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Infinite Tensor Products of Operators

By

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§1. Introduction

In the previous paper [8] we established a definition of an infinite tensor product $\otimes x_\iota$ of operators on $\otimes\mathscr{H}_\iota$ and studied its properties under the assumption: $\prod\|x_\iota\|<+\infty$. In some applications, for instance, to Tomita's theory [3] and to quantum field theory [13, 14], we are obliged to work with a weaker assumption.

In the present paper, we shall define an infinite tensor product $\otimes^{c'c}x_\iota$ of operators x_ι on $\mathscr{H}_\iota$ as a closed linear mapping from an incomplete infinite tensor product space $\otimes^c\mathscr{H}_\iota$ to another $\otimes^{c'}\mathscr{H}_\iota$. We do not make any assumption on $\|x_\iota\|$, allowing unbounded closed operators x_ι. The crucial assumption on (x_ι) is the existence of what we call a non-zero reference pair. This assumption turns out to be sufficiently general to allow various applications and yet sufficiently strong to yield significant results. Typical result is the following:

Theorem 1.1. *If x_ι is positive self-adjoint and $(\xi_{0\iota}, \eta_{0\iota})$ is a non-zero reference pair of (x_ι), then $(\xi_{0\iota})$ and $(\eta_{0\iota})$ belong to the same equivalence class c and $\otimes^{cc}x_\iota$ is essentially self-adjoint on the linear span of the product vectors $\otimes\xi_\iota$ such that $\xi_\iota=\xi_{0\iota}$ except for a finite number of ι and ξ_ι is in the domain of x_ι.*

Terminologies here are defined as follows:

Definition 1.1. A pair $(\xi_{0\iota}, \eta_{0\iota})$ is a *non-zero reference pair* of (x_ι) if the following conditions are fulfilled:

(a) $(\xi_{0\iota})$ and $(\eta_{0\iota})$ are C_0-sequences;

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$$\xi_{0\iota}\neq 0,\ \sum|\|\xi_{0\iota}\|^2-1|<+\infty,\ \eta_{0\iota}\neq 0,\ \sum|\|\eta_{0\iota}\|^2-1|<+\infty .$$

(b) $\xi_{0\iota}$ is in the domain of x_ι and $(x_\iota\xi_{0\iota})$ is a C-sequence;

$$\sum|\|x_\iota\xi_{0\iota}\|^2-1|<+\infty .$$

(c) $(x_\iota\xi_{0\iota})$ is equivalent to $(\eta_{0\iota})$;

$$\sum|(x_\iota\xi_{0\iota}|\eta_{0\iota})-1|<+\infty .$$

(d) $\eta_{0\iota}$ is in the domain of x_ι^* and $(x_\iota^*\eta_{0\iota})$ is a C-sequence;

$$\sum|\|x_\iota^*\eta_{0\iota}\|^2-1|<+\infty .$$

All assumptions except for (d) are obviously unavoidable if we want to define what can be denoted by $\otimes^{c'c}x_\iota$. The assumption (d) is crucial and enables all calculations go through.

The product operators $\otimes^{c'c}x_\iota$ for $c\equiv c(\xi_{0\iota})$ and $c'\equiv c(x_\iota\xi_{0\iota})$ is defined in three steps: On the product vector $\otimes\xi_\iota$ with $\xi_\iota=\xi_{0\iota}$ except for a finite number of ι and ξ_ι in the domain of x_ι, a mapping $\odot(x_\iota, \xi_{0\iota})$ is defined by

$$\odot(x_\iota,\ \xi_{0\iota})\otimes\xi_\iota=\otimes x_\iota\xi_\iota .$$

It is then proved to be extendable linearly to the linear span of such product vectors (denoted as $\odot(D(x_\iota),\ \xi_{0\iota})$). It is then proved to be closable and the closure is denoted by $\otimes^{c'c}x_\iota$. The assumption (d) is necessary for this closability (Remark 2.2).

All these discussions and the proof of the formula

$$(\otimes^{c'c}x_\iota)^*=\otimes^{cc'}x_\iota^*$$

are given in Section 2. This formula contains Theorem 1.1 as a special case $x_\iota^*=x_\iota$.

In Section 3, we give several conditions for the existence of a non-zero reference pair, one of which is closely related to Kolmogorov's three series theorem. Theorem 3.1 has a close connection with some results of Reed [13] and Streit [14].

In Section 4, we apply our result to a modular operators Δ_{ξ_ι} and show that $\otimes^c\Delta_{\xi_\iota}$ is a modular operator for $\otimes\xi_\iota$ where $(\xi_\iota)\in c$.

In Section 5, we apply our results to an infinite product μ of σ-finite measures μ_ι. Theorem in Section 3 gives us conditions for the equivalence $\mu\sim\nu$ when μ_ι is equivalent to a given probability measure ν_ι and ν is the product measure of ν_ι. One of the conditions reproduces a result of Hill [5].

The discussion in Section 5 is generalized to an infinite product of semi-finite faithful normal weights in Section 6. The result is used in a separate paper [6].

Notations: For standard definitions and notations for infinite tensor products of Hilbert spaces and von Neumann algebras, see [11]. Let I be an infinite index set and $J\subset\subset I$ indicates that J is a finite subset of I. S denotes the set of all C-sequences (ξ_ι) (i.e., $\sum|\|\xi_\iota\|^2-1|<+\infty$) and S_0 denotes the set of all C_0-sequences (ξ_ι) (i.e., $(\xi_\iota)\in S$, $\xi_\iota\neq 0$). The word "sequence" is used for (ξ_ι) even if I is uncountably infinite. $(\xi_\iota)\sim(\eta_\iota)$ denotes the condition $\sum|(\xi_\iota|\eta_\iota)-1|<+\infty$. It defines equivalence relations in S and in S_0. The equivalence class of (ξ_ι) is denoted by $c(\xi_\iota)$. The incomplete infinite tensor product $\mathscr{H}_c\equiv\otimes^c\mathscr{H}_\iota$ is spanned by $\otimes\xi_\iota$ with a fixed $c=c(\xi_\iota)$. The projection on $\mathscr{H}_c$ in the complete infinite tensor product $\otimes\mathscr{H}_\iota$ is denoted by p_c. Let (ξ_ι), $(\eta_\iota)\in S$ and $c\equiv c(\xi_\iota)$, $c'\equiv c(\eta_\iota)$. (ξ_ι) and c are u-equivalent (resp. p-equivalent) to (η_ι) and c', respectively, if $(\xi_\iota)\sim(u_\iota\eta_\iota)$ for some unitary (resp. partial isometry) $u_\iota\in M'_\iota$. This is denoted by $(\xi_\iota)\underset{u}{\sim}(\eta_\iota)$, $c\underset{u}{\sim}c'$ (resp. $(\xi_\iota)\underset{p}{\sim}(\eta_\iota)$, $c\underset{p}{\sim}c'$). If I is countable, $c\underset{p}{\sim}c'$ and $c\underset{u}{\sim}c'$ are equivalent, [1]. Let $p(c)$ denote the central carrier of p_c in $(\otimes M_\iota)'$. $p(c)$ is the sum of $p_{c'}$ with $c'\underset{p}{\sim}c$, [1]. For $x_\iota\in B(\mathscr{H}_\iota)$ with $\prod\|x_\iota\|<+\infty$, we can define an infinite tensor product $\otimes x_\iota$ of operators, which is bounded on $\otimes\mathscr{H}_\iota$. When $\mathscr{H}_c$ is invariant under $\otimes x_\iota$, the induction of $\otimes x_\iota$ to $\mathscr{H}_c$ is denoted by $\otimes^c x_\iota$ or $(\otimes_J x_\iota)\otimes(\otimes^c_{I\setminus J}x_\iota)$ for $J\subset\subset I$.

§2. Infinite Tensor Products of Operators

For an operator x (resp. y) with domain $D(x)$ (resp. $D(y)$), let $D(x)\odot D(y)$ denote the algebraic tensor product in $\overline{D(x)}\otimes\overline{D(y)}$ of $D(x)$ and $D(y)$, and $x\odot y$ the operator on $D(x)\odot D(y)$ defined by

$$(x\odot y)\xi\otimes\eta=x\xi\otimes y\eta$$

for all $\xi\in D(x)$ and $\eta\in D(y)$.

Lemma 2.1. *If x and y are essentially self-adjoint, then $x\odot y$ and $\bar{x}\odot\bar{y}$ are essentially self-adjoint and $\overline{x\odot y}=\overline{\bar{x}\odot\bar{y}}$.*

For self-adjoint operators x and y, we denote $\overline{x\odot y}$ by $x\otimes y$ in the following.

Throughout this and next sections x_ι is a non zero densely defined closed operator on a Hilbert space $\mathscr{H}_\iota$, $x_\iota=u_\iota|x_\iota|$ is the polar decomposition of x_ι, and $D(x_\iota)$ denotes the domain of x_ι.

For $(\xi_{0\iota})\in S_0$ with $\xi_{0\iota}\in D(x_\iota)$ and $(x_\iota\xi_{0\iota})\in S$, we denote by $\odot(D(x_\iota), \xi_{0\iota})$ the linear span of $\otimes\xi_\iota$ such that $\xi_\iota=\xi_{0\iota}$ for all but a finite number of $\iota\in I$ and $\xi_\iota\in D(x_\iota)$ for all $\iota\in I$.

Lemma 2.2. *Let $(\xi_{0\iota})\in S_0$ and $\xi_{0\iota}\in D(x_\iota)$ for all $\iota\in I$. If $(x_\iota\xi_{0\iota})\in S$, there exists a non zero operator x with domain $\odot(D(x_\iota), \xi_{0\iota})$ such that $x\otimes\xi_\iota=\otimes x_\iota\xi_\iota$ for all $\otimes\xi_\iota$ in $\odot(D(x_\iota), \xi_{0\iota})$.*

Proof. For $\xi=\sum_{k=1}^n\otimes\xi_{k\iota}$ in $\odot(D(x_\iota), \xi_{0\iota})$, there exists a $J\subset\subset I$ such that $\xi=\xi_J\otimes(\otimes_{I\setminus J}\xi_{0\iota})$ for $\xi_J\in\otimes_J\mathscr{H}_\iota$ and $\xi_J=\sum_{k=1}^n\otimes_J\xi_{k\iota}$. Since $(\xi_{0\iota})\in S_0$, if $\xi=0$ then $\xi_J=0$ and so $(\otimes_J x_\iota)\xi_J=0$. Therefore $\sum_{k=1}^n\otimes x_\iota\xi_{k\iota}=0$. Thus the mapping

$$\sum_{k=1}^n\otimes\xi_{k\iota}\longmapsto\sum_{k=1}^n\otimes x_\iota\xi_{k\iota}$$

is well defined. We denote it by x. Since $(x_\iota\xi_{0\iota})\in S$, there exists a $\otimes\xi_\iota$ in $\odot(D(x_\iota), \xi_{0\iota})$ with $(x_\iota\xi_\iota)\in S_0$. Therefore x is non zero.

Q.E.D.

Definition 2.1. Let $(\xi_{0\iota})\in S_0$ and $\prod\|x_\iota\xi_{0\iota}\|<+\infty$. An operator $\odot(x_\iota, \xi_{0\iota})$ on $\odot(D(x_\iota), \xi_{0\iota})$ is defined by

$$\odot(x_\iota, \xi_{0\iota})\equiv\begin{cases} x \text{ in Lemma 3.2} & \text{if } (x_\iota\xi_{0\iota})\in S, \\ 0 & \text{otherwise.}\end{cases}$$

The following lemma is immediate from Definition 1.1.

Lemma 2.3. *The following three conditions are equivalent:*
(i) $(\xi_{0\iota}, \eta_{0\iota})$ *is a non-zero reference pair of* (x_ι);
(ii) $\xi_{0\iota} \in D(x_\iota)$, $\eta_{0\iota} \in D(x_\iota^*)$, $(\xi_{0\iota}) \in S_0$, $(\eta_{0\iota}) \in S_0$, $(x_\iota \xi_{0\iota}) \in S$, $(x_\iota^* \eta_{0\iota}) \in S$ *and* $(x_\iota \xi_{0\iota}) \sim (\eta_{0\iota})$; *and*
(iii) $(\eta_{0\iota}, \xi_{0\iota})$ *is a non-zero reference pair of* (x_ι^*).

Example 2.1. For $0 < \varepsilon_\iota < 1$, $\iota \in I$, put

$$x_\iota \equiv \begin{pmatrix} \varepsilon_\iota^{-1} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \xi_\iota \equiv \eta_\iota \equiv \begin{pmatrix} \varepsilon_\iota^2 \\ 1 \end{pmatrix}.$$

If $\sum \varepsilon_\iota^2 < +\infty$, then $x_\iota > 0$, $(\xi_\iota) = (\eta_\iota) \in S_0$, $(x_\iota \xi_\iota) = (x_\iota \eta_\iota) \in S_0$ and $(x_\iota \xi_\iota) \sim (\eta_\iota)$. But $(x_\iota^2 \xi_\iota) \notin S$.

Lemma 2.4. *If* $(\xi_{0\iota}, \eta_{0\iota})$ *is a non-zero reference pair of* (x_ι), *then*
(i) $\odot(x_\iota, \xi_{0\iota}) \odot(D(x_\iota), \xi_{0\iota}) \subset \mathscr{H}_{c'}$ *for* $c' \equiv c(\eta_{0\iota})$;
(ii) $(\odot(x_\iota, \xi_{0\iota}))^* \supset \odot(x_\iota^*, \eta_{0\iota})$ *and* $\odot(x_\iota, \xi_{0\iota})$ *is closable; and*
(iii) *for the closure* x *of* $\odot(x_\iota, \xi_{0\iota})$, x^*x *is a self-adjoint operator on* $\mathscr{H}_c$ *for* $c \equiv c(\xi_{0\iota})$.

Proof. (i) It is clear from Lemma 2.2.

(ii) For all $\otimes \xi_\iota \in \odot(D(x_\iota), \xi_{0\iota})$ and $\otimes \eta_\iota \in \odot(D(x_\iota^*), \eta_{0\iota})$ we have

$$\begin{aligned}(\odot(x_\iota, \xi_{0\iota}) \otimes \xi_\iota | \otimes \eta_\iota) &= (\otimes x_\iota \xi_\iota | \otimes \eta_\iota) \\ &= \prod (x_\iota \xi_\iota | \eta_\iota) = \prod (\xi_\iota | x_\iota^* \eta_\iota) \\ &= (\otimes \xi_\iota | \otimes x_\iota^* \eta_\iota) = (\otimes \xi_\iota | \odot(x_\iota^*, \eta_{0\iota}) \otimes \eta_\iota).\end{aligned}$$

Since $\odot(D(x_\iota), \xi_{0\iota})$ is dense in $\mathscr{H}_c$ for $c \equiv c(\xi_{0\iota})$ and $\odot(D(x_\iota^*), \eta_{0\iota})$ is dense in $\mathscr{H}_{c'}$ for $c' \equiv c(\eta_{0\iota})$, it follows that $\odot(x_\iota^*, \eta_{0\iota}) \subset (\odot(x_\iota, \xi_{0\iota}))^*$.

(iii) Since x is a closed operator of $\mathscr{H}_c$ to $\mathscr{H}_{c'}$, x^* is an operator of $\mathscr{H}_{c'}$ to $\mathscr{H}_c$. Therefore x^*x is self-adjoint on $\mathscr{H}_c$. Q.E.D.

Lemma 2.5. *Let* $(\xi_{j\iota}, \eta_{j\iota})$ *be a non-zero reference pair of* (x_ι) *for* $j = 0, 1$. *If* $c(\xi_{0\iota}) = c(\xi_{1\iota})$, *then* $c(\eta_{0\iota}) = c(\eta_{1\iota})$ *and the closure of* $\odot(x_\iota, \xi_{0\iota})$ *is the closure of* $\odot(x_\iota, \xi_{1\iota})$.

Proof. Since $(\xi_{j\iota}, \eta_{j\iota})$ is a non-zero reference pair of (x_ι), we have $(x_\iota\xi_{0\iota})\sim(\eta_{0\iota})$ and $(\xi_{1\iota})\sim(x_\iota^*\eta_{1\iota})$. Since $(\xi_{0\iota})\sim(\xi_{1\iota})$ by assumption, we have $(\xi_{0\iota})\sim(x_\iota^*\eta_{1\iota})$ and hence $(x_\iota\xi_{0\iota})\sim(\eta_{1\iota})$. Therefore $(\eta_{0\iota})\sim(\eta_{1\iota})$. Let $c\equiv c(\xi_{0\iota})$ and $c'\equiv c(\eta_{0\iota})$. Since $(\xi_{0\iota})\sim(\xi_{1\iota})$ and $(x_\iota\xi_{0\iota})\sim(x_\iota\xi_{1\iota})$, there exists for any $\varepsilon>0$ a $J_1\subset\subset I$ such that

$$\|\otimes\xi_{1\iota}-(\otimes_J\xi_{1\iota})\otimes(\otimes_{I\setminus J}\xi_{0\iota})\|<\varepsilon$$

and

$$\|\otimes x_\iota\xi_{1\iota}-(\otimes_J x_\iota\xi_{1\iota})\otimes(\otimes_{I\setminus J} x_\iota\xi_{0\iota})\|<\varepsilon$$

for all $J_1\subset J\subset\subset I$. Therefore $\otimes\xi_{1\iota}$ is in the domain of $\overline{\odot(x_\iota, \xi_{0\iota})}$ and hence $\overline{\odot(x_\iota, \xi_{1\iota})}\subset\overline{\odot(x_\iota, \xi_{0\iota})}$. The converse inclusion is proved similarly. Q.E.D.

Definition 2.2. The closed operator in Lemma 2.4 is denoted by $\otimes^{c'c}x_\iota$. $\otimes^{cc}x_\iota$ is also denoted by $\otimes^c x_\iota$.

For a non-zero reference pair $(\xi_{0\iota}, \eta_{0\iota})$ of (x_ι) if $(\xi_\iota)\in S_0$ with $\otimes\xi_\iota\in\odot(D(x_\iota), \xi_{0\iota})$ and if $(\eta_\iota)\in S_0$ with $\otimes\eta_\iota\in\odot(D(x_\iota^*), \eta_{0\iota})$, then (ξ_ι, η_ι) is a non-zero reference pair of (x_ι).

We are now ready to prove the main theorem.

Proof of Theorem 1.1. Let $s(x_\iota)$ be the carrier projection of x_ι. Since $(\xi_{0\iota})\in S$, $(x_\iota^*\eta_{0\iota})\in S$, $\|s(x_\iota)\|=1$ and $(s(x_\iota)\xi_{0\iota})\sim(x_\iota^*\eta_{0\iota})$, it follows from Lemma 1 in [1] that $(s(x_\iota)\xi_{0\iota})\in S$. Therefore there is a $(\xi_\iota)\in S_0$ such that $\otimes\xi_\iota\in\odot(D(x_\iota), \xi_{0\iota})$ and $\otimes s(x_\iota)\xi_\iota\neq 0$. Since $(\xi_\iota, \eta_{0\iota})$ is a non-zero reference pair of (x_ι) and $\odot(x_\iota, \xi_{0\iota})=\odot(x_\iota, \xi_\iota)$, we may assume that $\otimes s(x_\iota)\xi_{0\iota}\neq 0$ by choosing such a (ξ_ι) as $(\xi_{0\iota})$.

Let x denote the operator $\otimes^{c'c}x_\iota$ for $c\equiv c(\xi_{0\iota})$ and $c'\equiv c(\eta_{0\iota})$. Let

$$\otimes_{I\setminus J}^{c'c}x_\iota=u(I\setminus J)y(I\setminus J)$$

be the polar decomposition of $\otimes_{I\setminus J}^{c'c}x_\iota$ for any $J\subset\subset I$. Put $y_J\equiv(\otimes_J x_\iota)\otimes y(I\setminus J)$. Since y_J is positive self-adjoint on $\mathscr{H}_c$,

$$|x|=(x^*x)^{1/2}=(y_J^*y_J)^{1/2}=y_J.$$

Putting $u\equiv u(I)$ and $u_J\equiv(\otimes_J s(x_\iota))\otimes u(I\backslash J)$, we have

$$u|x|=\otimes^{c'c}x_\iota=(\underset{J}{\otimes}x_\iota)\otimes(\underset{I\setminus J}{\otimes}{}^{c'c}x_\iota)$$

$$=(\underset{J}{\otimes}x_\iota)\otimes u(I\backslash J)y(I\backslash J)=u_J y_J\,.$$

The uniqueness of a polar decomposition implies that $u=u_J$ and u transforms $\mathscr{H}_c$ to $\mathscr{H}_{c'}$. Since $u_J=u\neq 0$, we have $u(I\backslash J)(\otimes_{I\setminus J}\xi_{0\iota})\neq 0$ for some J. Since $(s(x_\iota)\xi_{0\iota})\in S_0$, we have $u\otimes\xi_{0\iota}\neq 0$. Accordingly there exists a $(\zeta_\iota)\in S$ and a $\zeta\in\mathscr{H}_{c'}$ such that $\|\zeta_\iota\|=1$, $c(\zeta_\iota)=c'$, $(\zeta|\otimes\zeta_\iota)=0$ and

$$u\otimes\xi_{0\iota}=\lambda\otimes\zeta_\iota+\zeta$$

for $\lambda>0$. If $c\neq c'$, we have an ε in $(0, 1)$ such that for any $J_0\subset\subset I$ there exists a $J_1\subset\subset I\backslash J_0$ satisfying $|\prod_{J_1}(s(x_\iota)\xi_{0\iota}|\zeta_\iota)|<\varepsilon$. Choose $\lambda_0>1$ such that $\lambda_0^{-1}\leqq\prod_J\|\xi_{0\iota}\|\leqq\lambda_0$ for all J. Then there exists an $n\in\boldsymbol{N}$ with $\varepsilon^n<\lambda\lambda_0^{-1}$ and a $K\subset\subset I$ such that $|\prod_K(s(x_\iota)\xi_{0\iota}|\zeta_\iota)|<\varepsilon^n$. Since $\lambda=(u_J\otimes\xi_{0\iota}|\otimes\zeta_\iota)$, we have

$$|(u(I\backslash K)\underset{I\setminus K}{\otimes}\xi_{0\iota}|\underset{I\setminus K}{\otimes}\zeta_\iota)|=\lambda|\prod_K(s(x_\iota)\xi_{0\iota}|\zeta_\iota)|^{-1}>\lambda_0\,,$$

which is impossible. Thus $c=c'$.

For $\otimes\xi_\iota\in\odot(D(x_\iota), \xi_{0\iota})$, $J_2=\{\iota\in I\colon \xi_\iota\neq\xi_{0\iota}\}$ and $\varepsilon>0$, we can choose a $J_3\subset\subset I$ with $J_2\subset J_3$ such that

$$\|(\odot(x_\iota, \xi_{0\iota})-x_K)\otimes\xi_{0\iota}\|<\varepsilon$$

for any $J_3\subset K\subset\subset I$, where $x_K=(\otimes_K x_\iota)\otimes(\otimes^c_{I\setminus K}1_\iota)$. Since $\otimes_K x_\iota$ is self-adjoint and $\odot_K D(x_\iota)$ is its core by Lemma 2.1, we have $\eta_K^\pm\in\odot_K D(x_\iota)$ such that

$$\|(\underset{K}{\otimes}x_\iota\pm i1)\eta_K^\pm-\underset{K}{\otimes}\xi_\iota\|^2+\|\eta_K^\pm-(\underset{K}{\otimes}x_\iota\pm i1)^{-1}(\underset{K}{\otimes}\xi_\iota)\|^2<\varepsilon^2.$$

Put $\eta^\pm\equiv\eta_K^\pm\otimes(\otimes_{I\setminus K}\xi_{0\iota})$. From the above two inequalities we have

$$\|(\odot(x_\iota, \xi_{0\iota})\pm i1)\eta^\pm-\otimes\xi_\iota\|^-$$

$$\leqq\|(\odot(x_\iota, \xi_{0\iota})-x_K)\eta^\pm\|+\|(x_K\pm i1)\eta^\pm-\otimes\xi_\iota\|$$

$$\leqq \|(\underset{K}{\otimes} x_\iota)\eta_K^\pm\| \|(\underset{I\setminus K}{\odot}(x_\iota, \xi_{0\iota})-1)\underset{I\setminus K}{\otimes}\xi_{0\iota}\|$$

$$+\|\underset{I\setminus K}{\otimes}\xi_{0\iota}\| \|(\underset{K}{\otimes}x_\iota \pm i1)\eta_K^\pm - \underset{K}{\otimes}\xi_\iota\|$$

$$\leqq \varepsilon\{2(\|\underset{K}{\otimes}\xi_\iota\|+\varepsilon)\|\underset{K}{\otimes}x_\iota\xi_{0\iota}\|^{-1}+\|\underset{I\setminus K}{\otimes}\xi_{0\iota}\|\}.$$

Since there exists a $\lambda_1>1$ satisfying $\prod_J\|\xi_{0\iota}\|<\lambda_1$ and $\lambda_1^{-1}<\prod_J\|x_\iota\xi_{0\iota}\|$ for all $J\subset I$, we conclude that the deficiency indices of $\odot(x_\iota, \xi_{0\iota})$ are 0, 0 and hence it is essentially self-adjoint. Furthermore $\odot(D(x_\iota), \xi_{0\iota})$ is a core of $x=\otimes^c x_\iota$.

Each ξ_0 in $\odot(D(x_\iota), \xi_{0\iota})$ is of the form $\xi_J\otimes(\otimes_{I\setminus J}\xi_{0\iota})$ for some $J\subset\subset I$ and $\xi_J\in\otimes_J\mathscr{H}_\iota$. Since $\otimes_J x_\iota$ is positive, we have

$$(x\xi_0|\xi_0)=((\underset{J}{\otimes}x_\iota)\xi_J|\xi_J)\prod_{I\setminus J}(x_\iota\xi_{0\iota}|\xi_{0\iota})\geqq 0.$$

Since $\odot(D(x_\iota), \xi_{0\iota})$ is a core of x, x is positive. Q.E.D.

Remark 2.1. If x_ι is positive self-adjoint and if $\odot(x_\iota, \xi_{0\iota})$ is closable, then $(\xi_{0\iota}, \xi_{0\iota})$ is a non-zero reference pair of (x_ι).

We may assume that $(x_\iota\xi_{0\iota})\in S_0$. If $\odot(x_\iota, \xi_{0\iota})$ is closable, then $\odot_{I\setminus J}(x_\iota, \xi_{0\iota})$ is closable for any $J\subset\subset I$ and hence

$$\overline{\odot(x_\iota, \xi_{0\iota})}=(\underset{J}{\otimes}x_\iota)\otimes(\overline{\underset{I\setminus J}{\odot}(x_\iota, \xi_{0\iota})}). \tag{2.1}$$

Let

$$\overline{\underset{I\setminus J}{\odot}(x_\iota, \xi_{0\iota})}=v(I\setminus J)x(I\setminus J)$$

be the polar decomposition. It then follows from (2.1) that $v(I)=(\otimes_J 1_\iota)\otimes v(I\setminus J)$. Since $(x_\iota\xi_{0\iota})\in S_0$, we may apply the similar argument as in the proof of Theorem 1.1 to these partial isometries and obtain that $(\xi_{0\iota})\sim(x_\iota\xi_{0\iota})$. Thus $(\xi_{0\iota}, \xi_{0\iota})$ is a non-zero reference pair of (x_ι).

Example 2.2. For $\lambda>0$, put

$$x_\iota\equiv\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix} \text{ and } \xi_\iota\equiv\begin{pmatrix}(1+\lambda^2)^{-1/2}\\ \lambda(1+\lambda^2)^{-1/2}\end{pmatrix}.$$

Then $(\xi_\iota)\in S_0$ and $(x_\iota\xi_\iota)\in S_0$. Besides, if $\lambda\neq 1$, then $(x_\iota^2\xi_\iota)\notin S$, $(\xi_\iota)\nsim(x_\iota\xi_\iota)$ and $\odot(x_\iota, \xi_{0\iota})$ is not closable.

Lemma 2.6. *Let $(\xi_{0\iota}, \eta_{0\iota})$ be a non-zero reference pair of (x_ι) and let $x_\iota=u_\iota|x_\iota|$ be the polar decomposition of x_ι. Then*

(i) *$(u_\iota^*u_\iota\xi_{0\iota})\in S$ and $(u_\iota u_\iota^*\eta_{0\iota})\in S$;*

(ii) *$(\xi_{0\iota}, \eta_{0\iota})$ and $(\eta_{0\iota}, \xi_{0\iota})$ are non-zero reference pairs of (u_ι) and (u_ι^*), respectively;*

(iii) *$(\otimes^{c'c}u_\iota)^*=\otimes^{cc'}u_\iota^*$; and*

(iv) *if $(u_\iota^*u_\iota\xi_{0\iota})\in S_0$ and $(u_\iota u_\iota^*\eta_{0\iota})\in S_0$, $(u_\iota\xi_{0\iota}, u_\iota^*\eta_{0\iota})$ is a non-zero reference pair of (x_ι^*).*

Proof. (i) Since $(x_\iota^*\eta_{0\iota})\in S$, we have $(|x_\iota^*|\eta_{0\iota})\in S$. Since $(x_\iota^*\eta_{0\iota})\sim(\xi_{0\iota})$, we have $(|x_\iota^*|\eta_{0\iota})\sim(u_\iota\xi_{0\iota})$. Since $\|u_\iota\|=1$ and $(\xi_{0\iota})\in S$, it follows from Lemma 1 in [1] that $(u_\iota\xi_{0\iota})\in S$ and hence $(u_\iota^*u_\iota\xi_{0\iota})\in S$. Since $(x_\iota\xi_{0\iota})\in S$, we have $(u_\iota u_\iota^*\eta_{0\iota})\in S$.

(ii) $(u_\iota\xi_{0\iota})\in S$ and $(u_\iota^*\eta_{0\iota})\in S$ are shown in the above. Since $(|x_\iota^*|\eta_{0\iota})\sim(\eta_{0\iota})$ by Theorem 1.1, we have $(u_\iota\xi_{0\iota})\sim(\eta_{0\iota})$ and $(\xi_{0\iota})\sim(u_\iota^*\eta_{0\iota})$. Thus (ii) follows.

(iii) Since $\otimes^{c'c}u_\iota$ is bounded and since

$$((\otimes^{c'c}u_\iota)\otimes\xi_\iota|\otimes\eta_\iota)=(\otimes u_\iota\xi_\iota|\otimes\eta_\iota)$$

$$=\prod(u_\iota\xi_\iota|\eta_\iota)=\prod(\xi_\iota|u_\iota^*\eta_\iota)$$

$$=(\otimes\xi_\iota|\otimes u_\iota^*\eta_\iota)=(\otimes\xi_\iota|(\otimes^{cc'}u_\iota^*)\otimes\eta_\iota)$$

for all $\otimes\xi_\iota\in\odot(D(x_\iota), \xi_{0\iota})$ and $\otimes\eta_\iota\in\odot(D(x_\iota^*), \eta_{0\iota})$, we have (iii).

(iv) Since $x_\iota^*u_\iota\xi_{0\iota}=|x_\iota|\xi_{0\iota}$ and $x_\iota u_\iota^*\eta_{0\iota}=|x_\iota^*|\eta_{0\iota}$, $(u_\iota\xi_{0\iota}, u_\iota^*\eta_{0\iota})$ is a non-zero reference pair of (x_ι^*). Q.E.D.

Theorem 2.1. *Let $(\xi_{0\iota}, \eta_{0\iota})$ be a non-zero reference pair of (x_ι) and let $x_\iota=u_\iota|x_\iota|$ be the polar decomposition of x_ι. Then*

$$\otimes^{c'c}x_\iota=(\otimes^{c'c}u_\iota)(\otimes^{c}|x_\iota|) \tag{2.2}$$

$$=(\otimes^{c'}|x_\iota^*|)(\otimes^{c'c}u_\iota) \tag{2.3}$$

and (2.2) is the polar decomposition of $\otimes^{c'c}x_\iota$, where $c\equiv c(\xi_{0\iota})$ and

$c' \equiv c(\eta_{0\iota})$.

Proof. Since $D(x_\iota)=D(|x_\iota|)$, we have $\odot(D(x_\iota), \xi_{0\iota})=\odot(D(|x_\iota|), \xi_{0\iota})$. From Theorem 1.1 we have $(\xi_{0\iota})\sim(|x_\iota|\xi_{0\iota})$. Since $(\xi_{0\iota}, \eta_{0\iota})$ is a non-zero reference pair of (u_ι) by Lemma 2.6 and since $\|u_\iota\|\leqq 1$, we find that $(|x_\iota|\xi_{0\iota}, \eta_{0\iota})$ is also a non-zero reference pair of (u_ι) and that $\otimes^{c'c}u_\iota$ is the closure of $\odot(u_\iota, |x_\iota|\xi_{0\iota})$. We have

$$(\otimes^{c'c}x_\iota)\otimes\xi_\iota=\otimes x_\iota\xi_\iota=\otimes u_\iota|x_\iota|\xi_\iota$$

$$=(\otimes^{c'c}u_\iota)\otimes|x_\iota|\xi_\iota=(\otimes^{c'c}u_\iota)(\otimes^{c}|x_\iota|)\otimes\xi_\iota$$

for all $\otimes\xi_\iota\in\odot(D(x_\iota), \xi_{0\iota})$. Since $\odot(D(x_\iota), \xi_{0\iota})$ is a core of $\otimes^{c'c}x_\iota$ and $\otimes^{c}|x_\iota|$, we have (2.2).

Since $x_\iota=u_\iota|x_\iota|$ is a polar decomposition of x_ι, $u_\iota^*u_\iota$ is a projection onto the closure of the range of $|x_\iota|$. Since $(\xi_{0\iota})\sim(|x_\iota|\xi_{0\iota})$, $\otimes^{c}u_\iota^*u_\iota\mathscr{H}_\iota$ is the closed linear span of

$$\{\otimes|x_\iota|\xi_\iota:\ \otimes\xi_\iota\in\odot(x_\iota, \xi_{0\iota})\}.$$

Therefore the closure of the range of $\otimes^{c}|x_\iota|$ is the initial space of a partial isometry $\otimes^{c'c}u_\iota$. Thus (2.2) is the polar decomposition.

(2.3) is proved similarly. Since $(x_\iota\xi_{0\iota})\in S$ and $(x_\iota^*\eta_{0\iota})\in S$, we may assume that $\otimes x_\iota\xi_{0\iota}\neq 0$ and $\otimes x_\iota^*\eta_{0\iota}\neq 0$ by the same reason at the beginning part of the proof of Theorem 1.1. Therefore $(u_\iota^*u_\iota\xi_{0\iota})\in S_0$ and $(u_\iota u_\iota^*\eta_{0\iota})\in S_0$ as above. From Lemma 2.6 it follows that $(u_\iota\xi_{0\iota}, u_\iota^*\eta_{0\iota})$ is a non-zero reference pair of (x_ι^*) and hence from Theorem 1.1 that $(u_\iota\xi_{0\iota}, u_\iota\xi_{0\iota})$ is a non- zero reference pair of $(|x_\iota^*|)$. Since $|x_\iota|=u_\iota^*|x_\iota^*|u_\iota$, we have $u_\iota D(x_\iota)=u_\iota D(|x_\iota|)=D(|x_\iota^*|)$. This implies $(\otimes^{c'c}u_\iota)\odot(D(x_\iota), \xi_{0\iota})=\odot(D(|x_\iota^*|), u_\iota\xi_{0\iota})$. Hence we have

$$(\otimes^{c'c}x_\iota)\otimes\xi_\iota=\otimes|x_\iota^*|u_\iota\xi_\iota=(\otimes^{c'}|x_\iota^*|)\otimes u_\iota\xi_\iota$$

$$=(\otimes^{c'}|x_\iota^*|)(\otimes^{c'c}u_\iota)\otimes\xi_\iota$$

for all $\otimes\xi_\iota\in\odot(D(x_\iota), \xi_{0\iota})$. Since $\odot(D(x_\iota), \xi_{0\iota})$ is a core of $\otimes^{c'c}x_\iota$ and $(\otimes^{c'c}u_\iota)\odot(D(x_\iota), \xi_{0\iota})$ is a core of $\otimes^{c'}|x_\iota^*|$, we have (2.3).

Q.E.D.

Remark 2.2. If $\odot(x_\iota, \xi_{0\iota})$ is closable, then there exists a C_0-sequence $(\eta_{0\iota}) \in S_0$ such that $(\xi_{0\iota}, \eta_{0\iota})$ is a non-zero reference pair fo (x_ι). This is proved by combining Remark 1.1 and Theorem 2.1.

Theorem 2.2. *Under the same assumption as Theorem* 2.1,

$$(\otimes^{c'c} x_\iota)^* = \otimes^{cc'} x_\iota^* . \tag{2.4}$$

Proof. Using (2.3) and (iii) of Lemma 2.5, we have

$$(\otimes^{c'c} x_\iota)^* = (\otimes^{c'c} u_\iota)^* (\otimes^{c'} |x_\iota^*|) = (\otimes^{cc'} u_\iota^*)(\otimes^{c'} |x_\iota^*|) .$$

Since $x_\iota^* = u_\iota^* |x_\iota^*|$ and $(\eta_{0\iota}, \xi_{0\iota})$ is a non-zero reference pair of (x_ι^*) by Lemma 2.3, we have $\otimes^{cc'} x_\iota^* = (\otimes^{cc'} u_\iota^*)(\otimes^{c'} |x_\iota^*|)$ by (2.2). This completes the proof.

Theorem 2.3. *Let M_ι be a von Neumann algebra on $\mathscr{H}_\iota$ for each $\iota \in I$, and let x_ι be an operator affiliated with M_ι. If $(\xi_{0\iota}, \eta_{0\iota})$ is a non-zero reference pair of (x_ι) with $c(\xi_{0\iota}) = c(\eta_{0\iota}) = c$, then $\otimes^c x_\iota$ is affiliated with $\otimes^c M_\iota$.*

Proof. If $\xi \in D(\otimes^c x_\iota)$, there exists a sequence $\{\xi_n\}_{n=1}^\infty$ in $\odot(D(x_\iota), \xi_{0\iota})$ such that $\xi_n \to \xi$ and $(\otimes^c x_\iota)\xi_n \to (\otimes^c x_\iota)\xi$ in $\mathscr{H}_c$. According to Lemma 6.10 in [2], we have $(\otimes^c M_\iota)' = \otimes^c M_\iota'$ and hence $\otimes^c M_\iota'$ is generated by $\otimes^c v_\iota$ such that v_ι is a unitary in M_ι' and $v_\iota = 1$ except for a finite number of ι. For each ξ_n of the form $\xi_n = \sum_{j=1}^m \otimes \xi_{j\iota}$ with $\otimes \xi_{j\iota} \in \odot(D(x_\iota), \xi_{0\iota})$, we find $(\otimes^c v_\iota)\xi_n = \sum_{j=1}^m \otimes v_\iota \xi_{j\iota}$ in $\odot(D(x_\iota), \xi_{0\iota})$. This shows that $\odot(D(x_\iota), \xi_{0\iota})$ is invariant under such $\otimes v_\iota$ and hence $D(\otimes^c x_\iota)$ is invariant under $\otimes^c M_\iota'$. It follows that $\{(\otimes^c v_\iota)\xi_n\}_{n=1}$ is a Cauchy sequence in $\odot(D(x_\iota), \xi_{0\iota})$ in the sense of graph of $\otimes^c x_\iota$. Thus

$$\begin{aligned}(\otimes^c x_\iota)(\otimes^c v_\iota)\xi &= \lim_{n\to\infty} (\otimes^c x_\iota)(\otimes^c v_\iota)\xi_n \\ &= \lim_{n\to\infty} (\otimes^c v_\iota)(\otimes^c x_\iota)\xi_n = (\otimes^c v_\iota)(\otimes^c x_\iota)\xi ,\end{aligned}$$

which shows that $\otimes^c x_\iota$ is affiliated with $\otimes^c M_\iota$. Q.E.D.

§3. Conditions for the Existence of a Reference Pair

We shall give some conditions for the existence of a non-zero reference pair of invertible, positive and self-adjoint operators (x_ι) in the following theorem. With a slight modification on convergence, the condition (iv) is known as Kolmogorov's three series theorem and the condition (vi) is interpreted as follows: the product of characteristic functions is also a characteristic function.

Theorem 3.1. *Let x_ι be an invertible, positive and self-adjoint operator on $\mathscr{H}_\iota$ for $\iota \in I$ and $y_\iota \equiv \log x_\iota$. Let e_ι be the spectral projection of x_ι corresponding to the interval $[\lambda_0^{-1}, \lambda_0]$ for any fixed $\lambda_0 > 1$. The following six conditions are equivalent for $c \in C$:*

(i) *there exists a non-zero reference pair $(\xi_{0\iota}, \xi_{0\iota})$ of (x_ι) with $c = c(\xi_{0\iota})$;*

(ii) *$(e_\iota \xi_{1\iota}) \in S$, $(x_\iota e_\iota \xi_{1\iota}) \in S$ and $(e_\iota \xi_{1\iota}) \sim (x_\iota e_\iota \xi_{1\iota})$ hold for some $(\xi_{1\iota}) \in c$;*

(iii) *$(e_\iota \xi_\iota) \in S$, $(x_\iota e_\iota \xi_\iota) \in S$ and $(e_\iota \xi_\iota) \sim (x_\iota e_\iota \xi_\iota)$ hold for all $(\xi_\iota) \in c$;*

(iv) *$(e_\iota \xi_\iota) \in S$, $\sum \|y_\iota e_\iota \xi_\iota\|^2 < +\infty$ and $\sum |(y_\iota e_\iota \xi_\iota | \xi_\iota)| < +\infty$ hold for all $(\xi_\iota) \in c$;*

(v) *$\xi_{2\iota} \in D(y_\iota)$, $\sum \|y_\iota \xi_{2\iota}\|^2 < +\infty$ and $\sum |(y_\iota \xi_{2\iota} | \xi_{2\iota})| < +\infty$ hold for some $(\xi_{2\iota}) \in c$; and*

(vi) *$\otimes^c x_\iota^{it}$, $t \in \boldsymbol{R}$ is a strongly continuous one parameter unitary group.*

Proof. (i)⇒(ii). We put $\xi_{1\iota} \equiv \xi_{0\iota}$ for all ι. Since $(\xi_{0\iota}, \xi_{0\iota})$ is a non-zero reference pair of (x_ι), we have

$$\sum | \|x_\iota \xi_{1\iota}\|^2 - 1 | < +\infty \quad \text{and} \quad \sum |(x_\iota \xi_{1\iota} | \xi_{1\iota}) - 1| < +\infty,$$

which imply

$$\sum \|(1 - x_\iota)\xi_{1\iota}\|^2 < +\infty \quad \text{and} \quad \sum |((1 - x_\iota)\xi_{1\iota} | \xi_{1\iota})| < +\infty.$$

Since $(1 - \lambda_0^{-1})(1 - e_\iota) \leqq |1 - x_\iota|(1 - e_\iota)$, we have

$$((1 - e_\iota)\xi_{1\iota} | \xi_{1\iota}) \leqq (1 - \lambda_0^{-1})^{-2}((1 - x_\iota)^2(1 - e_\iota)\xi_{1\iota} | \xi_{1\iota})$$

and

$$|((1-x_\iota)e_\iota\xi_{1\iota}|\xi_{1\iota})|$$
$$\leqq|((1-x_\iota)\xi_{1\iota}|\xi_{1\iota})|+(1-\lambda_0^{-1})^{-1}((1-x_\iota)^2(1-e_\iota)\xi_{1\iota}|\xi_{1\iota}).$$

Since $\|(1-x_\iota)e_\iota\xi_{1\iota}\|\leqq\|(1-x_\iota)\xi_{1\iota}\|$ and $\|(1-x_\iota)(1-e_\iota)\xi_{1\iota}\|\leqq\|(1-x_\iota)\xi_{1\iota}\|$, it follows from

$$|\|e_\iota\xi_{1\iota}\|^2-1|\leqq\|(1-e_\iota)\xi_{1\iota}\|^2+|\|\xi_{1\iota}\|^2-1|$$

and

$$|\|x_\iota e_\iota\xi_{1\iota}\|^2-1|$$
$$\leqq|\|e_\iota\xi_{1\iota}\|^2-1|+2|((1-x_\iota)e_\iota\xi_{1\iota}|\xi_{1\iota})|+\|(1-x_\iota)e_\iota\xi_{1\iota}\|^2$$

that $(e_\iota\xi_{1\iota})\in S$, $(x_\iota e_\iota\xi_{1\iota})\in S$ and $(e_\iota\xi_{1\iota})\sim(x_\iota e_\iota\xi_{1\iota})$.

(ii)⇒(iii). $(e_\iota\xi_{1\iota})\in S$ implies $(\xi_{1\iota})\sim(e_\iota\xi_{1\iota})$. If $(\xi_\iota)\in c$, then $(\xi_\iota)\sim(\xi_{1\iota})$. Therefore $(e_\iota\xi_\iota)\in S$ by Lemma 1 in [1]. Since $(\xi_\iota)\sim(\xi_{1\iota})$, we have $\sum\|\xi_\iota-\xi_{1\iota}\|^2<+\infty$. Since

$$\|(1-x_\iota)e_\iota\xi_\iota\|^2\leqq2(\|(1-x_\iota)e_\iota\xi_{1\iota}\|^2+\|(1-x_\iota)e_\iota(\xi_\iota-\xi_{1\iota})\|^2)$$
$$\leqq2(\|(1-x_\iota)e_\iota\xi_{1\iota}\|^2+(\lambda_0-1)\|\xi_\iota-\xi_{1\iota}\|^2),$$

we have $\sum\|(1-x_\iota)e_\iota\xi_\iota\|^2<+\infty$. Since

$$|((1-x_\iota)e_\iota\xi_\iota|\xi_\iota)|$$
$$\leqq|((1-x_\iota)e_\iota\xi_{1\iota}|\xi_{1\iota})|+(\|(1-x_\iota)e_\iota\xi_\iota\|+\|(1-x_\iota)e_\iota\xi_{1\iota}\|)\|\xi_\iota-\xi_{1\iota}\|$$
$$\leqq|((1-x_\iota)e_\iota\xi_{1\iota}|\xi_{1\iota})|+\|(1-x_\iota)e_\iota\xi_\iota\|^2+\|(1-x_\iota)e_\iota\xi_{1\iota}\|^2+\|\xi_\iota-\xi_{1\iota}\|^2,$$

we have $\sum|((1-x_\iota)e_\iota\xi_\iota|\xi_\iota)|<+\infty$. Consequently, $(x_\iota e_\iota\xi_\iota)\in S$ and $(e_\iota\xi_\iota)\sim(x_\iota e_\iota\xi_\iota)$.

(iii)⇒(i). Since $(e_\iota\xi_\iota)\in S$, we may assume that $(e_\iota\xi_\iota)\in S_0$. Set $\xi_{0\iota}\equiv e_\iota\xi_\iota$. It then follows that $(\xi_{0\iota})\in c$ and that $(\xi_{0\iota}, \xi_{0\iota})$ is a non-zero reference pair of (x_ι).

(iii)⇒(iv). Since

$$-x_\iota^{-1}(1-x_\iota)=-(x_\iota^{-1}-1)\leqq y_\iota\leqq x_\iota-1$$

and

$$|y_\iota-(x_\iota-1)|e_\iota \leqq \lambda_1(x_\iota-1)^2 e_\iota$$

for some constant $\lambda_1>0$, we have

$$|(y_\iota e_\iota \xi_\iota|\xi_\iota)| \leqq |((x_\iota-1)e_\iota\xi_\iota|\xi_\iota)|+\lambda_1\|(x_\iota-1)e_\iota\xi_\iota\|^2$$

and

$$\|y_\iota e_\iota \xi_\iota\|^2 \leqq \|(x_\iota-1)e_\iota\xi_\iota\|^2+\|x_\iota^{-1}(1-x_\iota)e_\iota\xi_\iota\|^2$$

$$\leqq (1+\lambda_0^2)\|(1-x_\iota)e_\iota\xi_\iota\|^2 .$$

Since $(e_\iota\xi_\iota)\in S$, $(x_\iota e_\iota\xi_\iota)\in S$ and $(\xi_\iota)\sim(x_\iota e_\iota\xi_\iota)$ from (iii), the right hand sides of these inequalities are summable over $\iota\in I$. Thus (iv) follows.

(iv)$\Rightarrow$(iii). Since $-x_\iota y_\iota \leqq 1-x_\iota \leqq -y_\iota$ and $|1-x_\iota-(-y_\iota)|e_\iota \leqq \lambda_2 y_\iota^2 e_\iota$ for some constant $\lambda_2>0$, we have

$$|((1-x_\iota)e_\iota\xi_\iota|\xi_\iota)| \leqq |(y_\iota e_\iota\xi_\iota|\xi_\iota)|+\lambda_2\|y_\iota e_\iota\xi_\iota\|^2$$

and

$$\|(1-x_\iota)e_\iota\xi_\iota\|^2 \leqq \|y_\iota e_\iota\xi_\iota\|^2+\|x_\iota y_\iota e_\iota\xi_\iota\|^2 \leqq (1+\lambda_0^2)\|y_\iota e_\iota\xi_\iota\|^2 .$$

Thus we have (iii) from (iv).

(iv)$\Rightarrow$(v). Putting $\xi_{2\iota}=e_\iota\xi_\iota$, we have (v) from (iv).

(v)$\Rightarrow$(vi). If $(\xi_\iota)\in c$, then $(\xi_\iota)\sim(\xi_{2\iota})$. Since $\log\lambda_0(1-e_\iota)\leqq|y_\iota|(1-e_\iota)$, we have

$$((1-e_\iota)\xi_{2\iota}|\xi_{2\iota}) \leqq (\log\lambda_0)^{-2}\|y_\iota(1-e_\iota)\xi_{2\iota}\|^2.$$

Since $\|y_\iota(1-e_\iota)\xi_{2\iota}\|\leqq\|y_\iota\xi_{2\iota}\|$, we have $(e_\iota\xi_{2\iota})\in S$. Since $(\xi_\iota)\sim(e_\iota\xi_{2\iota})$, $(e_\iota\xi_\iota)\in S$. Since $(\xi_\iota)\sim(\xi_{2\iota})$, $\sum\|\xi_\iota-\xi_{2\iota}\|^2<+\infty$. Since

$$\|y_\iota e_\iota\xi_\iota\|^2 \leqq 2(\|y_\iota e_\iota\xi_{2\iota}\|^2+\|y_\iota e_\iota(\xi_\iota-\xi_{2\iota})\|^2)$$

$$\leqq 2(\|y_\iota\xi_{2\iota}\|^2+(\log\lambda_0)^2\|\xi_\iota-\xi_{2\iota}\|)^2 ,$$

we have $\sum\|y_\iota e_\iota\xi_\iota\|^2<+\infty$. Since

$$
\begin{aligned}
&|(y_\iota e_\iota \xi_\iota|\xi_\iota)| \\
&\leqq |(y_\iota e_\iota \xi_{2\iota}|\xi_{2\iota})| + (\|y_\iota e_\iota \xi_\iota\| + \|y_\iota e_\iota \xi_{2\iota}\|)\|\xi_\iota - \xi_{2\iota}\| \\
&\leqq |(y_\iota e_\iota \xi_{2\iota}|\xi_{2\iota})| + \|y_\iota e_\iota \xi_\iota\|^2 + \|y_\iota e_\iota \xi_{2\iota}\|^2 + \|\xi_\iota - \xi_{2\iota}\|^2,
\end{aligned}
$$

we have $\sum|(y_\iota e_\iota \xi_\iota|\xi_\iota)| < +\infty$.

(i)⇒(iv). There is a countable subset I_0 of I such that $\|\xi_{0\iota}\|=1$, $\|x_\iota\xi_{0\iota}\|=1$ and $(x_\iota\xi_{0\iota}|\xi_{0\iota})=1$ for all $\iota\in I\backslash I_0$. Therefore $\|x_\iota\xi_{0\iota}-\xi_{0\iota}\|^2=0$ and hence $x_\iota\xi_{0\iota}=\xi_{0\iota}$ for all $\iota\in I\backslash I_0$. Therefore $x_\iota^{it}\xi_{0\iota}=\xi_{0\iota}$ for $\iota\in I\backslash I_0$. Restricting the index set to I_0, we know that $(\xi_{0\iota}, \xi_{0\iota})$ is a non-zero reference pair of $(x_\iota;\ \iota\in I_0)$. Then $\otimes_{I_0}^{(\xi_{0\iota})}x_\iota^{it}$, $t\in\boldsymbol{R}$ is strongly continuous by [14] and [13]. Since $\odot(\mathscr{H}_\iota, \xi_{0\iota})$ is dense in $\mathscr{H}_c$ and since $\otimes^c x_\iota^{it}$ is bounded, it is strongly continuous unitary group in $t\in\boldsymbol{R}$.

(vi)⇒(i). Choose t_0 and t_1 in $\boldsymbol{R}$ such that t_0/t_1 is irrational. For any $(\xi_\iota)\in S_0$ with $c=c(\xi_\iota)$, there exists a countable subset I_1 of I such that $x_\iota^{it_0}\xi_\iota=\xi_\iota$ and $x_\iota^{it_1}\xi_\iota=\xi_\iota$ for all $\iota\in I\backslash I_1$. Then $x_\iota^{it}\xi_\iota=\xi_\iota$ for all $t\in t_0\boldsymbol{Z}+t_1\boldsymbol{Z}$ and $\iota\in I\backslash I_1$. Since $t_0\boldsymbol{Z}+t_1\boldsymbol{Z}$ is dense in $\boldsymbol{R}$ and since x_ι^{it} is strongly continuous in $t\in\boldsymbol{R}$, we find that $x_\iota^{it}\xi_\iota=\xi_\iota$ for all $t\in\boldsymbol{R}$ and $\iota\in I\backslash I_1$. Applying [14] and [13] for this countable I_1, we have (iv) and hence (i) for I_1. Therefore there exists a non-zero reference pair (ξ'_ι, ξ'_ι) of (x_ι) for I_1 and $(\xi'_\iota)\sim(\xi_\iota)$ for I_1. Define $(\xi_{0\iota})\in S_0$ for I by $\xi_{0\iota}\equiv\xi_\iota$ for $\iota\in I\backslash I_1$ and $\xi_{0\iota}\equiv\xi'_\iota$ for $\iota\in I_1$. Then $(\xi_{0\iota})\in S_0$, $c=c(\xi_{0\iota})$, $(x_\iota\xi_{0\iota})\in S$ and $(\xi_{0\iota})\sim(x_\iota\xi_{0\iota})$. Consequently, $(\xi_{0\iota}, \xi_{0\iota})$ is a non-zero reference pair of (x_ι) with $c=c(\xi_{0\iota})$. Q.E.D.

Remark 3.1. Let x_ι be invertible, positive and self-adjoint, and $y_\iota\equiv\log x_\iota$. If $(\xi_{0\iota}, \xi_{0\iota})$ is a non-zero reference pair of (x_ι), there exists a strong convergence vector $\otimes\xi_{2\iota}$ of (y_ι) with $(\xi_{0\iota})\sim(\xi_{2\iota})$ in the sense of Reed, [13].

Theorem 3.2. *If $(\xi_{0\iota}, \eta_{0\iota})$ is a non-zero reference pair of (x_ι), there exists a non-zero reference pair $(\xi_{1\iota}, \xi_{1\iota})$ of $(x_\iota^*x_\iota)$ with $(\xi_{1\iota})\sim(\xi_{0\iota})$ and*

$$
\otimes^c x_\iota^* x_\iota = (\otimes^{c'c}x_\iota)^*(\otimes^{c'c}x_\iota). \tag{3.1}
$$

Proof. If $(\xi_{0\iota}, \eta_{0\iota})$ is a non-zero reference pair of (x_ι), then $(\xi_{0\iota},$

$\xi_{0\iota}$) is a non-zero reference pair of $(|x_\iota|)$. Since $(\mathrm{Ker}\otimes^c x_\iota^* x_\iota)^\perp \subset (\mathrm{Ker} \otimes^{c'c} x_\iota)^\perp = \otimes^c (\mathrm{Ker}\, x_\iota)^\perp$, we can restrict our proof over $\otimes^c(\mathrm{Ker}\, x_\iota)^\perp$. By the implication (i)⇒(vi) of Theorem 3.1, $\otimes^c |x_\iota|^{it}$ is strongly continuous unitary group in $t \in \boldsymbol{R}$ for $c = c(\xi_{0\iota})$. Since $\otimes^c (x_\iota^* x_\iota)^{it} = \otimes^c |x_\iota|^{2it}$, by the implication (vi)⇒(i) of Theorem 3.1 we have a non-zero reference pair $(\xi_{1\iota}, \xi_{1\iota})$ of $(x_\iota^* x_\iota)$ with $c = c(\xi_{1\iota})$. Since $(x_\iota^* x_\iota \xi_{1\iota}) \in S$, we may assume that $(x_\iota^* x_\iota \xi_{1\iota}) \in S_0$. Since $D(x_\iota^* x_\iota) \subset D(|x_\iota|)$ and since

$$\|(|x_\iota| - 1)\xi_{1\iota}\|^2 \leqq \|(x_\iota^* x_\iota - 1)\xi_{1\iota}\|^2$$

$$= \|x_\iota^* x_\iota \xi_{1\iota}\|^2 - 2\|x_\iota \xi_{1\iota}\|^2 + \|\xi_{1\iota}\|^2,$$

we have $(\xi_{1\iota}) \sim (|x_\iota| \xi_{1\iota})$. Therefore $(\xi_{1\iota}, \xi_{1\iota})$ is also a non-zero reference pair of $(|x_\iota|)$. Since $(x_\iota^* x_\iota \xi_{1\iota}) \in S_0$ and $(x_\iota \xi_{1\iota}) \in S_0$, we find that $(x_\iota \xi_{1\iota}, \xi_{1\iota})$ is a non zero-reference pair of (x_ι^*) and

$$(\otimes^{c'c} x_\iota) \odot (D(x_\iota^* x_\iota), \xi_{1\iota}) \subset \odot (D(x_\iota^*), x_\iota \xi_{1\iota}).$$

Therefore $\odot(D(x_\iota^* x_\iota), \xi_{1\iota})$ is included in the domain of $(\otimes^{c'c} x_\iota)^* (\otimes^{c'c} x_\iota)$. Since (3.1) holds on $\odot(D(x_\iota^* x_\iota), \xi_{1\iota})$, we have

$$\otimes^c x_\iota^* x_\iota \subset (\otimes^{c'c} x_\iota)^* (\otimes^{c'c} x_\iota).$$

Since both sides are self-adjoint, (3.1) is obtained. Q.E.D.

Lemma 3.1. *If* $\lambda_\iota \geqq 0$, $\prod\{\lambda_\iota : \lambda_\iota \neq 0\} < +\infty$, $(\xi_\iota) \in S_0$ *and* $\sum \| |x_\iota| \xi_\iota - \lambda_\iota \xi_\iota \| < +\infty$, *then for any* $0 < \varepsilon < 2^{-1}$ *there exists a* $J \subset\subset I$ *such that for any* $K \subset\subset I \backslash J$

$$\| \underset{K}{\otimes} |x_\iota| \xi_\iota - \underset{K}{\otimes} \lambda_\iota \xi_\iota \| < \varepsilon.$$

Proof. Since $\lambda_\iota \geqq 0$, $\prod\{\lambda_\iota : \lambda_\iota \neq 0\} < +\infty$ and $(\xi_\iota) \in S_0$, there is a $\mu > 1$ with $\prod_J \|\lambda_\iota \xi_\iota\| < \mu$ for $J \subset\subset I$. Choose any $0 < \varepsilon < 2^{-1}$. Since $\sum \| |x_\iota| \xi_\iota - \lambda_\iota \xi_\iota \| < +\infty$, there exists a $J \subset\subset I$ such that for any $K \subset\subset I \backslash J$

$$\sum_K \| |x_\iota| \xi_\iota - \lambda_\iota \xi_\iota \| < (2\mu)^{-1} \varepsilon,$$

which implies

$$\|\underset{K}{\otimes}|x_\iota|\xi_\iota-\underset{K}{\otimes}\lambda_\iota\xi_\iota\|$$

$$=\|\underset{K}{\otimes}\{\lambda_\iota\xi_\iota+(|x_\iota|\xi_\iota-\lambda_\iota\xi_\iota)\}-\underset{K}{\otimes}\lambda_\iota\xi_\iota\|$$

$$=\|\sum_{\iota\in K}(|x_\iota|\xi_\iota-\lambda_\iota\xi_\iota)\otimes(\underset{\substack{\kappa\in K\\ \kappa\neq\iota}}{\otimes}\lambda_\kappa\xi_\kappa)$$

$$+\sum_{\substack{\iota,\iota'\in K\\ \iota\neq\iota'}}(|x_\iota|\xi_\iota-\lambda_\iota\xi_\iota)\otimes(|x_{\iota'}|\xi_{\iota'}-\lambda_{\iota'}\xi_{\iota'})\otimes(\underset{\substack{\kappa\in K\\ \kappa\neq\iota,\iota'}}{\otimes}\lambda_\kappa\xi_\kappa)$$

$$+\cdots+\underset{K}{\otimes}(|x_\iota|\xi_\iota-\lambda_\iota\xi_\iota)\|<\varepsilon.$$

Q.E.D.

In the following we designate the spectrum and the point spectrum of a closed operator x by $\sigma(x)$ and $\sigma_p(x)$, respectively.

Let $z=u|z|$ be the polar decomposition of z. Let e be the spectral projection of $|z|$ corresponding to $[\lambda_0-\varepsilon, \lambda_0+\varepsilon]$ for any given $\varepsilon>0$. If $\lambda_0\in\sigma(|z|)\backslash\sigma_p(|z|)$, there exists a non zero vector ξ such that $e\xi=\xi$, $u^*u\xi=\xi$, $z\xi\neq 0$, which implies

$$\||z|\xi-\lambda\xi\|<\varepsilon\|\xi\| \text{ and } \||z^*|u\xi-\lambda u\xi\|<\varepsilon\|\xi\|,$$

whenever $|\lambda-\lambda_0|<\varepsilon$.

$\boldsymbol{R}_+^*$ denotes the set of all positive numbers. Theorem 1.1 in [7] is then restated as follows: Let y_n, $n\in\boldsymbol{N}$ and y be invertible, positive and self-adjoint operators on a separable Hilbert space. Then the following conditions are equivalent when n tends to $+\infty$:

(i) $f(y_n)$ converges strongly to $f(y)$ for every $f\in C(\boldsymbol{R}_+^*)$ which vanishes at 0 and $+\infty$;

(ii) $f(y_n)$ converges strongly to $f(y)$ for every bounded $f\in C(\boldsymbol{R}_+^*)$; and

(iii) y_n^{it} converges strongly to y^{it} for all $t\in\boldsymbol{R}$.

Using this we have

Theorem 3.3. (i) *Assume that* $x_\iota\neq 0$, $x_\iota=u_\iota|x_\iota|$ *is the polar decomposition, and there is a* $\lambda_\iota\in\sigma(|x_\iota|)$ *for each* $\iota\in I$ *such that* $\prod\{\lambda_\iota:\lambda_\iota\neq 0\}<+\infty$ *and* $\{\iota\in I:\lambda_\iota\notin\sigma_p(|x_\iota|)\}$ *is countable. If* $\sum|\lambda_\iota-1|<+\infty$, *then there exists a non zero reference pair* $(\xi_{0\iota},\eta_{0\iota})$ *of* (x_ι) *satisfying* $u_\iota^*u_\iota\xi_{0\iota}=\xi_{0\iota}$, $\eta_{0\iota}=u_\iota\xi_{0\iota}$ *and*

(3.2) $$\sum \| |x_\iota|\xi_{0\iota} - \lambda_\iota \xi_{0\iota} \| < +\infty .$$

(ii) *If* $(\xi_{0\iota}, \eta_{0\iota})$ *is a non-zero reference pair of* (x_ι) *with* (3.2) *for some* $\lambda_\iota \geqq 0$, *then*

(3.3) $$(\otimes^{c'c} x_\iota)\xi = \lim_{J \subset\subset I} y_J \xi$$

for any $\xi \in D(\otimes^{c'c} x_\iota)$, *where* w_ι *is a partial isometry with the initial space* $\{\lambda\xi_{0\iota} : \lambda \in \boldsymbol{C}\}$ *and the final space* $\{\lambda u_\iota \xi_{0\iota} : \lambda \in \boldsymbol{C}\}$; $y_\iota \equiv x_\iota$, $\iota \in J$ *and* $y_\kappa \equiv \lambda_\kappa w_\kappa$, $\kappa \in I \backslash J$ *for each* $J \subset\subset I$; $y_J \equiv \otimes^{c'c} y_\iota$.

(iii) *Assume that* $\mathscr{H}_\iota$ *is separable and* x_ι *is invertible, positive and self-adjoint on* $\mathscr{H}_\iota$. *If* $(\xi_{0\iota}, \xi_{0\iota})$ *is a non-zero reference pair of* (x_ι) *satisfying* $(\xi_{0\iota}) \in c$, *then* $\otimes^c x_\iota^{it}$ *is unitary on* $\mathscr{H}_c$ *and*

(3.4) $$(\otimes^c x_\iota)^{it} = \otimes^c x_\iota^{it}$$

for all $t \in \boldsymbol{R}$.

Proof. (i) Let $I_p = \{\iota \in I : \lambda_\iota \notin \sigma_p(|x_\iota|)\}$ and $I_0 = \{\iota \in I : \lambda_\iota = 0\}$. Since I_p is countable, I_p is identified with $\boldsymbol{N}$. Let e_m, $m \in \boldsymbol{N}$ be the spectral projection of $|x_m|$ corresponding to $\{\lambda \in \boldsymbol{R}_+^* : |\lambda - \lambda_m| \leqq \varepsilon^{m+1}\}$ for any fixed $0 < \varepsilon < 2^{-1}$. By the discussion preceding to this theorem there is a unit vector ξ_{0m} such that $e_m \xi_{0m} = \xi_{0m}$, $u_m^* u_m \xi_{0m} = \xi_{0m}$, $x_m \xi_{0m} \neq 0$ and

$$\sum_{I_p} \| |x_\iota|\xi_{0\iota} - \lambda_\iota \xi_{0\iota} \| < +\infty .$$

For $\iota \in I \backslash I_p$ there is a unit vector $\xi_{0\iota}$ in $D(x_\iota)$ with $|x_\iota|\xi_{0\iota} = \lambda_\iota \xi_{0\iota}$. Therefore $(\xi_{0\iota}) \in S_0$ and (3.2) is obtained. Putting $\eta_{0\iota} \equiv u_\iota \xi_{0\iota}$ for all $\iota \in I$, we have $(\eta_{0\iota}) \in S$.

If $\sum |\lambda_\iota - 1| < +\infty$, namely, if $I_0 \subset\subset I$ and $\prod \{\lambda_\iota : \iota \notin I_0\} > 0$, then

$$\sum | \|x_\iota \xi_{0\iota}\| - 1 | \leqq \sum \| |x_\iota|\xi_{0\iota} - \lambda_\iota \xi_{0\iota} \| + \sum | \|\lambda_\iota \xi_{0\iota}\| - 1 | < +\infty ,$$

which implies $(x_\iota \xi_{0\iota}) \in S$ and $(x_\iota^* \eta_{0\iota}) = (|x_\iota|\xi_{0\iota}) \in S$. Since $\sum |\lambda_\iota - 1| < +\infty$, $(\lambda_\iota \xi_{0\iota}) \in S$, $(\bar{\lambda}_\iota \eta_{0\iota}) \in S$ and $(\bar{\lambda}_\iota \eta_{0\iota}) \sim (\eta_{0\iota})$. Since $(|x_\iota|\xi_{0\iota}) \sim (\lambda_\iota \xi_{0\iota})$ by (3.2), we have $(x_\iota \xi_{0\iota}) \sim (\bar{\lambda}_\iota \eta_{0\iota})$ and hence $(x_\iota \xi_{0\iota}) \sim (\eta_{0\iota})$. Therefore $(\xi_{0\iota}, \eta_{0\iota})$ is a non zero reference pair of (x_ι) with desired properties, if we replace $\xi_{0\iota}$ with $\iota \in I_0$ by any vector satisfying $u_\iota^* u_\iota \xi_{0\iota} = \xi_{0\iota}$.

(ii) We use the same notations I_p, I_0 and e_m as above. From (3.2), if $\sum|\lambda_\iota-1|=+\infty$, then $\prod\|x_\iota\xi_\iota\|=0$ for all $\otimes\xi_\iota\in\odot(D(x_\iota), \xi_{0\iota})$ and $y_J=0$. Thus (3.3) holds.

If $\sum|\lambda_\iota-1|<+\infty$, there is a δ in $(0, 2^{-1})$ such that $\delta<\prod_K\lambda_\iota<\delta^{-1}$ for any $K\subset I\setminus I_0$. Choose an $\varepsilon>0$ with $\varepsilon<\delta$. From the definition of e_m, we have $(\lambda_m-\varepsilon^{m+1})e_m\leqq|x_m|e_m$. Since $0<\prod_{m\in I_p\setminus I_0}(\lambda_m-\varepsilon^{m+1})<+\infty$, there exists a $0<\mu<1$ such that $\mu<\prod_K(\lambda_m-\varepsilon^{m+1})<\mu^{-1}$ for any $K\subset I_p\setminus I_0$. Since $e_m\xi_{0m}=\xi_{0m}$ for $m\in I_p$, if $K'\subset I_p\setminus I_0$, then

$$\mu(\underset{K'}{\otimes} w_\iota^* w_\iota)\leqq\mu(\underset{K'}{\otimes} e_m)\leqq\underset{K'}{\otimes}\{(\lambda_m-\varepsilon^{m+1})e_m\}\leqq\underset{K'}{\otimes}|x_\iota|,$$

and if $K''\subset \boldsymbol{I}\setminus(I_p\cup I_0)$, then

$$\delta(\underset{K''}{\otimes} w_\iota^* w_\iota)\leqq\underset{K''}{\otimes}(\lambda_\iota w_\iota^* w_\iota)\leqq\underset{K''}{\otimes}|x_\iota|.$$

Since $\delta\mu<\min\{\delta, \mu\}<1$ and $\otimes_{I_0}\lambda_\iota w_\iota^* w_\iota=0$, we have

$$(\delta\mu)^2 y_J^* y_J\leqq(\otimes^{c'c}x_\iota)^*(\otimes^{c'c}x_\iota) \tag{3.5}$$

on $D(\otimes^{c'c}x_\iota)$ for every $J\subset\subset I$. Since $\odot(D(x_\iota), \xi_{0\iota})$ is a core of $\otimes^{c'c}x_\iota$, there exists a sequence $\{\xi_n\}_{n=1}^\infty$ in $\odot(D(x_\iota), \xi_{0\iota})$ which converges to ξ in the sense of the graph of $\otimes^{c'c}x_\iota$. It follows from (3.5) that $\{\xi_n\}_{n=1}^\infty$ is a Cauchy sequence in the sense of the graph of y_J. Therefore, since y_J is closed, we have $\xi\in D(y_J)$. For the above $\varepsilon>0$ there exists an n_0 and a $J_0\subset\subset I$ such that for every $n\geqq n_0$ and for every $J\subset\subset I$ with $J_0\subset J$

$$\|(\otimes^{c'c}x_\iota)(\xi_n-\xi)\|<(2+2(\delta\mu)^{-1})^{-1}\varepsilon$$

and

$$\|(y_J-\otimes^{c'c}x_\iota)\xi_{n_0}\|<2^{-1}\varepsilon.$$

Then we have

$$\begin{aligned}&\|(y_J-\otimes^{c'c}x_\iota)\xi\|\\&\leqq\|y_J(\xi-\xi_{n_0})\|+\|(y_J-\otimes^{c'c}x_\iota)\xi_{n_0}\|+\|(\otimes^{c'c}x_\iota)(\xi_{n_0}-\xi)\|\\&\leqq(1+(\delta\mu)^{-1})\|(\otimes^{c'c}x_\iota)(\xi_{n_0}-\xi)\|+\|(y_J-\otimes^{c'c}x_\iota)\xi_{n_0}\|<\varepsilon\end{aligned}$$

for $\xi \in D(\otimes^{c'c} x_\iota)$.

(iii) Since $(\xi_{0\iota}, \xi_{0\iota})$ is a non-zero reference pair of (x_ι), we have $(x_\iota \xi_\iota) \in S$ and $(x_\iota \xi_\iota) \sim (\xi_\iota)$ for all non zero $\otimes \xi_\iota$ in $\odot(D(x_\iota), \xi_{0\iota})$. Since $(x_\iota \xi_\iota) \in S$, there exists a $\lambda > 1$ with $\prod_K \|x_\iota \xi_\iota\| < \lambda$ for all K. Since $(x_\iota \xi_\iota) \sim (\xi_\iota)$, it follows from Lemma 3.3 in [8] that for any ε in $(0, 1)$ there exists a $J_0 \subset\subset I$ such that

$$\|\bigotimes_J x_\iota \xi_\iota - \bigotimes_J \xi_\iota\| < \varepsilon/\lambda$$

for all $J \subset I \backslash J_0$. Thus

$$\|\otimes x_\iota \xi_\iota - (\bigotimes_K x_\iota \xi_\iota) \otimes (\bigotimes_{I \setminus K} \xi_\iota)\| = \|\bigotimes_K x_\iota \xi_\iota\| \, \|\bigotimes_{I \setminus K} x_\iota \xi_\iota - \bigotimes_{I \setminus K} \xi_\iota\| < \varepsilon$$

for all K with $J_0 \subset K \subset\subset I$.

Assume first that I is countable. Let $I = \boldsymbol{N}$ and $I_n = \{1, \ldots, n\}$. Denote $y \equiv \otimes^c x_\iota$ and $y_n \equiv y_{I_n}$ (or $y_n \equiv x_{I_n}$), where we take $\lambda_\iota w_\iota = 1$. Since y_n and y are self-adjoint, $\|(y_n - i1)^{-1}\| \leqq 1$ and $\|(y - i1)^{-1}\| \leqq 1$. Let $C(x) \equiv (x + i1)(x - i1)^{-1}$. Let $D \equiv \{(y - i1)\xi : \xi \in \odot(D(x_\iota), \xi_{0\iota})\}$. Since $\odot(D(x_\iota), \xi_{0\iota})$ is a core of y by Theorem 1.1, D is dense in $\mathscr{H}_c$. For any $\eta \in D$

$$\begin{aligned} & C(y_n)\eta - C(y)\eta \\ & \quad = (y_n + i1)\{(y_n - i1)^{-1} - (y - i1)^{-1}\}\eta + (y_n - y)(y - i1)^{-1}\eta \\ & \quad = (y_n + i1)(y_n - i1)^{-1}(y - y_n)(y - i1)^{-1}\eta + (y_n - y)(y - i1)^{-1}\eta . \end{aligned}$$

Since η is of the form $(y - i1)\xi$ for some $\xi \in \odot(D(x_\iota), \xi_{0\iota})$,

$$\|C(y_n)\eta - C(y)\eta\| \leqq 2\|(y_n - y)\xi\| ,$$

which converges to 0. Since D is dense in $\mathscr{H}_c$ and since $C(y_n)$ and $C(y)$ are bounded, $C(y_n)$ converges strongly to $C(y)$. Since $\mathscr{H}_c$ is separable and since y_n and y are positive and self-adjoint, $f(y_n)$ converges strongly to $f(y)$ for every bounded function $f \in C(\boldsymbol{R}_+^*)$ by [7, Theorem 1.1]. Since $f(\lambda) = \lambda^{it}$ for $\lambda \in \boldsymbol{R}_+^*$ and $t \in \boldsymbol{R}$ is a bounded continuous function in λ, it follows that

$$(\otimes^c x_\iota)^{it}\xi = \lim_{n \to \infty} (\bigotimes_{I_n} x_\iota^{it} \xi_\iota) \otimes (\bigotimes_{I \setminus I_n} \xi_{0\iota})$$

$$=\otimes x_\iota^{it}\xi_\iota=(\otimes^c x_\iota^{it})\xi$$

for any $\xi\in\odot(D(x_\iota),\,\xi_{0\iota})$ with $\xi=\otimes\xi_\iota$.

For a general I we choose a countable $I_0\subset I$ such that $x_\iota\xi_{0\iota}=\xi_{0\iota}$ for $\iota\in I\backslash I_0$. Since $I_0\cup J$ is countable, we have

$$(\otimes^c x_\iota)^{it}\xi=((\underset{I_0\cup J}{\otimes^{(\xi_{0\iota})}}x_\iota)^{it}\otimes(\underset{I\setminus(I_0\cup J)}{\otimes^{(\xi_{0\iota})}}x_\iota)^{it})\xi$$

$$=((\underset{I_0\cup J}{\otimes^{(\xi_{0\iota})}}x_\iota^{it})\otimes(\underset{I\setminus(I_0\cup J)}{\otimes^{(\xi_{0\iota})}}x_\iota^{it}))\xi=(\otimes^c x_\iota^{it})\xi$$

for any $\xi\in\odot(D(x_\iota),\,\xi_{0\iota})$ of the form $\xi=\xi_J\otimes(\otimes_{I\setminus J}\xi_{0\iota})$ for some $J\subset\subset I$ and $\xi_J\in\otimes_J\mathscr{H}_\iota$. Thus we have (3.4). Q.E.D.

Remark 3.2. If $(\xi_{0\iota},\,\eta_{0\iota})$ is a non-zero reference pair of (x_ι) with (3.2) and if $\sum|\lambda_\iota-1|<+\infty$, we have

$$(\otimes^{c'c}x_\iota)=\lim_{J\subset\subset I}x_J\xi \tag{3.6}$$

for any $\xi\in\odot(D(x_\iota),\,\xi_{0\iota})$, where $x_J=(\otimes_J x_\iota)\otimes(\otimes^{c'c}_{I\setminus J}\lambda_\iota u_\iota)$ for each $J\subset\subset I$.

Remark 3.3. Assume the same assumption as the above (iii). Let $(\xi_{0\iota},\,\xi_{0\iota})$ be a non-zero reference pair of (x_ι) satisfying $(\xi_{0\iota})\in c$. Put $y_\iota\equiv\log x_\iota$ and $\pi_\iota^c(y_\iota)\equiv y_\iota\otimes(\otimes^c_{I\setminus\{\iota\}}1_\kappa)$. Then

$$\log\otimes^c x_\iota=\sum\pi_\iota^c(y_\iota),$$

where the sum of the right hand side is taken in the sense of Streit, [14].

Lemma 3.2. *Let z be a positive and self-adjoint operator. If $\varepsilon>0$ and $\|z\xi-\lambda\xi\|\leqq\varepsilon\|\xi\|$ for some non zero $\xi\in D(z)$, then there exists a $\lambda_0\in\sigma(z)$ such that $|\lambda-\lambda_0|\leqq\varepsilon$ and $\|z\xi-\lambda_0\xi\|\leqq2\varepsilon\|\xi\|$.*

Proof. Let e be the spectral projection of z corresponding to $[\lambda-\varepsilon,\lambda+\varepsilon]$. Put $\xi_0=e\xi$. If $\xi_0=0$, then

$$\|z\xi-\lambda\xi\|=\|(z-\lambda1)(1-e)\xi\|>\varepsilon\|(1-e)\xi\|=\varepsilon\|\xi\|,$$

which is impossible. Therefore $\xi_0\neq0$. Hence $[\lambda-\varepsilon,\,\lambda+\varepsilon]\cap\sigma(z)$ is non empty and for any λ_0 in this intersection we have

$$\|z\xi-\lambda_0\xi\| \leqq \|z\xi-\lambda\xi\| + \|(\lambda-\lambda_0)\xi\| \leqq 2\varepsilon\|\xi\| .$$

Q.E.D.

Corollary 3.1. *If* $x_\iota \neq 0$ *for all* $\iota \in I$ *and if* $\sum \| |x_\iota|\xi_{1\iota} - \xi_{1\iota}\| < +\infty$ *for some* $(\xi_{1\iota}) \in S_0$, *then there is a* $\lambda_\iota \in \sigma(|x_\iota|)$ *for each* $\iota \in I$ *such that* $0 < \prod \lambda_\iota < +\infty$ *and* $\{\iota \in I : \lambda_\iota \notin \sigma_p(|x_\iota|)\}$ *is countable.*

Proof. By Lemma 3.2, there exists a $\lambda_\iota \in \sigma(|x_\iota|)$ such that $|1-\lambda_\iota| \leqq \| |x_\iota|\xi_{1\iota} - \xi_{1\iota}\|$ and $\| |x_\iota|\xi_{1\iota} - \lambda_\iota \xi_{1\iota}\| \leqq 2\| |x_\iota|\xi_{1\iota} - \xi_{1\iota}\| \|\xi_{1\iota}\|$. Then $\sum |1-\lambda_\iota| < +\infty$ and $\sum \| |x_\iota|\xi_{1\iota} - \lambda_\iota\xi_{1\iota}\| < +\infty$. Except for a countable number of $\iota \in I$, we have $|x_\iota|\xi_{1\iota} = \lambda_\iota \xi_{1\iota}$. Q.E.D.

Example 3.1. For $0 < \varepsilon_\iota < 1$, $\iota \in I$, put

$$x_\iota \equiv \begin{pmatrix} 1+\varepsilon_\iota & 0 \\ 0 & 1-\varepsilon_\iota \end{pmatrix} \quad \text{and} \quad \xi_\iota \equiv \begin{pmatrix} 2^{-1/2} \\ 2^{-1/2} \end{pmatrix}.$$

If $\sum \varepsilon_\iota^2 < +\infty$ and $\sum \varepsilon_\iota = +\infty$, then $(\xi_\iota) \in S_0$, $(x_\iota \xi_\iota) \in S_0$, $(x_\iota^2 \xi_\iota) \in S_0$ and $\sum \|x_\iota \xi_\iota - \xi_\iota\|^2 < +\infty$. Thus we have a situation where we have a non-zero reference pair $(\xi_{0\iota}, \xi_{0\iota})$ of (x_ι) and yet there is no $\{\lambda_\iota \in \sigma(x_\iota) : \iota \in I\}$ satisfying $0 < \prod \lambda_\iota < +\infty$.

For (ξ_ι) and (η_ι) in S, $(\xi_\iota) \underset{w}{\sim} (\eta_\iota)$ denotes the condition $\sum | |(\xi_\iota|\eta_\iota)| - 1| < +\infty$, which is the weak equivalence due to von Neumann [11].

Remark 3.4. If $(\xi_\iota) \in S_0$, $(\eta_\iota) \in S_0$ and $\sum \|\xi_\iota - \eta_\iota\|^2 < +\infty$, then $(\xi_\iota) \underset{w}{\sim} (\eta_\iota)$.

Indeed, since $(\xi_\iota) \in S_0$ and $(\eta_\iota) \in S_0$, we have $\sup \|\xi_\iota\| < +\infty$ and $\sup \|\eta_\iota\| < +\infty$, so that $\sum | \|\xi_\iota\| \|\eta_\iota\| - 1| < +\infty$. Since $\sum \|\xi_\iota - \eta_\iota\|^2 < +\infty$, we have $\sum |\mathrm{Re}(\xi_\iota|\eta_\iota) - 1| < +\infty$. Therefore

$$\begin{aligned} \sum \{\mathrm{Im}(\xi_\iota|\eta_\iota)\}^2 &= \sum \{ |(\xi_\iota|\eta_\iota)|^2 - |\mathrm{Re}(\xi_\iota|\eta_\iota)|^2 \} \\ &\leqq \sum \{ \|\xi_\iota\|^2 \|\eta_\iota\|^2 - |\mathrm{Re}(\xi|_\iota \eta_\iota)|^2 \} \\ &\leqq 2(\sup \|\xi_\iota\| \|\eta_\iota\|) \sum (| \|\xi_\iota\| \|\eta_\iota\| - 1| + |\mathrm{Re}(\xi_\iota|\eta_\iota) - 1|) < +\infty \end{aligned}$$

and there exists a $J \subset\subset I$ such that $2^{-1} < \mathrm{Re}(\xi_\iota|\eta_\iota) < 2$ for $\iota \in I \backslash J$. Since

$$|(\xi_\iota|\eta_\iota)| = \mathrm{Re}(\xi_\iota|\eta_\iota)|1+\{\mathrm{Im}(\xi_\iota|\eta_\iota)\}^2\{\mathrm{Re}(\xi_\iota|\eta_\iota)\}^{-2}|^{1/2}$$

$$\leqq \mathrm{Re}(\xi_\iota|\eta_\iota)+\{\mathrm{Im}(\xi_\iota|\eta_\iota)\}^2$$

for all $\iota \in I\backslash J$, we have

$$\sum||(\xi_\iota|\eta_\iota)|-1|$$

$$\leqq \sum||(\xi_\iota|\eta_\iota)|-\mathrm{Re}(\xi_\iota|\eta_\iota)|+\sum|\mathrm{Re}(\xi_\iota|\eta_\iota)-1|<+\infty .$$

Remark 3.5. Let $(\xi_\iota)\in S_0$ and $(\eta_\iota)\in S_0$. Define $(\xi_\iota)\underset{n}{\sim}(\eta_\iota)$ for some fixed $n\geqq 1$ by $\sum\|\xi_\iota-\eta_\iota\|^n<+\infty$. Then "$\underset{n}{\sim}$" is an equivalence relation. If $(\xi_\iota)\underset{1}{\sim}(\eta_\iota)$, then $(\xi_\iota)\sim(\eta_\iota)$. If $(\xi_\iota)\sim(\eta_\iota)$, then $(\xi_\iota)\underset{2}{\sim}(\eta_\iota)$. If $(\xi_\iota)\underset{2}{\sim}(\eta_\iota)$, then $(\xi_\iota)\underset{w}{\sim}(\eta_\iota)$. In general, if $(\xi_\iota)\underset{n}{\sim}(\eta_\iota)$ for $n\geqq 2$, then $\sum||(\xi_\iota|\eta_\iota)|-1|^{n/2}< +\infty$.

§4. Modular Operator

Let $\mathscr{H}_\iota$ denote the completion of a left Hilbert algebra $\mathfrak{A}_\iota$, which is supposed to have a normalized idempotent element $\xi_{0\iota}$ with $\xi_{0\iota}^{\#}=\xi_{0\iota}$.

Definition 4.1. An infinite tensor product of left Hilbert algebras $\mathfrak{A}_\iota$ is an involutive algebra of all $\otimes\xi_\iota$ in $\otimes\mathscr{H}_\iota$ with $\xi_\iota\in\mathfrak{A}_\iota$ and $\{\iota\in I: \xi_\iota\neq\xi_{0\iota}\}\subset\subset I$ whose involution and product are defined by

$$(\otimes\xi_\iota)^{\#}=\otimes\xi_\iota^{\#} \quad\text{and}\quad (\otimes\xi_\iota)(\otimes\eta_\iota)=\otimes\xi_\iota\eta_\iota .$$

This is denoted by $\odot(\mathfrak{A}_\iota, \xi_{0\iota})$.

Lemma 4.1. *$\odot(\mathfrak{A}_\iota, \xi_{0\iota})$ is a left Hilbert algebra.*

Proof. Let $\mathfrak{A}=\odot(\mathfrak{A}_\iota, \xi_{0\iota})$. Since $\xi_{0\iota}=\xi_{0\iota}^{\#}=\xi_{0\iota}^2$, it follows that $(\xi\eta|\zeta)=(\eta|\xi^{\#}\zeta)$ for ξ, η and ζ in $\mathfrak{A}$ and that for each $\xi\in\mathfrak{A}$, the mapping: $\eta\in\mathfrak{A}\mapsto\xi\eta\in\mathfrak{A}$ is continuous. Since $\mathfrak{A}_\iota^2$ is dense in $\mathfrak{A}_\iota$ and $\xi_{0\iota}^2=\xi_{0\iota}$, $\mathfrak{A}^2$ is dense in $\mathfrak{A}$. Define S_ι and S by $S_\iota\xi_\iota=\xi_\iota^{\#}$ for $\xi_\iota\in\mathfrak{A}_\iota$ and $S(\otimes_\iota\xi)=\otimes S_\iota\xi_\iota$ for $\otimes\xi_\iota\in\mathfrak{A}$. Since S_ι is closable in $\mathscr{H}_\iota$ and $\xi_{0\iota}=\xi_{0\iota}^{\#}$, it follows that $(\xi_{0\iota}, \xi_{0\iota})$ is a non-zero reference pair of $\bar{S}_\iota$. Therefore S is closable by Lemma 2.4. Q.E.D.

Remark 4.1. In order that $\odot(\mathfrak{A}_\iota, \xi_{0\iota})$ is a left Hilbert algebra, we have only to assume the existence of $\xi_{0\iota}\in\mathfrak{A}_\iota$ for each $\iota\in I$ which satisfies that $(\xi_{0\iota})\in S_0$, $(\xi_{0\iota}^2)\in S_0$, $(\xi_{0\iota})\sim(\xi_{0\iota\iota}^2)$ and that $(\xi_{0\iota}, \xi_{0\iota})$ is a non-zero reference pair of $(\bar{S}_\iota)$. In this case we can define $\otimes^c\bar{S}_\iota$ and $\otimes^c S_\iota^*$, which fulfill $\otimes^c S_\iota^*\bar{S}_\iota=(\otimes^c S_\iota^*)(\otimes^c\bar{S}_\iota)$ for $c\equiv c(\xi_{0\iota})$.

It is clear from the definition that $\odot(\mathfrak{A}_\iota, \xi_{0\iota})$ is dense in $\otimes^c\mathscr{H}_\iota$. If we define a left representation π of $\odot(\mathfrak{A}_\iota, \xi_{0\iota})$ on $\otimes^c\mathscr{H}_\iota$ by

$$\pi(\otimes\xi_\iota)\otimes\eta_\iota=\otimes\xi_\iota\eta_\iota,$$

then $\pi(\odot(\mathfrak{A}_\iota, \xi_{0\iota}))''=\otimes^c\pi_\iota(\mathfrak{A}_\iota)''$, where π_ι is a left representation of $\mathfrak{A}_\iota$ on $\mathscr{H}_\iota$. This is proved by the similar argument as the proof of Corollary 3.3 in [9].

Let $\mathfrak{B}_\iota$ denote a Tomita algebra dense in $\mathscr{H}_\iota$ with the modular automorphism $\Delta_\iota(z)$ for $\iota\in I$. If $\Delta_\iota(z)\xi_{0\iota}=\xi_{0\iota}$ for all $\iota\in I$ and $z\in\boldsymbol{C}$, we can define a modular automorphism $\Delta(z)$ on $\odot(\mathfrak{B}_\iota, \xi_{0\iota})$ by

$$\Delta(z)(\otimes\xi_\iota)=\otimes\Delta_\iota(z)\xi_\iota$$

for $\otimes\xi_\iota$ in $\odot(\mathfrak{B}_\iota, \xi_{0\iota})$. Here we denote by Δ_ι the modular operator on $\mathscr{H}_\iota$ associated with the modular automorphism $\Delta_\iota(z)$, $z\in\boldsymbol{C}$. Since $(\xi_{0\iota}, \xi_{0\iota})$ is a non-zero reference pair of (Δ_ι), we can define by Theorem 1.1 a positive self-adjoint operator $\Delta=\otimes^c\Delta_\iota$ in $\otimes^c\mathscr{H}_\iota$ for $c\equiv c(\xi_{0\iota})$. Here we suppose that $\mathscr{H}_\iota$ is separable for all $\iota\in I$. Since $\odot(\mathfrak{B}_\iota, \xi_{0\iota})$ is a core of $\otimes^c\Delta_\iota$, we have $\Delta^{it}=\otimes^c\Delta_\iota^{it}$ by Theorem 3.3. It then follows from the uniqueness of modular operator that Δ is the modular operator associated with $\Delta(z)$, $z\in\boldsymbol{C}$.

Lemma 4.2. *Suppose that $\mathscr{H}_\iota$ is separable for all $\iota\in I$. If $\Delta_\iota(z)\xi_{0\iota}=\xi_{0\iota}$ for all $\iota\in I$ and $z\in\boldsymbol{C}$, $\odot(\mathfrak{B}_\iota, \xi_{0\iota})$ is a Tomita algebra and $\Delta=S^*\bar{S}$.*

Proof. Since $S_\iota^*\xi_{0\iota}=S_\iota^*\bar{S}_\iota\xi_{0\iota}=\Delta_\iota\xi_{0\iota}=\Delta_\iota(1)\xi_{0\iota}=\xi_{0\iota}$, we have $S^*\bar{S}(\otimes\xi_\iota)=\otimes S_\iota^*\bar{S}_\iota\xi_\iota=\otimes\Delta_\iota\xi_\iota$ for $\otimes\xi_\iota$ in $\odot(\mathfrak{B}_\iota, \xi_{0\iota})$.

Corollary 4.1. *Let $\omega_\iota\equiv\omega_{\xi_{0\iota}}$ and $\omega\equiv\otimes\omega_\iota$ on $\otimes^c M_\iota$ for $c=c(\xi_{0\iota})$. If $\mathfrak{A}_\iota$ is separable for all $\iota\in I$, then $\sigma_t^\omega=\otimes^c\sigma_t^{\omega_\iota}$.*

The separability assumption of $\mathfrak{A}_\iota$ in the above corollary will be omitted in Lemma 6.1.

§5. Infinite Product of σ-finite Measures

We shall apply the results of §3 to the infinite product of σ-finite measure spaces and give a similar result as Hill's.

Throughout this section we assume the index set I to be countably infinite.

Let $(\Omega_\iota, \mathscr{F}_\iota, \nu_\iota)$, $\iota \in I$ be a probability space. Put $(\Omega, \mathscr{F}) = \prod(\Omega_\iota, \mathscr{F}_\iota)$, $\nu = \prod \nu_\iota$, $\mathscr{H}_\iota = L^2(\Omega_\iota, \mathscr{F}_\iota, \nu_\iota)$ and $Z_\iota = L^\infty(\Omega_\iota, \mathscr{F}_\iota, \nu_\iota)$. Then $(\Omega, \mathscr{F}, \nu)$ is a probability space. When a vector ξ in $\mathscr{H}_\iota$ belongs to Z_ι, we write the operator by $\pi_\iota(\xi)$. For an η in $\mathscr{H}_\iota$ we denote by ω_η a measure on Ω_ι or a positive linear form on Z_ι dfiened by $\omega_\eta(x) = (x\eta|\eta)$ for all $x \in Z_\iota$.

Let μ_ι be a σ-finite measure on $(\Omega_\iota, \mathscr{F}_\iota)$ with $\mu_\iota \ll \nu_\iota$ and $h_\iota = d\mu_\iota/d\nu_\iota$. For $\xi_\iota \in D(h_\iota^{1/2})$ with $h_\iota^{1/2}\xi_\iota \neq 0$, we define $\xi_{0\iota} = \|\xi_\iota\|^{-1}\xi_\iota$ and $\eta_{0\iota} = \|h_\iota^{1/2}\xi_\iota\|^{-1} h_\iota^{1/2}\xi_\iota$. Then $\omega_{\eta_{0\iota}}$ is a probability measure on Ω_ι and $\omega_{\eta_{0\iota}} \ll \nu_\iota$. Therefore we can define a σ-finite measure μ_J for $J \subset\subset I$ on Ω by

$$\mu_J = (\bigotimes_J \|h_\iota^{1/2}\xi_\iota\|^{-2}\mu_\iota) \otimes (\bigotimes_{I \setminus J} \omega_{\eta_{0\iota}}).$$

Then μ_J is a semi-finite normal trace on $\otimes^{c'} Z_\iota$ for all c' with $c' \underset{u}{\sim} c(\eta_{0\iota})$.

Proposition 5.1. *With the above notations, assume that* $0 < \prod \|\pi(\xi_\iota)\| < +\infty$ *and* $\mu_\iota \ll \nu_\iota$. *If* $(\eta_{0\iota}, \eta_{0\iota})$ *is a non-zero reference pair of* $(\pi(\xi_\iota))$, *then* $\mu \equiv \sup_{J \subset\subset I} \mu_J$ *is a σ-finite measure on* Ω, *which is singular to* $\otimes \omega_{\eta_\iota}$ *whenever* $(\eta_\iota) \in S$ *and* $(\eta_\iota) \underset{u}{\nsim} (\eta_{0\iota})$. *Moreover* μ *is a semi-finite normal trace on* $\otimes^{c'} Z_\iota$ *for all* c' *with* $c' \underset{u}{\sim} c(\eta_{0\iota})$.

Proof. If $(\eta_{0\iota}, \eta_{0\iota})$ is a non zero reference pair of $(\pi(\xi_\iota))$, $\otimes^{c'}\pi(\xi_\iota)$ is in $\otimes^{c'} B(\mathscr{H}_\iota)$ for $c' = c(\eta_{0\iota})$. Since $0 < \prod \|\pi(\xi_\iota)\| < +\infty$, $\prod_{I \setminus J} \|\pi(\xi_\iota)\|$ for $J \subset\subset I$ converges to 1 as J tends to I. Since $\|h_\iota^{1/2}\xi_\iota\|^2 \omega_{\eta_{0\iota}} \leqq \|\pi(\xi_\iota)\|^2 \mu_\iota$, $\{(\prod_{I \setminus J} \|\pi(\xi_\iota)\|^{-2})\mu_J : J \subset\subset I\}$ is an increasing net of σ-finite measures on Ω. Put

$$\mu \equiv \lim_{J \subset\subset I} \{(\prod_{I \setminus J} \|\pi(\xi_\iota)\|^{-2})\mu_J\}.$$

Then for $c'=c(\eta_{0\iota})$

$$\mu(\otimes^{c'}|\pi(\xi_\iota)|^2)=\sup_{J\subset\subset I}\prod_{I\setminus J}(\|\pi(\xi_\iota)\|^{-2}\|\pi(\xi_\iota)\eta_{0\iota}\|^2)=1\,.$$

Since the set of $(\otimes_J x_\iota)\otimes(\otimes^{c'}_{I\setminus J}\|\pi(\xi_\iota)\|^{-2}\pi(\xi_\iota))$ for any $x_\iota\in Z_\iota$ and $J\subset\subset I$ is weakly total in $\otimes^{c'}Z_\iota$, it follows that μ is semi-finite. Since each Z_ι is countably decomposable and I is countable, $\otimes^{c'}Z_\iota$ is countably decomposable and hence μ is σ-finite.

If $(\eta_\iota)\in S_0$ and $(\eta_\iota)\underset{u}{\not\sim}(\eta_{0\iota})$, then the central carriers of $p_{c'}$ for $c'\equiv c(\eta_{0\iota})$ and $p_{c''}$ for $c''\equiv c(\eta_\iota)$ in $\otimes Z_\iota$ are orthogonal by Theorem (2) in [1]. Therefore μ and $\otimes\omega_{\eta_\iota}$ are mutually singular. Q.E.D.

Definition 5.1. Let μ_ι be a σ-finite measure with $\mu_\iota\ll\nu_\iota$ and $h\equiv d\mu_\iota/d\nu_\iota$. For $\xi_\iota\in D(h_\iota^{1/2})$ with $0<\prod\|\pi(\xi_\iota)\|<+\infty$ and $h_\iota^{1/2}\xi_\iota\neq 0$, let $\eta_{0\iota}\equiv\|h_\iota^{1/2}\xi_\iota\|^{-1}h_\iota^{1/2}\xi_\iota$ and $(\eta_{0\iota},\eta_{0\iota})$ be a non-zero reference pair of $(\pi(\xi_\iota))$. The σ-finite measure μ in Proposition 5.1 is denoted by $\mu^{(\xi_\iota)}$, since it depends on $(\xi_\iota)\in S_0$.

Theorem 5.1. *Let ν_ι, ν, μ_ι, h_ι be as before and let $\mu_\iota\sim\nu_\iota$* (resp. $\mu_\iota\ll\nu_\iota$). *Assume that $\xi_\iota\in D(h_\iota^{1/2})$, $0<\prod\|\pi(\xi_\iota)\|<+\infty$ and $(\eta_{0\iota},\eta_{0\iota})$ is a non-zero reference pair of $(\pi(\xi_\iota))$. Let $h_{0\iota}\equiv\|\xi_\iota\|^2\|h_\iota^{1/2}\xi_\iota\|^{-2}h_\iota$ and e_ι be the spectral projection of $h_{0\iota}^{1/2}$ corresponding to $[\lambda^{-1},\lambda]$ for any fixed $\lambda>1$. Then the following nine conditions are equivalent for $c\equiv c(\xi_{0\iota})$:*

(i) $\mu^{(\xi_\iota)}\sim\nu$ (resp. $\mu^{(\xi_\iota)}\ll\nu$);

(ii) *$(\xi_\iota)\in S$ and (ξ_ι,ξ_ι') is a non-zero reference pair of $(h_{0\iota}^{1/2})$;*

(iii) *$(\xi_\iota)\in S_0$ and $(\xi_\iota)\sim(h_{0\iota}^{1/2}\xi_\iota)$;*

(iv) *$(\xi_\iota)\in S_0$ and $(\xi_{1\iota},\xi_{1\iota})$ is a non-zero reference pair of $(h_{0\iota}^{1/2})$ for some $(\xi_{1\iota})\in c$;*

(v) *$(\xi_\iota)\in S$, $(e_\iota\xi_{2\iota})\in S$, $(h_{0\iota}^{1/2}e_\iota\xi_{2\iota})\in S$ and $(e_\iota\xi_{2\iota})\sim(h_{0\iota}^{1/2}e_\iota\xi_{2\iota})$ hold for some $(\xi_{2\iota})\in c$;*

(vi) *$(\xi_\iota)\in S$, $(e_\iota\eta_\iota)\in S$, $(h_{0\iota}^{1/2}e_\iota\eta_\iota)\in S$ and $(e_\iota\eta_\iota)\sim(h_{0\iota}^{1/2}\eta_\iota)$ hold for all $(\eta_\iota)\in c$ with $s(h_\iota)\eta_\iota=\eta_\iota$;*

(vii) *$(\xi_\iota)\in S$, $(e_\iota\eta_\iota)\in S$, $\sum\|\log h_{0\iota}^{1/2}e_\iota\eta_\iota\|^2<+\infty$ and $\sum|(\log h_{0\iota}^{1/2}e_\iota\eta_\iota|\eta_\iota)|<+\infty$ hold for all $(\eta_\iota)\in c$ with $s(h_\iota)\eta_\iota=\eta_\iota$;*

(viii) *$(\xi_\iota)\in S$, $\xi_{3\iota}\in D(h_\iota)$, $\sum\|\log h_{0\iota}^{1/2}\xi_{3\iota}\|^2<+\infty$ and $\sum|(\log h_{0\iota}^{1/2}\xi_{3\iota}|\xi_{3\iota})|<+\infty$ hold for some $(\xi_{3\iota})\in c$ with $s(h_\iota)\eta_\iota=\eta_\iota$; and*

(ix) $(\xi_\iota)\in S$, *and* $\otimes^c h_{0\iota}^{it}$, $t\in\boldsymbol{R}$ *is strongly continuous one parameter unitary* (resp. *partial isometry*) *group.*

Here $s(h_\iota)$ *is a projection to* $(\mathrm{Ker}\, h_\iota)^\perp$.

The proof of this theorem will be given after the following Proposition 5.2.

Proposition 5.2. *Under the same assumption as in Proposition* 5.1, *let* $(\eta_{0\iota}, \eta_{0\iota})$ *be a non-zero reference pair of* $(\pi(\xi_\iota))$. *Then* $\mu^{(\xi_\iota)}\sim\nu$ (resp. $\mu^{(\xi_\iota)}\ll\nu$) *if and only if* $(\xi_\iota)\in S_0$ *and* $(\xi_{0\iota})\sim(\eta_{0\iota})$. *In this case* $d\mu^{(\xi_\iota)}/d\nu=(\prod\|\xi_\iota\|^{-2})\otimes^c h_{0\iota}$ *for* $c=c(1_\iota)$ *and* $h_{0\iota}=\|\xi_\iota\|^2\|h_\iota^{1/2}\xi_\iota\|^{-2}h_\iota$.

Proof. Suppose that $(\xi_\iota)\in S_0$ and $(\xi_{0\iota})\sim(\eta_{0\iota})$. $(\xi_\iota)\in S_0$ implies $(\xi_\iota)\sim(\xi_{0\iota})$ and hence $(\xi_\iota)\sim(\eta_{0\iota})$. Since $(\pi(\xi_\iota)\eta_{0\iota})\in S$ and $(\xi_\iota)\sim(\eta_{0\iota})$, we have $(\pi(\xi_\iota)\eta_{0\iota})\sim(1_\iota)$. It then follows that

$$(\xi_\iota)\sim(\xi_{0\iota})\sim(\eta_{0\iota})\sim(\pi(\xi_\iota)\eta_{0\iota})\sim(1_\iota).$$

Since $(\xi_{0\iota})\sim(\eta_{0\iota})$, $(\xi_{0\iota}, \xi_{0\iota})$ is a non-zero reference pair of $(h_{0\iota}^{1/2})$ and $h\equiv\otimes^c h_{0\iota}$ is obtained for $c=c(1_\iota)$. Let $\mathfrak{n}_\iota=L^2(\Omega_\iota, \mu_\iota)\cap L^\infty(\Omega_\iota, \mu_\iota)$ and $\mathfrak{m}_\iota$ be the linear span of $\mathfrak{n}_\iota^*\mathfrak{n}_\iota$. For any $\otimes^c x_\iota$ in $\otimes^c Z_\iota$ with $x_\iota\in\mathfrak{m}_\iota^+$ we have

$$\nu(h((\underset{J}{\otimes}x_\iota)\otimes(\underset{I\setminus J}{\otimes^c}x_\iota|\pi(\xi_\iota)|^2)))=(\prod\|\xi_\iota\|^2)\mu_J(\otimes^c x_\iota)$$

and hence

$$\begin{aligned}\nu(h(\otimes^c x_\iota))&=\lim\nu(h((\underset{J}{\otimes}x_\iota)\otimes(\underset{I\setminus J}{\otimes^c}x_\iota|\pi(\xi_\iota)|^2)))\\&=\lim(\prod\|\xi_\iota\|^2)\mu_J(\otimes^c x_\iota)=(\prod\|\xi_\iota\|^2)\mu^{(\xi_\iota)}(\otimes^c x_\iota).\end{aligned}$$

Therefore $\mu^{(\xi_\iota)}\ll\nu$ and $d\mu^{(\xi_\iota)}/d\nu=(\prod\|\xi_\iota\|^{-2})\otimes^c h_{0\iota}$. If $\mu_\iota\sim\nu_\iota$, then $h_{0\iota}$ is invertible and hence $d\mu^{(\xi_\iota)}/d\nu$ is also invertible or $\mu^{(\xi_\iota)}\sim\nu$.

Conversely, suppose that $\mu^{(\xi_\iota)}\ll\nu$. From Proposition 5.1 it follows that $(\eta_{0\iota})\underset{u}{\sim}(1_\iota)$ or $(\eta_{0\iota})\sim(u_\iota 1_\iota)$ for some unitary u_ι in $Z_\iota'=Z_\iota$ for each $\iota\in I$. Since $(\eta_{0\iota}, \eta_{0\iota})$ is a non zero reference pair of $(\pi(\xi_\iota)^*)$ by Lemma 2.3, we have $(\pi(\xi_\iota)^*\eta_{0\iota})\sim(u_\iota 1_\iota)$. Therefore $(\xi_\iota)\in S_0$ and $(\eta_{0\iota})\sim(u_\iota\xi_\iota)$. Since $(\xi_\iota)\sim(\xi_{0\iota})$, we have $(\eta_{0\iota})\sim(u_\iota\xi_{0\iota})$. Since $(\xi_{0\iota}, u_\iota\xi_{0\iota})$ is a non zero reference pair of $(h_{0\iota})$, it follows from Theorem 1.1 that $(\xi_{0\iota})\sim(u_\iota\xi_{0\iota})$.

Therefore $(\xi_{0\iota})\sim(\eta_{0\iota})$. Q.E.D.

Proof of Theorem 5.1. (i)⇒(ii). By Proposition 5.2 $(\xi_{0\iota}, \xi_{0\iota})$ is a non-zero reference pair of $(h_{0\iota}^{1/2})$ and $(\xi_\iota)\in S_0$. It follows that (ξ_ι, ξ_ι) is a non-zero reference pair of $(h_{0\iota}^{1/2})$.

(ii)⇒(i). Put $(\xi_{0\iota}\equiv\|\xi_\iota\|^{-1}\xi_\iota$. Then $(\xi_{0\iota}, \xi_{0\iota})$ is a non-zero reference pair of $(h_{0\iota}^{1/2})$. (i) follows from Proposition 5.2.

(ii)⇔(iii)⇒(iv). Clear.

(iv)⇒(iii). Since $(\xi_{1\iota}, \xi_{1\iota})$ is a non-zero reference pair of $(h_{0\iota}^{1/2})$, we have $(h_{0\iota}^{1/2}\xi_{1\iota})\in S$ and $(\xi_{1\iota})\sim(h_{0\iota}^{1/2}\xi_{1\iota})$. Since $c(\xi_{1\iota})=c(\xi_{0\iota})$, we have $(\xi_{1\iota})\sim(h_{0\iota}^{1/2}\xi_{0\iota})$. Therefore $(\xi_\iota)\sim(\xi_{0\iota})\sim(\xi_{1\iota})\sim(h_{0\iota}^{1/2}\xi_{0\iota})\sim(h_{0\iota}^{1/2}\xi_\iota)$.

(iv)⇔(v)⇔(vi)⇔(vii)⇔(viii)⇔(ix). By Theorem 3.1.

Remark 5.1. For each $J\subset\subset I$ a σ-finite measure $\mu(I\setminus J)=(\prod_J \|h_\iota^{1/2}\xi_\iota\|^{-2})\mu^{(\xi_\iota:\iota\in I\setminus J)}$ on $(\prod_{I\setminus J}\Omega_\iota, \prod_{I\setminus J}\mathscr{F}_\iota)$ satisfies that $\mu^{(\xi_\iota)}=(\prod_J\mu_\iota)\times\mu(I\setminus J)$. Therefore $\mu^{(\xi_\iota)}$ is a product measure of $\{\mu_\iota : \iota\in I\}$ in the sense of Hill. In Proposition 5.2, if we choose a measurable $\Omega'_\iota\subset\Omega_\iota$ with $0<\mu_\iota(\Omega'_\iota)<+\infty$ and define $\xi_\iota=\chi_{\Omega'_\iota}$, then $0<\prod\|\pi(\xi_\iota)\|<+\infty$ and $(\eta_{0\iota}, \eta_{0\iota})$ is a non-zero reference pair of $(\pi(\xi_\iota))$. Therefore $\mu^{(\xi_\iota)}\sim\nu$ if and only if $(\xi_\iota)\in S_0$ and $(\xi_{0\iota})\sim(\eta_{0\iota})$. This is a result of Hill. It should however be noted that we can not omitt the condition $(\xi_\iota)\in S_0$ as the following example shows.

Let $I\equiv\boldsymbol{N}$. Let $\Omega_n\equiv\boldsymbol{R}$ for $n\in I$, ν_n be a normal distribution with mean 0 and variance 1, and μ_n be the Lebesgue measure. Put $\Omega'_n\equiv[-\lambda_n, \lambda_n]$, $\lambda_n>0$ for all $n\in\boldsymbol{N}$ and $\xi_n\equiv\chi_{\Omega'_n}$. Then

$$(\xi_{0n}|\eta_{0n})=\int_{-\lambda_n/\sqrt{2}}^{\lambda_n/\sqrt{2}}\exp\left(-\frac{x^2}{2}\right)dx\left\{\lambda_n\int_{-\lambda_n}^{\lambda_n}\exp\left(-\frac{x^2}{2}\right)dx\right\}^{-1/2}.$$

By choosing λ_n sufficiently small, we have $(\xi_{0\iota})\sim(\eta_{0\iota})$. However $(\xi_\iota)\notin S_0$ and hence $(\xi_{0\iota})\nsim(1_\iota)$.

§6. An Infinite Product of Semi-finite Weights

Following the similar argument as the preceding section, we shall give a definition of an infinite tensor product of semi-finite faithful normal weights. I is not necessarily countable.

We begin by recalling the tensor product of semi-finite faithful normal weights ψ_1 on $(M_1)_+$ and ψ_2 on $(M_2)_+$. Let $\mathfrak{A}_j$ denote the full left Hilbert algebra of (M_j, ψ_j) obtained by the GNS construction for $j=1, 2$. Let $\mathfrak{A}$ denote the full left Hilbert algebra formed from the algebraic tensor product of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. If π is the left representation of $\mathfrak{A}$, then $M_1 \otimes M_2$ is isomorphic to $\pi(\mathfrak{A})''$. Through this isomorphism, the tensor product $\psi_1 \otimes \psi_2$ of ψ_1 and ψ_2 is defined as the canonical weight of $\pi(\mathfrak{A})''$.

As Theorem 15.3 in [15] holds for a semi-finite faithful normal weight ψ on M_+ in place of a faithful normal positive linear form ϕ_0 on M by a slight improvement of the proof, we have that the necessary and sufficient condition for $a\psi a^* \leqq \psi$, $a \in \mathfrak{n}_\psi$ is that $\|\Delta_\psi^{-1/2} \pi_\psi(a) \Delta_\psi^{1/2}\| \leqq 1$, $a \in \mathfrak{n}_\psi$. Here $\mathfrak{n}_\psi$ denote the set of all $x \in M$ with $\psi(x^*x) < +\infty$, π_ψ the GNS representation of M induced by ψ and Δ_ψ the modular operator.

Let ξ_ι be a unit vector in $\mathscr{H}_\iota$ which is cyclic and separating for M_ι, and $\phi_\iota \equiv \omega_{\xi_\iota}$ on M_ι. Let ψ_ι be a semi-finite faithful normal weight on $(M_\iota)_+$ such that $\psi_\iota = h_\iota^{1/2} \phi_\iota h_\iota^{1/2}$ for some invertible, positive and self-adjoint operator h_ι affiliated with the centralizer $(M_\iota)_{\phi_\iota}$. Put $\mathfrak{n}_\iota \equiv \{x \in M_\iota : \psi_\iota(x^*x) < +\infty\}$. Let $e_\iota(n)$ denote the spectral projection of h_ι corresponding to $[0, n]$ for $n \in \boldsymbol{N}$. Let J_{ξ_ι} and Δ_{ξ_ι} be a modular conjugation and a modular operator of (M_ι, ϕ_ι), respectively. Put $j_{\xi_\iota}(x) \equiv J_{\xi_\iota} x J_{\xi_\iota}$ for $x \in M_\iota$. For each $x \in \mathfrak{n}_\iota$ we have

$$\begin{aligned} x h_\iota^{1/2} e_\iota(n) \xi_\iota &= x J_{\xi_\iota} \Delta_{\xi_\iota}^{1/2} h_\iota^{1/2} e_\iota(n) \xi_\iota = x J_{\xi_\iota} h_\iota^{1/2} e_\iota(n) \xi_\iota \\ &= x j_{\xi_\iota}(h_\iota^{1/2} e_\iota(n)) \xi_\iota = j_{\xi_\iota}(h_\iota^{1/2} e_\iota(n)) x \xi_\iota . \end{aligned}$$

Since $\{e_\iota(n+1) - e_\iota(n)\}_{n \in \boldsymbol{N}}$ are orthogonal and since

$$\sup_n \| j_{\xi_\iota}(h_\iota^{1/2} e_\iota(n)) x \xi_\iota \|^2 = \sup_n \| x h_\iota^{1/2} e_\iota(n) \xi_\iota \|^2 \leqq \psi_\iota(x^*x) < +\infty ,$$

it follows that $\{x h_\iota^{1/2} e_\iota(n) \xi_\iota\}_{n=1}^\infty$ is a Cauchy sequence. We denote the limit $j_{\xi_\iota}(h_\iota^{1/2}) x \xi_\iota$ by $x h_\iota^{1/2} \xi_\iota$ symbolically.

For a fixed $x_{0\iota} \in \mathfrak{n}_\iota$ with $x_{0\iota} \neq 0$. put

$$\xi_{0\iota} \equiv \|x_{0\iota} \xi_\iota\|^{-1} x_{0\iota} \xi_\iota \quad \text{and} \quad \eta_{0\iota} \equiv \|x_{0\iota} h_\iota^{1/2} \xi_\iota\|^{-1} x_{0\iota} h_\iota^{1/2} \xi_\iota .$$

Define a semi-finite normal weight ψ_J on $(\otimes^{c'}M_\iota)_+$ for c' with $c' \underset{p}{\sim} c(\eta_{0\iota})$ by

$$\psi_J = (\underset{J}{\otimes} \|x_{0\iota}h_\iota^{1/2}\xi_\iota\|^{-2}\psi_\iota) \otimes (\underset{I\setminus J}{\otimes} \omega_{\eta_{0\iota}}).$$

Proposition 6.1. *With the above notations, assume that* $0 < \prod \|x_{0\iota}\| < +\infty$. *If* $(\eta_{0\iota}, \eta_{0\iota})$ *is a non-zero reference pair of* $(x_{0\iota})$ *and if* $x_{0\iota} \in \mathfrak{n}_\iota$ *with* $\|\Delta_{\psi_\iota}^{-1/2}\pi_{\psi_\iota}(x_{0\iota})\Delta_{\psi_\iota}^{1/2}\| \leqq \|x_{0\iota}\|$, *then* $\psi = \lim_{J\subset\subset I}\psi_J$ *is a semi-finite faithful normal weight on* $(\otimes^{c'}M_\iota)_+$ *for all* c' *with* $c' \underset{p}{\sim} c(\eta_{0\iota})$.

Proof. Since $x_{0\iota} \in \mathfrak{n}_\iota$ and $\|\Delta_{\psi_\iota}^{-1/2}\pi_{\psi_\iota}(x_{0\iota})\Delta_{\psi_\iota}^{1/2}\| \leqq \|x_{0\iota}\|$, we have $x_{0\iota}\psi_\iota x_{0\iota}^* \leqq \|x_{0\iota}\|^2\psi_\iota$ and hence $\|x_{0\iota}h_\iota^{1/2}\xi_\iota\|^2\omega_{\eta_{0\iota}} \leqq \|x_{0\iota}\|^2\psi_\iota$. Therefore $\{(\prod_{I\setminus J}\|x_{0\iota}\|^{-2})\psi_J : J \subset\subset I\}$ is an increasing net of semi-finite normal weights on $(\otimes^{c'}M_\iota)_+$. Putting

$$\psi \equiv \sup_{J\subset\subset I} \{(\prod_{I\setminus J} \|x_{0\iota}\|^{-2})\psi_J\}$$

on $(\otimes^{c'}M_\iota)_+$, we know that ψ is a normal weight on $(\otimes^{c'}M_\iota)_+$ and that

$$\psi(\otimes^{c'}|x_{0\iota}|^2) = \sup_{J\subset\subset I} \prod_{I\setminus J} (\|x_{0\iota}\|^{-2}\|x_{0\iota}\eta_{0\iota}\|^2) = 1\,.$$

The semi-finiteness of ψ is then proved by the similar way as Proposition 5.1. Let S_ι denote the carrier of $\omega_{\eta_{0\iota}}$ in M_ι and u_ι be a partial isometry in M_ι' such that $u_\iota^*u_\iota\eta_{0\iota} = \eta_{0\iota}$ and $c' = c(u_\iota\eta_{0\iota})$. Since $S_\iota u_\iota\eta_{0\iota} = u_\iota\eta_{0\iota}$ and since the carriers $(\otimes_J 1_\iota)\otimes(\otimes_{I\setminus J}^{c'}S_\iota)$ of ψ_J in $\otimes^{c'}M_\iota$ are majorized by the carrier of ψ for all $J \subset\subset I$, ψ is faithful on $\otimes^{c'}M_\iota$.

Q.E.D.

Definition 6.1. The semi-finite faithful normal weight on $(\otimes^{c'}M_\iota)_+$ obtained in Proposition 6.1 is denoted by $\psi^{(x_{0\iota})}$.

$\psi^{(x_{0\iota})}$ is considered as an infinite tensor product of normal weights ψ_ι. We will show some conditions for $\psi^{(x_{0\iota})}$ to live on $\otimes^c M_\iota$ in Theorem 6.1 after the following proposition.

Proposition 6.2. *Let* ξ_ι, ϕ_ι, ψ_ι, h_ι *and* $x_{0\iota}$ *be as above. Let* $\phi \equiv \otimes\phi_\iota$ *on* $\otimes^c M_\iota$ *for* $c \equiv c(\xi_\iota)$ *and* $\psi \equiv \psi^{(x_{0\iota})}$ *on* $\otimes^{c'}M_\iota$ *for some* $c' \underset{p}{\sim} c(\eta_{0\iota})$. *Then*

(i) $c \underset{p}{\sim} c'$ *if and only if* $(x_{0\iota}\xi_\iota) \in S_0$ *and* $(\xi_{0\iota}) \underset{p}{\sim} (\eta_{0\iota})$;

and

(ii) *under* (i), $\psi = \psi \circ \sigma_t^\phi$ *for all* $t \in \boldsymbol{R}$.

Proof. (i) Suppose that $(x_{0\iota}\xi_\iota) \in S_0$ and $(\xi_{0\iota}) \underset{p}{\sim} (\eta_{0\iota})$. $(\xi_{0\iota}) \underset{p}{\sim} (\eta_{0\iota})$ implies $(\xi_{0\iota}) \sim (u_\iota \eta_{0\iota})$ for some partial isometry u_ι in M'_ι with $u_\iota^* u_\iota \eta_{0\iota} = \eta_{0\iota}$. $(x_{0\iota}\xi_\iota) \in S_0$ implies $(x_{0\iota}\xi_\iota) \sim (\xi_{0\iota}) \sim (u_\iota \eta_{0\iota})$. Since $(\eta_{0\iota}, \eta_{0\iota})$ is a non-zero reference pair of $(x_{0\iota})$, we have $(x_{0\iota}^* \eta_{0\iota}) \sim (\eta_{0\iota})$. Since $0 < \prod \|x_{0\iota}^*\| < +\infty$, by Lemma 1 in [1] we have $(u_\iota x_{0\iota}^* \eta_{0\iota}) \in S$ and $(u_\iota x_{0\iota}^* \eta_{0\iota}) \sim (u_\iota \eta_{0\iota})$. Since $(x_{0\iota}\xi_\iota) \sim (u_\iota \eta_{0\iota})$, we have $(\xi_\iota) \sim (u_\iota x_{0\iota}^* \eta_{0\iota}) \sim (u_\iota \eta_{0\iota})$ and hence $c \underset{p}{\sim} c(\eta_{0\iota}) \underset{p}{\sim} c'$.

Conversely, suppose that $c \underset{p}{\sim} c'$. Since $c' \underset{p}{\sim} c(\eta_{0\iota})$, there exist partial isometries u_ι in M'_ι so that $u_\iota^* u_\iota \eta_{0\iota} = \eta_{0\iota}$ and $(\xi_\iota) \sim (u_\iota \eta_{0\iota})$. Since $(\eta_{0\iota}, \eta_{0\iota})$ is a non-zero reference pair of $(x_{0\iota})$, we have $(u_\iota x_{0\iota}^* \eta_{0\iota}) \sim (u_\iota \eta_{0\iota}) \sim (\xi_\iota)$. Since $0 < \prod \|x_{0\iota}\| < +\infty$, Lemma 1 in [1] implies that $(x_{0\iota}\xi_\iota) \in S$. Since ξ_ι is separating, $(x_{0\iota}\xi_\iota) \in S_0$ and $(u_\iota \eta_{0\iota}) \sim (x_{0\iota}\xi_\iota)$. Thus $(\xi_{0\iota}) \sim (u_\iota \eta_{0\iota})$.

In order to prove (ii) we need to prepare the following lemma. Before going into the proof, we recall Theorem 14.4 in [16]. This is restated as follows: Let ψ be a semi-finite faithful normal weight on M_+ and σ_t, $t \in \boldsymbol{R}$ a one parameter group of $*$-automorphisms of M. If a weakly dense $*$-subalgebra M_0 of M is invariant under σ_t, $t \in \boldsymbol{R}$ and if a pair of ψ and σ satisfies the KMS-condition for M_0, then $\sigma = \sigma^\psi$.

Lemma 6.1. *Let* $\phi \equiv \otimes \phi_\iota$ *on* $\otimes^c M_\iota$ *for* $c \equiv c(\xi_\iota)$.

$$\sigma_t^\phi(\otimes^c x_\iota) = \otimes^c \sigma_t^{\phi_\iota}(x_\iota)$$

for $\otimes^c x_\iota$ *in* $\otimes^c M_\iota$.

Proof. For any non zero $\otimes^c x_\iota$ in $\otimes^c M_\iota$, since $(\sigma_t^{\phi_\iota}(x_\iota)\xi_\iota | \xi_\iota) = (x_\iota \xi_\iota | \xi_\iota)$ and $(x_\iota \xi_\iota) \sim (\xi_\iota)$, we can define a one parameter group of $*$-automorphisms $\sigma_t = \otimes^c \sigma_t^{\phi_\iota}$ of $\otimes^c M_\iota$ by

$$\sigma_t(\otimes^c x_\iota) = \otimes^c \sigma_t^{\phi_\iota}(x_\iota)$$

for $t \in \boldsymbol{R}$. Let D denote the set of product vectors $\otimes \eta_\iota$ with $\{\iota \in I:$

$\eta_\iota \neq \xi_\iota\} \subset\subset I$. Since $(\sigma_t(x)\xi|\eta)$ is continuous in $t \in \boldsymbol{R}$ for $\xi, \eta \in D$ and $x \in \otimes^c M_\iota$. Since D is strongly total in $\mathscr{H}_c$, σ_t is weakly continuous in $t \in \boldsymbol{R}$.

For any $x \equiv (\otimes_J x_\iota) \otimes (\otimes^c_{I \setminus J} 1_\iota)$ and $y \equiv (\otimes_K y_\iota) \otimes (\otimes^c_{I \setminus K} 1_\iota)$ in $\otimes^c M_\iota$, we have a bounded function $F_\iota(z)$ holomorphic in and continuous on $0 \leqq \operatorname{Im} z \leqq 1$ such that

$$F_\iota(t) = \phi_\iota(\sigma_t^{\phi_\iota}(x_\iota) y_\iota) \quad \text{and} \quad F_\iota(t+i) = \phi_\iota(y_\iota \sigma_t^{\phi_\iota}(x_\iota))$$

for $t \in \boldsymbol{R}$. Therefore, by $\phi = \otimes \phi_\iota$, there is a bounded function $F(z) = \prod_{J \cup K} F_\iota(z)$ holomorphic in and continuous on $0 \leqq \operatorname{Im} z \leqq 1$ such that

$$F(t) = \phi(\sigma_t(x) y) \quad \text{and} \quad F(t+i) = \phi(y \sigma_t(x)).$$

Since the $*$-subalgebra of all finite linear combinations of $(\otimes_J x_\iota) \otimes (\otimes^c_{I \setminus J} 1_\iota)$ with $x_\iota \in M_\iota$ and $J \subset\subset I$ is weakly dense in $\otimes^c M_\iota$ and is invariant under σ_t, $t \in \boldsymbol{R}$, it follows from the discussion preceding to this lemma that $\sigma_t = \sigma_t^\phi$ for all $t \in \boldsymbol{R}$. Q.E.D.

Proof of (ii) *in Proposition* 6.2. Since $\psi_\iota = \psi_\iota \circ \sigma_t^{\phi_\iota}$, for any $x \in \otimes^c M_\iota$ of the form $(\otimes_K x_\iota) \otimes (\otimes^c_{I \setminus K} 1_\iota)$ we have

$$\begin{aligned}
(\psi \circ \sigma_t^\phi)(x) &= \psi(\otimes^c \sigma_t^{\phi_\iota}(x_\iota)) \\
&= \lim_{K \subset J \subset\subset I} \prod_J \|x_{0\iota} h_\iota^{1/2} \xi_\iota\|^{-2} \psi_\iota(\sigma_t^{\phi_\iota}(x_\iota)) \\
&= \lim_{K \subset J \subset\subset I} \prod_J \|x_{0\iota} h_\iota^{1/2} \xi_\iota\|^{-2} \psi_\iota(x_\iota) \\
&= \psi(\otimes^c x_\iota) = \psi(x).
\end{aligned}$$

Q.E.D.

Theorem 6.1. *Let* $\xi_\iota, \phi_\iota, \psi_\iota, h_\iota$ *and* $x_{0\iota}$ *as before. Let* $\phi \equiv \otimes \phi_\iota$ *on* $\otimes^c M_\iota$ *for* $c \equiv c(\xi_\iota)$ *and* $\psi \equiv \psi^{(x_{0\iota})}$ *on* $\otimes^{c'} M_\iota$ *for* $c' \underset{p}{\sim} c(\eta_{0\iota})$.

(1) *Let* $\lambda_\iota \equiv \|x_{0\iota}\xi_\iota\| \|x_{0\iota} h_\iota^{1/2} \xi_\iota\|^{-1}$, $h_{0\iota} \equiv \lambda_\iota^2 h_\iota$ *and* e_ι *a spectral projection of* $h_{0\iota}^{1/2}$ *corresponding to* $[\lambda^{-1}, \lambda]$ *for any fixed* $\lambda > 1$. *It is sufficient for* ψ *to be a semi-finite faithful normal weight on* $(\otimes^c M_\iota)_+$ *that one of the following conditions holds*:

(i) $(x_{0\iota}\xi_\iota) \in S_0$ *and* $(\xi_{0\iota}) \underset{p}{\sim} (\eta_{0\iota})$;

(ii) $(\xi_{1\iota}, \xi_{1\iota})$ *is a non-zero reference pair of* $(h_{0\iota}^{1/2})$ *for some* $(\xi_{1\iota}) \in c$;

(iii) $(e_\iota \xi_\iota) \in S$, $(h_{0\iota}^{1/2} e_\iota \xi_\iota) \in S$ *and* $(e_\iota \xi_\iota) \sim (h_{0\iota}^{1/2} e_\iota \xi_\iota)$;

(iv) $(e_\iota \xi_\iota) \in S$, $\sum \| \log h_{0\iota}^{1/2} e_\iota \xi_\iota \|^2 < +\infty$ *and* $\sum |(\log h_{0\iota}^{1/2} e_\iota \xi_\iota | \xi_\iota)| < +\infty$; *and*

(v) $\otimes^c h_{0\iota}^{it}$, $t \in \boldsymbol{R}$ *is a strongly continuous one parameter unitary group.*

Under conditions from (ii) *to* (v) $h \equiv (\prod \| x_{0\iota} \xi_\iota \|^{-2}) \| \otimes^c h_{0\iota}$ *is affiliated with* $(\otimes^c M_\iota)_\phi$ *and* $\psi = \phi \circ h$. *In particular, if* $x_{0\iota} \in M_{\psi_\iota}$, *then* $\sigma_t^\psi = \otimes^c \sigma_t^{\psi_\iota}$.

(2) *If* $x_{0\iota}$ *commutes with* h_ι *for all* $\iota \in I$, *every condition in* (1) *is necessary for* ψ *to be a semi-finite faithful normal weight on* $(\otimes^c M_\iota)_+$.

Proof. (1) By Proposition 6.2, (i) is a sufficient condition.

If one of the conditions from (ii) to (v) holds, we can define $\otimes^c h_{0\iota}^{1/2}$ by Theorem 3.1 and $\otimes^c h_{0\iota} = (\otimes^c h_{0\iota}^{1/2})^2$ by Theorem 3.2. We have for all non zero $\otimes^c x_\iota$ in $(\mathfrak{m}_\psi)_+$,

$$\begin{aligned}\psi(\otimes^c x_\iota) &= \sup \{ (\prod_{I \setminus J} \| x_{0\iota} \|^{-2}) \psi_J(\otimes^c x_\iota) \} \\ &= \lim \prod_J \{ \| x_{0\iota} h_\iota^{1/2} \xi_\iota \|^{-2} \psi_\iota(x_\iota) \} \\ &= \lim \prod_J \{ \| x_{0\iota} \xi_\iota \|^{-2} \phi_\iota(h_{0\iota} x_\iota) \} \\ &= \phi(h \otimes^c x_\iota).\end{aligned}$$

From (ii) of Proposition 6.2, we know that h is affiliated with $(\otimes^c M_\iota)_\phi$ and $\psi = \phi \circ h$. Since h is invertible, ψ is faithful.

Suppose that $x_{0\iota}$ is in $(M_\iota)_{\psi_\iota}$. By virtue of Lemma 6.1 we have $\sigma_t^\phi = \otimes^c \sigma_t^{\phi_\iota}$. Define a $*$-automorphism σ_t of $\otimes^c M_\iota$ by

$$\sigma_t(x) = (\otimes^c h_{0\iota}^{it}) \sigma_t^\phi(x) (\otimes^c h_{0\iota}^{-it})$$

for $x \in \otimes^c M_\iota$. Since $\sigma_t^{\psi_\iota}(y) = h_{0\iota}^{it} \sigma_t^\phi(y) h_{0\iota}^{-it}$ for $y \in M_\iota$, we have

$$\sigma_t(\otimes^c x_\iota) = \otimes^c \sigma_t^{\psi_\iota}(x_\iota), \qquad t \in \boldsymbol{R}$$

and σ_t is weakly continuous by Theorem 3.1. For any $x \equiv (\otimes_J x_\iota) \otimes$

$(\otimes^c_{I\setminus J}|x_{0\iota}|)$ and $y\equiv(\otimes_K y_\iota)\otimes(\otimes^c_{I\setminus K}|x_{0\iota}|)$ with x_ι and y_ι in $\mathfrak{n}_{\psi_\iota}$ we have a bounded function $F_\iota(z)$ holomorphic in and continuous on $0\leqq \mathrm{Im}\, z\leqq 1$ such that

$$F_\iota(t)=\dot{\psi}_\iota(\sigma_t^{\psi_\iota}(x_\iota)y_\iota) \quad \text{and} \quad F_\iota(t+i)=\dot{\psi}_\iota(y_\iota\sigma_t^{\psi_\iota}(x_\iota))$$

for $t\in \boldsymbol{R}$, where $\dot{\psi}_\iota$ is the linear extension of ψ_ι to $\mathfrak{m}_{\psi_\iota}$. Therefore, since $x_{0\iota}\in(M_\iota)_{\psi_\iota}$ and $c'\underset{p}{\sim}c(\eta_{0\iota})$, there is a bounded function

$$F(z)=\prod_{J\cup K}\psi_\iota(|x_{0\iota}|^2)^{-1}F_\iota(z)$$

holomorphic in and continuous on $0\leqq \mathrm{Im}\, z\leqq 1$ such that

$$F(t)=\dot{\psi}(\sigma_t(x)y) \quad \text{and} \quad F(t+i)=\dot{\psi}(y\sigma_t(x)).$$

Thus $\sigma_t=\sigma_t^{\psi}$ and hence $\sigma_t^{\psi}=\otimes^c\sigma_t^{\psi_\iota}$ for all $t\in\boldsymbol{R}$.

(2) By means of the proof of necessity of (i) in Proposition 6.2, we have $(x_{0\iota}\xi_\iota)\sim(\xi_{0\iota})\sim(u_\iota\eta_{0\iota})$ for some unitary u_ι in M'_ι. Then $(\xi_{0\iota}, u_\iota^*\xi_{0\iota})$ is a non-zero reference pair of $(h_{0\iota}^{1/2})$. Hence by Theorem 3.1 we have every condition in (1). Q.E.D.

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Note added in proof. The separability assumption of $\mathscr{H}_\iota$ in (iii) of Theorem 3.3 can be omitted by using Remark 3.10 in the following paper:

Araki, H. and Woods, E. J., Topologies induced by representations of the canonical commutation relations. *Reports on Math. Phys.* **4** (1973), 227–254.

Therefore Lemma 4.2 and Corollary 4.1 hold without the separability assumption and Lemma 6.1 is clear from Corollary 4.1.