=0$ whenever $\pi_*(X)\otimes Q=0$. So we see

(1.1) *$E^*(X_\wedge Y)$ is Hausdorff for any CW-spectrum Y in the case* III).

1.2. Chase [24] proved that over a coherent ring Λ direct products of flat Λ-modules are flat. Using this we show

Lemma 1. *Let Λ be a coherent ring and $\{A_n\}$ be an inverse sequence of flat Λ-modules with $\varprojlim_n{}^1 A_n=0$. Then $\varprojlim_n A_n$ is a flat Λ-module.*

Proof. There is an exact sequence

$$0 \longrightarrow \varprojlim A_n \longrightarrow \prod A_n \longrightarrow \prod A_n \longrightarrow 0$$

of Λ-modules as $\varprojlim^1 A_n=0$. $\prod A_n$ is flat by the result of Chase [24], so we see immediately that $\varprojlim A_n$ is flat.

Remark. If a coherent ring Λ satisfies the property that every finitely presented Λ-module has finite projective dimension as a Λ-module (for example $\Lambda=\pi_*(MU)$), we can slightly generalize Lemma 1. Thus $\varprojlim A_\alpha$ is a flat Λ-module for an inverse system $\{A_\alpha\}$ of flat Λ-modules with $\varprojlim^p A_\alpha=0$ for all $p\geqq 1$.

Let E be a ring-spectrum with $\pi_*(E)$ a coherent ring and X a CW-spectrum. If G is a Q-module, then there is an isomorphism

$$(EG)^*(X)\cong \prod_p H^p(X;\pi_*(E)\otimes G)\cong \prod_p \operatorname{Hom}(H_p(X),\pi_*(E)\otimes G)$$

by Dold's theorem [19] (or see [III, (1.3)]. $\operatorname{Hom}(H_p(X),\pi_*(E)\otimes G)$ is a direct product of copies of $\pi_*(E)\otimes G$, and hence it is a flat $\pi_*(E)$-module. Thus we get

(1.2) *$(EG)^*(X)$ is a flat $\pi_*(E)$-module.*

(1.2) implies that $(EG)^*(X)\underset{\pi_*(E)}{\otimes}E^*(\ )$ forms a cohomology theory on the category of finite CW-spectra. And the multiplication

$$(EG)^*(X)\underset{\pi_*(E)}{\otimes}E^*(Y)\longrightarrow (EG)^*(X\wedge Y) \tag{1.3}$$

is an isomorphism for any finite CW-spectrum Y.

For the case I) we obtain the following answer under certain restrictions on $\pi_*(E)$.

Proposition 2. *Let R be a subring of Q and E be a ring-spectrum such that $\pi_*(E)$ is a coherent ring and it is of finite type as an R-module. If $E^*(X)$ is Hausdorff, then so is $E^*(X\wedge Y)$ for any finite CW-spectrum Y.*

Proof. Consider the following commutative square

$$\begin{array}{ccc}
(E\hat{Z})^*(X)\underset{\pi_*(E)}{\otimes}E^*(Y) & \xrightarrow{\kappa_1\otimes 1} & (E\hat{Z}/Z)^*(X)\underset{\pi_*(E)}{\otimes}E^*(Y)\\
\downarrow{\scriptstyle\hat{\mu}} & & \downarrow{\scriptstyle\bar{\mu}}\\
(E\hat{Z})^*(X\wedge Y) & \xrightarrow{\kappa_2} & (E\hat{Z}/Z)^*(X\wedge Y).
\end{array}$$

The top horizontal map $\kappa_1\otimes 1$ is an epimorphism by means of [III, Theorem 1], and the right multiplication $\bar{\mu}$ is an isomorphism because of

(1.3). So we find that the bottom map κ_2 is an epimorphism. Thus $E^*(X_\wedge Y)$ is Hausdorff.

1.3. Let R be a subring of Q and E be a ring-spectrum such that the R-module $\pi_*(E)$ is a coherent ring. Further let Y be a finite CW-spectrum and W be a connective CW-spectrum such that $H_*(W)\otimes R$ is a free R-module. First recall [23, Proposition 17] (or see [28]) that

(1.4) *$E^*(Y)$ is a finitely presented $\pi_*(E)$-module for any finite CW-spectrum Y.*

$H_n(W)\otimes R$ is a direct sum of copies of R, i.e. $H_n(W)\otimes R\cong\sum R_\alpha$. Using the universal coefficient theorem and (1.4) an easy calculation shows

$$(1.5)\quad E^*(Y)\underset{\pi_*(E)}{\otimes}H^n(W;\pi_*(E))\cong E^*(Y)\underset{\pi_*(E)}{\otimes}\mathrm{Hom}_R(H_n(W)\otimes R,\pi_*(E))$$

$$\cong E^*(Y)\underset{\pi_*(E)}{\otimes}\prod_\alpha \mathrm{Hom}_R(R_\alpha,\pi_*(E))\cong\prod_\alpha E^*(Y)_\alpha.$$

Thus $E^*(Y)\underset{\pi_*(E)}{\otimes}H^n(W;\pi_*(E))$ is a direct product of copies of $E^*(Y)$. So we obtain that $H^n(W;\pi_*(E))$ is a flat $\pi_*(E)$-module.

Here we need to add another assumption on $\pi_*(E)$ that it is flat as an R-module. Let $\{F^nE^*(W)\}$ be the usual decreasing filtration of $E^*(W)$ defined by skeletons. Since the Atiyah-Hirzebruch spectral sequence for $E^*(W)$ collapses [III, (3.3)], there is an exact sequence

$$(1.6)\quad 0\to H^n(W;\pi_*(E))\to E^*(W)/F^{n+1}E^*(W)\to E^*(W)/F^nE^*(W)\to 0$$

for each degree n. Moreover we have

$$(1.7)\quad E^*(W)\cong\varprojlim_n E^*(W)/F^nE^*(W)\quad\text{and}\quad \varprojlim_n{}^p E^*(W)/F^nE^*(W)=0$$

for all $p\geqq 1$. By induction on n and use of Lemma 1 we get

(1.8) *$E^*(W)$ and $E^*(W)/F^nE^*(W)$ are flat $\pi^*(E)$-modules,*

because $H^n(W;\pi_*(E))$ is flat. In virtue of (1.8) the multiplication

$$(1.9)\quad E^*(Y)\underset{\pi_*(E)}{\otimes}E^*(W)\longrightarrow E^*(Y_\wedge W)$$

becomes an isomorphism for any finite CW-spectrum Y.

$E^*(Y)$ has a projective resolution of finitely generated $\pi_*(E)$-modules by means of (1.4). Then, by [8, Theorem 2] we have two spectral sequences $\{E_r\}$, $\{\bar{E}_r\}$ associated with the same graded $\pi_*(E)$-module such that

$$E_2^{p,q} = \varprojlim_n{}^p \operatorname{Tor}_{-q}^{\pi_*(E)}(E^*(Y), E^*(W)/F^nE^*(W))$$

and

$$\bar{E}_2^{p,q} = \operatorname{Tor}_{-p}^{\pi_*(E)}(E^*(Y), \varprojlim_n{}^q E^*(W)/F^nE^*(W)).$$

By use of (1.7) and (1.8) we compute

$$E_2^{p,q} = 0 \quad \text{unless} \quad q = 0, \quad \text{and}$$

$$\bar{E}_2^{p,q} = 0 \quad \text{unless} \quad (p, q) = (0, 0).$$

This implies

$$(1.10) \quad E^*(Y_\wedge W) \cong E^*(Y) \underset{\pi_*(E)}{\otimes} E^*(W) \cong \varprojlim_n E^*(Y) \underset{\pi_*(E)}{\otimes} E^*(W)/F^nE^*(W),$$

and $$\varprojlim_n{}^p E^*(Y) \underset{\pi_*(E)}{\otimes} E^*(W)/F^nE^*(W) = 0$$

for all $p \geqq 1$.

Using the above results we prove the following satisfactory result for the case IV).

Theorem 1. *Let E be a ring-spectrum with a coherent ring $\pi_*(E)$ which is free and of finite type as an R-module, and W be a connective CW-spectrum such that $H_*(W) \otimes R$ is a free R-module where R is a subring of Q. If $E^*(X)$ is Hausdorff, then $E^*(X_\wedge W)$ is so, too.*

Proof. Let $\{X^\lambda\}$ be the set of all finite subspectra of X. By [III, Theorem 1] we notice that $\varprojlim{}^p E^*(X^\lambda) = 0$ for all $p \geqq 1$.

Consider the two spectral sequences $\{'E_r\}$ and $\{''E_r\}$ associated with $\varprojlim_{\lambda, n}{}^* E^*(X^\lambda) \underset{\pi_*(E)}{\otimes} E^*(W)/F^nE^*(W)$ such that

$$'E_2^{p,q} = \varprojlim_\lambda{}^p \varprojlim_n{}^q E^*(X^\lambda) \underset{\pi_*(E)}{\otimes} E^*(W)/F^nE^*(W)$$

and

$$''E_2^{p,q} = \varprojlim_n{}^p \varprojlim_\lambda{}^q E^*(X^\lambda) \underset{\pi_*(E)}{\otimes} E^*(W)/F^nE^*(W)$$

(see [8] or [I]). The exact sequence (1.6) yields exact sequences

$$0 \longrightarrow E^*(X^\lambda) \underset{\pi_*(E)}{\otimes} H^n(W; \pi_*(E)) \longrightarrow E^*(X^\lambda) \underset{\pi_*(E)}{\otimes} E^*(W)/F^{n+1}E^*(W)$$
$$\longrightarrow E^*(X^\lambda) \underset{\pi_*(E)}{\otimes} E^*(W)/F^nE^*(W) \longrightarrow 0$$

by the aid of (1.8). Using (1.5) we compute

$$\varprojlim_\lambda{}^k E^*(X^\lambda) \underset{\pi_*(E)}{\otimes} H^n(W; \pi_*(E)) \cong \varprojlim_\lambda{}^k \prod_\alpha E^*(X^\lambda)_\alpha \cong \prod_\alpha \varprojlim_\lambda{}^k E^*(X^\lambda)_\alpha = 0,$$

for all $k \geqq 1$. Applying inverse limit functors $\varprojlim{}^k$ on the above exact sequences and using induction on n we have that

$$\varprojlim_\lambda E^*(X^\lambda) \underset{\pi_*(E)}{\otimes} E^*(W)/F^{n+1}E^*(W) \longrightarrow \varprojlim_\lambda E^*(X^\lambda) \underset{\pi_*(E)}{\otimes} E^*(W)/F^nE^*(W)$$

is an epimorphism, and

$$\varprojlim_\lambda{}^k E^*(X^\lambda) \underset{\pi_*(E)}{\otimes} E^*(W)/F^nE^*(W) = 0$$

for all $k \geqq 1$. We use the previous result (1.10) and the above ones to calculate the $'E_2$- and $''E_2$-terms. Thus we see

$$'E_2^{p,q} = 0 \quad \text{unless} \quad q = 0, \quad \text{and}$$
$$''E_2^{p,q} = 0 \quad \text{unless} \quad (p, q) = (0, 0).$$

This implies

$$\varprojlim_\lambda{}^p E^*(X^\lambda \wedge W) \cong {}'E_2^{p,0} = 0$$

for all $p \geqq 1$.

So it follows from [III, Proposition 2] that

$$*) \qquad E^*(X \wedge W) \cong \varprojlim_\lambda E^*(X^\lambda \wedge W).$$

On the other hand, by Proposition 2 we see

$$**) \qquad E^*(X^\lambda \wedge W) \cong \varprojlim_\mu E^*(X^\lambda \wedge W^\mu)$$

where $\{W^\mu\}$ is the set of all finite subspectra of W. Putting $*$) and $**$) together we conclude that $E^*(X_\wedge W)$ is Hausdorff.

1.4. Let E be a CW-spectrum such that $\pi_*(E)$ is free and of finite type as an R-module and Y be a CW-spectrum with finite skeletons such that $E^*(Y)$ is Hausdorff. Further we assume that $E^*(Y)$ satisfies *Condition R* described in [III], i.e., *for each $\alpha\in E^*(Y)$ there exists a connective CW-spectrum W_α with $H_*(W_\alpha)\otimes R$ a free R-module and a map f_α: $Y\to W_\alpha$ such that $\alpha\in\mathrm{Im}\{f_\alpha^*: E^*(W_\alpha)\to E^*(Y)\}$*. Then we have a connective CW-spectrum W with $H_*(W)\otimes R$ a free R-module and a map f: $Y\to W$ which induces a monomorphism f_*: $\pi_*(Y)\otimes Q\to\pi_*(W)\otimes Q$ [III, Theorem 7].

Consider the following commutative square

$$\begin{array}{ccc} \pi_*(X)\otimes\pi_*(Y)\otimes Q & \xrightarrow{\mu_1} & \pi_*(X_\wedge Y)\otimes Q \\ {\scriptstyle 1\otimes f_*}\downarrow & & \downarrow{\scriptstyle (1_\wedge f)_*} \\ \pi_*(X)\otimes\pi_*(W)\otimes Q & \xrightarrow{\mu_2} & \pi_*(X_\wedge W)\otimes Q. \end{array}$$

The multiplications μ_1 and μ_2 are isomorphisms. So it follows from the injectivity of $1\otimes f_*$ that the right vertical map $(1_\wedge f)_*$: $\pi_*(X_\wedge Y)\otimes Q\to \pi_*(X_\wedge W)\otimes Q$ is a monomorphism.

Therefore, Theorem 1 combined with [III, Theorem 2] shows

Theorem 2. *Let R be a subring of Q and E be a ring-spectrum with a coherent ring $\pi_*(E)$ which is free and of finite type as an R-module. Assume that Y is a CW-spectrum with finite skeletons and that $E^*(Y)$ satisfies Condition R. If both $E^*(X)$ and $E^*(Y)$ are Hausdorff, then this is also true for $E^*(X_\wedge Y)$.*

Let MU denote the unitary Thom spectrum. As is well known, $\pi_*(MU)\otimes R$ is a coherent ring (see [28]). Further $(MUR)^*(X)$ satisfies Condition R for an arbitrary CW-spectrum X [III, (4.4)]. As a corollary of Theorem 2 we obtain

Theorem 3. *Let R be a subring of Q and Y a CW-spectrum with finite skeletons. If $(MUR)^*(X)$ and $(MUR)^*(Y)$ are Hausdorff, then*

$(MUR)^*(X_\wedge Y)$ *is so, too.* (Cf., [26]).

2. Cohomology Theories H^*, K^* and W-groups

2.1. For an abelian group A we denote by TA the torsion subgroup of A, i.e., $TA \cong \mathrm{Tor}(A, Q/Z)$. We shall require the following elementary lemmas on abelian groups.

Lemma 3. *Let $f: A \to B$ be a homomorphism such that the tensored homomorphisms with Q and Q/Z are isomorphisms. Then f induces an isomorphism $A/TA \to B/TB$.*

Proof. Applying "five lemma" in the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A/TA & \longrightarrow & A\otimes Q & \longrightarrow & A\otimes Q/Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f\otimes 1} & & \downarrow{\scriptstyle f\otimes 1} & & \\ 0 & \longrightarrow & B/TB & \longrightarrow & B\otimes Q & \longrightarrow & B\otimes Q/Z & \longrightarrow & 0. \end{array}$$

with exact rows, we get that $f': A/TA \to B/TB$ is an isomorphism.

Lemma 4. $A\otimes B/T(A\otimes B) \cong A/TA\otimes B/TB$.

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T(A\otimes B) & \longrightarrow & A\otimes B & \longrightarrow & A\otimes B\otimes Q \\ & & & & \downarrow & & \downarrow \\ & & & & A/TA\otimes B/TB & \longrightarrow & A/TA\otimes B/TB\otimes Q \end{array}$$

with an exact row. The left vertical map is an epimorphism, and the right one is an isomorphism. And the bottom horizontal map is a monomorphism. A diagram chasing argument shows that

$$0 \longrightarrow T(A\otimes B) \longrightarrow A\otimes B \longrightarrow A/TA\otimes B/TB \longrightarrow 0$$

is exact.

Let R be a subring of Q. An abelian group A is called a *W^R-group* if it satisfies $\mathrm{Ext}(A, R)=0$. By definition,

(2.1) *an arbitrary direct sum of W^R-groups is a W^R-group, and every* *subgroup of a W^R-group is so, too.*

Particularly the torsion subgroup TA is a W^R-group if A is so. Since a similar discussion to [II, (5.4)] shows

$$\mathrm{Ext}(TA, R)=0 \quad \text{if and only if } TA\otimes R=0,$$

(2.2) *$A\otimes R$ is a flat R-module (i.e., it is torsion free as an abelian group) whenever A is a W^R-group.*

By definition it is evident

(2.3) *A is a W^R-group if $A\otimes R$ is a free R-module.*

Stein and later on Rotman proved that the converse of (2.3) is valid under the countability restriction on A in the case $R=Z$ (see [25]). We give a new proof based on Gray's result [12] in the general case R.

Proposition 5. *If A is a countable W^R-group, then $A\otimes R$ is a free R-module.*

Proof. We may write A as a union of a countable sequence of finitely generated subgroups:

$$A_1\subset A_2\subset\cdots\subset A_n\subset\cdots,\ A=\bigcup_n A_n.$$

Note that $A_n\otimes R$ is a free R-module for each n. With the aid of [II, (1.4)] we obtain an isomorphism

$$\varprojlim_n{}^1 \mathrm{Hom}(A_n, M)\cong \mathrm{Ext}(A, M)$$

for any R-module M because $\mathrm{Ext}(A_n, M)\cong \mathrm{Ext}_R^1(A_n\otimes R, M)=0$. The assumption that $\mathrm{Ext}(A, R)=0$ means $\varprojlim{}^1 \mathrm{Hom}(A_n, R)=0$. According to Gray [12] this is equivalent to the property that the inverse system $\{\mathrm{Hom}(A_n, R)\}$ satisfies the Mittag-Leffler condition (ML). Since $\mathrm{Hom}(A_n, R)\otimes M\cong \mathrm{Hom}(A_n, M)$, the inverse system $\{\mathrm{Hom}(A_n, M)\}$ satisfies (ML) for any R-module M. As is well known, this implies that $\varprojlim{}^1 \mathrm{Hom}(A_n, M)=0$. Therefore $\mathrm{Ext}(A, M)=0$ for any R-module M. Thus $A\otimes R$ is a free R-module.

Next we discuss a certain relation between tensor product $\otimes$ and

W^R-groups.

Lemma 6. i) *Assume that $A\otimes R$ is a free R-module* (*hence A is a W^R-group*). *If B is a W^R-group, then the tensor product $A\otimes B$ is so.*

ii) *Let A and B be non-zero and torsion free. If the tensor product $A\otimes B$ is a W^R-group, then both A and B are so.*

Proof. i) By [II, (1.9)] there is an exact sequence

$$0\to \mathrm{Ext}(A, \mathrm{Hom}(B, R))\to \mathrm{Ext}(A\otimes B, R)\to \mathrm{Hom}(A, \mathrm{Ext}(B, R))\to 0$$

because $\mathrm{Hom}(\mathrm{Tor}(A, B), R)=0$. From hypotheses on A and B $\mathrm{Ext}(A, \mathrm{Hom}(B, R))=\mathrm{Hom}(A, \mathrm{Ext}(B, R))=0$. So we see that $A\otimes B$ is a W^R-group.

ii) Let A_λ be a non-zero finitely generated subgroup of A. Consider the following exact sequence

$$\mathrm{Hom}(A, \mathrm{Ext}(B, R))\to \mathrm{Hom}(A_\lambda, \mathrm{Ext}(B, R))\to \mathrm{Ext}(A/A_\lambda, \mathrm{Ext}(B, R)).$$

Using the previous exact sequence again, $\mathrm{Ext}(A\otimes B, R)=0$ implies that $\mathrm{Hom}(A, \mathrm{Ext}(B, R))=0$. On the other hand, by use of [II, (1.9)] we compute that $\mathrm{Ext}(A/A_\lambda, \mathrm{Ext}(B, R))\cong \mathrm{Ext}(\mathrm{Tor}(A/A_\lambda, B), R)=0$. Consequently $\mathrm{Hom}(A_\lambda, \mathrm{Ext}(B, R))=0$. Since A_λ is non-zero and free we get immediately

$$\mathrm{Ext}(B, R)=0.$$

Similarly we have $\mathrm{Ext}(A, R)=0$.

2.2. Now we restrict our interest to the Eilenberg-MacLane spectrum H and the BU-spectrum K. Let us denote by E either H or K. The cohomology theory E^* has a relation with the homology theory E_* by the following universal coefficient sequence (see [16]): *There is a natural short exact sequence*

$$(2.4)\qquad 0\to \mathrm{Ext}(E_{n-1}(X), G)\to (EG)^n(X)\to \mathrm{Hom}(E_n(X), G)\to 0$$

for any coefficient group G.

Using the universal coefficient sequence (2.4) we give a condition on

$E_*(X)$ under which $(ER)^*(X)$ is Hausdorff in the cases $E=H$, K.

Theorem 4. *Let E denote either H or K. Let R be a subring of Q and X a CW-spctrum. $(ER)^{n+1}(X)$ is Hausdorff if and only if $E_n(X)/T(E_n(X))$ is a W^R-group.*

Proof. We may assume that R is a proper subring of Q. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Ext}(E_n(X)\otimes Q, R) & \to & ER^n(X; \hat{Z}/Z) & \to & \mathrm{Hom}(E_{n+1}(X)\otimes Q, R) & \to 0 \\ & \downarrow & & \downarrow \delta & & \downarrow & \\ 0 \to & \mathrm{Ext}(E_n(X), R) & \to & ER^{n+1}(X) & \to & \mathrm{Hom}(E_{n+1}(X), R) & \to 0 \end{array}$$

in which $ER^*(\ ; \hat{Z}/Z)$ is defined by $ER^n(X; \hat{Z}/Z)=ER^{n+2}(X_\wedge S_l)$ using the co-Moore space S_l of type $(\hat{Z}/Z, 2)$ constructed in [II]. Two rows are exact by the universal coefficient sequence (2.4). Moreover, Hom $(E_{n+1}(X)\otimes Q, R)=0$. The left vertical map admits a factorization

$$\mathrm{Ext}(E_n(X)\otimes Q, R) \longrightarrow \mathrm{Ext}(E_n(X)/TE_n(X), R) \longrightarrow \mathrm{Ext}(E_n(X), R)$$

such that the former is an epimorphism and the latter is a monomorphism. According to [III, (2.5)] (rather than [III, Theorem 1]), $ER^{n+1}(X)$ is Hausdorff if and only if $\delta: ER^n(X; \hat{Z}/Z) \to ER^{n+1}(X)$ is trivial. So we see easily that $ER^{n+1}(X)$ is Hausdorff if and only if $\mathrm{Ext}(E_n(X)/TE_n(X), R)=0$, i.e., $E_n(X)/TE_n(X)$ is a W^R-group.

As an immediate corollary of Theorem 4, we get

Corollary 7. *Let E denote either H or K. If $E_n(X)/T(E_n(X))\otimes R$ is a free R-module, then $(ER)^{n+1}(X)$ is Hausdorff.*

2.3. Let M_q be the mapping cone of the map $S^1 \to S^1$ of degree q as in [1]. Assume that the Hopf map $\eta: S^3 \to S^2$ induces $\eta_*=0: E_*(S^3) \to E_*(S^2)$. The cofibration sequence $S^1 \to M_q \to S^2$ yields an exact sequence

(2.5) $0 \longrightarrow E_n(X)\otimes Z_q \xrightarrow{\rho} E_{n+1}(X_\wedge M_q) \xrightarrow{\partial} \mathrm{Tor}(E_{n-1}(X), Z_q) \longrightarrow 0.$

The dual argument to [1, Proposition 2.2] shows

(2.6) *$E_*(X_\wedge M_q)$ is a Z_q-module for any $q>1$.*

Let d denote the gratest common measure of q and r. By virtue of (2.6) we have the following commutative square

$$\begin{array}{ccc} E_n(X)\otimes Z_q\otimes Z_r & \xrightarrow{\rho\otimes 1} & E_{n+1}(X_\wedge M_q)\otimes Z_r \\ \downarrow & & \downarrow{\scriptstyle \overline{(q/d)}_*} \\ E_n(X)\otimes Z_d & \xrightarrow{\rho} & E_{n+1}(X_\wedge M_d). \end{array}$$

Obviously the left canonical map is an isomorphism. So we find by the aid of (2.5) that the top horizontal map $\rho\otimes 1$ is a monomorphism. Thus the above sequence (2.5) is a pure exact sequence. Since $E_n(X)\otimes Z_q$ is bounded, it follows from [25, Theorem 14] that

(2.7) *the above exact sequence* (2.5) *is Z-split.* (Cf., [1]).

The complex homology K-theory K_* has Bott's isomorphism

$$\beta: K_n(X) \longrightarrow K_{n+2}(X). \tag{2.8}$$

So we can define the Z_2-graded homology K-theory $K_\#$ by putting

$$K_\#(X)=K_0(X)\oplus K_1(X)$$

and identifying $K_{2n}(X)$ with $K_0(X)$ and $K_{2n+1}(X)$ with $K_1(X)$ *via* Bott's isomorphism β.

Here we again restrict ourselves to $E=H$ or K. Denote by $E_\#$ either H_* or $K_\#$. Recall that the Künneth formula holds in our case $E_\#=H_*$, $K_\#$ (see [16, 23]): *There is a natural short exact sequence*

$$0\longrightarrow E_\#(X)\otimes E_\#(Y) \xrightarrow{\mu} E_\#(X_\wedge Y)\longrightarrow \mathrm{Tor}(E_\#(X), E_\#(Y))\longrightarrow 0. \tag{2.9}$$

As is easily seen,

$$\mu\otimes 1: E_\#(X)\otimes E_\#(Y)\otimes Q\longrightarrow E_\#(X_\wedge Y)\otimes Q \tag{2.10}$$

is an isomorphism.

Next, consider the following commutative square

$$\begin{array}{ccc} E_\#(X)\otimes E_\#(Y)\otimes Z_r & \xrightarrow{\mu\otimes 1} & E_\#(X_\wedge Y)\otimes Z_r \\ \downarrow{\scriptstyle 1\otimes\rho} & & \downarrow{\scriptstyle \rho} \\ E_\#(X)\otimes E_{\#+1}(Y_\wedge M_r) & \xrightarrow{\mu} & E_{\#+1}(X_\wedge Y_\wedge M_r)\,. \end{array}$$

The left vertical map $1\otimes\rho$ is a monomorphism by virtue of (2.7) and so is the bottom multiplication μ because of (2.9). Hence the top horizontal map $\mu\otimes 1$ is a monomorphism. Thus

(2.11) *the above Künneth sequence* (2.9) *is a pure exact sequence.* (Cf., [27]).

This implies that the multiplication μ induces a monomorphism (and an isomorphism by means of (2.10))

(2.12) $$\mu\otimes 1: E_{\#}(X)\otimes E_{\#}(Y)\otimes Q/Z \longrightarrow E_{\#}(X_{\wedge}Y)\otimes Q/Z.$$

We use Lemmas 3 and 4 to obtain

Lemma 8. *Let $E_{\#}$ denote either H_* or $K_{\#}$. Then the multiplication μ induces an isomorphism*

$$E_{\#}(X)/TE_{\#}(X)\otimes E_{\#}(Y)/TE_{\#}(Y)\cong E_{\#}(X_{\wedge}Y)/TE_{\#}(X_{\wedge}Y).$$

Using a new criterion Theorem 4 for Hausdorff-ness of $H^*(X)$ and $K^*(X)$, and Lemmas 6 and 8 we obtain

Theorem 5. *Let E denote either H or K. Let R be a subring of Q and X, Y CW-spectra.*

i) *Assume that $E_*(Y)/T(E_*(Y))\otimes R$ is a free R-module. If $(ER)^*(X)$ is Hausdorff, then $(ER)^*(X_{\wedge}Y)$ is so, too.*

ii) *If $(ER)^*(X_{\wedge}Y)$ is Hausdorff, then either $\pi_*(X)$ or $\pi_*(Y)$ is a torsion group, or both $(ER)^*(X)$ and $(ER)^*(Y)$ are Hausdorff.*

$E_*(Y)$ becomes a countable abelian group when Y has finite skeletons. By the aid of Proposition 5 we have a corollary of Theorem 5 i).

Corollary 9. *Let E denote either H or K and Y be a CW-spectrum with finite skeletons. If $(ER)^*(X)$ and $(ER)^*(Y)$ are Hausdorff, then this is also true for $(ER)^*(X_{\wedge}Y)$.*

References

[1]-[8], [9]-[15] and [16]-[22] are listed at the end of papers [I], [II] and [III].
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