Publ. RIMS, Kyoto Univ.
10 (1974), 11-49

Characterization of Inner $*$-Automorphisms of W^*-Algebras

By

Hans J. BORCHERS*

1. Introduction

For some problems in physics I would like to have a characterization of $*$-automorphisms of a C^*-algebra which can be realized by unitary operators in the enveloping von Neumann algebra. Looking at this problem I realized that one should first treat the same problem, as an "exercise", for W^*-algebras. The simplification is due to the fact that an inner automorphism lies also on a one-parametric group, while, for a permanently ewakly inner automorphism this property is not known.

The technique emplyoed here is a further development of a method I have used in a recent paper [5] in order to give a new and constructive proof of the theorem of Kadison [6] and Sakai [9] on derivations and my own result [3] on groups with semibounded spectrum. The same technique as used in [5] has been developed independently by Arveson [1] and also by Pedersens [8]. The advantage of this method is due to the fact that it gives a rather explicit construction of the spectral resolution of the unitary operator we are looking for.

The technique is derived from the concept of creation- and annihilation-operators which is used in physics. These operators define a shift operation on the spectrum of the unitary operator we are looking for. The aim is, of course to cnostruct a spectral resolution. This means one has to associate to every projection in the center of the invariant elements a subset of the torus. The difficulty is due to the fact that an inner automorphism defines the unitary operator only up to a unitary in the center. This means the mapping from the projections to the

---

Communicated by H. Araki, July 19, 1973.
* Institut für Theoretische Physik, Universität Gottingen, Göttingen, Germany.

subset of the torus is not unique. But assume there exists such a map then we can form for two projections the difference set which is unique. That this can be done is the content of section 2. These difference sets contain also enough information in order to give a classification of automorphisms, section 3. Having associated to every pair a difference set one only needs to fix one point of the spectrum in order to localize the projections on the torus. That this can be done and gives rise to a unitary operator with the correct properties will be shown in section 4. In section 5, we exploit these techniques in order to give several necessary and sufficient conditions for an automorphism to be inner, including a generalization of the result of Kadison and Ringrose [6]. Some further applications are also discussed. In the last section we give a brief outline of the extension of our method to general locally compact abelian groups.

Acknowledgements

This work has been carried out at the Research Institute for Mathematical Sciences Kyoto University. The author would like to thank H. Araki and all other members of the institute for their warm hospitality, and the Japan Society for the Promotion of Science for financial support.

2. Notations and Preliminarie

By the couple (R, α) we denote a von Neumann algebra R and a $*$-automorphism α of R which is automatically a normal automorphism of R. The group $\{\alpha^n, n \in \boldsymbol{Z}\}$ is a representation of the additive group $\boldsymbol{Z}$. Its dual-group we denote by T. Since T is isomorphic to the unit circle we identify it also with $[0, 2\pi)$ so that 0 means the identity of T. A closed interval we write often as $[a, b]$ with the prescription that one shall start at e^{ia} on T and shall proceed in the positive direction until we get to the point e^{ib}. This means e.g. that $[a, b]$ and (b, a) are complementary sets. Then for $\varepsilon > 0$ the sets $[-\varepsilon, \varepsilon]$ are symmetrical, intervals around the identity of T.

2.1. Definitions: We introduce the following notations:

a) $R_0 = \{x \in R;\ \alpha x = x\}$ which is the set of invariant elements.

b) Z denotes the center of R.
Z_0 denotes the center of R_0.

c) P_0 denotes the set of projections in Z_0 (we introduce for this a separate symbol since we work exclusively with this set).

d) Let $x \in R$ and $f \in l^1$ then we denote

$$x(f) = \sum_n \alpha^n x f(n).$$

e) Let I be a closed interval of T then we denote by $l_I^1 = \{f \in l^1;$ support $\mathscr{F}^{-1} f \subset I\}$, where $\mathscr{F}$ denotes the Fourier-operator.

f) Let E be a projection in R_0 then we denote by $\phi_I E$ the union of the ranges of $\{x(f)E;\ x \in R, f \in l_I^1\}$ i.e. $\phi_I E$ is the smallest projection G in R such that $Gx(f)E = x(f)E$ holds for all $x \in R$ and $f \in l_I^1$.

The operator ϕ_I assigns to every projection $E \in R_0$ a unique different one. This operation is not linear, and for some pairs I and E, $\phi_I E$ might be zero. Before investigating its properties in detail we show:

2.2. Lemma: *Let G be a projector in R_0 and E its central support in R_0, then we have for all $I \subset T$*

$$\phi_I G = \phi_I E \in P_0$$

Proof: In order to prove the first relation, we remark that R_0 contains the identity. Hence we get $R \cdot R_0 = R$. Furthermore the elements in R_0 are invariant. This implies for $x \in R$, $y \in R_0$ the relation $(xy)(f) = x(f) \cdot y$. Thus we get:

$$\{(x \cdot y)(f) G\} = \{x(f) \cdot y \cdot G\} x \in R \qquad y \in R_0 .$$

Since we have $\overline{R_0 G \mathscr{H}} = E\mathscr{H}$ (where $\mathscr{H}$ denotes the Hilbert-space on which R is acting) we get

$$\phi_I G = \phi_I E .$$

In order to prove the second relation, notice that $G \in R_0$ and is therefore invariant. So we get:

$$\alpha(x(f)\cdot G)=\alpha(x(f))\cdot G=(\alpha x)(f)\cdot G.$$

Since α is an automorphism of R follows $\phi_I G$ is invariant. Let now $H\in R_0$ then we get:

$$Hx(f)G=(Hx)(f)\cdot G\in\{x(f)G;\ x\in R\}.$$

This means $\phi_I G\mathscr{H}$ is invariant under R_0. But since it belongs to R_0, the projector $\phi_I G$ belongs to the center of R_0.

This lemma shows that, with respect to the operations ϕ_I, we can restrict ourselves to Z_0. Since we only work with projections a separate notation for them is justified.

2.3. Theorem: *Consider ϕ_I as a mapping from P_0 into P_0 then it has the following properties*:

a. α) $E_1\geqq E_2$ implies $\phi_I E_1\geqq\phi_I E_2$.

β) Put $E_1\vee E_2=E_1+E_2-E_1\cdot E_2$ then we get

$$\phi_I(E_1\vee E_2)=(\phi_I E_1)\vee\phi_I E_2.$$

γ) *Let Γ be an ordered index set, let E_γ be increasing i.e. $E_\beta\leqq E_\gamma$ for $\beta<\gamma$ and let $E=\operatorname{str}\lim_{\gamma\in\Gamma}E_\gamma$ then we have*

$$\phi_I E=\operatorname{str}\lim_{\gamma\in\Gamma}\phi_I E_\gamma.$$

b. α) *The relation $I_1\subset I_2$ implies*

$$\phi_{I_1}E\leqq\phi_{I_2}E.$$

β) *Let I° denote the inner points of I. Assume $I_1\cap I_2\neq\emptyset$ and $(I_1\cup I_2)^\circ=I_1^\circ\cup I_2^\circ$ then we get*

$$\phi_{I_1\cup I_2}E=\phi_{I_1}E\vee\phi_{I_2}E$$

γ) *Let I_1 and I_2 be two arbitrary intervals, then we get*

$$\phi_{I_1^\circ}\phi_{I_2}E\leqq\phi_{I_1+I_2}E.$$

(Here we have written the group action of T with a $+$sign.)

c. α) *Assume $I^\circ\ni 0$ then we get*:

$$\phi_I E \geqq E.$$

β) *Let* $I=T$ *then we get*:

$$\phi_T E \in P_0 \cap Z.$$

Furthermore $\phi_T E$ *is the carrier of* E *in* $P_0 \cap Z$.

d. α) *Let* $F \in P_0 \cap Z$ *then we get*:

$$\phi_I(F \cdot E) = F \cdot \phi_I E.$$

β) *Let* E_1 *be such that* $E_1 \cdot \phi_I E = 0$ *then follows*:

$$E \cdot \phi_{I^{-1}} E_1 = 0.$$

Proof: a. α) Is trivial.

β) From α) we get $\phi_I E_1 \vee \phi_I E_2 \leqq \phi_I(E_1 \vee E_2)$. Assume $H \in P_0$ and $H \cdot \phi_I E_1 = H \cdot \phi_I E_2 = 0$ then we get for $x \in R$ and $f \in l_I^1$: $Hx(f)\{E_1\mathscr{H} + E_2\mathscr{H}\} = 0$. But this implies $Hx(f)(E_1 \vee E_2)\mathscr{H} = 0$. Consequently we have $\phi_I(E_1 \vee E_2) \leqq (\phi_I E_1) \vee (\phi_I E_2)$.

γ) Since $E_\gamma \leqq E$ we get from α) the relation $\phi_I E_\gamma \leqq \phi_I E$. Assume $H \in P_0$ is such that $H\phi_I E_\gamma = 0$ for all $\gamma \in \Gamma$ then follows $Hx(f)E_\gamma = 0$ for all $\gamma, x \in R, f \in l_I^1$ and consequently $Hx(f)E = 0$. This implies $H \cdot \phi_I E = 0$ and thus $\phi_I E \leqq \operatorname{str}\lim_{\gamma \in \Gamma} \phi_I E_\gamma$. But by the first line $\operatorname{str}\lim_{\gamma \in \Gamma} \phi_I E_\gamma \leqq \phi_I E$.

b. α) This follows directly from the relation

$$l_{I_1}^1 \subset l_{I_2}^1 \qquad \text{for} \quad I_1 \subset I_2.$$

β) First we have from b. α) the relations $\phi_{I_1 \cup I_2} E \geqq \phi_{I_j} E, j=1, 2$ and thus $\phi_{I_1 \cup I_2} E \geqq \phi_{I_1} E \vee \phi_{I_2} E$. If $f \in l_{I_1 \cup I_2}^1$ then we find functions $f_j \in l_{I_j}^1$, $j=1, 2$ with $f = f_1 + f_2$ since the two intervals are over lapping. So we get for $x \in R$ the equation

$$x(f)E = x(f_1)E + x(f_2)E$$

and from this $\phi_{I_1 \cup I_2} E \leqq \phi_{I_1} E \vee \phi_{I_2} E$

γ) Assume $f \in l_{I_1}^1$, $g \in l_{I_2}^1$ and $x, y \in R$ then we find

$$x(f)y(g) = \sum_{n,m} f(n)g(m)(\alpha^n x)(\alpha^m y)$$

$$=\sum_{n,m} f(n)g(n+m)\alpha^n(x\alpha^m y)$$

$$=\sum_m \sum_n f(n)g(n+m)\alpha^n(x\alpha^m y)\,.$$

Since $f\in l^1_{I_1}$ and $g\in l^1_{I_2}$ follows that for every fixed m the function $f(n)g(n+m)\in l^1_{I_1+I_2}$. This implies

$$(1-\phi_{I_1+I_2}E)x(f)y(y)E$$
$$=\sum_m(1-\phi_{I_1+I_2}E)\sum_n f(n)g(n+m)\alpha^n(x\alpha^m y)E=0.$$

This implies

$$\phi_{I_1}\circ\phi_{I_2}E\leqq\phi_{I_1+I_2}E$$

c. α) Since zero is an inner point of I exists a function $f\in l^1_I$ such that $\sum_n f(n)=1$. From this we get $1(f)=1$. And consequently

$$\phi_I E\geqq 1(f)E=E$$

β) If $I=T$ then we get from c. α) $\phi_T E\geqq E$. On the other hand the function $f=\begin{cases}1 & \text{for } n=0\\ 0 & \text{otherwise}\end{cases}$

is an element of $l^1_T=l^1$ and consequently we get $\{x(f);\ x\in R, f\in l^1\}=R$. This implies $\phi_T E\mathscr{H}=\overline{RE\mathscr{H}}$ and hence $\phi_T E$ is the $P_0\cap Z$ carrier of E.

d. α) Let $F\in P_0\cap Z$ then we get for $x\in R$, $f\in l^1_I$ and $E\in P_0$

$$x(f)F\cdot E\cdot\mathscr{H}=Fx(f)E\mathscr{H} \text{ and consequently}$$

$$\phi_I(FE)=F\phi_I E.$$

β) Let $E_1, E\in P_0$ and assume $E_1\phi_I E=0$, then we have for $x\in R$, $f\in l^1_I$

$$E_1 x(f)E=0\,.$$

Taking the adjoint of this equation we get $E(x(f)^*)E_1=0$. Now $\{x(f)\}^*=x^*(\bar f)$ which implies $Ex^*(\bar f)E_1=0$. Since $f\to\bar f$ maps l^1_I onto $l^1_{I^{-1}}$ we get $E\phi_{I^{-1}}E_1=0$.

This ends the proof of the theorem.

We should emphasize again that ϕ_I is not a one-to-one map. One gets a good idea of ϕ_I by choosing a unitary operator $u \in R$ and defining α by the relation $\alpha x = uxu^*$. If E_I is a spectral projection of u belonging to the interval $I \subset T$ then $\phi_{I_1} E_I$ is majorized by the spectral projection belonging to the interval $I + I_1$. Therefore in the case where the spectrum of u has a gap there exist pairs (E, I) such that $\phi_I E = 0$. Moreover it can happen that ϕ_I is the null-operator, for instance if α generates the group $\boldsymbol{Z}_2$ and I does not contain the points 0 and π then $\phi_I = 0$.

It is our aim to assign to every projector in P_0 a set of T. If this could be done, then the operator ϕ_I gives us information about the relative positions of these sets belonging to two projections. Since we know the operators ϕ_I we start with these relative positions.

2.4. Definition: Denote by I_ε the interval $\left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]$, $\varepsilon > 0$. Let $E_1, E_2 \in P_0$ then we define

$$n(E_1, E_2) = \{b \in T;\ \exists \varepsilon > 0 \text{ with } E_2 \phi_{b + I_\varepsilon} E_1 = 0\}$$

2.5. Proposition: *The sets* $n(E_1, E_2)$ *have the following properties*:

a) $n(E_1, E_2)$ *is open.*

b) $n(E_2, E_1) = n^{-1}(E_1, E_2)$.

c) $E'_1 \leqq E_1$ *and* $E'_2 \leqq E_2$ *implies*

$$n(E_1, E_2) \subset n(E'_1, E'_2).$$

d) *For* $F \in P_0 \cap Z$ we get

$$n(FE_1, E_2) = n(E_1, FE_2) = n(FE_1, FE_2).$$

e) $n(E_1 \vee E_2, E_3 \vee E_4) = n(E_1, E_3) \cap n(E_1, E_4) \cap n(E_2, E_3) \cap n(E_2, E_4)$

f) *Let* E_α *be an increasing family of projections and define* $E = \text{str.}\lim_\alpha E_\alpha$, *then we get*

$$n(E_1, E) = \{\bigcap_\alpha n(E_1, E_\alpha)\}^\circ$$

g) *Let* K *be a connected component of* $n(E_1, E_2)$, *and let* I *be the* *closure of K. Assume* $I \neq T$ *then we get*

$$E_2 \cdot \phi_I E_1 = 0$$

h) *Let* F_1 *and* F_2 *be the* $P_0 \cap Z$ *carriers of* E_1, E_2 *then* $n(E_1, E_2) = T$ *if and only if* $F_1 \cdot F_2 = 0$

Proof: a) Let $b \in n(E_1, E_2)$ then exists $\varepsilon > 0$ with $E_2 \phi_{b+I_\varepsilon} E_1 = 0$. If $b' \in (b+I_\varepsilon)^0$ then exists $\varepsilon' > 0$ with $b' + I_{\varepsilon'} \subset b + I_\varepsilon$. Hence we get from 2.3. b. α)

$$E_2 \phi_{b'+I_{\varepsilon'}} E_1 \leqq E_2 \phi_{b+I_\varepsilon} E_1 = 0 .$$

This means $b' \in n(E_1, E_2)$.

b) Let $b \in n(E_1, E_2)$, then exists $\varepsilon > 0$ with $E_2 \phi_{b+I_\varepsilon} E_1 = 0$. Thus we get from 2.3. d. β)

$$E_1 \phi_{(b+I_\varepsilon)^{-1}} E_2 = 0 .$$

Since $(b+I_\varepsilon)^{-1} = b^{-1} + I_\varepsilon$ we get $b^{-1} \in n(E_2, E_1)$ and consequently $n^{-1}(E_1, E_2) \subset n(E_2, E_1)$ and $n^{-1}(E_2, E_1) \subset n(E_1, E_2)$ which implies $n^{-1}(E_1, E_2) = n(E_2, E_1)$

c) This follows directly from 2.3. a. α)

d) From 2.3. d. α) follows:

$$E_2 \phi_I F E_1 = E_2 F \phi_I E_1 = F E_2 \phi_I F E_1, \text{ which implies d)}$$

e) Because of b) it is sufficient to prove the relation

$$n(E_1, E_3 \vee E_4) = n(E_1, E_3) \cap n(E_1, E_4) .$$

c) implies that the left hand side is contained in the intersection on the right. So it remains to prove the converse. Let $b \in n(E_1, E_3) \cap n(E_1, E_4)$ then exists $\varepsilon > 0$ with $E_3 \phi_{b+I_\varepsilon} E_1 = E_4 \phi_{b+I_\varepsilon} E_1 = 0$ and consequently $E_3 \vee E_4 \phi_{b+I} E_1 = 0$. This means $n(E_1, E_3) \cap n(E_1, E_4) \subset n(E_1, E_3 \vee E_4)$.

f) Since $E \geqq E_\alpha$ follows from *c)* $n(E_1, E) \subset n(E_1, E_\alpha)$, and hence $n(E_1, E) \subset \{\bigcap_\alpha n(E_1, E_\alpha)\}^\circ$. Assume b is contained in the right hand side, exists $\varepsilon > 0$ such that $b + I_\varepsilon$ is also contained in it. Thus we get $E_\alpha \phi_{b+I_\varepsilon} E_1 = 0$ for all α and consequently $E \phi_{b+I_\varepsilon} E_1 = 0$. This means

$$n(E_1, E_2) \supset \{\bigcap_{\alpha} n(E_1, E_\alpha)\}^\circ .$$

g) Let K be a connected component of $n(E_1, E_2)$. Let I_1 be a connected closed subset of K, then I_1 is compact. Hence exists a finite number of points $b_i \in I_1$ and ε_i such that $\cup b_i + I^\circ_{\varepsilon_i}$ covers I_1 and $E_2 \phi_{b_i + I_{\varepsilon_i}} E_1 = 0$. This implies by 2.3. b. β) that $E_2 \phi_{\cup_i b_i + I_{\varepsilon_i}} E_1 = 0$. From this we get $E_2 x(f) E_1 = 0$ for all $x \in R$ and all $f \in l^1_I$ such that $\operatorname{supp} \mathscr{F}^{-1} f \subset I^\circ$. Since these functions are dense in l^1_I and $f \rightarrow x(f)$ is continuous in f in the l^1 topology follows $E_2 x(f) E_1 = 0$ for $f \in l^1_I$. This implies $E_2 \phi_I x_1 = 0$.

h) From 2.3. c. β) follows $\phi_T E_1 = F_1$. If $n(E_1, E_2) = T$ then we get from g) $\phi_T E_1 E_2 = F_1 E_2 = 0$.

This ends the proof of the proposition.

3. Classification of *-Automorphisms

In this section we give a complete classification of automorphisms.

A. The class K_∞

Our investigation is based on the sets $n(E_1, E_2)$. The first obstruction which might happen is that these sets are empty. This leads us to the first

3.1. Definition: We say (R, α) belongs to the class K_∞ if for all pairs $E_1, E_2 \in P_0$ we have $n(E_1, E_2) = \emptyset$ or $= T$.

Equivalently (R, α) belongs to K_∞ if for all pairs $E_1, E_2 \in P_0$ the set $n(E_1, E_2)$ is invariant under the whole rotation-group of the torus.

The second definition we have put in, since it is in analogy to the defining relations of the other classes. Next we consider an arbitrary automorphisms (R, α).

3.2. Proposition: *Let (R, α) be arbitrary, then exists a unique projection $F_\infty \in P_0 \cap Z$ such that $(F_\infty R, \alpha)$ belongs to K_∞ and for every $F \in P_0 \cap Z$ with $F_\infty \cdot F = 0$ follows (FR, α) does not belong to K_∞. This means $((1 - F_\infty)R, \alpha)$ contains no invariant part which belongs to K_∞.*

Proof: Let $\mathscr{F}=\{F\in P_0\cap Z;\ (FR,\alpha)\in K_\infty\}$.

Assume first $F\in\mathscr{F}$ and $F_1\in P_0\cap Z$ with $F_1\leqq F$ then follows for $E_1, E_2\leqq F_1$ that also $E_1, E_2\leqq F$. Hence $n(E, E_2)$ must be $\emptyset$ or T. This means $F_1\in\mathscr{F}$. Let now $F_1, F_2\in\mathscr{F}$ and $E_1, E_2\leqq F_1\vee F_2$ then follows from 2.5. e)

$$n(E_1, E_2)=n(E_1, F_1E_2)\cap n(E_1, F_2E_2)$$

This implies $n(E_1, E_2)$ is $\emptyset$ or T since it holds for the right hand side. Consequently $F_1\vee F_2\in\mathscr{F}$. Let $F_\alpha\in\mathscr{F}$ be an increasing family of projections and define $F=\operatorname{str}\lim_\alpha F_\alpha$. Assume $E_1, E_2\leqq F$ then we get by 2.5. f) $n(E_1, E_2)=\{\bigcap_\alpha n(E_1, E_\alpha E_2)\}^\circ$. Since now $n(E_1, F_\alpha E_2)=n(F_\alpha E_1, F_\alpha E_2)$ is $\emptyset$ or T follows $n(E_1, E_2)$ is $\emptyset$ or T. Hence F belongs to $\mathscr{F}$. Using now Zorn's lemma we see that $\mathscr{F}$ contains a unique maximal element F_∞ such that $F\cdot F_\infty=F$ for all $F\in\mathscr{F}$. From the construction follows furthermore that $(1-F_\infty)R$ contains no invariant part belonging to K_∞.

Next we want to give a better characterization of the part which does not belong to K_∞.

3.3. Lemma: *For any pair (R, α) the following statements are equivalent.*

1. $F_\infty=0$.
2. *For every $F\in P_0\cap Z$ exists a pair of projection $E_1, E_2\leqq F$ such that*

$$n(E_1, E_2)\neq T \quad \textit{and} \quad n(E_1, E_2)\neq\emptyset.$$

3. *For every $E\in P_0$ exists a projection $E_1\leqq E$, $E_1\in P_0$ and a projection $E_2\in P_0$ such that*

$$n(E_1, E_2)\neq T \quad \textit{and} \quad n(E_1, E_2)\neq\emptyset.$$

4. *For every $E\in P_0$ exists a projection $E_1\leqq E$, $E_1\in P_0$, $E_1\neq 0$ such that*

$$n(E_1, E_1)\neq\emptyset.$$

Remark that $n(E_1, E_1)$ is also not T. Since $E_1\neq 0$ $n(E_1, E_1)$ must not contain 0 by 2.5. g)

Proof: 1.⇒2. Assume 1 holds but not 2. Then not 2 implies there exists an element $F \in P_0 \cap Z$ that for any pair $E_1, E_2 \leqq F$ we must have $n(E_1, E_2) = T$ or $\emptyset$. But this means (FR, α) belongs to K_∞, contradicting the assumption that $F_\infty = 0$

2.⇒3. Assume 2 holds and let $E \in P_0$ be given. Denote by $F(E)$ the support of E in $P_0 \cap Z$, then from 2 follows that there exist G and $E_2 \in P_0$ and G, $E_2 \leqq F(E)$ with $n(G, E_2) \neq T$ and $n(G, E_2) \neq \emptyset$. Multiplying all projection by $F(E_2)$ then we can assume $F(E) = F(E_2)$. Since $n(G, E_2) \neq T$ follows $F(G) \neq 0$. This implies $n(G, E) \neq T$. This implies for $b \notin n(G, E)$ and $\varepsilon > 0$ that $E_1 = E\phi_{b+I_\varepsilon} G \neq 0$. Choose now a and ε in such a way that $b + a + I_{2\varepsilon} \subset n(G, E_2)$ then we get:

$$E_2 \cdot \phi_{a+I_\varepsilon} E_1 = E_2 \phi_{a+I_\varepsilon}(E\phi_{b+I_\varepsilon} G) \leqq E_2 \phi_{a+I_\varepsilon^\circ} \phi_{b+I_\varepsilon} G \leqq E_2 \phi_{a+b+I_{2\varepsilon}} G = 0$$

by 2.5. c), 2.3. b. γ), and 2.5. g). This implies $n(E_1, E_2) \neq \emptyset$. Since $E_1 \neq 0$ and $F(E_1) \leqq F(E_2)$ by construction follows from 2.5. h) that $n(E_1, E_2) \neq T$

3.⇒4. Assume 3, and let E be given, then exists a pair $G \leqq E$ and E_2 such that $n(G, E_2) \neq T$ and $\neq \emptyset$. By 2.5. b) this also holds for $n(E_2, G)$. Let $b \notin n(E_2, G)$, then $E_1 = G\phi_{b+I_\varepsilon} E_2 \neq 0$. Choose a and ε such that $a + b + I_{2\varepsilon} \subset n(E_2, G)$ then we get again by 2.5. c), 2.3. b. γ), 2.5. g) and 2.5. h)

$$E_1 \phi_{a+I_\varepsilon} E_1 = (G\phi_{b+I_\varepsilon} E_2) \phi_{a+I_\varepsilon}(G\phi_{b+I_\varepsilon} E_2) \leqq G\phi_{a+I_\varepsilon}(G\phi_{b+I_\varepsilon} E_2)$$

$$\leqq G \cdot \phi_{a+I_\varepsilon^\circ} \phi_{b+I_\varepsilon} E_2 \leqq G\phi_{a+b+I_{2\varepsilon}} E_2 = 0 .$$

This implies $n(E_1, E_1) \neq \emptyset$. Since $E_1 \neq 0$ we have also $n(E_1, E_1) \neq T$.

4.⇒1. Is trivial.

B. The classes K_n, $n \in \boldsymbol{N}$.

Before defining these classes we need some preparation and motivation. We start with the technical part

3.4. Lemma: *Denote by* $n_\varepsilon(E_1, E_2)$ *those points of* $n(E_1, E_2)$ *which have distance greater than* ε *from the boundary of* $n(E_1, E_2)$. *Assume* $b \notin n(E_1, E_2)$ *then* $E' = E_2 \phi_{b+I_\varepsilon} E_1 \neq 0$. *From this we get* $n(E', E') \neq T$ *and*

$n(E', E') \supset n(E_2, E_2) \cup n_{2\varepsilon}(E_1, E_1) \cup \{n_\varepsilon(E_1, E_2) - b\} \cup \{n_\varepsilon(E_2, E_1) + b\}$

Proof: (i) Assume $a \in n(E_2, E_2)$ and ε' such that $a + I_{\varepsilon'} \subset n(E_2, E_2)$ then we get:

$$E' \cdot \phi_{a+I_{\varepsilon'}} E' = E_2 \cdot (\phi_{b+I_\varepsilon} E_1) \cdot \phi_{a+I_{\varepsilon'}}(E_2 \phi_{b+I_\varepsilon} E_1)$$

$$\leqq E_2 \cdot \phi_{a+I_{\varepsilon'}} E_2 = 0.$$

(ii) Assume $a + I_{\varepsilon'} \subset n_{2\varepsilon}(E_1, E_1)$, then we find:

$$E' \cdot \phi_{a+I_{\varepsilon'}} E' = E_2(\phi_{b+I_\varepsilon} E_1) \phi_{a+I_{\varepsilon'}}(E_2 \phi_{b+I_\varepsilon} E_1) \leqq (\phi_{b+I_\varepsilon} E_1) \phi_{a+I^\circ_{\varepsilon'}} \phi_{b+I_\varepsilon} E_1$$

$$\leqq (\phi_{b+I_\varepsilon} E_1) \cdot \phi_{a+b+I_{\varepsilon+\varepsilon'}} E_1 \,.$$

In order to show that this vanishes we make use of 2.3. d. β). $a + I_{\varepsilon'} \subset n_{2\varepsilon}(E_1, E_1)$ implies:

$0 = E_1 \cdot \phi_{a+I_{\varepsilon'+2\varepsilon}} E_1 \geqq E_1 \cdot \phi_{b^{-1}+I_\varepsilon} \circ \phi_{a+b+I_{\varepsilon'+\varepsilon}} E_1$ and hence

$(\phi_{b+I_\varepsilon} E_1) \phi_{a+b+I_{\varepsilon'+\varepsilon}} E_1 = 0$

(iii) Let $a + I_{\varepsilon'} \subset \{n_\varepsilon(E_1, E_2) - b\}$, then follows $a + b + I_{\varepsilon'+\varepsilon} \subset n(E_1, E_2)$. From this we get:

$$E' \cdot \phi_{a+I_{\varepsilon'}} E' = E_2(\phi_b + I_\varepsilon E_1) \phi_{a+I_{\varepsilon'}}(E_2 \phi_{b+I_\varepsilon} E_1)$$

$$\leqq E_2 \phi_{a+I_{\varepsilon'}} \circ \phi_{b+I_\varepsilon} E_1 \leqq E_2 \cdot \phi_{a+b+I_{\varepsilon'+\varepsilon}} E_1 = 0$$

(iv) The last term follows from (iii) by the relation $n(E_1, E_1) = n^{-1}(E_1, E_1)$.

It is our aim to classify the sets $n(E, E)$. In order to see what kind of situations might occur we look first at extremal cases.

3.5. Lemma: *Assume* $E \neq 0$ *and* $n(E, E) \neq \emptyset$. *Suppose furthermore that for every* E_1 *with* $n(E_1, E) \neq T$ *we get for all* $b \notin n(E_1, E)$ *and all* $\varepsilon > 0$ *the relation* $\{n_\varepsilon(E_1, E) - b\} \subset n(E, E)$, *then we have*:

(i) *There exists* $n \in \boldsymbol{N}$ *such that*

$$T \backslash n(E, E) = \left\{0, \frac{2\pi}{n} n, \ldots \frac{m}{n} 2\pi, \ldots \frac{n-1}{n} \cdot 2\pi\right\}.$$

(ii) *For every* E_1 *we get* $n(E_1, E)$ *is invariant under rotations by the angles* $\frac{m}{n}2\pi$, $m=0, 1,\dots n-1$.

Proof: (i) For proving the first statement we put $E_1=E$. Since T is compact there exists a connected ccomponent $I=(a, b)$ of $n(E, E)$ with maximal length. Since a does not belong to $n(E, E)$ we get from the assumption: $\{(a+\varepsilon, b-\varepsilon)-a)\subset n(E, E)$. Since $n(E, E)$ is symmetric follows also $((b-a-\varepsilon)^{-1}, -\varepsilon)\subset n(E, E)$. Since ε was arbitrary follows 0 is an isolated point of $T\setminus n(E, E)$. This implies $(0, c)\subset n(E, E)$ with $c=b-a$. Since c and hence c^{-1} does not belong to $n(E, E)$ follows that also $(0, c)+c=(c, 2c)\subset n(E, E)$. By repeating the argument we get

$$(mc, (m+1)c)\subset n(E, E) \qquad m=1.\ 2\dots$$

Since T is copact, exists a smallest n such that $n\cdot c=2\pi$. This implies $T\setminus n(E, E)$ contains at most the points $\left\{\frac{m}{n}2\pi;\ m=0, 1,\dots n-1\right\}$. None of these points can belong to $n(E, E)$, since by construction there exists no component of $n(E, E)$ with length greater than c. Since $E\neq 0$ follows from 2.3. c. α) that $0\notin n(E, E)$ which implies that n most be at least one.

(iii) Now let E_1 be arbitrary. If $n(E_1, E)=T$ then the statement is correct. Assume now $b\notin n(E_1, E)$ then follows from the assumption $n_\varepsilon(E_1, E)-b\subset n(E, E)$ or equivalently $n_\varepsilon(E_1, E)-b\not\subset\left\{\frac{m}{n}2\pi\right\}$ or $n_\varepsilon(E_1, E)\not\supset b+\left\{\frac{m}{n}2\pi\right\}$. Since ε was arbitrary follows $b\notin n(E_1, E)$ implies also $b+\frac{m}{n}2\pi\notin n(E_1, E)$, consequently $T\setminus n(E_1, E)$ is invariant and thus also $n(E_1, E)$ is invariant under the rotations by $\frac{m}{n}2\pi$.

From this lemma we extract the following

3.6. Definition: A pair (R, α) belongs to the class K_n, $n\in\boldsymbol{N}$, if the following two conditions are fulfilled:

(i) For every pair $E_1, E_2\in P_0$ the set $n(E_1, E_2)$ is invariant under the rotations by the angles $\frac{m}{n}2\pi$, $m=0, 1,\dots n-1$.

(ii) For every $F \in P_0 \cap Z$ and every $0<a<\frac{\pi}{n}$ exists an $E \in P_0$ with $E \leqq F$ and $n(E, E) \subset \left(a, \frac{2\pi}{n}-a\right)$.

3.7. Proposition: *Let (R, α) be arbitrary, then exists a unique projection $F_n \in P_0 \cap Z$ such that $(F_n R, \alpha)$ belongs to the class K_n, but for no projection $F \in P_0 \cap Z$ with $F \cdot F_n = 0$, (FR, α) belongs to K_n.*

Proof: Define $\mathscr{F} = \{F \in P_0 \cap Z;\ (FR, \alpha)$ belongs to the class $K_n\}$. Now we show:

(i) $F \in \mathscr{F}$ and $F_1 \leqq F$ with $F_1 \in P_0 \cap Z$ then $F_1 \in \mathscr{F}$. This is clear from the definition of K_n.

(ii) $F_1, F_2 \in \mathscr{F}$ implies $F_1 \vee F_2 \in \mathscr{F}$. Let $E_1, E_2 \leqq F_1 \vee F_2$ then follows $n(E_1, E_2) = n(F_1 E_1, E_2) \cap n((E_2 - F_1 F_2) E_1, E_2)$. Since both sets on the right hand side have the invariance property it follows also for $n(E_1, E_2)$. Now let $F < F_1 \vee F_2$, then $F \cdot F_1$ or FF_2 is not zero, say FF_1. If $F \in P_0 \cap Z$ then also $F \cdot F_1$. Hence exists for a given $0<a<\frac{\pi}{n}$ a projection E with $n(E, E) \supset \left(a, \frac{2\pi}{n}-a\right)$. Consequently $F_1 \vee F_2 \in \mathscr{F}$.

(iii) Let now $F_\alpha \in \mathscr{F}$ be an increasing family and let $F = \operatorname{str\,lim}_\alpha F_\alpha$. We want to show that F belongs to $\mathscr{F}$. Take $E_1, E_2 \leqq F$ then we have by 2.5. f) $n(E_1, E_2) = \{\bigcap_\alpha n(E_1, F_\alpha E_2)\}^\circ$. Since all $n(E_1, F_\alpha E_2)$ have the invariance property, it also follows for $n(E_1, E_2)$. Next let $F_1 \in P_0 \cap Z$ such that $0 \neq F_1 \leqq F$. Then exists α such that $F_1 F_\alpha \neq 0$, and hence we find E with $0 \neq E \leqq F_1 F_\alpha \leqq F_1$ such that $n(E, E) \supset \left(a, \frac{n}{2}-a\right)$ with $0<a<\frac{\pi}{n}$. This implies $F \in \mathscr{F}$.

Now by Zorn's lemma $\mathscr{F}$ contains a unique maximal element F_n such that $FF_n = F$ for al $F \in \mathscr{F}$. For any $F \in P_0 \cap Z$ with $FF_n = 0$ the pair (FR, α) does not belong to K_n by construction of the family $\mathscr{F}$.

Before we discuss the meaning of the different classes we show that the classification is complete.

3.8. Theorem: *Let R be a von Neumann algebra and α a $*$-auto-*

morphism of R. Then exist projections F_n, $n\in N$ and F_∞ belonging to $P_0\cap Z$ which are mutually orthogonal such that

$$F_\infty+\sum_{n\in N}F_n=1.$$

and such that (F_nR, α) belongs to K_n for $n=1, 2,\dots$ and ∞.

Proof: First we will define a decomposition, and afterwards we will show in two separate lemmas that this decomposition coincides with the decomposition into the classes K_n and K_∞.

We define families $\mathscr{F}_n$ by the following conditions: $F\in P_0\cap Z$ belongs to $\mathscr{F}_n$ if there exists an element $0\neq E\in P_0$ such that

(i) The $P_0\cap Z$ support of E is F.

(ii) In $n(E, E)$ exists a connected component I with length $l(I)>\frac{2\pi}{n+1}$.

From the construction follows $\mathscr{F}_n\supset\mathscr{F}_{n-1}$. Next we define projection $H_n\in P_0\cap Z$ by the relation $H_n=\bigvee_{F\in\mathscr{F}_n}F$. Since $\mathscr{F}_{n+1}\supset\mathscr{F}_n$ follows $H_{n+1}\geqq H_n$. Therefore exists $H_\infty=\operatorname{str}\lim_n H_n$.

Now define $G_\infty=1-H_\infty$ and $G_n=H_n-H_{n-1}$. From the construction follows $G_\infty+\sum_n G_n=1$. It remains to prove that G_∞ and G_n coincides with F_∞ and F_n defined in 3.2. and 3.7. We start with G_∞. Let $0\neq E\leqq G_\infty$ then we have $n(E, E)=\emptyset$. Because, if $n(E, E)\neq\emptyset$ then its $P_0\cap Z$ support $F(E)$ would belong to some family $\mathscr{F}_n$. Now lemma 3.3 implies that for $E_1, E_2\leqq G_\infty$ follows $n(E_1, E_2)=T$ or $\emptyset$. But this shows $G_\infty\leqq F_\infty$. Conversely if $F\in P_0\cap Z$ with $G_\infty F=0$ then exists a number n such that $F\cdot H_n\neq 0$ and by definition of H_n exists a projector $F_1\in\mathscr{F}_n$ with $F\cdot F_1\neq 0$. But by definition of F_1 exists E with $F(E)=F_1$ and $n(E, E)\neq\emptyset$ hence also $n(E, FE)\neq\emptyset$ and consequently $F\cdot F_\infty=0$. This implies $F_\infty\leqq G_\infty$ and thus $F_\infty=G_\infty$.

For proving the identity between G_n and F_n we need some preparations which we put into the form of two lemmas.

3.9. Lemma: (i) *Let F belong to $\mathscr{F}_n$ and let $\frac{2\pi}{n}>a>\frac{2\pi}{n+1}$, then there exists a projection $0\neq E\in P_0$ with its $P_0\cap Z$ support $\leqq F$ and $n(E, E)$ contains a connected component I with $l(I)\geqq a$*

(ii) *H_n belongs to $\mathscr{F}_n$, and there exists E with support $E=H_n$ and*

$n(E, E)$ contains a connected component I with $l(I) \geqq a$

Proof: (i) This part is based on lemma 3.4 and 3.5. Let $F \in \mathscr{F}_n$. For every $0 \neq E \in P_0$ with $E \leqq F$ define $l(E)$ to be the maximum of the length of the connected components of $n(E, E)$. Define $\bar{l}(F) = \sup_{0 \neq E \leqq F} l(E)$. Since F belongs to $\mathscr{F}_n$ follows $l(E) > \frac{2\pi}{n+1}$. By definition of $\bar{l}(F)$ exists for any $\varepsilon > 0$ a projector $0 \neq E \leqq F$ with $l(E) \leqq \bar{l}(F) + \varepsilon$. Assume this component I is $\left(c + \frac{\varepsilon'}{2}, c + \bar{l}(F) - \frac{\varepsilon'}{2}\right)$ with $0 \leqq \varepsilon' \leqq \varepsilon$, then follows from the fact that there exists no $E \neq 0$ with $l(E) > \bar{l}(F)$ that also $(\varepsilon', \bar{l}(F) - \varepsilon') \subset n(E, E)$. If $b \notin n(E, E)$ would belong to this interval we would find $\varepsilon_1 > 0$ with $\varepsilon' < b - \varepsilon_1$ and $b + \varepsilon_1 < \bar{l}(F) - \varepsilon'$ and hence $\left(c + \frac{\varepsilon'}{2}, c + \bar{l}(F) - \frac{\varepsilon'}{2}\right) + I_{\varepsilon_1} + b$ is the interval $\left(c + b + \frac{\varepsilon'}{2} + \varepsilon_1, c + \bar{l}(F) + b - \frac{\varepsilon'}{2} - \varepsilon_1\right)$. But we have $c + b + \frac{\varepsilon'}{2} + \varepsilon_1 < c + \bar{l}(F) - \frac{\varepsilon'}{2}$ and $c + \bar{l}(F) + b - \frac{\varepsilon'}{2} - \varepsilon_1 > c + \bar{l}(F) + \frac{\varepsilon'}{2}$. This would imply $E' = E \cdot \phi_{b + I_{\varepsilon_1}} E \neq 0$ and $l(E') > \bar{l}(F)$.

Repeating this argument we find

$$n(E, E) \supset (k\bar{l}(F) + (k+1)\varepsilon, \ (k+1)\bar{l}(F) - (k-1)\varepsilon)$$

Since such relation holds for all $\varepsilon > 0$ exists a number m with $m\bar{l}(F) = 2\pi$ or $\bar{l}(F) = \frac{2\pi}{m}$. Since $\bar{l}(F) > \frac{2\pi}{n+1}$ follows $m \leqq n$.

(ii) For any given a with $\frac{2\pi}{n+1} < a < \frac{2\pi}{n}$ define a family of pairs

$\mathscr{F}_a = \Big\{(F, E);\ F \in P_0 \cap Z$ and $F \leqq H_n$, $E \in P_0$ and its $P_0 \cap Z$ carrier $F(E)$ is equal to F, and $n(E, E) \subset \left(\frac{\pi}{n} - \frac{a}{2}, \frac{\pi}{n} + \frac{a}{2}\right)\Big\}$

From the construction given in the proof of (i) follows that $\mathscr{F}_a$ is not empty and, moreover, every $F \leqq H_n$ contains $F_1 \neq 0$ such that $(F_1, E) \in \mathscr{F}_a$ for a suitable E.

In $\mathscr{F}_a$ we introduce a semi-ordering by the relations $(F_1, E_1) < (F_2, E_2)$ if $F_1 \leqq F_2$ and $E_1 = F_1 E_2$. Let now $\{(F_\alpha, E_\alpha)\}$ be a stirctly ordered increasing set and define $F = \operatorname{str}\lim_\alpha F_\alpha$ and $E = \operatorname{str}\lim_\alpha E_\alpha$.

Since $E \geqq E_\alpha$ follows $F(E) \geqq F_\alpha$ and hence $F(E) \geqq F$. But since $E \leqq F$ follows $F(E)=F$. Furthermore we get $E_\alpha = F_\alpha \cdot E$ and from this $n(E_\alpha, E_\alpha) = n(E, F_\alpha E) = n(E, E_\alpha)$. Using 2.5. f) we get:

$$n(E, E) = \{\bigcap_\alpha n(E, E_\alpha)\}^\circ = \{\bigcap_\alpha n(E_\alpha, E)\}^\circ \supset \left(\frac{\pi}{n} - \frac{a}{2}, \frac{\pi}{n} + \frac{a}{2}\right).$$

This means every strictly ordered family has an upper bound, and consequently by Zorn's lemma exist maximal elements in $\mathscr{F}_a$. Let now (F, E) such a maximal element and assume $F \neq H_n$ then $H_n - F \neq 0$ and there exists $F_1 \neq 0$ such that $(F_1, E_1) \in \mathscr{F}_a$ for an appropriate E. Since $F_1 \cdot F = 0$ follows $(F+F_1, E+E_1)$ is strictly greater than (F, E). This contradicts the maximality and therefore we get $H_n = F$ for every maximal element (F, E). This proves the lemma. In addition we remark that for any given pair (F_1, E_1) there exists a maximal element (H_n, E) such that $E_1 = F_1 E$.

Next we turn to the projection $G_n = H_n - H_{n-1}$. We know that it fulfills the second part of the definition 3.6. In order to show that G_n and F_n coincide it remains to check the periodicity condition. This we will do in a separate lemma.

3.10. Lemma: *Let F belong to $P_0 \cap Z$ and assume for every $\frac{2\pi}{n} > a > \frac{2\pi}{n+1}$ exists $E \in P_0$ with $F(E) = F$ and $n(E, E)$ has a connected component I with length $l(I) \geqq a$. Assume furthermore for every $0 \neq E_1 \leqq F$ no connected component I of $n(E_1, E_1)$ can have length $l(I) > \frac{2\pi}{n}$. Then $n(E_1, E_2)$ is invariant under rotations by the angles $\frac{m}{n} 2\pi$, $m = 0, 1, \dots n-1$, provided $E_1, E_2 \leqq F$.*

Proof: The proof is based on a repeated use of lemma 3.4. This lemma does not say anything if $n=1$, therefore we will assume $n>1$. (α) Assume the conditions of the lemma are fulfilled and give $\varepsilon > 0$, then exists a projection $E \in P_0$, $F(E) = F$ such that $n(E, E) \supset \left(\frac{m}{n} 2\pi + \varepsilon, \frac{m+1}{n} 2\pi - \varepsilon\right)$, $m = 0, 1 \dots n-1$.

Proof: Let $\left[\frac{n}{2}\right]$ be the smallest entire number greater or equal to $\frac{n}{2}$, and let $\varepsilon_1 < \varepsilon / \left[\frac{n}{2}\right]$. By the remark at the end of the proof of the last lemma exists a projection $E \in P_0$ with $F(E) = F$ and $n(E, E) \supset \left(\varepsilon_1, \frac{2\pi}{n} - \varepsilon_1\right)$. Assume $b \in \left(\frac{2\pi}{n} + 2\varepsilon_1, \frac{2}{n} \cdot 2\pi - 2\varepsilon_1\right)$ then exists $\delta > 0$ such that $b + I_\delta$ is contained in the same interval. Since $n(E, E)$ is symmetric we have $n(E, E) \supset \left(-\frac{2\pi}{n} + \varepsilon_1, -\varepsilon_1\right)$. Now $\left(-\frac{2\pi}{n} + \varepsilon_1, -\varepsilon_1\right)_\delta + b$ is the interval $\left(-\frac{2\pi}{n} + \varepsilon_1 + \delta + b, -\varepsilon_1 - \delta + b\right)$. By choice of b we have $-\frac{2\pi}{n} + \varepsilon_1 + \delta + b < \frac{2\pi}{n} - \varepsilon_1$ and $-\varepsilon_1 - \delta + b < \frac{2\pi}{n} + \varepsilon_1$. Hence we get $l\left[\left(\varepsilon_1, \frac{2\pi}{n} - \varepsilon_1\right) \cup \left(-\frac{2\pi}{n} + \varepsilon_1, -\varepsilon_1\right)_\delta + b\right] > \frac{2\pi}{n}$. Using 3.4 and the assumptions of this lemma we get $E\phi_{-b+I_\delta}E = 0$ hence $n(E, E) \supset \left(\frac{2\pi}{n} + 2\varepsilon_1, \frac{2}{n} \cdot 2\pi - 2\varepsilon_1\right)$. If we repeat this argument we loose at every step ε_1. Hence we get

$$n(E, E) \supset \left(\frac{m}{n} 2\pi + (m+1)\varepsilon_1, \frac{m+1}{n} 2\pi - (m+1)\varepsilon_1\right).$$

Since $n(E, E)$ is symmetric we get the desired result by the choice of ε_1.
(β) Assume next there exists $E_1, E_2 \in P_0$, $E_1, E_2 \leqq F$ such that $n(E_1, E_2)$ is not invariant under rotations by the angles $\frac{m}{n} 2\pi$ then exists $0 \neq E' \in P_0$, $E' \leqq F$ and $\varepsilon > 0$ and $m \leqq \left[\frac{n}{2}\right]$ such that $n(E', E') \supset \frac{m}{n} 2\pi + I_\varepsilon$.

Proof: Since $n(E_1, E_2)$ not is periodic exists $b \notin n(E_1, E_2)$ and m such that $b + \frac{m}{n} \cdot 2\pi \in n(E_1, E_2)$. Since $n(E_1, E_2)$ is open exists $\varepsilon > 0$ such that $b + \frac{m}{n} \cdot 2\pi + I_\varepsilon \subset n(E_1, E_2)$. Take $\varepsilon_1 < \frac{\varepsilon}{3}$ and defien $E' \cdot E_2 = \phi_{b+I_{\varepsilon_1}} E_1$ which is unequal to zero. Lemma 3.4 implies $n(E', E') \supset \{n_{\varepsilon_1}(E_1, E_2) - b\} \supset \frac{m}{n} \cdot 2\pi + I_{\varepsilon_1}$ by choice of ε_1. Since $n(E', E')$ is symmetrical follows $n(E', E') \supset \frac{n-m}{n} 2\pi + I_{\varepsilon_1}$.
(γ) Assume the conclusion of the lemma is false, then by (β) exists $E' \neq 0$ $E' \leqq F$, $\varepsilon > 0$ and m with $n(E', E') \supset \frac{m}{n} \cdot 2\pi + I_\varepsilon$. Choose $\varepsilon_1 < \frac{\varepsilon}{5}$ then by (α) exists $E \neq 0$ with $F(E) = F$ and $n(E, E) \supset \left(\frac{m}{n} 2\pi + \varepsilon_1, \frac{m+1}{n} 2\pi - \right.$

$\left.\varepsilon_1\right)$ $m=1, 2,\dots, n-1$. Since $E'\neq 0$, $E'\leqq F=F(E)$ follows by 2.5. h) that $n(E', E)\neq T$. Hence exists $b\notin n(E', E)$ and consequently $E\cdot\phi_{b+I_{\varepsilon_1}}E'=E_1\neq 0$. But lemma 3.4. implies

$$n(E_1, E_1)\supset n(E, E)\cup n_{2\varepsilon_1}(E', E')\supset\left(\frac{m-1}{n}\cdot 2\pi+\varepsilon_1, \frac{m+1}{n}2\pi-\varepsilon_1\right)$$

by construction of ε_1. The length of this interval is $\frac{2}{n}\cdot 2\pi-2\varepsilon_1$ which is greater than $\frac{2\pi}{n}$ if ε_1 is sufficiently small. But this contradicts the assumptions of the lemma which therefore holds.

Proof of the Theorem (continuation). From lemma 3.9 and 3.10 follows that $(G_nR, \alpha)\in K_n$. Since G_n is the largest element in $P_0\cap Z$ fulfilling 3.9 and 3.10 follows $G_n=F_n$ and hence the Theorem 3.8 is proved.

4. Construction of Spectral Resolutions and Interpretation of the Classes

In this section we want to give an interpretation of the different classes in terms of inner and outer automorphisms. The main tool for this is the construction of spectral resolutions. We first will state the result and afterwards prepare the proof in several lemmas.

4.1. Theorem: *Let R be a von Neumann algebra and α be a $*$-automorphism of R then (R, α) belongs to K_n, $n\in N$ if and only if for any $F\in P_0\cap Z$ the automorphisms α^i restricted to FR are inner for $i\equiv 0 \bmod n$ and outer for $i\not\equiv 0 \bmod n$. (R, α) belongs to K_∞ if and only if for every $F\in P_0\cap Z$ α^i restricted to FR is outer for all $i\neq 0$.*

In order to simplify the discussion assume for the moment that R is a factor and (R, α) belongs to K_1. In this case we want to construct a unitary operator $U\in R$ which implements α. Our aim is to construct the spectral resolution of U which is located on T since U is unique only up to a phase-factor we have to fix an arbitrary point of T. Defining a spectral resolution located on T means that we have to assign to every $E\in P_0$ a subset of T. If this can be done then a filter of projection can also be used for fixing a point of T. Therefore our aim is to construct such a filter.

4.2. Lemma: *Let (R, α) belong to class K_n and let $0<\varepsilon<\frac{\pi}{n}$, then exists a decreasing family of projections $E_\varepsilon \in P_0$ such that*

(α) $E_{\varepsilon_1} \leqq E_{\varepsilon_2}$ *for* $\varepsilon_1<\varepsilon_2$

(β) *The* $P_0 \cap Z$ *support of* E *which we denote by* $F(E)$ *is equal to* 1

(γ) $n(E_\varepsilon, E_\varepsilon) \supset \left(\frac{m}{n} 2\pi+\varepsilon, \frac{m+1}{n} 2\pi-\varepsilon\right)$, $m=0, 1, \ldots n-1$

A family of projections with these properties will be called a point-fixing family f.

Proof: Since by Definition 3.6 all $n(E_1, E_2)$ are periodic it is sufficient to look at the interval $\left[0, \frac{2\pi}{n}\right]$. Since it is not required that all E_ε are different for different ε, it is sufficient to construct E_ε for some sequence tending to zero, e.g. on a geometric progression. Given ε_0 then we know from lemm 3.9. the existence of a projector E_{ε_0} with the properties (β) and (γ). Assume we have constructed E_ε for $\varepsilon=\varepsilon_0 2^{-i}$, $i=1, 2, \ldots n$ then we have to construct it for $i=n+1$. We write $\varepsilon_0 \cdot 2^{-n}=\delta$ for abbreviation. Let $b=\sup\{a ;\, 0 \leqq a<\delta,\, a \notin n(E_\delta, E_\delta)\}$ and define $E_1=E_\delta \cdot \phi_{b+I_{b/2}} E_\delta \neq 0$.

From lemma 3.4., we get:

$$n(E_1, E_1) \supset n(E_\delta, E_\delta) \cup \{n_{b/2}(E_\delta, E_\delta)-b\} \cup \{n_{b/2}(E_\delta, E_\delta)+b\}.$$

This means

$$n(E_1, E_1) \supset\left(\frac{b}{2}, \frac{2\pi}{n}-\frac{b}{2}\right) \quad \text{with} \quad \frac{b}{2} \leqq \varepsilon_0 2^{-(n+1)}.$$

Now let $F(E_1)$ be the $P_0 \subset Z$ support of E_1 and $F_2=1-F_1$ then we have:

$$0=F_2 \cdot E_1=F_2 E_\delta \cdot \phi_{b+I_{b/2}} F_2 E_\delta .$$

This implies

$$n(F_2 E_\delta, F_2 E_\delta) \supset\left(\frac{b}{2}, \frac{2\pi}{n}-\frac{b}{2}\right).$$

Putting $E_{\delta/2}=E_1+F_2 E_\delta$, we have $n(E_{\delta/2}, E_{\delta/2}) \supset\left(\frac{\delta}{2}, \frac{2\pi}{n}-\frac{\delta}{2}\right)$ and

$F(E_{\delta/2})=1$. This proves the lemma

4.3. Definition: Let f be a point-fixing family and $E\in P_0$, then we define the support of E, symbol $S_f(E)$, by the relation

$$S_f(E)=\text{complement of }\{\bigcup_{\varepsilon>0} n(E_\varepsilon, E)\}$$

Since we choose the family f once and for all, we write simply $S(E)$.

4.4. Proposition: *Let (R, α) belong to K_n then the support $S(E)$ has the following properties*:

a. *$S(E)$ is periodic with periodicity* $\frac{2\pi}{n}$

b. $S(E_\varepsilon)\subset$ period $[-\varepsilon, \varepsilon]$, *where* period $[a, b]$ *is defined as* $\bigcup_{0\leqq m\leqq n-1}\left[\frac{m}{n}2\pi+a, \frac{m}{n}2\pi+b\right]$

c. $S(\phi_{b+I\varepsilon_1}E_\varepsilon)\subset\text{period}\,[+b-\varepsilon-\varepsilon_1,\ +b+\varepsilon+\varepsilon_1]$

d. *Let* $E=E_1\vee E_2$ *then* $S(E)=S(E_1)\cup S(E_2)$

e. $S(E)=\phi$ *if and only if* $E=0$

f. *The relation* $S(E_1)\cap S(E_2)=\emptyset$ *implies* $E_1\cdot E_2=0$

g. *Let* E_α *be increasing and* $E=\text{str.}\lim_\alpha E_\alpha$ *then we get* $S(E)=\text{closure}\ \{\bigcup_\alpha S(E_\alpha)\}$

h. *Let* $G\subset T$ *be a closed set, then exists a unique maximal projection* E_G *such that* $S(E_G)\subset G$ *and such that for any projection* E_1 *with* $(1-E_G)E_1\neq 0$ *follows* $S(E_1)$ *is not a subset of* G.

Proof: a. This follows directly from Definition 3.6 and the definition of $S(E)$

b. For $\varepsilon_1<\varepsilon$ we have $E_{\varepsilon_1}\leqq E_\varepsilon$. This implies (see 2.5. c)) $n(E_{\varepsilon_1}, E_\varepsilon)\subset n(E_\varepsilon, E_\varepsilon)$ and consequently $S(E_\varepsilon)\subset\text{period}\,[-\varepsilon, \varepsilon]$

c. Assume $a\notin\text{period}\,[+b-\varepsilon-\varepsilon_1,\ +b+\varepsilon+\varepsilon_1]$ then exists $\delta>0$ such that $a+I_\delta$ does not interect with the same set. This also implies $a-b+I_\delta+I_{\varepsilon_1}\cap\text{period}\,[-\varepsilon, \varepsilon]=\emptyset$. For $\varepsilon_2<\varepsilon$ we get:

$$E_{\varepsilon_2}\phi_{a+I_\delta}\circ\phi_{b+I\varepsilon_1}E_\varepsilon\leqq E_\varepsilon\cdot\phi_{-a+I_\delta}\circ\phi_{b+I\varepsilon_1}E_\varepsilon\leqq E_\varepsilon\phi_{-a+b+I_{\delta+\varepsilon_1}}E_\varepsilon=0,$$

since $-a+b+I_{\delta+\varepsilon_1}\subset n(E_\varepsilon, E_\varepsilon)$. This implies (see 2.3. d. β)) $a\in$

$n(E_{\varepsilon_2}, \phi_{b+I\varepsilon_1}E_\varepsilon)$ for all $\varepsilon_2 < \varepsilon$. This proves the statement by the definition of $S(E)$.

d. From 2.5. e) we have $n(E_\varepsilon, E) = n(E_\varepsilon, E_1) \cap n(E_\varepsilon, E_2)$, and hence by the monotony in ε we find:

$$\bigcup_\varepsilon n(E_\varepsilon, E) = \bigcup_\varepsilon \{n(E_\varepsilon, E_1) \cap n(E_\varepsilon, E_2)\} = \{\bigcup_\varepsilon n(E_\varepsilon, E_1)\} \cap \{\bigcup_\varepsilon n(E_\varepsilon, E_2)\}.$$

Consequently we have $\{\bigcup_\varepsilon n(E_\varepsilon, E)\} = \{\bigcup_\varepsilon n(E_\varepsilon, E_1)\} \cap \{\bigcup_\varepsilon n(E_\varepsilon, E_2)\}$ which is equivalent to statement d.

e. If $E=0$ then $n(E_\varepsilon, E) = T$ for all ε and hence $S(E) = \emptyset$. If $S(E) = \emptyset$ then follows $\bigcup_\varepsilon n(E_\varepsilon, E) = T$. Since T is compact and $n(E_\varepsilon, E)$ is open exists ε_0 such that $n(E_{\varepsilon_0}, E) = T$. Since $F(E_\varepsilon) = 1$ we get $E=0$ by 2.5. b)

f. We get from d the relation $S(E_1, E_2) \subset S(E_i)$ $i=1, 2$. Hence $S(E_1) \cap S(E_2) = \emptyset$ implies $S(E_1\ E_2) = \emptyset$ and consequently $E_1 \cdot E_2 = 0$ follows from e.

g. From $E_\alpha \leqq E$ follows $S(E_\alpha) \subset S(E)$ and hence $\text{cl}\{\bigcup_\alpha S(E_\alpha)\} \subset S(E)$. Assume now $b \notin cl\{\bigcup_\alpha S(E_\alpha)\}$ then exists $\delta > 0$ such that $b + I_\delta \cap cl\{\bigcup_\alpha S(E_\alpha)\} = \emptyset$ which implies $b + I_\delta \cap S(E_\alpha) = \emptyset$ for all α. From C. follows $S(\phi_{b+I_{\delta/2}} E_{\delta/2}) \cap S(E_\alpha) = \emptyset$ and consequently from f. $E_\alpha \phi_{b+I_{\delta/2}} E_{\delta/2} = 0$ for all α. But this implies $E\phi_{b+I_{\delta/2}} E_{\delta/2} = 0$ and therefore $b \in n(E_{\delta/2}, E)$ which implies $b \notin S(E)$. This gives the relation $S(E) \subset cl\{S(E) \subset cl\{\bigcup_\alpha S(E_\alpha)\}$ which proves g.

h. Define $F_G\{E;\ S(E) \subset G\}$. Then d., g., and Zorn's lemma implies that F_G contains a unique maximal element. The definition of F_G implies that also the second stetement is fulfilled.

Statement h. of the last proposition permits us to define a spectral resolution.

4.5. Definition: Let $(R, \alpha) \in K_n$ then we define for $\lambda \in [0, 2\pi)$ the spectral projector E_λ to be the unique maximal element belonging to the set period $\left[0, \frac{\lambda}{n}\right]$ which is described in 4.4. h.

4.6. Lemma: *E_λ has the following properties*:

a. $E_\lambda \leqq E_\mu$ *for* $\lambda < \mu$

b. $\operatorname{str}\lim_{\lambda\to 2\pi} E_\lambda = 1$

c. $E_\lambda = \operatorname{str}\lim_{\substack{\mu>\lambda\\ \mu\to\lambda}} E_\mu$

d. *Assume* I_1 *and* $I_2 \subset I_1$ *are two connected intervals, define* $E(I)$ *as usual then we have for all* $x \in R$: *support* $\mathscr{F}_l^{-1}\{E(I_1)\alpha^{nl}xE(I_2)\} \subset$ closure $I_1 - I_2$. *The index* l *indicates that we have to take the Fourier transform with respect to* l.

Proof: a. $\lambda < \mu$ implies period $\left[0, \frac{\lambda}{n}\right] \subset$ period $\left[0, \frac{\mu}{n}\right]$ and hence by definition of the spectral family $E_\lambda \leqq E_\mu$

b. Let $E' = 1 - \operatorname{str}\lim_{\lambda\to 2\pi} E_\lambda$, and let $b + I_\varepsilon \subset \left(0, \frac{2\pi}{n}\right)$, and $E_{\varepsilon/2} \in f$, the point-fixing family. Then we have by 4.4. c. $S(\phi_{b+I_{\varepsilon/2}} E_{\varepsilon/2}) \subset$ period $\{b + I_\varepsilon\}$, and hence $\phi_{b+I_{\varepsilon/2}} E_{\varepsilon/2} \leqq E_\lambda$ for some λ.
This implies $E'\phi_{b+I_{\varepsilon/2}} E_{\varepsilon/2} = 0$ and consequently $S(E') =$ period $\{0\}$. From the maximality condition imposed on E_λ follows $E' = 0$

c. We have $S(\operatorname{str}\lim_{\lambda\to\mu+0} E_\lambda) \subset \bigcap_{\lambda>\mu}$ period $\left[0, \frac{\lambda}{n}\right] =$ period $\left[0, \frac{\mu}{n}\right]$. But this implies $\operatorname{str}\lim_{\lambda\to\mu+0} E_\lambda \leqq E_\mu$. Since the other inclusion is trivial, follows c.

d. As in b. it is easy to check that $S(E(I))$ is contained in $cl\left\{\text{period} \frac{1}{n}\right\} I$.
This implies (4.4. c. and f.) $E(I_1)\phi_{b+I_\varepsilon} E(I_2) = 0$ if $cl\left\{\text{period} \frac{1}{n} I_1\right\} \cap$ $cl\left\{b + I + \frac{1}{n} I_2\right\} = \emptyset$, and hence $n(E(I_2), E(I_1)) \subset \left\{\text{complement} \left[\text{period} \left(\frac{1}{n} I_1 - \frac{1}{n} I_2\right)\right]\right\}^0$

Let us denote by C the closure of the complement of $I_1 - I_2$, and take $f \in l_C^1$. Define $f^n(l) = \begin{cases} f\left(\frac{l}{n}\right) & \text{for } l \equiv 0 \bmod n. \\ 0 & \text{elsewhere.} \end{cases}$ From the relation $(\mathscr{F}^{-1} f^n)(a) = (\mathscr{F}^{-1} f)(n \cdot a)$ follows $f^n(l) \in l^1_{\frac{1}{n}C}$. Since $\frac{1}{n} C \subset n(E(I_2), E(I_1))$ we get by use of 2.5. g) and the definition of ϕ the equation

$$E(I_1)x(f^n)E(I_2) = 0, \qquad x \in R \quad \text{and} \quad f \in l_C^1.$$

This implies statement d.

We are now prepared to prove the main part of theorem 4.1.

4.7. Proposition: *Let* (R, α) *belong to* K_n *and let* E_λ, $0 \leqq \lambda < 2\pi$ *be the spectral family defined in* 4.5. *Define the unitary operator* $u \in R$ *by the equation* $u = \int_0^{2\pi} e^{i\lambda} \mathrm{d}E_\lambda$, *then we get for every* $x \in R$

$$\alpha^n x = u x u^* .$$

More precisely $u \in Z_0$.

Proof: Define $h = \int_0^{2\pi} \lambda \mathrm{d}E_\lambda$ and approximate it by the operators

$$h_N = \sum_{i=1}^{N} \frac{i}{N} 2\pi (E_{\frac{i}{N}2\pi} - E_{\frac{i-1}{N}2\pi})$$

then we have $\|h - h_N\| \leqq \frac{2\pi}{N}$. Define $u_N = e^{ih_N}$ then $u^l - u_N^l$ is bounded on the real axis by two and hence we get from Schwarz lemma and the Phragmen-Lindelöf theorem the estimate (see [11], 5.2. and 5.6.)

$$\|u^l - u_N^l\| = \|1 - e^{il(h_N - h)}\| \leqq |l| 2e \frac{2\pi}{N}.$$

Next consider the expression $u_N^{*l} \alpha^{nl} x u_N^l$, $x \in R$, as function of l. Replacing u_N by its definition we get:

$$u_N^{*l} \alpha^{nl} x u_N^l = \sum_{j,k=1}^{N} e^{-il\frac{j}{N}2\pi} (E_{\frac{j}{N}2\pi} - E_{\frac{j-1}{N}2\pi}) \alpha^{nl} x (E_{\frac{k}{N}2\pi} - E_{\frac{k-1}{N}2\pi}) e^{il\frac{k}{N}2\pi}.$$

Using lemma 4.6. d. we find

$$\text{support } \mathscr{F}_l^{-1} \Big\{ e^{-il\frac{j}{n}2\pi} (E_{\frac{j}{n}2\pi} - E_{\frac{j-1}{N}2\pi}) \alpha^{nl} x (E_{\frac{k}{N}2\pi} - E_{\frac{k-1}{N}2\pi}) e^{il\frac{k}{N}2\pi} \subset \Big[-\frac{2\pi}{N}, +\frac{2\pi}{N} \Big] \Big\}.$$

Since this holds for every term of the sum its also true for the whole sum. The Fourier integral defines an interpolation

$$(u_N^* \alpha^n x u_N)(z) = \int_{-\frac{2\pi}{n}}^{+\frac{2\pi}{n}} e^{iza} (\mathscr{F}_l^{-1} (u_N^{*l} \alpha^{nl} u_N))(a) \mathrm{d}a$$

which is an intire function of exponential type $\frac{2\pi}{N}$. Since it is bounded by $\|x\|$ on all entire values of z, there exists a constant K depending only on N such that the interpolation is bounded by $K\|x\|$ (Cartwright's theorem, see Boas [2] theorem 10. 3. 2.) This constant is majorized by $4+2e\log(1/_{1-\frac{2}{N}})$ and hence we can replace K by 5 if we choose $N\geqq 14$. From this we get $F_N(z)=x-(u_N^*\alpha^N x u_N)(z)$ is an entire function of exponential type $\frac{2\pi}{N}$ with $\|F_N(z)\|\leqq 6\|x\|$ for real z and $N\geqq 14$ which vanishes at the origin. Hence from Schwarz lemma and the Phragmén-Lindlöf theorem we get the estimate

$$\|F_N(z)\| \leqq |z|\,\|x\|6e\frac{2\pi}{N}$$

provided $N\geqq 14$ and $|z|\leqq \frac{N}{2\pi}$. This shows $F_N(z)$ converges to zero uniformly on every compact. Since u_N approaches u we get the desired result.

Let us now prove the converse statement.

4.8. Lemma: *Let* (R, α) *be given and assume for every* $F\in P_0\cap z$ α^n *restricted to* $F\cdot R$ *is an inner automorphism but for any* $0<m<n$ α^m *restricted to* $F\cdot R$ *is not inner, then* (R, α) *belongs to the class* K_n.

Proof: Let $f\in l^1$ then we define the functions f_k, $k=0,\dots n-1$ by the equation,

$$f_k(l)=\sum_{m=0}^{n-1}\frac{1}{n}\exp i\left\{\frac{m}{n}2\pi(l-k)\right\}f(l)=\frac{1}{n}\,\frac{1-e^{i2\pi(l-k)}}{1-e^{i2\pi\frac{l-k}{n}}}f(l)$$

$$=\begin{cases} f(l) \text{ for } l=k \mod n \\ 0 \text{ elsewhere.}\end{cases}$$

This implies $f(l)=\sum_{k=0}^{n-1}f_k(l)$. Now we compute

$$\sum_l e^{-ial}f(k+nl)=e^{i\frac{k}{n}a}\sum_l e^{-i\frac{a}{n}(k+nl)}f(k+nl)=e^{i\frac{k}{n}a}\sum_{l'}e^{-i\frac{a}{n}l'}f_k(l')$$

$$=e^{i\frac{k}{n}a}\sum_l e^{-i\frac{a}{n}l}\frac{1}{n}\sum_{m=0}^{n-1}e^{i\frac{m}{n}2\pi(l-k)}f(l)$$

$$= e^{ik\frac{a}{n}} \frac{1}{n} \sum_{m=0}^{n-1} e^{-i\frac{km}{n}2\pi} \sum_{l} e^{-i\frac{a-2\pi m}{n}l} f(l)$$

$$= e^{ik\frac{a}{n}} \frac{1}{n} \sum_{m=0}^{n-1} e^{-i\frac{km}{2}2\pi} (\mathscr{F}^{-1}f)\left(\frac{a-2\pi m}{n}\right).$$

Let now $f \in l_I^1$ then $f(k+nl)$ belongs for fixed k and n to l_{nI}^1. This implies for $x \in R$

$$\sum_{l} f(l)\alpha^l x = \sum_{k=0}^{n-1} \sum_{l} f(k+nl)\alpha^{nl}\alpha^k x$$

and hence we find for $E \in P_0$ the relation

$$\phi_I^\alpha E \leqq \phi_{nI}^{\alpha^n} E,$$

where the upper index indicates the automorphism with which the map ϕ_I is constructed. By assumption α^n is an inner automrophism, and hence exists a unitary operator $u = \int_0^{2\pi} e^{i\lambda} \mathrm{d}E_\lambda$ implementing α^n. Let now I_1 and I_2 be two intervals then we get

$$\text{support } \mathscr{F}^{-1}(E(I_1)u^l x u^{-l} E(I_2)) \subset \text{closure } I_1 - I_2$$

and hence $n^{\alpha^n}(E_\varepsilon, E_\varepsilon) \supset (\varepsilon, 2\pi - \varepsilon)$. From this follows $n^\alpha(E_\varepsilon, F_\varepsilon) \subset \left(\frac{\varepsilon}{n}, \frac{2\pi}{n} - \frac{\varepsilon}{n}\right)$. This means for every $\varepsilon > 0$ exists an $E \neq 0$ with $P_0 \cap Z$ support $= 1$ and $n(E, E)$ contains a connected interval of length $\geqq \frac{2\pi}{n} - 2\varepsilon$. On the other hand there can not exist an $E \neq 0$ such that $n(E, E)$ contains an interval of length greater then $\frac{2\pi}{n}$ otherwise α restricted to $F(E)R$ has the property that α^m is inner for some $m < n$. Because theorem 3.8 permits us to decompose $F(E)$ is $\Sigma F_k(E)$ when k divides n and proposition 4.7. allows us to construct unitary operators in $F_k(E)R$ implementing α^k. But this contradicts the assumptions of this lemma. Finally lemma 3.10 shows that all conditions for defining K_n are fulfilled.

Proof of the theorem: Let n be a number and $d(n)$ all its divisors including 1 but not n. Let (R, α) be arbitrary then exists a unique maximal projection H_n in $P_0 \cap Z$ such that α^n restricted to $H_n R$ is inner. Define $G_n = H_n - \bigvee_{k \in d(n)} H_k$. From the definition follows that all the

G_n are mutual orthogonal. Define $G_\infty = 1 - \sum_n G_n$. Compare this decomposition with that given by theorem 3.8. Proposition 4.7. and lemma 4.8. shows that G_n and F_n coincide. Hence G_∞ and F_∞ coincide by construction. But G_∞ can also be defined to be the maximal element of $P_0 \cap Z$ such that for every $F \leqq G_\infty$ no power of α restricted to $F \cdot R$ is inner. But this proves the theorem.

5. Characterization of Inner $*$-Automorphisms

In this section we want to give characterizations of inner automorphisms of a von Neumann algebra. Moreover we want to apply our technique to various problems related to automorphisms and derivations. To this end we introduce the following

5.1. Notations: Let β be a linear operator acting on a Banach-Space B then we denote

a. $\|\beta\|$ the usual operator norm

$$\|\beta\| = \sup\{\|\beta x\|;\ x \in B \quad \|x\| \leqq 1\}$$

b. $\mathrm{sp}(\beta)$ the spectrum of β which is the complement of the points $z \in C$ such that $(\beta - z)$ has a bounded inverse. If α is a $*$-automorphism of a W^* or C^* algebra then $\mathrm{sp}(\alpha)$ is located on the unit circle.

c. $\rho(\beta)$ denote the spectral norm of β

$$\rho(\beta) = \sup\{|z|;\ z \in \mathrm{sp}(\beta)\}$$

It is well known that $\rho(\beta) \leqq \|\beta\|$ and that $\rho(\beta) = \lim_{n \to \infty} \|x^n\|^{1/n}$.

d. Let B_1 be a closed linear subspace of B invariant under β, then we denote by $\beta|_{B_1}$ the restriction of β to B_1.

e. Let α be a $*$-automorphism of a C^*- or W^*-algebra then its spectrum is located on the unit circle T we denote by $g(\alpha) = T \backslash \mathrm{sp}(\alpha)$ and call it the gap of the spectrum. Since $\mathrm{sp}(\alpha)$ is closed follows $g(\alpha)$ is open.

Giving characterizations of inner automorphisms we will include in the list also the result of Kadison and Ringrose [7]. The main results of this section are as follows

5.2. Theorem: *Let α be a $*$-automorphism of a W^* algebra R, then the following statements are equivalent*:

a. *α is an inner automorphism.*

b. *α lies in the connected component of the $*$-automorphism group of R furnished with the norm-topology.*

c. *There exists an invariant projection $E\in R$ with central support 1 such that*

$$\rho\{(\alpha-1)_{|ERE}\}<\sqrt{3}$$

d. *There exists an invariant projection $E'\in R$ with central support 1 such that*

$$\|(\alpha-1)_{|E'RE'}\|<2$$

Both numbers $<\sqrt{3}$ in c and <2 in d cannot be improved.

5.3. Theorem: *For $*$-automorphisms we have the following relation between $\rho(\alpha-1)$ and $\|\alpha-1\|$.*

a. *If $\rho(\alpha-1)<\sqrt{3}$ then we get*

$$\|\alpha-1\|=\rho(\alpha-1)$$

b. *If $\sqrt{3}\leqq\rho(\alpha-1)\leqq 2$ then $\|\alpha-1\|$ is a double-valued function of $\rho(\alpha-1)$ and takes the values*

$$\|\alpha-1\|=\begin{cases}\text{either} & \rho(\alpha-1)\\ \text{or} & 2\end{cases}.$$

Both values are taken by special examples.

5.4. Theorem: A. *Let α be a $*$-automorphism of a W^*-algebra R then we have*

(i) $\mathrm{sp}(\alpha)$ *is invariant under the substitution*

$$z\longrightarrow z^{-1}$$

(ii) *If (R,α) belongs to the class K_n then it is also invariant under the substitution*

$$z \longrightarrow e^{i2\pi\frac{m}{n}}z \qquad m=0, 1,\dots n-1$$

(iii) *If the spectrum of* α *has a gap, then, in the standard decomposition (theorem* 3.8) *only those* F_n *can bc different from zero for which*

$$\exp\left\{i2\pi\frac{m}{n}\right\} \in \mathrm{sp}(\alpha) \qquad \text{for all } m=0, 1,\dots, n-1$$

(iv) *Assume* $\|\alpha-2\|<2$, *then exists a unique unitary operator in* R *with the properties: For every central projection* F

$$\mathrm{sp}(uF) \subset \mathrm{sp}\{\alpha_{|FRF}\} \cap \{\mathrm{Im}\, z \geqq 0\}$$

such that $\alpha x = uxu^*$ *for every* $x \in R$.

B. *Let* δ *be a real derivation of* R *thne we have*

(i) $\mathrm{sp}\,\delta$ *is invariant under the substitution*

$$z \longrightarrow -z$$

(ii) *There exists a unique positive operator* $A \in R$ *such that for every central projection* F *we have*

$$\mathrm{sp}(FA) \subset \mathrm{sp}\left\{\frac{\delta}{i}_{|FRF}\right\} \cap \{z \geqq 0\}$$

This last result generalizes a result of Pedersen [8].

For the proofs of these three theorems, we need some preparations. We start with

5.5. Lemma: *Let* α *be a* $*$*-automorphism of a* W^**-algebra* R *and let* ϕ_I *be the map defined in* 2.2., *then* $\phi_I=0$ *if* $I \subset g(\alpha)$ *and only if* $I^\circ \subset g(\alpha)$ (I° *the interior points of* I).

Proof: Assume first $I \subset g(\alpha)$. Since I is closed and $g(\alpha)$ is open exists a constant M such that $\|(\alpha-z)^{-1}\| \leqq M$ for all z in some compact set including I.

Now let $f \in l_I^1$ and $\hat{f} = \mathscr{F}^{-1} f$ then we have $\mathrm{supp}\,\hat{f} \subset I$ and $|\hat{f}(a)| \leqq \|f\|$ and furthermore $f(n) = \int e^{-ina}\hat{f}(a)\mathrm{d}a$. This gives

$$\sum_n f(n)\alpha^n = \sum_n \int e^{ina}\hat{f}(a)\mathrm{d}a\alpha^n = \sum_{-\infty}^{-1}\int e^{ina}\hat{f}(a)\mathrm{d}a\alpha^n + \sum_0^{\infty}\int e^{ina}\hat{f}(a)\mathrm{d}a\alpha^n$$

Since $f\in l_1$ follows $\int e^{-ina-rn}\hat{f}(a)\mathrm{d}a = f_r^+(n)$ converges to $f^+(n)$ for $n\geqq 0$ and $r\to +0$ in l^1 and $\int e^{-ina+rn}\hat{f}(a)\mathrm{d}a = f_r^-(n)$ converges to $f^-(n)$ for $n<0$ and $r\to +0$ in $l_1(f^+(n)=f(n)$ for $n\geqq 0$ and 0 for $n<0$, $f^-(n)=f(n)-f^+(n))$. Hence exists for any $\varepsilon>0$ a number $r_0(\varepsilon)>0$ such that $\|f_r^+(n)-f^+(n)\|\leqq\varepsilon\|f^+(n)\|$ and $\|f_r^-(n)-f^-(n)\|\leqq\varepsilon\|f^-(n)\|$ for $0\leqq r\leqq r_0$. Inserting now f_r^+ for f^+ and f_r^- for f^- then we can interchange the summation with the integration. So we get

$$\|\Sigma f(n)\alpha^n\| - \varepsilon\|f\| \leqq \|\sum_{n=-\infty}^{-1}\int e^{-ina+nr}\hat{f}(a)\mathrm{d}a\alpha^n + \sum_0^{\infty}\int e^{-ina-nr}\hat{f}(a)\mathrm{d}a\alpha^n\|$$

$$= \left\|\int \mathrm{d}a\hat{f}(a)e^{+ia-r}\alpha^{-1}(1-e^{ia-r}\alpha^{-1})^{-1} + \int \mathrm{d}a\hat{f}(a)(1-e^{-ia-r}\alpha)^{-1}\right\|$$

$$= \left\|\int \mathrm{d}a\hat{f}(a)\{e^{+ia-r}(\alpha-e^{+ia-r})^{-1} - e^{+ia+r}(\alpha-e^{+ia+r})^{-1}\}\right\|$$

$$= \left\|\int \mathrm{d}a\hat{f}(a)\,e^{+ia}(e^{-r}-e^{r})\alpha(\alpha-e^{ia-r})^{-1}(\alpha-e^{ia+r})^{-1}\right\|$$

$$\leqq 2\sinh r\cdot\|f\|L(I)M^2$$

where $L(I)$ denotes the length of the interval I. This implies

$$\|\sum_n f(n)\alpha\| \leqq \|f\{\varepsilon + 2\sinh r\cdot M(M+1)L(I)\}\,.$$

Since ε and r are arbitrarily small follows $\Sigma f(n)\alpha^n = 0$.

Let now conversely $\phi_I=0$ then follows $\phi_I 1=0$ and hence by definition of ϕ_I follows $\Sigma f(n)\alpha^n x=0$ for all $f\in l_I^1$ and all $x\in R$, and hence $\Sigma f(n)\alpha^n=0$ for all $f\in l_I^1$. Hence we get with the same calculation

$$\left\|\int \mathrm{d}af(a)\{e^{ia-r}(\alpha-e^{ia-r})^{-1} - e^{ia+r}(\alpha-e^{ia+r})^{-1}\}\right\| \leqq \varepsilon\left\|f\right\|$$

for $r<r_0(\varepsilon)$. This implies by the edge of the wedge theorem that $e^{ia-r}(\alpha-e^{ia-r})^{-1}$ is analytic in I°. Hence $I^\circ\subset g(\alpha)$.

From this we immediately get

5.6. Corollary: *Let* α *be a* $*$*-automorphism then we have*

(i) $\mathrm{sp}(\alpha)$ *is invariant under the substitution* $z \to z^{-1}$

(ii) *If* $(R, \alpha) \subset K_n$ *then* $\mathrm{sp}(\alpha)$ *is invariant under the substitution*

$$z \longrightarrow e^{i2\pi\frac{m}{n}}z \qquad m=0, 1,\dots, n-1$$

and $\exp\left\{i2\pi\frac{m}{n}\right\} \in \mathrm{sp}(\alpha)$ *for* $m=0, 1,\dots, n-1$

(iii) *If* $\rho(\alpha-1)<\sqrt{3}$ *then* α *is inner and there exists an example with* $\rho(\alpha-1)=\sqrt{3}$ *and* α *is outer.*

Proof: Let $z \in g(\alpha)$ then exists an closed intreval $I \subset g(\alpha)$ such that $z \in I^{\circ}$. Since we have $\phi_I=0$ follows for every $E \in P_0$ that $E\phi_I 1=0$ and hence by Theorem 2.3 we have $\phi_{I^{-1}}E=0$. Since this holds for all $E \in P_0$ follows $\phi_{I^{-1}}=0$. Since $z \in I^{\circ}$ follows $z^{-1} \in (I^{-1})^{\circ}$ and hence by lemma 5.5 we get $z^{-1} \in g(\alpha)$.

Now let $(R, \alpha) \in K_n$ then we have from lemma 3.10, for all E_1, $E_2 \in P_0$ that $n(E_1, E_2)$ is invariant under the rotations by the angles $2\pi\frac{m}{n}$, $m=0, 1,\dots, n-1$. Now let $z \in g(\alpha)$ then for some $\varepsilon>0$ we have $z+I_\varepsilon \subset g(\alpha)$ hence $\phi_{z+I_\varepsilon}E=0$ which means $n(E_1, E_2) \supset g(\alpha)$ for all E_1, E_2. This implies $\phi_{z\cdot e^{i2\pi\frac{m}{n}}+I_\varepsilon}E=0$ for all E and hence $z\cdot e^{i2\pi\frac{m}{n}} \in g(\alpha)$.

Theorem 2.3 implies $\phi_I E \geqq E$ if $1 \subset I^{\circ}$ hence $1 \in \mathrm{sp}\,\alpha$ and since $(\alpha, R) \in K_n$ follows from the previous result $e^{i2\pi\frac{m}{n}} \in \mathrm{sp}(\alpha)$.

Let now $\rho(\alpha-1)<\sqrt{3}$ then $\mathrm{sp}(\alpha)$ is contained in the interval $[e^{-il}, e^{+il}]$ with

$$\rho^2(\alpha-1)=2-2\cos l$$

which means $l<\frac{2\pi}{3}$. Let now $\left[\frac{n}{2}\right]$ the smallest entire number $\geqq\frac{n}{2}$ then we have for $n\geqq 2$, $\left[-\frac{n}{2}\right]/n\leqq\frac{2}{3}$ and $\left[\frac{n}{2}\right]/n\geqq\frac{1}{2}$ which implies

$$\pi\leqq 2\pi\left[\frac{n}{2}\right]/n\leqq\frac{4\pi}{3}.$$

This means in the standard decomposition (Theorem 3.8) can appear only the term F_1 this means α is inner.

Let now M be a factor of type II or III and α be the mapping $x\otimes y\otimes z\to z\otimes x\otimes y$. By a recent result of Sakai [10] this defines an outer automorphism of $M\otimes M\otimes M$. Since $\alpha^3=1$ follows the spectrum of α are exactly the three third roots of 1. This implies $\rho(\alpha-1)=\sqrt{3}$.

As next we show

5.7. Lemma: *Let α be a $*$-automorphism of a W^*-algebra R and E be an invariant projection with central support equal to one. Suppose $\alpha_{|ERE}$ is an inner automorphism then follows α is inner.*

Proof: Let identify R with some normal faithful representation of R on some Hilbert space H. By the assumption exists an operator $v\in ERE$ with $vv^*=v^*v=E$ and $vxv^*=\alpha x$ for every $x\in ERE$. Define an operator u on H by the equation

$$u\sum_i x_i y_i f_i=\sum_i(\alpha x_i)y_i v f_i$$

$$x_i\in R,\ y_i\in R' \quad \text{and} \quad f_i\in E\cdot H$$

We get the estimate (since $f_i=Ef_i$):

$$\|u\sum_i x_i y_i f_i\|^2=\sum_{ij}(f_j,\ v^*\alpha(x_j^*x_i)vy_j^*y_i f_j)$$

$$=\sum_{ij}(f_j,\ v^*\alpha(Ex_j^*x_iE)vy_j^*y_i f_i)$$

$$=\sum_{ij}(f_j,\ Ex_j^*x_i y_j^* y_i f)=\|\sum_i x_i y_i f_i\|^2.$$

This means u is an isometric operator. Since E has central support 1 follows that the domain and the range is dense in H which means the closure of u is a unitary operator.

From the defining equation of u one sees that $u\in R$ and furthermore the relation $(\alpha x)u=ux$. Since u is unitary follows that u implements the automorphism.

As last step for the demonstration of Theorem 5.2 we prove the following

5.8. Lemma: *Let α be an inner $*$-automorphism of a W^*-algebra and $\varepsilon>0$ arbitrary, then exists an invariant projection E with central* *support one such that* $\|(\alpha-1)_{|ERE}\| \leqq \varepsilon$.

Proof: We start from the construction of section 4. Since we have a point fixing family follows that for every $\delta>0$ that $E([-\delta, +\delta])$ has central support 1. Choose now $\delta=\frac{\varepsilon}{4e}$ and $E=E([-\delta, +\delta])$ then $uE=E\cdot e^{iAE}$ with $\|AE\| \leqq \delta$. Now $u\cdot E$ implements the automorphism $\alpha_{|ERE}$. Let now $\tilde{\alpha}_t x=e^{itAE}xe^{-itAE}$ for $x\in ERE$. $\tilde{\alpha}_t$ is an entire analytic function with $\|\tilde{\alpha}_t\| \leqq e^{2\delta|\mathrm{Im}\, t|}$. Now $\tilde{\alpha}_t-1$ is zero for $t=0$ and fulfills the estimate $\|\tilde{\alpha}_t-1\| \leqq 2e^{2\delta|\mathrm{Im}\, t|}$. Using now Schwarz lemma for the circle $|2\delta t| \leqq 1$ we get $\|\tilde{\alpha}_t-1\| \leqq 4e\cdot\delta|t|$, $|2\delta t| \leqq 1$ for $t=1$ we have $\tilde{\alpha}_{t-1}=(\alpha-1)_{|ERE}$ and hence $\|(\alpha-1)_{|ERE}\| \leqq 4e\delta=\varepsilon$.

Proof of theorem 5.2. If a is fulfilled, then lemma 5.8 shows that d is fullfilled and also c since $\rho\{(\alpha-1)_{|ERE}\} \leqq \|(\alpha-1)_{|ERE}$. If c holds, then corollary 5.6 shows that $\alpha_{|ERE}$ is inner and consequently α is inner by lemma 5.7. If d holds then $\alpha_{|ERE}$ is inner by the result of Kadison and Ringrose ([7] theorem 7) and hence α is inner by lemma 5.7. If α is an automorphism fulfilling b, then α is inner by either c or d. If a holds then $\alpha x=e^{iA}xe^{-iA}$ with some $A\in R$ and $\|A\|<T$. Hence $\alpha_t x=e^{itA}xe^{-itA}$ is a norm continuous one-parametric group of inner automorphisms. This means b is fulfilled. Since there exist an outer automorphism with $\rho(\alpha-1)=\sqrt{3}$ (See the example given at the end of the proof of corollary 5.6. We must also have $\|\alpha-1\|=2$ which means that the region of validity of 5.2 can not be improved.

Our next aim is the proof of theorem 5.4, but we first need some preparation.

5.9. Lemma: *Let* $\rho(\alpha-1)<\sqrt{3}$ *(this implies* α *is inner by Theorem* 5.2) *and define for* $-\frac{\pi}{3}<\mu<\frac{2\pi}{3}$ *the interval*

$$I_\mu=\left[\frac{2\pi}{3}+\mu,\ -\mu\right]$$

and E_μ*=maximal projection* $E\in P_0$ *such that* $\phi_{I_\mu}E_\mu=0$. *Theorem* 2.3 *implies that this maximal projection exists and is unique.* E_μ *has the following properties*:

a. $\mu<0$ *implies* $E_\mu=0$

b. *$\mu_1>\mu_2$ implies $E_{\mu_1}\geqq E_{\mu_2}$*
c. *For every central projection F define $l_F>0$ by $2(1-\cos l_F)=\rho^2\{(\alpha-1)_{|FRF}\}$, then $F\cdot E_\mu=F$ for $\mu>l_F$*
d. *For $\mu>0$ we have central support of $E_\mu=1$*
e. *$\mu>0$ then we have $T\backslash n(E_\mu, E_\mu)\subset[-\mu, +\mu]$.*

The properties b, d and e show that $\{E_\mu\}$ is a point-fixing family. Let $S(E)$ be the support of E defined by $\{E_\mu\}$ (see Definition 4.3) then we have

f. For every $E\in P_0$ and every central projection F we have

$$S(E\cdot F)\subset \mathrm{sp}\,\{\alpha_{|FRF}\}\cap\{\mathrm{Im}\,z\geqq 0\}$$

Proof: a. For $\mu<0$ follows $1\subset I_\mu^0$ and hence by Theorem 2.3. c. d) we have $\phi_{I_\mu}E\geqq E$ for all $E\in P_0$ which implies $E_\mu=0$

b. $\mu_1>\mu_2$ implies $I_{\mu_1}\subset I_{\mu_2}$ and hence by 2.3. $\phi_{I_{\mu_1}}E_{\mu_2}\leqq\phi_{I_{\mu_2}}E_{\mu_2}=0$ which implies $E_{\mu_2}\leqq E_{\mu_1}$

c. Let $\mu>l_F$ then follows $I_\mu\subset g(\alpha_{|FRF})$ and hence by Lemma 5.5 $\phi_{I_\mu}FE=0$ for all E which implies $E_\mu\geqq F$.

d. Let $\mu>0$ and F_μ the central carrier of E_μ. Assume $F_\mu\neq 1$ then follows $\phi_{I_\mu}E\neq 0$ for all $0\neq E\leqq(1-F)$. This implies $\phi_{I_\mu^{-1}}(1-F_\mu)=(1-F_\mu)$ (otherwise if there exists an $E\leqq(1-F_\mu)$ with $E\phi_{I_\mu^{-1}}(1-F_\mu)=0$ would imply $0=(1-F_\mu)\phi^\mu_{I\mu}E=\phi_{I_\mu}E$, contradicitng the assumption). Since $\phi_{I_\mu^{-1}}(1-F_\mu)=\phi_{\left[\mu,\frac{2\pi}{3}\right]}(1-F_\mu)\vee\phi_{\left[\frac{2\pi}{3}-\varepsilon,\frac{4\pi}{3}-\mu\right]}(1-F_\mu)$ by 2.3, and the second expression vanishes for sufficiently small ε by Lemma 5.5, we get $\phi_{\left[\mu,\frac{2\pi}{3}\right]}(1-F_\mu)=(1-F_\mu)$. Assume now $\mu<l_{1-F\mu}$ and $2l_{1-F\mu}-\mu+\dfrac{3\varepsilon}{2}<\dfrac{4\pi}{3}$ then we have $\left[\mu,\dfrac{2\pi}{3}\right]+(l_{1-F\mu}-\mu)+\varepsilon+I_{\varepsilon/2}\subset g(\alpha)$ and hence by 5.5 and theorem 2.3. $0=\phi\{_{\left[\mu,\frac{2\pi}{3}\right]+l_{1-F\mu}-\mu+\varepsilon+I_{\varepsilon/2}}\}(1-F_\mu)\geqq\phi_{l_{1-F\mu}-\mu+\varepsilon+I_{\varepsilon/2}}\phi_{\left[\mu,\frac{2\pi}{3}\right]}(1-F_\mu)=\phi_{l_{1-F\mu}-\varepsilon+I_{\varepsilon/2}}(1-F_\mu)$. This implies by Lemma 5.5 $l_{1-F\mu}-\mu+\varepsilon\subset g(\alpha_{|(1-F\mu)R(1-F\mu)})$. This contradicts the definition of $l_{1-F\mu}$.

e. Let $\dfrac{2\pi}{3}>b>\mu$ and $b-\varepsilon>\mu$ then $\phi_{-b+I_\varepsilon}E_\mu\leqq\phi_{I_\mu}E_\mu=0$. If $\dfrac{2\pi}{3}\leqq b\leqq\dfrac{4\pi}{3}$ then $b\in g(\alpha)$ and hence $\phi_{b+I_\varepsilon}=0$. Since $n(E_\mu, E_\mu)$ is invariant under $b\to -b$ follows $(\mu, 2\pi-\mu)\subset n(E_\mu, E_\mu)$.

f. Let $b<0$ and $\varepsilon<\frac{|b|}{2}$ then $\phi_{b+I_\varepsilon}E_{\frac{|b|}{2}}\leqq\phi_{I_{\frac{|b|}{2}}}E_{\frac{|b|}{2}}=0$ and hence $b\notin S(E)$ for any E. Let now $b+I_\varepsilon\subset g(\alpha_{|FRF})$ then $\phi_{b+I_\varepsilon}F=0$ and hence $b\notin S(EF)$. This means $S(FE)\subset \mathrm{sp}(\alpha_{|FRF})\cap\{\mathrm{Im}\, z\geqq 0\}$.

Proof of theorem 5.4. Part A (i), (ii) and (iii) are proved in corollary 5.6. Assume now $\|\alpha-1\|<2$, then exists according to Kadison and Ringrose [7] (Theorem 7) a unique real derivation with $\|\delta\|<\frac{\pi}{2}$ such that $\alpha=e^\delta$. If we consider $\alpha_t=e^{t\delta}$ then for $t\leqq\frac{2}{3}$ we have $\rho(\alpha_t-1)<\sqrt{3}$. Now according to Lemma 5.9 exists unitary operators u_t with $\mathrm{sp}\, u_t\subset \mathrm{sp}\,\alpha_t\cap\{\mathrm{Im}\, z\geqq 0\}$ which implement the automorphisms. By construction of u_t follows that u_t is a one-parametric group and hence the result extends also to $t=1$. This proves the first part of A (iv). Let δ be a derivation then we can pass to the one-parametric group $e^{t\delta}=\alpha_t$ for $|t|<\frac{2\pi}{3}\|\delta\|^{-1}$ we have $\rho(\alpha_t-1)<\sqrt{3}$ and thus part B (i) and the first part of (ii) are consequences of A (i) and the first part of A (iv). In order to prove the uniqueness it is sufficient to do this for the derivation. Let δ be a real derivation then exists a positive operator d with sp $idF\subset \mathrm{sp}\{\delta_{|FRF}\}$ and $\delta=\mathrm{ad}\, id$ since d is positive follows $\|\delta_{|FRF}\|\leqq \|d\cdot F\|\leqq\rho(\delta_{|FRF})\leqq\|\delta_{|FRF}\|$. This shows that $\|\delta_{|FRE}\|=\|dF\|=\rho(\delta_{|FRF})$. Assume there exists a second positive operator d' with $\delta=ad\, id'$ and $\mathrm{sp}\, id'F\subset\mathrm{sp}\{\delta_{|FRF}\}$ then we have $d-d'\in Z$. Assume the spectrum of $d-d'$ contains a point $x\neq 0$ then exists a spectral projection F with $\mathrm{sp}\, F(d-d')\subset\left[\frac{x}{2},\frac{3}{2}x\right]$. Assume first $x>0$ follows from $Fd=Fd'+F(d-d')$ that $\|Fd'\|+\frac{x}{2}<\|Fd\|$ or $\|\delta_{|FRF}\|\leqq\|Fd'\|\leqq\|Fd\|-\frac{x}{2}=\|\delta_{|FRF}\|-\frac{x}{2}$. This is absurd and hence $d-d'$ is not positive. Assume next x is negative then we have $Fd-F(d-d')=Fd'$ and hence $\mathrm{sp}\, Fd+\frac{|x|}{2}\subset \mathrm{sp}\, Fd'\subset\mathrm{sp}\{\delta_{|FRF}\}$. This means $\|Fd\|=\rho(\delta_{|FRF})-\frac{|x|}{2}$ which is also absurd and hence $d=d'$.

Proof of theorem 5.3. Let δ be a real derivation with $\|\delta\|<\pi$. The function $t\to e^{t\delta}-e^{-t\delta}$ is an entire function of t and we get $\|e^{t\delta}-e^{-t\delta}\|\leqq 2\sum_{u=0}^{\infty}\frac{1}{(n+1)!}\|t\delta\|^{2u+1}=2\sin\mathrm{h}\,\|t\delta\|$. Since $e^{t\delta}$ is a *-automorphism finds $\phi_I^\gamma \beta E = \beta \phi_I^\alpha E$ and hence for $E_1, E_2 \in P_0^\alpha$ $n^\alpha(E_1, E_2) = n^\gamma(\beta E_1, \beta E_2)$. for real t we have $\|e^{t\delta} - e^{-t\delta}\| \leqq 2$ for t real. Using that $e^{t\delta}x - e^{-t\delta}x$ is selfadjoint for x selfadjoint we get (see Boas [2] 6. 2. 6) $\|e^{t\delta} - e^{-t\delta}\| \leqq 2 \cosh \|\delta\| \operatorname{Im} t$. Since this function vanishes at the origin we can define

$$F(t) = \frac{e^{t\delta} - e^{-t\delta}}{\sin t\|\delta\|}$$

in the strip $|\operatorname{Re} t| < \frac{\pi}{\|\delta\|}$. Since $|\sin x + iy| = \sqrt{\cosh^2 y - \cos^2 x}$ follows $\|F(t)\|$ is bounded in the strip $|\operatorname{Re} t| \leqq \frac{\pi}{2\|\delta\|}$ and $\|F(t)\| \leqq 2$ for $|\operatorname{Re} t| = \frac{\pi}{2\|\delta\|}$. This implies (see e.g. Titchmarch [11] 5. 6. 5.) $\|F(t)\| \leqq 2$ in $|\operatorname{Re} t| \leqq \frac{\pi}{2\|\delta\|}$. From this we get:

$$\|e^\delta - 1\| = \|e^{\frac{\delta}{2}}(e^{\frac{\delta}{2}} - e^{-\frac{\delta}{2}})\| \leqq \|e^{\frac{\delta}{2}} - e^{-\frac{\delta}{2}}\| = \left\|F\left(\frac{1}{2}\right)\right\| \cdot \sin\frac{\|\delta\|}{2} \leqq 2\sin\frac{\|\delta\|}{2}.$$

From theorem 5.4 we know $\|\delta\| = \rho(\delta)$. Since we have $\rho^2(e^\delta - 1) = 2 - 2\cos\rho(\delta) = 4\sin^2\frac{\rho(\delta)}{2}$ we find $\|e^\delta - 1\| \leqq \rho(e^\delta - 1)$ which implies that both expressions are equal. If now $\rho(\alpha - 1) < \sqrt{3}$ then α lies on a one parametric group $e^{t\delta}$ with $\rho(\delta) < \pi\frac{2}{3}$. Hence we have $\|\alpha - 1\| = \rho(\alpha - 1)$. If $\sqrt{3} \leqq \rho(\alpha - 1) \leqq 2$ then $\|\alpha - 1\|$ might be 2. If $\|\alpha - 1\| < 2$ then by the result of Kadison and Ringrose we have $\alpha = e^\delta$ with $\|\delta\| < \pi$ and hence $\|\alpha - 1\| = \rho(\alpha - 1)$. Let now M be a factor of type II or III and α_0 the outer automorphism of $M \otimes M \otimes M$ at the end of corollary 5.6. Let $\frac{2\pi}{3} \leqq l \leqq \pi$ and d be a positive invariant (under α_0) operator with $\operatorname{sp} d = [0, l]$ then we have $\|ad\, id\| = l$. Let $\alpha_1 = e^{ad\, id}$ then we have $\rho(\alpha_1 - 1) = \rho(\alpha_0\alpha_1 - 1) = 2\sin\frac{l}{2}$ but $\|\alpha_1 - 1\| = 2\sin\frac{l}{2}$ and $\|\alpha_0\alpha_1 - 1\| = 2$, since α_0 is an outer automorphism.

6. Some Final Remarks

A. If α and β are two $*$-automorphisms of the same W^*-algebra R and one defines $\gamma = \beta\alpha\beta^{-1}$ and denotes the quantities used in this paper with an upper index denoting the automorphism which we are considering, then one finds easily from the definitions $R_0^\gamma = \beta R_0^\alpha$, for $E \in P_0^\alpha$ one

This means for a fixed W^*-algebra R the classes $\mathrm{Aut}_n = \{\alpha, (R, \alpha) \in K_n\}$ are invariant sets.

B. With a little care most of our technique can also be used for arbitrary locally compact abelian groups which we will denote by G. We want to give a brief discussion of this case.

a. Since we are looking for continuous group-representations it is according to an earlier paper [4] neceassry that G acts strongly continuous on the pre-dual R_* of R, i.e. for every $\varepsilon > 0$ and every $\varphi \in R_*$ exists a neighbourhood U of the groupidentity with $\|(\alpha - 1)\varphi\| < \varepsilon$, $\alpha \in U$. If this is the case integration with respect to the invariant Haar-measure of expressions $\int f(g)\alpha g \times dg$ are well defined as weak integral.

b. Let $\hat{G}$ be the dual group of G then we can define $\phi_S E$ for elements in R_0 for sets S with the properties (i) $S = \overline{S^\circ}$ and (ii) for every $\hat{g} \in S^\circ$ exists a function $f \in l_G^1$ with support $\mathscr{F}^{-1} f \subset S$ and $(\mathscr{F}^{-1} f) g \neq 0$. With this provision and a little care about the union of two such sets all results of section 2 are true in this general situation.

c. For the definition of the classes it is easier to start from the group G itself. Let H be a closed subgroup of G then we give the following

Definition: A group $\alpha(G)$ of $*$-automorphisms belong to the class K_H if and only if for every Projection $F \in P_0 \cap Z$ the subgroup of automorphisms α_h such that $\alpha_{h|FRF}$ is inner, and we can find a continuous group-representation $u(h) \in R$ of H, implementing H, is exactly H.

d. Since in general a group G has continuously many different subgroups, a standard decomposition of R is in general not possible or at least not so easy. Therefore we will stick to the case where R is a factor then $\alpha(G)$ lies always in one of the classes.

e. For $E_1, E_2 \in P_0$ we can investigate the expressions $n(E_1, E_2)$ on $\hat{G}$. Let now $\hat{H}(E_1, E_2)$ be the subgroup of $\hat{G}$ such that $\hat{g} \in n(E_1, E_2)$ implies $\hat{h}\hat{g} \in n(E_1, E_2)$ for all $\hat{h} \in \hat{H}(E_1, E_2)$ and $\hat{H}_0 = \bigcap_{E_1, E_2} \hat{H}(E_1, E_2)$. If R is a factor then $\hat{H}_0$ is well defined and is a closed subgroup of

G. This then gives a second definition of classes $K_{\hat{H}}$ and one would like to show that these classifications are isomorphic.

f. If $\alpha(G)$ belongs to $K_{\hat{H}_0}$ then the definition of a point-fixing family is possible on $\hat{G}/\hat{H}_0$ and hence we get also a spectral measure on $\hat{G}/\hat{H}_0$.

g. Let H be the subgroup of G such that $(h, \hat{h})=1$ for all $\hat{h}\in\hat{H}_0$ then $\hat{G}/\hat{H}_0$ is isomorphic to $\hat{H}$. From this the following statement is immediate. If $\alpha(G)$ belongs to K_H, then it need not belong to $K_{\hat{H}}$ in general. The converse statement seems to be true, but, I don't know how to prove in general. Having a spectral measure on $\hat{G}/\hat{H}_0$ it is of cause easy to construct a grouprepresentation $u(h)$ of H. But to show that $u(h)$ implements α_h I know to do only in special cases. This is based on the fact that our proof uses entire analytic functions. So we can give the proof for the cases $\hat{G}/\hat{H}_0=\boldsymbol{R}$, T, $\boldsymbol{Z}$, $\boldsymbol{Z}_n$ and finite direct products of these special cases.

h. These special cases just mentioned cover most of the interesting problems. This leads e.g. to the conclusion that derivations are inner and the case with semibounded spectrum treated in [5] leads to inner automorphisms.

i. Passing from one single automorphism to an abelian group of automorphisms, then the following possibility does occur and complicates the situation: Every single automorphism is inner, but we have only a continuous group-representation up to a multiplier. Take for example a representation of the C. C. R. of one degree of freedom, then the Weyl operators induce an automorphism-group which is isomorphic to $\boldsymbol{R}^2$. (I learned this example from A. Connes.)

References

[1] Arveson, W.: On groups of automorphisms of operator algebras, Preprint (1973).

[2] Boas, R. P.: *Entire Functions*, Academic Press Inc. New York (1954).

[3] Borchers, H. J.: Energy and momentum as observables in quantum field theory, *Comm. math. Phys.* **2** (1966).

[4] ————: Über C^*-Algebren mit Lokalkompakten Symmetriegruppen, *Nachr.*

d. Göttinger Akad. Nr **1** (1973).
[5] ————: Über Ableitungen von C^*-Algebren, *Nachr. d. Göttinger Akad.* Nr **2** (1973).
[6] Kadison, R. V.: Derivations of operator algebras, *Ann. Math.* **83** (1966).
[7] ———— and Ringrose, J.: Derivations and automorphisms of operator algebras, *Comm. math. Phys.* **4** (1967).
[8] Olesen, D. and Pedersen, G. K.: Derivations of C^*-algebras have semi-continuous generators, Preprint (1973).
[9] Sakai, S.: Derivations of W^*-algebras, *Ann. Math.* **83** (1966).
[10] ————: Automorphisms and tensor products of operator algebras, Preprint (1973).
[11] Titchmarch, E. C.: *The Theory of Functions*, Oxford University Press, 2. ed. (1939).