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On Cohomology Theories of Infinite CW-complexes, III

By

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Let h^* be an additive cohomology theory and X be a based CW-complex which is the union of all finite subcomplexes X^λ. The subgroups $F^\lambda h^*(X)=\mathrm{Ker}\{h^*(X)\to h^*(X^\lambda)\}$ gives a topology in the cohomology group $h^*(X)$. In the second paper [II] with the same title we investigated conditions on h^* and X under which $h^*(X)$ is Hausdorff. The purpose of the present paper is to continue the investigation.

A based CW-complex X is regarded as a (-1)-connected CW-spectrum in the stable category [16, 17]. Then every additive cohomology theory h^* defined on the category of based CW-complexes is represented by a suitable CW-spectrum E. In the sequel we shall work in the *stable category of CW-spectra* rather than in the category of based CW-complexes. An (additive) cohomology theory is written E^* in place of h^*. Now it seems natural that a cohomology theory $(EG)^*$ with coefficient G is defined by using a Moore spectrum of type G following Adams [16], but not by using a co-Moore spectrum of type G as in [II].

In the previous paper we restricted ourselves to the case when h^* is of finite type as an abelian group. We shall slightly relax the restriction. Thus we shall discuss mainly conditions that $E^*(X)$ is Hausdorff, under the assumption that $\pi_*(E)$ is *of finite type as an R-module* where R is a subring of the rational numbers Q.

First we extend some results of [II] to a CW-spectrum X and a cohomology theory E^* such that $\pi_*(E)$ is of finite type as an R-module. Then we find that Hausdorff-ness of $E^*(X)$ is closely related to the exact sequence

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$$\longrightarrow E^n(X) \longrightarrow (E\hat{Z})^n(X) \longrightarrow (E\hat{Z}/Z)^n(X) \longrightarrow E^{n+1}(X) \longrightarrow$$

corresponding to the coefficient sequence $0 \to Z \to \hat{Z} \to \hat{Z}/Z \to 0$ (Theorem 1). This is a very important criterion for Hausdorff-ness of $E^*(X)$.

This criterion gives sufficient conditions on X under which $E^*(X)$ is Hausdorff (Theorem 2). And also we show with the aid of it that Hausdorff-ness of $E^*(X)$ is shared by $E^*(X^p)$ where X^p denotes the p-skeleton of X (Theorem 4). Using the same criterion we discuss Hausdorff-ness of $E^*(W)$ for W having a free R-module $H_*(W)\otimes R$, under a certain restriction on $\pi_*(E)$ (Theorem 5).

Next we give some criteria for Hausdorff-ness of $E^*(Y)$ for Y having finite skeletons (Theorems 6 and 7). Then we apply the above results to $E^*=MU^*$, complex cobordism, or K^*, complex K-cohomology. In particular, we get that $E^*(BG)$ is Hausdorff for an arbitrary compact Lie group G if $\pi_*(E)$ is free and of finite type as an R-module.

In [2] we constructed a spectral sequence for a based CW-complex X which is a version of the Milnor's short exact sequence. In Appendix we shall extend the spectral sequence to that for CW-spectrum, whose existence allows us to study Hausdorff-ness of $E^*(X)$ of a CW-spectrum X as well as a based CW-complex.

Several results of [II] are repeated, since they are presented here in greater generality.

Throughout this paper we understand by H_*, H^* the reduced ordinary homology and cohomology theories.

1. Cohomology Theories of CW-spectra

1.1. Suppose given a CW-spectrum E. Then we define the homology and cohomology groups of CW-spectra X with coefficient in E by

$$E_n(X)=\{S, E\wedge X\}_n, \quad E^n(X)=\{X, E\}_{-n}.$$

We remark that every additive homology or cohomology theory defined on the category of based CW-complexes is representable with some CW-spectrum E.

Let X be a CW-spectrum and $\mathfrak{U}_X=\{X^\lambda\}$ the set of all finite subspectra of X ordered by inclusions which is directed. We introduce subgroups

$F^\lambda E^n(X)$ of $E^n(X)$ by

$$F^\lambda E^n(X)=\mathrm{Ker}\{E^n(X)\to E^n(X^\lambda)\}\,.$$

The inverse system $\{F^\lambda E^n(X)\}$ gives $E^n(X)$ the structure of a topological group. The inclusions $i_\lambda : X^\lambda \subset X$ induces a continuous homomorphism

$$\pi=\varprojlim_\lambda i_\lambda^*\colon E^n(X)\to\varprojlim_\lambda E^n(X^\lambda)$$

where $\varprojlim E^n(X^\lambda)$ is topologized by the inverse limit topology.

According to Adams [9, Theorem 1.8],

(1.1) *$\pi\colon E^*(X)\to\varprojlim_\lambda E^*(X^\lambda)$ is an epimorphism for any CW-spectrum X.*

The proof in [9] is actually given for a based CW-complex X, but it is easily extended to a CW-spectrum.

By the aid of (1.1) we obtain

Proposition 1. *The following conditions are equivalent:*

i) $S^n(X)=\bigcap_\lambda F^\lambda E^n(X)=\{0\}$,

ii) *$E^n(X)$ is Hausdorff,*

iii) *$E^n(X)$ is complete and Hausdorff,*

iv) *$\pi\colon E^n(X)\to\varprojlim_\lambda E^n(X^\lambda)$ is an isomorphism.*

The proof is just the same as that of [II, Proposition 2].

Let X be any CW-spectrum which is the union of a direct system of subspectra X_α over a directed set, i.e., $X=\cup X_\alpha$. Then we can give a version of the Milnor's short exact sequence [6] (see also [21]) in a form of a spectral sequence:

(1.2) *There exists a spectral sequence $\{E_r^{p,q}\}$ associated with $E^*(X)$ such that*

$$E_2^{p,q}=\varprojlim_\alpha{}^p E^q(X_\alpha)\,.$$

And the edge homomorphism of the spectral sequence

$$E^n(X)\to E_\infty^{0,n}\subset E_2^{0,n}=\varprojlim_\alpha E^n(X_\alpha)$$

coincides with the natural epimorphism π.

In the case of a based CW-complex X the spectral sequence mentioned above was constructed in [2] (or see [16]). But we can construct it for a CW-spectrum as well as a based CW-complex. For the sake of completeness we shall give a proof in Appendix.

Using the spectral sequence (1.2) a standard argument shows

Proposition 2. *If* $\varprojlim_{\lambda}{}^{p} E^*(X^\lambda)=0$ *for all* $p\geqq 1$ *where* X^λ *runs over all finite subspectra of* X, *then* $E^*(X)$ *is Hausdorff.*

If $\pi_n(E)$ is a finite abelian group for each degree n, then so is $E^n(X^\lambda)$. Since [I, Corollary 5] implies that $\varprojlim{}^p E^*(X^\lambda)=0$ for all $p\geqq 1$, we have

Proposition 3. *Assume that* $\pi_n(E)$ *is a finite abelian group for each* n. *Then* $E^*(X)$ *is Hausdorff for any* CW*-spectrum* X.

Let us denote by Q the field of rational numbers. Assume that $\pi_*(E)$ is a Q-module. Then Dold's theorem [19] (or see [II, Theorem 6]) insists that there exists a natural isomorphism

$$E^n(X)\cong \prod_k H^k(X; \pi_{k-n}(E)) \tag{1.3}$$

for any CW-spectrum X.

Making use of [II, (1.4)] we compute

$$\varprojlim_{\lambda}{}^{p} E^*(X^\lambda)\cong \varprojlim_{\lambda}{}^{p}\mathrm{Hom}(\pi_*(X^\lambda),\quad \pi_*(E))=0 \tag{1.4}$$

for all $p\geqq 1$. So we get

Proposition 4. *If* $\pi_*(E)$ *is a* Q*-module, then* $E^*(X)$ *is Hausdorff for any* CW*-spectrum* X.

1.2. Let G be an abelian group. We can always construct a Moore spectrum $M(G)$ of type G, i.e., $\pi_r(MG)=0$ for $r<0$, $H_0(MG)\cong G$ and $H_r(MG)=0$ for $r>0$. Given any homomorphism $\phi: G\to G'$ of abelian groups, we can find a corresponding map $f: M(G)\to M(G')$ of Moore spectra with $H_*(f)\cong\phi$. Therefore there exists a cofibering sequence

$$M(G)\to M(G')\to M(G'') \tag{1.5}$$

associated with a short exact sequence $0\to G\to G'\to G''\to 0$.

For any CW-spectrum E we define the corresponding spectrum with coefficient group G by

$$EG = E \wedge M(G)\,.$$

The homology and cohomology theories represented by EG are written EG_*, EG^*. For them we have universal coefficient sequences as follows [16, Proposition 6.6]:

(1.6) i) *There exists an exact sequence*

$$0 \longrightarrow E_n(X)\otimes G \longrightarrow (EG)_n(X) \longrightarrow \mathrm{Tor}(E_{n-1}(X), G) \longrightarrow 0$$

for any CW-spectrum X, and
ii) *if X is a finite CW-spectrum or G is finitely generated, there exists an exact sequence*

$$0 \longrightarrow E^n(X)\otimes G \longrightarrow (EG)^n(X) \longrightarrow \mathrm{Tor}(E^{n+1}(X), G) \longrightarrow 0.$$

If there exists a co-Moore spectrum $M^*(G)$ of type G, then we may define a cohomology theory $E^*(\ ;G)$ by

$$E^*(X; G) = E^*(X \wedge M^*(G))\,.$$

The two cohomology theories EG^* and $E^*(\ ;G)$ with the same coefficient G don't necessarily coincide with each other. (For example, see (2.4)). So we have to distinguish between them. However we can find a natural isomorphism between EG^* and $E^*(\ ;G)$ in virtue of S-duality [21, Theorem 13.2] whenever G is finitely generated.

Let us denote by $\hat{Z}$ the completion of Z with respect to all of its subgroups, i.e., $\hat{Z}\cong \mathrm{Ext}(Q/Z, Z)$. The cofibration (1.5) yields an exact sequence

$$(1.7) \qquad \longrightarrow E^n(X) \xrightarrow{\ \iota\ } (E\hat{Z})^n(X) \xrightarrow{\ \kappa\ } (E\hat{Z}/Z)^n(X) \xrightarrow{\ \delta\ } E^{n+1}(X) \longrightarrow$$

corresponding to the coefficient sequence $0\to Z\to\hat{Z}\to\hat{Z}/Z\to 0$ (cf., [II, (4.4)]).

Since both $\hat{Z}$ and $\hat{Z}/Z$ are torsion free, (1.6) implies that there are natural isomorphisms

$$(1.8) \qquad E^*(Y)\otimes\hat{Z}\cong(E\hat{Z})^*(Y), \quad E^*(Y)\otimes\hat{Z}/Z\cong(E\hat{Z}/Z)^*(Y)$$

for any finite CW-spectrum Y. Moreover, by means of Dold's theorem (1.3) there exists a natural isomorphism

$$(E\hat{Z}/Z)^n(X) \cong \prod_k H^k(X; \pi_{k-n}(E) \otimes \hat{Z}/Z) \tag{1.9}$$

for any CW-spectrum X, because $\pi_*(E; \hat{Z}/Z) \cong \pi_*(E) \otimes \hat{Z}/Z$ is a Q-module.

In the following commutative diagram

$$\begin{array}{ccccccc} \longrightarrow E^n(X) & \xrightarrow{\iota} & (E\hat{Z})^n(X) & \xrightarrow{\kappa} & (E\hat{Z}/Z)^n(X) & \longrightarrow & \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_3} & & \\ 0 \longrightarrow \varprojlim E^n(X^\lambda) & \longrightarrow & \varprojlim (E\hat{Z})^n (X^\lambda) & \longrightarrow & \varprojlim (E\hat{Z}/Z)^n(X^\lambda) & \longrightarrow, & \end{array}$$

the upper row is exact by (1.7) and the lower one is exact by virtue of (1.8). All vertical maps are epimorphisms because of (1.1) and in particular Proposition 4 says that π_3 is an isomorphism.

A diagram chasing argument shows

Proposition 5. *$E^n(X)$ is Hausdorff if and only if $(E\hat{Z})^n(X)$ is Hausdorff and $\iota: E^n(X) \to (E\hat{Z})^n(X)$ is a monomorphism.*

2. Localization Z_l

2.1. Let R be a subring of the rational numbers Q with unit. Recall that it is just the *integers localized at* l where l is the set of all primes which are not invertible in R, and it is frequently denoted by Z_l. Putting $R' = Z_{l'}$ where l' is the set of primes such that $l \cap l' = \{\phi\}$ and $l \cup l' = \{\text{all primes}\}$, an easy argument shows that

$$0 \longrightarrow Z \longrightarrow R \oplus R' \longrightarrow Q \longrightarrow 0 \tag{2.1}$$

is exact.

Consider the following commutative diagram

$$\begin{array}{ccccccc} \longrightarrow E^n(X) & \longrightarrow & E(R \oplus R')^n(X) & \longrightarrow & (EQ)^n(X) & \longrightarrow & \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_3} & & \\ 0 \longrightarrow \varprojlim E^n(X^\lambda) & \longrightarrow & \varprojlim E(R \oplus R')^n(X^\lambda) & \longrightarrow & \varprojlim (EQ)^n(X^\lambda) & \longrightarrow & \end{array}$$

with exact rows. The right vertical map π_3 is an isomorphism by Proposition 4. If we assume that $E^n(X)$ is Hausdorff, i.e., π_1 is an isomorphism, then the middle one π_2 becomes a monomorphism with an application of "four lemma" and hence it is an isomorphism because of (1.1). Thus $E(R\oplus R')^n(X)$ is Hausdorff. So we obtain

Proposition 6. *If $E^n(X)$ is Hausdorff, then $(ER)^n(X)$ is so for any subring R of Q.*

2.2. Assume that $\pi_*(E)$ is an R-module where R is a subring of Q, i.e., $R=Z_l$. (For example, we might have $E=FR$). $\pi_*(E)\otimes\hat{R}'$ is a Q-module because so is $R\otimes\hat{R}'\cong R\otimes\mathrm{Ext}(R/Z, Z)$, where $R'=Z_{l'}$ and $l\cap l'=\{\phi\}$, $l\cup l'=\{\text{all primes}\}$. By Proposition 4 $(E\hat{R}')^*(X)$ is always Hausdorff for any CW-spectrum X. On the other hand, recall that

$$\hat{R}=\hat{Z}_l\cong\prod_{p\in l}\hat{Z}_p \quad\text{and}\quad \hat{Z}\cong\hat{R}\oplus\hat{R}'. \tag{2.2}$$

Then we get immediately

Lemma 7. *Assume that $\pi_*(E)$ is an R-module. Then $(E\hat{Z})^n(X)$ is Hausdorff if and only if $(E\hat{R})^n(X)$ is so.*

A graded R-module A is said to be of *finite type as an R-module* if each A_n is a finitely generated R-module.

Here we show that $(E\hat{Z})^*(X)$ is Hausdorff under certain finiteness assumption on $\pi_*(E)$.

Proposition 8. *Assume that $\pi_*(E)$ is of finite type as an R-module. Then $(E\hat{Z})^*(X)$ is Hausdorff for any CW-spectrum X.*

Proof. By Proposition 2 and Lemma 7 it is sufficient to show that

$$\varprojlim_{\lambda}{}^s(E\hat{R})^*(X^\lambda)=0$$

for all $s\geqq 1$ where X^λ runs over all finite subspectra of X. Note that $E^*(X^\lambda)$ is of finite type as an R-module. Using (1.6), (2.2) and [I, Corollary 5] an easy calculation shows

$$\varprojlim_{\lambda}{}^s(E\hat{R})^*(X^\lambda)\cong\varprojlim_{\lambda}{}^s E^*(X^\lambda)\otimes_R\prod_{p\in l}\hat{Z}_p\cong\varprojlim_{\lambda}{}^s\prod_{p\in l}E^*(X^\lambda)\otimes_R\varprojlim_{k} Z_{p^k}$$
$$\cong\prod_{p\in l}\varprojlim_{\lambda,k}{}^s E^*(X^\lambda)\otimes_R Z_{p^k}=0.$$

Now we obtain an important criterion for $E^*(X)$ being Hausdorff.

Theorem 1. *Let R be a subring of Q and E be a CW-spectrum such that $\pi_*(E)$ is of finite type as an R-module. Fix a degree n and let X be an arbitrary CW-spectrum. Then the following conditions are equivalent:*

i) *$E^n(X)$ is Hausdorff,*

ii) *$\varprojlim_{\lambda}{}^1 E^{n-1}(X^\lambda)=0$ where X^λ runs over all finite subspectra of X,*

iii) *$\iota: E^n(X)\to(E\hat{Z})^n(X)$ is a monomorphism.*

Proof. Combining Proposition 5 with Proposition 8 we see that i) and iii) are equivalent. On the other hand, the spectral sequence (1.2) yields a short exact sequence

$$0\longrightarrow \varprojlim_{\lambda}{}^1 E^{n-1}(X^\lambda)\longrightarrow E^n(X)\xrightarrow{\ \pi\ }\varprojlim_{\lambda} E^n(X^\lambda)\longrightarrow 0,$$

because $\varprojlim{}^p E^*(X^\lambda)=0$ for all $p\geqq 2$ [14, Theorem 2] (or see [II]). This implies that i) and ii) are equivalent.

Remark. The condition ii) means that $\varprojlim{}^p E^n(X^\lambda)=0$ for all $p\geqq 1$.

Let M_q, $\check{M}$ and S_l be the co-Moore spaces of type $(Z_q, 2)$, $(\hat{Z}, 2)$ and $(\hat{Z}/Z, 2)$ given in [II]. By use of them we define cohomology theories with coefficients Z_q, $\hat{Z}$ and $\hat{Z}/Z$ which are written $E^*(\ ;Z_q)$, $E^*(\ ;\hat{Z})$ and $E^*(\ ;\hat{Z}/Z)$ respectively. Assume that $\pi_*(E)$ is of finite type as an R-module where R is a proper subring of Q, i.e., $R=Z_l$, $l\neq\{\phi\}$. Then a parallel discussion to [II, Propositions 8 and 9] shows that

$$\rho: E^*(X;\hat{Z})\longrightarrow\prod_{p\in l}\varprojlim_{k} E^*(X;Z_{p^k}) \tag{2.3}$$

is an isomorphism for any CW-spectrum X, and there are natural isomorphisms

$$E^*(Y;\hat{Z})\cong E^*(Y)\otimes\hat{R} \quad\text{and}\quad E^*(Y;\hat{Z}/Z)\cong E^*(Y)\otimes\hat{R}/Z \tag{2.4}$$

for any finite CW-spectrum Y, because $E^*(M_{p^k})=0$ for each $p\notin l$. More precisely speaking, we have natural isomorphisms between $(E\hat{R})^*$ and $E^*(\ ;\hat{Z})$ and between $(E\hat{R}/Z)^*$ and $E^*(\ ;\hat{Z}/Z)$.

Further we continue a parallel discussion to [II, 4] so that we get the following result.

(2.5) *Assume that R is a proper subring of Q and $\pi_*(E)$ is of finite type as an R-module. Then $E^n(X)$ is Hausdorff if and only if $\iota: E^n(X) \to E^n(X; \hat{Z})$ is a monomorphism.*

2.3. Let A be an abelian group and G a non-zero Q-module. An easy argument shows

(2.6) $$A \otimes Q = 0 \quad \textit{if and only if} \quad \mathrm{Hom}(A, G) = 0.$$

The universal coefficient theorem yields an isomorphism

(2.7) $$H^*(X; G) \cong \mathrm{Hom}(H_*(X), G)$$

for any CW-spectrum X. So this implies

(2.8) $$H_n(X) \otimes Q = 0 \quad \textit{if and only if} \quad H^n(X; G) = 0.$$

Let $f: X \to Y$ be a map of CW-spectra. In the following commutative diagram

$$\begin{array}{ccccccc} & & \mathrm{Hom}(H_*(Y), G) & & & & \\ & & \downarrow & \searrow & & & \\ 0 \longrightarrow & & \mathrm{Hom}(\mathrm{Im} f_*, G) & \to & \mathrm{Hom}(H_*(X), G) & \longrightarrow \mathrm{Hom}(\mathrm{Ker} f_*, G) & \longrightarrow 0, \end{array}$$

the row is exact and the vertical map is an epimorphism. By the aid of (2.6) and (2.7) we get immediately

(2.9) *$f_*: H_n(X) \otimes Q \to H_n(Y) \otimes Q$ is a monomorphism if and only if $f^*: H^n(Y; G) \to H^n(X; G)$ is an epimorphism.*

Lemma 9. *Assume that $\pi_*(E)$ is a non-zero Q-module.*

i) *$\pi_*(X) \otimes Q = 0$ if and only if $E^*(X) = 0$.*

ii) *Let $f: X \to Y$ be a map. $f_*: \pi_*(X) \otimes Q \to \pi_*(Y) \otimes Q$ is a monomorphism if and only if $f^*: E^*(Y) \to E^*(X)$ is an epimorphism.*

This is immediate, using (2.8), (2.9) and Dold's theorem (1.3).

Taking $E\hat{Z}/Z$ as E in the above lemma and using Theorem 1 we obtain

Theorem 2. *Let R be a subring of Q and E be a CW-spectrum such that $\pi_*(E)$ is of finite type as an R-module.*

i) *Let* X *be a* CW*-spectrum. If* $\pi_*(X)\otimes Q=0$, *then* $E^*(X)$ *is Hausdorff.*

ii) *Let* $f:X\to Y$ *be a map of* CW*-spectra which induces a monomorphism* $f_*:\pi_*(X)\otimes Q\to\pi_*(Y)\otimes Q$. *If* $E^n(Y)$ *is Hausdorff for some degree* n, *then* $E^n(X)$ *is so, too.*

2.4. Let $\phi: F\to E$ be a map of CW-spectra for which $\phi_*: \pi_*(F)\otimes Q\to \pi_*(E)\otimes Q$ is an epimorphism. Then it induces an epimorphism

$$(F\hat{Z}/Z)^*(X)\to(E\hat{Z}/Z)^*(X)$$

for any CW-spectrum X, by virtue of Dold's theorem (1.9). From Theorem 1 (and Proposition 5) we obtain

Theorem 3. *Let* $\phi: F\to E$ *be a map of* CW*-spectra which induces an epimorphism* $\phi_*:\pi_*(F)\otimes Q\to\pi_*(E)\otimes Q$, *and assume that* $\pi_*(E)$ *is of finite type as an* R*-module where* R *is a subring of* Q. *If* $F^n(X)$ *is Hausdorff, then* $E^n(X)$ *is so for the same* CW*-spectrum* X *and degree* n.

Let us denote by K, KO the BU- and BO-spectrum and by MU, MSO and MO the Thom spectrum for U, SO and O. The corresponding cohomology theories K^*, KO^* are "complex" and "real" K-cohomologies, and MU^*, MSO^* and MO^* "complex", "oriented" and "unoriented" cobordisms. The inclusions $U(n)\subset SO(2n)\subset O(2n)$ yield realifications

$$r: K\longrightarrow KO,\quad s:MU\longrightarrow MSO.$$

It is known that $r_*:\pi_*(K)\otimes Q\to\pi_*(KO)\otimes Q$ and $s_*: \pi_*(MU)\otimes Q\to\pi_*(MSO)\otimes Q$ are epimorphisms, so we have

(2.10) i) $(KOR)^n(X)$ *is Hausdorff if* $(KR)^n(X)$ *is so, and*
ii) $(MSOR)^n(X)$ *is Hausdorff if* $(MUR)^n(X)$ *is so.*

On the other hand, $\pi_n(MO)$ is a finite abelian group for each degree n. By Proposition 3 we note

(2.11) $(MOR)^*(X)$ *is always Hausdorff for any* CW*-spectrum* X.

Let k be the connective BU-spectrum and denote by k^* the connective K-cohomology. The Thom map $\mu_c: MU\to K$ is lifted to a morphism ζ:

$MU \to k$, i.e., it coincides with the composite morphism

$$MU \xrightarrow{\zeta} k \xrightarrow{\lambda} K$$

of ring-spectra. Further the usual morphism μ: $MU \to H$ admits a factorization

$$MU \xrightarrow{\zeta} k \xrightarrow{\eta} H$$

in which H is the Eilenberg-MacLane spectrum, and both ζ and η induce epimorphisms in homotopy.

(2.12) *If* $(MUR)^n(X)$ *is Hausdorff, then this is also true for* $(HR)^n(X)$, $(kR)^n(X)$.

Let X be a connective CW-spectrum, i.e., $(m-1)$-connected for some m. Recall that the homomorphism of coefficients

$$\lambda_*: \pi_i(k) \longrightarrow \pi_i(K)$$

is an isomorphism for each non-negative integer i. Making use of Dold's theorem (1.9) we see easily that the map λ: $k \to K$ induces an isomorphism

$$(k\hat{Z}/Z)^n(X) \longrightarrow (K\hat{Z}/Z)^n(X)$$

for each degree n, $n \leqq m$.

Remark that complex K-cohomology K^* possesses the Bott periodicity, i.e., the Bott homomorphism

$$\beta: K^n(X) \longrightarrow K^{n-2}(X) \tag{2.13}$$

is an isomorphism for each degree n. Therefore, for any connective CW-spectrum X we get

(2.14) $(KR)^*(X)$ *is Hausdorff whenever* $(kR)^*(X)$ *is so.*

3. Atiyah-Hirzebruch Spectral Sequences

3.1. Let X be a connective CW-spectrum, and X^p denote its p-skeleton. Observe the Atiyah-Hirzebruch spectral sequences $\{\hat{E}_r\}$ and $\{\bar{E}_r\}$ associated to the skeleton filtration of X^p for the cohomology theories $(E\hat{Z})^*$

and $(E\hat{Z}/Z)^*$. We have the following commutative diagram

$$\begin{array}{ccccc} H^p(X^p;\pi_{-q}(E)\otimes\hat{Z})=\hat{E}_2^{p,p} & \longrightarrow & \bar{E}_2^{p,q}= H^p(X^p;\pi_{-q}(E)\otimes\hat{Z}/Z) \\ \downarrow & & \downarrow \\ \hat{E}_\infty^{p,q} & \longrightarrow & \bar{E}_\infty^{p,q} \\ \cap & & \cap \\ (E\hat{Z})^{p+q}(X^p) & \xrightarrow{\kappa^p} & (E\hat{Z}/Z)^{p+q}(X^p)\,. \end{array}$$

The top horizontal map and two vertical ones are obviously epimorphisms. In particular the right one becomes an isomorphism because the spectral sequence $\{\bar{E}_r\}$ collapses in virtue of Dold's theorem (1.9). So the middle horizontal map is an epimorphism, and this implies that the composite map

$$(3.1)\quad (E\hat{Z})^n(X^p) \xrightarrow{\kappa^p} (E\hat{Z}/Z)^n(X^p) \cong \prod_k H^k(X^p;\pi_{k-n}(E)\otimes\hat{Z}/Z) \longrightarrow H^p(X^p;\pi_{p-n}(E)\otimes\hat{Z}/Z)$$

is an epimorphism.

Lemma 10. *Let X be a connective CW-spectrum. If $\kappa\colon (E\hat{Z})^n(X)\to(E\hat{Z}/Z)^n(X)$ is an epimorphism, then this is true for each p-skeleton X^p.*

Proof. Consider the following commutative square

$$\begin{array}{ccc} (E\hat{Z})^n(X) & \xrightarrow{\kappa} & (E\hat{Z}/Z)^n(X)\cong\prod_k H^k(X;\pi_{k-n}(E)\otimes\hat{Z}/Z) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i^*} \\ (E\hat{Z})^n(X^p) & \xrightarrow{\kappa^p} & (E\hat{Z}/Z)^n(X^p)\cong\prod_k H^k(X^p;\pi_{k-n}(E)\otimes\hat{Z}/Z) \end{array}$$

where $i\colon X^p\subset X$ is the inclusion. Take any element $x\in(E\hat{Z}/Z)^n(X^p)$, i.e., $x=\{x_k\}\in\prod_{k\leq p}H^k(X^p;\pi_{k-n}(E)\otimes\hat{Z}/Z)$. Because of (3.1) we may choose an element $y\in(E\hat{Z})^n(X^p)$ with $\kappa^p(y)=x_p$. On the other hand, there exists an element $z\in(E\hat{Z}/Z)^n(X)$ with $i^*(z)=x-x_p$. Therefore it follows from the surjectivity of κ that $\kappa^p\colon (E\hat{Z})^n(X^p)\to(E\hat{Z}/Z)^n(X^p)$ is surjective.

Now we show that Hausdorff-ness of $E^*(X)$ is shared by $E^*(X^p)$.

Theorem 4. *Let E be a CW-spectrum such that $\pi_*(E)$ is of finite type as an R-module where R is a subring of Q, and X be a connective CW-spectrum. Fix a degree n. Then $E^n(X)$ is Hausdorff if and only if $E^n(X^p)$ are Hausdorff for all p and in addition $\varprojlim_p{}^1 E^{n-1}(X^p)=0$.*

Proof. First consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varprojlim^1 E^{n-1}(X^p) & \longrightarrow & E^n(X) & \xrightarrow{\pi_1} & \varprojlim E^n(X^p) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \iota & & \downarrow \varprojlim \iota^p & & \\ 0 & \longrightarrow & \varprojlim^1 (E\hat{Z})^{n-1}(X^p) & \longrightarrow & (E\hat{Z})^n(X) & \xrightarrow{\pi_2} & \varprojlim (E\hat{Z})^n(X^p) & \longrightarrow & 0 \end{array}$$

involving Milnor's short exact sequences (two rows). Recall (2.2) that $\hat{Z} \cong \hat{R} \oplus \hat{R}'$. By the aid of (2.3) and [I, Proposition 6] we compute

$$\varprojlim_p{}^1 (E\hat{R})^*(X^p) \cong \varprojlim_p{}^1 E^*(X^p; \hat{Z}) \cong \varprojlim_p{}^1 \varprojlim_q E^*(X^p; Z_q) \cong \varprojlim_{p,q}{}^1 E^*(X^p; Z_q) = 0.$$

On the other hand, since $\pi_*(E) \otimes \hat{R}'$ is a Q-module we have

$$\varprojlim_p{}^1 (E\hat{R}')^*(X^p) \cong \varprojlim_p{}^1 \mathrm{Hom}\,(\pi_*(X^p),\ \ \pi_*(E) \otimes \hat{R}') = 0$$

by use of [II, (1.4)]. Thus $\varprojlim^1 (E\hat{Z})^*(X^p) = 0$. So π_2 becomes an isomorphism.

The "only if" part: Since $\iota: E^n(X) \to (E\hat{Z})^n(X)$ is a monomorphism, $\varprojlim^1 E^{n-1}(X^p) = 0$ and by Lemma 10 $\iota^p: E^n(X^p) \to (E\hat{Z})^n(X^p)$ is a monomorphism, i.e., $E^n(X^p)$ is Hausdorff.

The "if" part: The injectivity of ι^p for each p yields that $\varprojlim \iota^p$: $\varprojlim E^n(X^p) \to \varprojlim (E\hat{Z})^n(X^p)$ is injective. And $\pi_1: E^n(X) \to \varprojlim E^n(X^p)$ is an isomorphism because $\varprojlim^1 E^{n-1}(X^p) = 0$. Hence $\iota: E^n(X) \to (E\hat{Z})^n(X)$ is a monomorphism, i.e., $E^n(X)$ is Hausdorff.

3.2. Let R be a subring of Q and W be a connective CW-spectrum such that $H_*(W) \otimes R$ is a free R-module. Assume that $\pi_*(E)$ is a flat R-module. Note that an R-module is flat if and only if it is torsion free as an abelian group. We observe the Atiyah-Hirzebruch spectral sequence $\{E_r\}$ for $E^*(W)$.

First recall that the inclusion $Z \subset Q$ induces a natural homomorphism

$$ch: E^*(X) \longrightarrow (EQ)^*(X) \cong \prod_k H^k(X; \pi_{k-n}(E) \otimes Q) \tag{3.2}$$

for any CW-spectrum X, called the *Chern-Dold character*.

Let $\{'E_r\}$ denote the Atiyah-Hirzebruch spectral sequence for $(EQ)^*(W)$. The Chern-Dold character $ch: E^*(W) \to (EQ)^*(W)$ yields a morphism

$$\{E_r\} \longrightarrow \{'E_r\}$$

of spectral sequences. Consider the following commutative square

$$
*)\qquad
\begin{array}{ccc}
H^*(W;\pi_*(E)) & \longrightarrow & \mathrm{Hom}(H_*(W),\pi_*(E)) \\
\downarrow & & \downarrow \\
H^*(W;\pi_*(E)\otimes Q) & \longrightarrow & \mathrm{Hom}(H_*(W),\pi_*(E)\otimes Q)\,.
\end{array}
$$

Since $\mathrm{Ext}(H_*(W),\pi_*(E))\cong \mathrm{Ext}^1_R(H_*(W)\otimes R,\pi_*(E))=0$ and $\mathrm{Ext}(H_*(W),\pi_*(E)\otimes Q)=0$, the duality homomorphisms (two horizontal maps) become isomorphisms by applying the universal coefficient theorem. The right vertical map is a monomorphism because $\pi_*(E)$ is torsion free. This means that $E_2\to{}'E_2$ is a monomorphism.

On the other hand, the spectral sequence $\{'E_r\}$ collapses in virtue of Dold's theorem (1.3). Therefore we find

(3.3) *the Atiyah-Hirzebruch spectral sequence* $\{E_r\}$ *for* $E^*(W)$ *collapses.*

Lemma 11. *Assume that* $\pi_*(E)$ *is a flat* R*-module. If* W *is a connective* CW*-spectrum such that* $H_*(W)\otimes R$ *is a free* R*-module, then* $\iota\colon E^*(W)\to (E\hat{Z})^*(W)$ *is a monomorphism.*

Proof. Let $\{E_r\}$ and $\{\hat{E}_r\}$ be the Atiyah-Hirzebruch spectral sequences for $E^*(W)$ and $(E\hat{Z})^*(W)$ respectively. Since both $\pi_*(E)$ and $\pi_*(E\hat{Z})\cong\pi_*(E)\otimes\hat{Z}$ are torsion free, the spectral sequences $\{E_r\}$ and $\{\hat{E}_r\}$ collapse by (3.3). Then we note that they are strongly convergent [2, Proposition 9]. Moreover we see that $H^*(W;\pi_*(E))\to H^*(W;\pi_*(E)\otimes\hat{Z})$ is a monomorphism, replacing Q by $\hat{Z}$ in the previous diagram $*$). This means that $\iota\colon E^*(W)\to(E\hat{Z})^*(W)$ induces a monomorphism $\iota_2\colon E_2\to\hat{E}_2$ and hence so is $\iota_\infty\colon E_\infty\to\hat{E}_\infty$.

Consider the following commutative diagram

$$
\begin{array}{ccccc}
0\longrightarrow & F^pE^*(W)/F^{p+1}E^*(W) & \longrightarrow & E^*(W)/F^{p+1}E^*(W) & \\
& \downarrow & & \downarrow & \\
0\longrightarrow & F^p(E\hat{Z})^*(W)/F^{p+1}(E\hat{Z})^*(W) & \longrightarrow & (E\hat{Z})^*(W)/F^{p+1}(E\hat{Z})^*(W) & \\
& & \longrightarrow & E^*(W)/F^pE^*(W) & \longrightarrow 0 \\
& & & \downarrow & \\
& & \longrightarrow & (E\hat{Z})^*(W)/F^p(E\hat{Z})^*(W) & \longrightarrow 0
\end{array}
$$

with exact rows in which $\{F^pE^*(W)\}$ and $\{F^p(E\hat{Z})^*(W)\}$ are the usual decreasing filtrations of $E^*(W)$ and $(E\hat{Z})^*(W)$ defined by skeletons. The left vertical map is a monomorphism because $\iota_\infty\colon E_\infty^{p,*}\to\hat{E}_\infty^{p,*}$ is so. Hence an induction on p shows that

$$E^*(W)/F^pE^*(W)\longrightarrow(E\hat{Z})^*(W)/F^p(E\hat{Z})^*(W)$$

is a monomorphism for each p. Remark that

$$E^*(W)\cong\varprojlim_p E^*(W)/F^pE^*(W),\quad (E\hat{Z})^*(W)\cong\varprojlim_p(E\hat{Z})^*(W)/F^p(E\hat{Z})^*(W)$$

since the spectral sequences $\{E_r\}$ and $\{\hat{E}_r\}$ are strongly convergent. Then we pass to inverse limit and get that

$$\iota\colon E^*(W)\longrightarrow(E\hat{Z})^*(W)$$

is a monomorphism.

Lemma 11 combined with Theorem 1 shows Hausdorff-ness of $E^*(W)$ for W having a free R-module $H_*(W)\otimes R$.

Theorem 5. *Let R be a subring of Q and E be a CW-spectrum such that $\pi_*(E)$ is free and of finite type as an R-module. If W is a connective CW-spectrum with $H_*(W)\otimes R$ a free R-module, then $E^*(W)$ is Hausdorff.* (Cf., [3] and [20]).

Putting Theorem 2 ii) and Theorem 5 together we have

Corollary 12. *Let E be as in the above theorem. Assume that there exists a connective CW-spectrum W such that $H_*(W)\otimes R$ is a free R-module and a map $f\colon X\to W$ which induces a monomorphism $f_*\colon\pi_*(X)\otimes Q\to\pi_*(W)\otimes Q$. Then $E^*(X)$ is Hausdorff.*

4. *CW*-spectra with Finite Skeletons

4.1. From now on we shall restrict ourselves to CW-spectra with finite skeletons, i.e., each p-skeleton is a finite CW-spectrum.

First we define a decreasing filtration $\{C_s^{p,n-p}\}$ of $E^n(X^p)$ by

$$C_s^{p,n-p}=\operatorname{Im}\{E^n(X^{p+s-1})\to E^n(X^p)\}$$

for each s, $1\leqq s\leqq\infty$ where we use the convention $X=X^{\infty}$.

Let Y be a CW-spectrum with finite skeletons Y^p. Assume that $\pi_*(E)$ is of finite type as an R-module.

Lemma 13. *There exists $s_0=s_0(p, n)<\infty$ such that $C_\infty^{p,n-p}=C_{s_0}^{p,n-p}$ if and only if there exists $r_0=r_0(p, n)<\infty$ such that $C_\infty^{p,n-p}\otimes Q=C_{r_0}^{p,n-p}\otimes Q$.*

Proof. The "only if" part is evident.

The "if" part: Tensoring with Q the decreasing sequence

$$E^n(Y^p)=C_1^{p,n-p}\supset\cdots\supset C_{r_0}^{p,n-p}\supset\cdots\supset C_\infty^{p,n-p}$$

of finitely generated R-modules, by assumption we have

$$C_1^{p,n-p}\otimes Q\supset\cdots\supset C_{r_0}^{p,n-p}\otimes Q=\cdots=C_\infty^{p,n-p}\otimes Q.$$

This means that the group $C_{r_0}^{p,n-p}/C_\infty^{p,n-p}$ is finite because it is a finitely generated R-module. Therefore we can find $s_0=s_0(p, n)\geqq r_0$ for which $C_\infty^{p,n-p}=C_{s_0}^{p,n-p}$.

Here we introduce the natural homomorphism

$$ch(l)\colon E^n(X)\longrightarrow\prod_{k\leqq l}H^k(X;\pi_{k-n}(E)\otimes Q)\tag{4.1}$$

for each l, defined by the composition

$$E^n(X)\xrightarrow{ch}(EQ)^n(X)\cong\prod_k H^k(X;\pi_{k-n}(E)\otimes Q)\longrightarrow\prod_{k\leqq l}H^k(X;\pi_{k-n}(E)\otimes Q).$$

We give some criteria for Hausdorff-ness of $E^*(Y)$ for Y having with finite skeletons.

Theorem 6. *Let E be a CW-spectrum such that $\pi_*(E)$ is of finite type as an R-module where R is a subring of Q, and Y be a CW-spectrum with finite skeletons. Fix a degree n and let $\{E_r\}$ denote the Atiyah-Hirzebruch spectral sequence for $E^*(Y)$. Then the following conditions are equivalent* (cf., [22]):

i) *$E^{n+1}(Y)$ is Hausdorff,*

iv) *the inverse system $\{E^n(Y^p)\}$ satisfies the Mittag-Leffler condition,*

v) *for each p there exists $r_0=r_0(p, n)<\infty$ such that*

$$E_{r_0}^{p,n-p}=E_r^{p,n-p} \qquad \text{for all } r,\ r_0\leqq r<\infty,$$

vi) *for each* p *there exists* $r_0=r_0(p,n)<\infty$ *such that*

$$E_\infty^{p,n-p}=E_{r_0}^{p,n-p},$$

vii) *The homomorphism* $ch(l)\otimes Q\colon E^n(Y)\otimes Q\to\prod_{k\leqq l}H^k(Y;\pi_{k-n}(E)\otimes Q)$ *induced by the Chern-Dold character* $ch\colon E^n(Y)\to\prod_k H^k(Y;\pi_{k-n}(E)\otimes Q)$ *is an epimorphism for each* l.

Proof. The proof of the equivalence of i) and iv)–vi) is the same as that of [II, Theorem 5].

So we shall prove that vi) is equivalent to vii), using the following commutative diagram

$$\begin{array}{ccccc}
E^n(Y)\otimes Q & \xrightarrow{i^*_{l+s}\otimes Q} & E^n(Y^{l+s})\otimes Q & \xrightarrow{i^*_{l,l+s}\otimes Q} & E^n(Y^l)\otimes Q \\
\downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c_{l+s}} & & \downarrow{\scriptstyle c_l} \\
\prod_{k\leqq l}H^k(Y;\pi_{k-n}(E)\otimes Q) & \xrightarrow[i^*_{l+s}]{} & \prod_{k\leqq l}H^k(Y^{l+s};\pi_{k-n}(E)\otimes Q) & \xrightarrow[i^*_{l,l+s}]{} & \prod_{k\leqq l}H^k(Y^l;\pi_{k-n}(E)\otimes Q)
\end{array}$$

for $s\geqq 1$. In the above diagram all vertical maps are the homomorphisms $ch(l)\otimes Q$ induced by $ch(l)$, and $i_{l,l+s}\colon Y^l\subset Y^{l+s}$ and $i_{l+s}\colon Y^{l+s}\subset Y$ are the inclusions. Notice that the homomorphism

$$ch\otimes Q\colon E^n(X)\otimes Q\longrightarrow\prod_k H^k(X;\pi_{k-n}(E)\otimes Q)$$

is an isomorphism for any finite CW-spectrum X. So the vertical maps c_{l+s}, $s\geqq 0$, become epimorphisms, and in particular c_l becomes an isomorphism. The bottom horizontal map i^*_{l+s}, $s\geqq 1$, is obviously an isomorphism.

vi)$\to$vii): By [2, Lemma 7 ii)] and Lemma 13 we may assume that $C_{r_0}^{l+1,n-l-1}\otimes Q=C_\infty^{l+1,n-l-1}\otimes Q$. Take any element $x\in\prod_{k\leqq l}H^k(Y;\pi_{k-n}(E)\otimes Q)$. Then we can choose an element $y_{l+1}\in C_{r_0}^{l+1,n-l-1}\otimes Q\subset E^n(Y^{l+1})\otimes Q$ such that $c_{l+1}(y_{l+1})=i^*_{l+1}(x)$. From hypothesis $y_{l+1}\in C_\infty^{l+1,n-l-1}\otimes Q$, so we find an element $y \in E^n(Y) \otimes Q$ with $i^*_{l+1}(c(y)) = i^*_{l+1}(x)$. Since the injectivity of i^*_{l+1} shows $c(y) = x$, c is an epimorphism.

vii)→vi): Take any element $y_l \in C_2^{l,n-l} \otimes Q \subset E^n(Y^l) \otimes Q$, i.e., $y_l \in \mathrm{Im}\{E^n(Y^{l+1}) \otimes Q \to E^n(Y^l) \otimes Q\}$. Since i^*_{l+1} is an isomorphism, there exists an element $x \in \prod_{k \leq l} H^k(Y; \pi_{k-n}(E) \otimes Q)$ with $i^*_l(x) = c_l(y_l)$. By assumption that c is surjective we get an element $y \in E^n(Y) \otimes Q$ such that $c_l(i^*_l \otimes Q(y)) = c_l(y_l)$, and hence $i^*_l \otimes Q(y) = y_l$. Consequently we obtain

$$C_\infty^{l,n-l} \otimes Q = C_2^{l,n-l} \otimes Q.$$

Using [2, Lemma 7 ii)] and Lemma 13 again, this becomes equivalent to vi).

4.2. We now introduce a condition on $E^*(X)$.

Condition R. *For each $\alpha \in E^*(X)$ there exists a connective CW-spectrum W_α with $H_*(W_\alpha) \otimes R$ a free R-module and a map f_α: $X \to W_\alpha$ such that $\alpha \in \mathrm{Im}\{f^*_\alpha$: $E^*(W_\alpha) \to E^*(X)\}$.*

In order to study still more Hausdorff-ness of $E^*(Y)$ for Y with finite skeletons, we shall require the following

Lemma 14. *Let Y be a CW-spectrum with finite skeletons. If $H^*(Y; Q)$ satisfies Condition R, then there is a connective CW-spectrum W such that $H_*(W) \otimes R$ is a free R-module and a map f: $Y \to W$ which induces an epimorphism f^*: $H^*(W; Q) \to H^*(Y; Q)$.*

Proof. Assume that Y is $(m-1)$-connected. First for each k, $k \geqq m$, we shall construct a $(k-1)$-connected CW-spectrum W_k with $H_*(W_k) \otimes R$ a free R-module and a map f_k: $Y \to W_k$ such that $f^*_k: H^k(W_k; Q) \to H^k(Y; Q)$ is an epimorphism.

Let $\{y\}$ be a system of generators of $H^k(Y; Q)$. Note that it is a finite set. For each generator y there exists a map f_y: $Y \to W_y$ by hypothesis. By considering the direct product of the composite map f'_y: $Y \to W_y \to W_y/W_y^{k-1}$, we get a map

$$f_k: Y \longrightarrow \prod_y W_y \longrightarrow \prod_y W_y/W_y^{k-1}.$$

Putting $W_k=\prod_y W_y/W_y^{k-1}$, it is $(k-1)$-connected. The natural map

$$\bigvee_y W_y/W_y^{k-1}\longrightarrow\prod_y W_y/W_y^{k-1}=W_k$$

is a homotopy equivalence because y runs over a finite set. Therefore $H_*(W_k)\otimes R\cong\sum_y H_*(W_y/W_y^{k-1})\otimes R$, and it is a free R-module. Moreover we can easily see that f_k: $Y\to W_k$ induces an epimorphism f_k^*: $H^k(W_k;Q)\to H^k(Y;Q)$, using the following commutative diagram

$$\begin{array}{ccccc} H^k(W_y/W_y^{k-1};Q) & \longrightarrow & H^k(W_y;Q) & & \\ \downarrow & & \downarrow & \searrow f_y^* & \\ H^k(W_k;Q) & \longrightarrow & H^k(\prod_y W_y:Q) & \longrightarrow & H^k(Y;Q) \end{array}$$

in which the top horizontal map is an epimorphism.

We put $W=\prod_{m\leqq k}W_k$. By [21, Theorem 12.8] W is homotopy equivalent to $\vee W_k$ because W_k is $(k-1)$-connected. Hence W is $(m-1)$-connected and $H_*(W)\otimes R$ is a free R-module. We define a map f: $Y\to W$ by the direct product

$$f=\prod_k f_k\colon Y\longrightarrow W=\prod_k W_k.$$

Then f^*: $H^*(W;Q)\to H^*(Y;Q)$ is evidently an epimorphism because so is f^*: $H^k(W_k;Q)\to H^k(Y;Q)$.

Under some hypothesis on $E^*(Y)$ we give another criterion for Hausdorffness of $E^*(Y)$ for Y having finite skeletons.

Theorem 7. *Let E be a CW-spectrum such that $\pi_*(E)$ is free and of finite type as an R-module where R is a subring of Q and Y be a CW-spectrum with finite skeletons. Assume that $E^*(Y)$ satisfies Condition R. Then the following conditions are equivalent* (cf., [20]):

i) *$E^*(Y)$ is Hausdorff,*

viii) *$H^*(Y;Q)$ satisfies Condition R,*

ix) *there is a connective CW-spectrum W such that $H_*(W)\otimes R$ is a free R-module and a map $f:Y\to W$ which induces a monomorphism f_*: $\pi_*(Y)\otimes Q\to\pi_*(W)\otimes Q$.*

Proof. We prove the implications: i)→viii)→ix)→i). The implications viii)→ix) and ix)→i) follow from Lemmas 9 and 14, and Corollary 12.

We use Condition R on $E^*(Y)$ to show that i)→viii). Assume that $\pi_r(E)\neq 0$, and fix a non-zero homomorphism $\pi_r(E)\otimes Q\to Q$. By Theorem 6 $ch(n)\otimes Q$: $E^{n-r}(Y)\otimes Q\to\prod_{k\leqq n}H^k(Y;\pi_{k-n+r}(E)\otimes Q)$ is an epimorphism. So the composite map

$$\tilde{ch}(n)\otimes Q: E^{n-r}(Y)\otimes Q\longrightarrow\prod_{k\leqq n}H^k(Y;\pi_{k-n+r}(E)\otimes Q)\longrightarrow\tilde{H}^n(Y;Q)$$

is an epimorphism, too. Hence, for an arbitrary element $y\in H^n(Y;Q)$ there exists $\alpha=\alpha(y)\in E^{n-r}(Y)$ such that $(\tilde{ch}(n)\otimes Q)(\alpha\otimes 1/N)=y$ with $N\neq 0$. Under Condition R on $E^*(Y)$ we choose a map f_α: $Y\to W_\alpha$ and $\beta\in E^{n-r}(W_\alpha)$ such that $f_\alpha^*(\beta)=\alpha$. From the naturality of $\tilde{ch}(n)\otimes Q$ it follows that $f_\alpha^*(\tilde{ch}(n)\otimes Q(\beta\otimes 1/N))=y$. Consequently we obtain the required map f_α: $Y\to W_\alpha$.

4.3. Finally we study examples of cohomology theories E^* which satisfy Condition R.

Recall that every cohomology theory E^* is given by $E^n(X)=\{X, E\}_{-n}$. So an arbitrary element $x\in E^n(X)$ is represented by a map f_x: $X\to S^nE$. As is easily seen, we have

(4.2) *$E^*(X)$ satisfies Condition R for any CW-spectrum X, if $H_*(E)\otimes R$ is a free R-module.*

Remark that every CW-spectrum is homotopy equivalent to a CW-spectrum associated with a Ω-spectrum [21, Theorem 14.4]. Let E be a CW-spectrum associated with a Ω-spectrum $\{E_p\}$. Then there exists an isomorphism $E^p(X)\cong[X, E_p]$ for any based CW-complex X [21, Theorem 14.5]. This implies

(4.3) *$E^*(X)$ satisfies Condition R for any based CW-complex X, if $H_*(E_p)\otimes R$ are free R-modules for sufficiently large p.*

As is well known, $H_*(MU)$, $\pi_*(MU)$ and $\pi_*(K)$ are free and of finite type as Z-modules, but $H_*(K)$ is a Q-module. However K is the CW-spectrum associated with the Ω-spectrum $\{BU\}$, and $H_*(BU)$ is free as a Z-module.

(4.2) and (4.3) combined with the above results show

(4.4) i) $(MUR)^*(X)$ *satisfies Condition* R *for any* CW*-spectrum* X,
ii) $(KR)^*(X)$ *satisfies Condition* R *for any based* CW*-complex* X.

Applying Theorem 7 to $E=MUR$ or KR and using Corollary 12 we get

Theorem 8. *Let* R *be a subring of* Q *and* E *be a* CW*-spectrum such that* $\pi_*(E)$ *is free and of finite type as an* R*-module.*

i) *Let* Y *be a* CW*-spectrum with finite skeletons. If* $(MUR)^*(Y)$ *is Hausdorff, then so is* $E^*(Y)$.

ii) *Let* Y *be a based* CW*-complex with finite skeletons. If* $(KR)^*(Y)$ *is Hausdorff, then so is* $E^*(Y)$.

As a corollary of Theorem 8 i) and ii) we have

Corollary 15. *Let* Y *be a based* CW*-complex with finite skeletons. Then* $(MUR)^*(Y)$ *is Hausdorff if and only if* $(KR)^*(Y)$ *is so.* (Cf., (2.12) and (2.14)).

Let G be a compact Lie group. We denote by BG a classifying space for G, taken as a based CW-complex with finite skeletons. It was proved by Buhštaber-Miščenko [18] that $K^*(BG)$ is Hausdorff. From this fact and Theorem 8 ii) we conclude

Corollary 16. *Let* E *be a* CW*-spectrum such that* $\pi_*(E)$ *is free and of finite type as an* R*-module. Then* $E^*(BG)$ *is Hausdorff for an arbitrary compact Lie group* G.

As is well known, $H_*(BG)\otimes Q=0$ whenever G is a finite group. By the aid of Theorem 2 we remark

(4.5) $E^*(BG)$ *is Hausdorff for any finite group* G, *even if* $\pi_*(E)$ *doesn't satisfy the condition stated in Corollary* 16.

Appendix

We shall construct the spectral sequence mentioned in (1.2).

Let I be a partially ordered set and $\mathscr{C}=\{X_\alpha, f_{\alpha\beta}\}$ be a direct system of CW-spectra and skeletal maps indexed by I. As in [I] we associate with I a semi-simplicial complex $I_*=\{I_n\}_{n\geqq 0}$. Let I'_n denote the set of all non-degenerate n-simplexes of I and put

$$\overline{B\mathscr{C}}_n=\bigvee_{\sigma\in I'_n} X_\sigma$$

where $X_\sigma=X_{\alpha_0}$ and α_0 is the leading vertex of σ for each $\sigma=\{\alpha_0,\dots,\alpha_n\}\in I'_n$.

First we construct an increasing sequence

$$B\mathscr{C}_0\subset B\mathscr{C}_1\subset\cdots\subset B\mathscr{C}_n\subset\cdots \tag{A.1}$$

such that

$$B\mathscr{C}_n/B\mathscr{C}_{n-1}\cong S^n\overline{B\mathscr{C}}_n. \tag{A.2}$$

We start with $B\mathscr{C}_0=\overline{B\mathscr{C}}_0=\bigvee_\alpha X_\alpha$ and proceed inductively. Assume that we have constructed an increasing sequence

$$B\mathscr{C}_0\subset B\mathscr{C}_1\subset\cdots\subset B\mathscr{C}_{n-1}$$

and for each m, $1\leqq m\leqq n-1$, skeletal maps $\rho_m\colon \overline{B\mathscr{C}}_m\wedge\dot{\Delta}^{m,+}\to B\mathscr{C}_{m-1}$ and $\pi_m\colon \overline{B\mathscr{C}}_m\wedge\Delta^{m,+}\to B\mathscr{C}_m$ such that

$$\begin{array}{ccc}\overline{B\mathscr{C}}_m\wedge\dot{\Delta}^{m,+} & \xrightarrow[\rho_m]{} & B\mathscr{C}_{m-1}\\ \cap & & \cap\\ \overline{B\mathscr{C}}_m\wedge\Delta^{m,+} & \xrightarrow[\pi_m]{} & B\mathscr{C}_m\end{array}$$

is push out.

Let us denote by $F_i\colon \Delta^{n-1}\to\dot{\Delta}^n\subset\Delta^n$, $0\leqq i\leqq n$, the standard i-th face map and by $\phi_{i,\sigma}\colon X_\sigma\to X_{F_i\sigma}$, $0\leqq i\leqq n$, the maps defined by $\phi_{0,\sigma}=f_{\alpha_0\alpha_1}$ and $\phi_{i,\sigma}=id$ for $0<i\leqq n$. We define a skeletal map

$$\rho_n\colon \overline{B\mathscr{C}}_n\wedge\dot{\Delta}^{n,+}\longrightarrow B\mathscr{C}_{n-1}$$

by

$$\rho_n(x, F_iu)=\pi_{n-1}(\phi_{i,\sigma}x, u)$$

for $x\in X_\sigma$ and $u\in\Delta^{n-1}$. Then, according to [21, Theorem 7.21] there exists a CW-spectrum $B\mathscr{C}_n$ having $B\mathscr{C}_{n-1}$ as a subspectrum and an extension

$\pi_n: \overline{B\mathscr{C}}_n \wedge \Delta^{n,+} \to B\mathscr{C}_n$ such that

$$\begin{array}{ccc} \overline{B\mathscr{C}}_n \wedge \dot{\Delta}^{n,+} & \xrightarrow[\rho_n]{} & B\mathscr{C}_{n-1} \\ \cap & & \cap \\ \overline{B\mathscr{C}}_n \wedge \Delta^{n,+} & \xrightarrow[\pi_n]{} & B\mathscr{C}_n \end{array}$$

is push out. Moreover the induced map

$$S^n \overline{B\mathscr{C}}_n \cong \overline{B\mathscr{C}}_n \wedge \Delta^{n,+} / \overline{B\mathscr{C}}_n \wedge \dot{\Delta}^{n,+} \longrightarrow B\mathscr{C}_n / B\mathscr{C}_{n-1}$$

becomes an isomorphism because the above square is push out.

Let $B\mathscr{C}$ denote the direct limit formed from the increasing sequence $\{B\mathscr{C}_n\}_{n\geqq 0}$. We now observe the spectral sequences $\{E^r\}$ and $\{E_r\}$ for $E_*(B\mathscr{C})$ and $E^*(B\mathscr{C})$ associated with the filtration $\{B\mathscr{C}_n\}$ of $B\mathscr{C}$. From definition and (A.2) we obtain

$$E^1_{p,q} = E_{p+q}(B\mathscr{C}_p / B\mathscr{C}_{p-1}) \cong \sum_{\sigma \in I'_p} E_q(X_\sigma)$$

and

$$E_1^{p,q} = E^{p+q}(B\mathscr{C}_p / B\mathscr{C}_{p-1}) \cong \prod_{\sigma \in I'_p} E^q(X_\sigma).$$

By the standard argument as in Atiyah-Hirzebruch spectral sequences we compute the E^2- and E_2-terms

(A.3) $$E^2_{p,q} \cong \varinjlim_{\alpha}{}_p E_q(X_\alpha), \quad E_2^{p,q} = \varprojlim_{\alpha}{}^p E^q(X_\alpha).$$

The edge maps coincide with the natural homomorphisms

(A.4) $$\varinjlim_{\alpha} E_n(X_\alpha) \longrightarrow E_n(B\mathscr{C}), \ E^n(B\mathscr{C}) \longrightarrow \varprojlim_{\alpha} E^n(X_\alpha)$$

induced by the inclusions $\iota_\alpha: X_\alpha \subset B\mathscr{C}_0 \subset B\mathscr{C}$.

Assume that the underlying ordered set I is directed. Then direct limit functor $\varinjlim$ is an exact functor. So $\varinjlim_p E_*(X_\alpha) = 0$ for all $p \geqq 1$. From this we see that the above homology spectral sequence $\{E^r\}$ collapses and there is an isomorphism

(A.5) $$\varinjlim \iota_{\alpha *}: \varinjlim_{\alpha} E_n(X_\alpha) \cong E_n(B\mathscr{C}).$$

Let X be a CW-spectrum and $\mathscr{C} = \{X_\alpha\}$, $X = \cup X_\alpha$, be a direct system of subspectra (and inclusions) over a directed set. By an induction process we can extend the canonical map $B\mathscr{C}_0 = \vee X_\alpha \to X$ induced by the inclusions $i_\alpha: X_\alpha \subset X$ to a map $\varpi: B\mathscr{C} \to X$. Consider the following commutative triangle

$$\begin{array}{ccc} & \xrightarrow{\varinjlim \iota_{\alpha *}} & \pi_*(B\mathscr{C}) \\ \varinjlim \pi_*(X_\alpha) & & \downarrow \varpi_* \\ & \xrightarrow{\varinjlim i_{\alpha *}} & \pi_*(X) \end{array} .$$

As is well known $\varinjlim i_{\alpha *}$ is an isomorphism, and so is $\varinjlim \iota_{\alpha *}$ because of (A.5). Hence $\varpi: B\mathscr{C} \to X$ induces an isomorphism in homotopy. Thus

(A.6) *$\varpi: B\mathscr{C} \to X$ is a homotopy equivalence.*

Consequently we obtain

Theorem. *Let E and X be CW-spectra and $\mathscr{C} = \{X_\alpha\}$ a direct system of subspectra of X with $X = \cup X_\alpha$ over a directed set. Then there exists a spectral sequence $\{E_r\}$ associated with $E^*(X)$ by a suitable filtration such that*

$$E_2^{p,q} = \varprojlim_\alpha{}^p E^q(X_\alpha).$$

References

[1]-[8] and [9]-[15] are listed at the end of papers [I] and [II].

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