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On the First Initial-Boundary Value Problem of the Generalized Burgers' Equation

By

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§1. Introduction and Notations

E. Hopf discussed in details on the Cauchy problem of Burgers' equation in his famous paper [5]. Since then, many papers on the equation and its related topics have been published. However, they have not treated the initial-boundary value problem of it. The author previously discussed on the first initial-boundary value problem of this equation in [17]. Recently, N. Itaya has shown the existence and the uniqueness, in a certain sense, of the temporally global solution of the Cauchy problem of the following generalized Burgers' equation:

$$(1.1)^1 \quad \left\{ \begin{array}{l} \dfrac{\partial v}{\partial t}(x,t)=\dfrac{\mu}{\rho(x,t)}\dfrac{\partial^2}{\partial x^2}v(x,t)-v(x,t)\dfrac{\partial}{\partial x}v(x,t)\,, \\ \dfrac{\partial \rho}{\partial t}+\dfrac{\partial}{\partial x}(\rho v)=0, \qquad (\mu \text{ is a positive constant}) \end{array} \right. \quad (1.1)^2$$

in [11] (cf. [9], [10]). Stimulated by his work, the author attempts to discuss on the first initial-boundary value problem of (1.1) in $[0, X]\subset R^1$, especially from the view-point of the temporally global behavior of the solution of (1.1).

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Notations. The functions considered in this paper should be understood to be defined in $[0, X]$ or $[0, X]\times[0, T]$ $(0<X<+\infty,\ 0<T<+\infty)$ and continuously differentiable as many times as necessary.

$$(1.2)\quad \begin{cases} \Omega=(0, X),\ \bar{\Omega}=[0, X],\ S_T^0=\{0\}\times[0, T], \\ S_T^X=\{X\}\times[0, \mathrm{T}],\ S_T=S_T^0\cup S_T^X,\ \Gamma_T=S_T\cup\Omega\times\{0\}, \\ Q_T=\Omega\times(0, T),\ \bar{Q}_T=\bar{\Omega}\times[0, T]. \end{cases}$$

$$(1.3)\quad \begin{cases} |u(x)|^{(0)}\equiv\sup_{\bar{\Omega}}|u(x)|, \\ |u(x)|^{(\alpha)}\equiv\sup_{\bar{\Omega},x\neq x'}\dfrac{|u(x)-u(x')|}{|x-x'|^{\alpha}}. \end{cases}$$

$$(1.3)'\quad \begin{cases} \|u(x)\|^{(n)}\equiv\sum_{i=0}^{n}|D_x^i u(x)|^{(0)}, \\ \|u(x)\|^{(n+\alpha)}\equiv\|u(x)\|^{(n)}+|D_x^n u(x)|^{(\alpha)} \qquad (n=0, 1,\dots). \end{cases}$$

$$(1.4)\quad \begin{cases} |v(x, t)|_T^{(0)}\equiv\sup_{\bar{Q}_T}|v(x, t)|, \\ |v(x, t)|_{x,T}^{(\alpha)}\equiv\sup_{\bar{Q}_T,x\neq x'}\dfrac{|v(x, t)-v(x', t)|}{|x-x'|^{\alpha}}, \\ |v(x, t)|_{t,T}^{(\alpha/2)}\equiv\sup_{\bar{Q}_T,t\neq t'}\dfrac{|v(x, t)-v(x, t')|}{|t-t'|^{\alpha/2}}, \\ |v(x, t)|_T^{(\alpha)}\equiv|v(x, t)|_{x,T}^{(\alpha)}+|v(x, t)|_{t,T}^{(\alpha/2)}. \end{cases}$$

$$(1.4)'\quad \begin{cases} \|v\|_T^{(n)}\equiv\sum_{2r+s=0}^{n}|D_t^r D_x^s v|_T^{(0)}, \\ \|v\|_T^{(n+\alpha)}\equiv\|v\|_T^{(n)}+\sum_{2r+s=n}|D_t^r D_x^s v|_{x,T}^{(\alpha)}+ \\ \qquad +\sum_{2r+s=(n-1)\vee 0}^{n}|D_t^r D_x^s v|_{t,T}^{(\alpha/2)},\ (a\vee b\equiv\max(a, b)), \\ <v>_T^{(n,\alpha)}\equiv\sum_{s=0}^{n}|D_x^s v|_T^{(0)}+\sum_{s=0}^{n-1}|D_x^s v|_{t,T}^{(\alpha/2)}, \\ '<v>_T^{(n,\alpha)}\equiv|D_x^n v|_T^{(\alpha)}, \\ \ll v\gg_T^{(n,\alpha)}\equiv<v>_T^{(n,\alpha)}+'<v>_T^{(n,\alpha)}, \end{cases}$$

where r and s are non-negative integers.

$$
(1.5)\qquad \begin{cases} H^n \equiv \{u(x) \mid \|u\|^{(n)} < +\infty\}, \\ H^{n+\alpha} \equiv \{u(x) \mid \|u\|^{(n+\alpha)} < +\infty\}, \\ H_T^{n+\alpha} \equiv \{v(x, t) \mid \|v\|_T^{(n+\alpha)} < +\infty\}, \\ \hat{H}_T^{n+\alpha} \equiv \{v(x, t) \mid \ll v \gg_T^{(n+\alpha)} < +\infty\}, \\ B_T^n \equiv \{v(x, t) \mid \sum_{r+s=0}^{n} |D_t^r D_x^s v|_T^{(0)} < +\infty\}, \\ B_T^{n+\alpha} \equiv \{v(x, t) \mid \sum_{r+s=0}^{n} |D_t^r D_x^s v|_T^{(0)} + \sum_{r+s=n} |D_t^r D_x^s v|_T^{(\alpha)} < +\infty\}. \end{cases}
$$

Other notations, not described above, will be explained where they appear.

§2. Preliminaries

We assume for (1.1) the following initial-boundary conditions:

$$
(2.1)\qquad v(x, 0) = v_0(x) \in H^{2+\alpha}, \quad \rho(x, 0) = \rho_0(x) \in H^1,
$$

$$
(0 < \bar{\rho}_0 \equiv \inf_{\bar{\Omega}} \rho_0(x) \leq \rho_0(x) \leq \bar{\bar{\rho}}_0 \equiv |\rho_0(x)|^{(0)})
$$

$$
(2.2)\qquad v(0, t) = v(X, t) = 0,
$$

and for $v(x, t)$ the following compatible condition:

$$
(2.3)\qquad v_{xx}(x, t)_{|S_T^0} = v_{xx}(x, t)_{|S_T^X} = 0.
$$

Let (v, ρ) be a solution in $H_T^{2+\alpha} \times B_T^1$ of (1.1) satisfying the initial-boundary conditions (2.1) and (2.2), and $\bar{x}(\tau; x, t)$ be the solution curve of the characteristic equation for $(1.1)^2$ as a linear equation in ρ:

$$
(2.4)\qquad \begin{cases} \dfrac{d}{d\tau} \bar{x}(\tau; x, t) = v(\bar{x}(\tau; x, t), \tau) \quad (0 \leqq \tau \leqq t \leqq T), \\ \bar{x}(t; x, t) = x. \end{cases}
$$

Since $v \in H_T^{2+\alpha}$, the solution curve for (2.4) starting at an arbitrary point

$(x, t) \in \bar{Q}_T$ is unique. By (2.4) we have

$$(2.5) \qquad \bar{x}_x(\tau; x, t) = \frac{\partial \bar{x}}{\partial x}(\tau; x, t) = \exp\{-\int_\tau^t v_x(\bar{x}(\tau'; x, t), \tau')d\tau'\}.$$

If $v(x, t) \in H_T^{2+\alpha}$ is given in $(1.1)^2$, then $\rho(x, t)$ or $\rho_v(x, t)$ is uniquely determined by the formula:

$$(2.6) \qquad \begin{aligned} \rho(x, t) \equiv \rho_v(x, t) &= \rho_0(\bar{x}(0; x, t))\bar{x}_x(0; x, t) \\ &= \rho_0(\bar{x}(0; x, t)) \exp\{-\int_0^t v_x(\bar{x}(\tau'; x, t), \tau')d\tau'\}. \end{aligned}$$

For simplicity, we put

$$(2.7) \qquad \begin{cases} \bar{\rho}(\tau; x, t) = \rho(\bar{x}(\tau; x, t), \tau), \\ \bar{v}(\tau; x, t) = v(\bar{x}(\tau; x, t), \tau), \text{ etc.} \end{cases}$$

By $(1.1)^2$, (2.4) and (2.6), the following fundamental lemma holds (cf. [11]).

Lemma 2.1. *If* (v, ρ) *is a solution of* (1.1) *in* $H_T^{2+\alpha} \times B_T^1$ *with* (2.1) *and* (2.2), *then the following equation holds*:

$$(2.8) \qquad \mu\int_0^t \bar{v}_{xx}(\tau; x, t)\bar{x}_x(\tau; x, t)d\tau = \rho(x, t)\{v(x, t) - v_0(x_0(x, t))\},$$

where

$$(2.9) \qquad x_0(x, t) = \bar{x}(0; x, t).$$

Concerning $\rho(x, t)$, we have, by (2.6) and simple calculations,

Lemma 2.2. *If* (v, ρ_v) *and* (w, ρ_w) *are solutions of* (1.1) *with* (2.1) *and* (2.2) *in* $H_T^{2+\alpha} \times B_T^1$ (cf. (2.6)). *then*

$$(2.10) \qquad \left|\frac{1}{\rho_v} - \frac{1}{\rho_w}\right|_{T_0}^{(0)} \leqq C_1(T_0; v, w)\|v - w\|_{T_0}^{(1)} (0 \leqq T_0 \leqq T),$$

where $C_1(T_0; v, w) \downarrow 0$ *as* $T_0 \downarrow 0$.

From (2.6), $(v, \rho) \in H_T^{2+\alpha} \times B_T^1$ implies $\mu/\rho \in H_T^\alpha$.

Let

(2.11) $Z^0(x-\xi, t; \xi, \tau; \mu/\rho)=\frac{1}{2\sqrt{\pi}}(\mu/\rho(\xi, \tau))^{-1/2}(t-\tau)^{-1/2}\times$

$$\times\exp\{-(x-\xi)^2/4\frac{\mu}{\rho}(\xi, \tau)(t-\tau)\} \quad (0\leqq\tau\leqq t\leqq T)$$

be the parametrix of the linear parabolic equiaton:

$$\frac{\partial w}{\partial t}(x, t)=\frac{\mu}{\rho}(\xi, \tau)\frac{\partial^2 w}{\partial x^2}(x, t)\,. \tag{2.12}$$

Then the parametrix of (2.12) with (2.2) is given by

$$Z(x-\xi, t; \xi, \tau; \mu/\rho)=\sum_{n=-\infty}^{\infty}\{Z^0(x-\xi+2nX, t; \xi, \tau; \mu/\rho)- \tag{2.13}$$

$$-Z^0(x+\xi+2nX, t; \xi, \tau; \mu/\rho)\}\,.$$

As is well known, Z^0 has the following properties:

Lemma 2.3.

(i) $|D_x^m Z^0(x-\xi, t; \xi, \tau; \mu/\rho)|\leqq C_2^{(m)}(t-\tau)^{-\frac{1+m}{2}}\times$

$$\times\exp\{-(x-\xi)^2/8\left|\frac{\rho}{\mu}\right|_T^{(0)}(t-\tau)\}\,,$$

(ii) *for* $t>t'>\tau$,

$$|D_x^m Z^0(x-\xi, t; \xi, \tau; \mu/\rho)-D_x^m Z^0(x-\xi, t'; \xi, \tau; \mu/\rho)|\leqq C_3^{(m)}(t-t')\times$$

$$\times(t'-\tau)^{-\frac{3+m}{2}}\exp\{-(x-\xi)^2/8\left|\frac{\mu}{\rho}\right|_T^{(0)}(t-\tau)\}\,,$$

(iii) $|D_x^m Z^0(x-\xi, t; \xi, \tau; \mu/\rho)-D_{x'}^m Z^0(x'-\xi, t; \xi, \tau; \mu/\rho)|\leqq C_4^{(m)}|x-x'|\times$

$$\times(t-\tau)^{-\frac{2+m}{2}}\exp\{-(x''-\xi)^2/8\left|\frac{\mu}{\rho}\right|_T^{(0)}(t-\tau)\}\,,$$

where

$$x''=\begin{cases} x & (\text{if } |x-\xi|<|x'-\xi|) \\ x' & (\text{otherwise}), \end{cases}$$

(iv) $|D_t^k D_z^m Z^0(z, t; \xi, \tau; \mu/\rho)-D_t^k D_z^m Z^0(z, t; \xi', \tau; \mu/\rho)|\leqq C_5^{(k,m)}|\xi-\xi'|^\alpha\times$

$$\times(t-\tau)^{-\frac{1+m+2k}{2}}\exp\{-z^2/8\left|\frac{\mu}{\rho}\right|_T^{(0)}(t-\tau)\}\,.$$

Using the relation (2.13) and Lemma 2.3, Z is estimated as follows:

Lemma 2.4.

$$|D_x^m Z(x-\xi,\, t;\, \xi,\, \tau;\, \mu/\rho)| \leqq C_6^{(m)}(t-\tau)^{-\frac{1+m}{2}} \times$$

$$\times \exp\left\{-(x-\xi)^2/16\left|\frac{\mu}{\rho}\right|_T^{(0)}(t-\tau)\right\}.$$

Proof. In general, $-X \leqq x \leqq 2X$ implies $\frac{x^2}{2}+\frac{n^2X^2}{2} \leqq (x+2nX)^2$ except the case that $X < x \leqq 2X$ and $n=-1$. We have for any n,

$$|D_x^m Z^0(x-\xi+2nX,\, t;\, \xi,\, \tau;\, \mu/\rho)| \leqq C_2^{(m)}(t-\tau)^{-\frac{1+m}{2}} \times$$

$$\times \exp\{-(x-\xi)^2/16\left|\frac{\mu}{\rho}\right|_T^{(0)}(t-\tau)\} \times$$

$$\times \exp\{-X^2n^2/16\left|\frac{\mu}{\rho}\right|_T^{(0)}(t-\tau)\},$$

and for any $n(\neq -1)$, $|D_x^m Z^0(x+\xi+2nX,\, t;\, \xi,\, \tau;\, \mu/\rho)|$ has the same bound. Hence we have

$$\text{(2.14)}\quad \sum_{n=-\infty}^{\infty} |D_x^m Z^0(x-\xi+2nX, t; \xi, \tau; \mu/\rho)| \leqq C_2^{(m)}\left\{1+\frac{4(\pi|\mu/\rho|_T^{(0)}T)^{1/2}}{X}\right\} \times$$

$$\times (t-\tau)^{-\frac{1+m}{2}} \exp\{-(x-\xi)^2/16\left|\frac{\mu}{\rho}\right|_T^{(0)}(t-\tau)\},$$

$$\text{(2.15)}\qquad (\sum_{n=-\infty}^{-1}+\sum_{n=0}^{\infty})|D_x^m Z^0(x+\xi+2nX,\, t;\, \xi,\, \tau;\, \mu/\rho)| \leqq$$

$$\leqq C_2^{(m)}\left\{1+\frac{4(\pi|\mu/\rho|_T^{(0)}T)^{1/2}}{X}\right\}(t-\tau)^{-\frac{1+m}{2}} \times$$

$$\times \exp\{-(x-\xi)^2/16\left|\frac{\mu}{\rho}\right|_T^{(0)}(t-\tau)\}.$$

In a direct way for $n=-1$, we obtain

$$\text{(2.16)}\qquad |D_x^m Z^0(x+\xi-2X,\, t;\, \xi,\, \tau;\, \mu/\rho)| \leqq C_2^{(m)}(t-\tau)^{-\frac{1+m}{2}} \times$$

$$\times \exp\{-(x-\xi)^2/16\left|\frac{\mu}{\rho}\right|_T^{(0)}(t-\tau)\}.$$

Thus, by (2.14), (2.15) and (2.16), we have finally

$$|D_x^m Z(x-\xi, t; \xi, \tau; \mu/\rho)| \leqq C_6^{(m)}(t-\tau)^{-\frac{1+m}{2}} \times$$
$$\times \exp\{-(x-\xi)^2/16\left|\frac{\mu}{\rho}\right|_T^{(0)}(t-\tau)\},$$

where $C_6^{(m)} = C_2^{(m)}\left\{3+\frac{8(\pi|\mu/\rho|_T^{(0)}T)^{1/2}}{X}\right\}$. Q.E.D.

Making use of the same procedure as in the proof of Lemma 2.4, we have

Lemma 2.5.

(i) *For* $t>t'>\tau$,

$$|D_x^m Z(x-\xi, t; \xi, \tau; \mu/\rho) - D_x^m Z(x-\xi, t'; \xi, \tau; \mu/\rho)| \leqq C_7^{(m)}(t-t') \times$$
$$\times (t'-\tau)^{-\frac{2+m}{2}} \exp\{-(x-\xi)^2/16\left|\frac{\mu}{\rho}\right|_T^{(0)}(t-\tau)\},$$

(ii) $$|D_x^m Z(x-\xi, t; \xi, \tau; \mu/\rho) - D_{x'}^m Z(x'-\xi, t; \xi, \tau; \mu/\rho)| \leqq C_8^{(m)}|x-x'| \times$$
$$\times (t-\tau)^{-\frac{2+m}{2}} \exp\{-(x-\xi)^2/16\left|\frac{\mu}{\rho}\right|_T^{(0)}(t-\tau)\},$$

(iii) $$|D_t^k D_z^m Z(z, t; \xi, \tau; \mu/\rho) - D_t^k D_z^m Z(z, t; \xi', \tau; \mu/\rho)| \leqq C_9^{(k,m)}|\xi-\xi'|^\alpha \times$$
$$\times (t-\tau)^{-\frac{1+m+2k}{2}} \exp\{-z^2/16\left|\frac{\mu}{\rho}\right|_T^{(0)}(t-\tau)\}.$$

Furthermore, by Lemma 2.5, we have

Lemma 2.6.

$$\left|\int_0^X D_t^k D_x^m Z(x-\xi, t; \xi, \tau; \mu/\rho)\,d\xi\right| \leqq C_{10}(t-\tau)^{-\frac{2k+m-\alpha}{2}}, \quad \text{for } 2k+m>0.$$

It is easily seen that the fundamental solution $\Gamma(x, t; \xi, \tau; \mu/\rho)$ of

$$\frac{\partial v}{\partial t}(x, t) = \frac{\mu}{\rho(x, t)} \frac{\partial^2 v}{\partial x^2}(x, t) \qquad (T \geqq t > 0) \tag{2.17}$$

with (2.2) is given in the form

$$(2.18)\qquad \Gamma(x, t; \xi, \tau; \mu/\rho)=Z(x-\xi, t; \xi, \tau; \mu/\rho)+$$
$$+\int_{\tau}^{t}d\sigma\int_{0}^{X}K(x, t; y, \sigma; \mu/\rho)\Phi(y, \sigma; \xi, \tau; \mu/\rho)dy.$$

The function Φ satisfies a Volterra-type integral equation:

$$(2.19)\qquad \Phi(x, t; \xi, \tau; \mu/\rho)=K(x, t; \xi, \tau; \mu/\rho)+$$
$$+\int_{\tau}^{t}d\sigma\int_{0}^{X}K(x, t; y, \sigma; \mu/\rho)\Phi(y, \sigma; \xi, \tau; \mu/\rho)dy,$$

where

$$(2.20)\qquad K(x, t; \xi,\tau; \mu/\rho)=\left\{\frac{\mu}{\rho(x, t)}-\frac{\mu}{\rho(\xi, \tau)}\right\}D_x^2 Z(x-\xi, t; \xi, \tau; \mu/\rho).$$

The function Φ is given in the form

$$(2.21)\qquad \Phi(x, t; \xi, \tau; \mu/\rho)=\sum_{m=0}^{\infty}K_m(x, t; \xi, \tau; \mu/\rho),$$

where

$$(2.22)\qquad \begin{cases} K_0(x, t; \xi, \tau; \mu/\rho)=K(x, t; \xi, \tau; \mu/\rho), \\ K_m(x, t; \xi, \tau; \mu/\rho)=\int_{\tau}^{t}d\sigma\int_{0}^{X}K(x, t; y, \sigma; \mu/\rho)K_{m-1}(y, \sigma; \xi, \tau; \mu/\rho)dy. \end{cases}$$

We shall prove the convergence of the series in (2.21) and estimate Γ similarly to the way in which we did Z in Lemmas 2.5 and 2.6.

Lemma 2.7. $|K(x, t; \xi, \tau; \mu/\rho)|\leqq C_{11}(t-\tau)^{-\frac{3-\alpha}{2}}\times$
$$\times\exp\left\{x(-\xi)^2/32\left|\frac{\mu}{\rho}\right|_T^{(0)}(t-\tau)\right\}.$$

Proof. This follows directly from the Hölder continuity of $\frac{\mu}{\rho}$ and Lemma 2.4 ($m=2$). Q.E.D.

Proceeding similarly to evaluate K_1, K_2, etc., for any integer $m\geqq 0$ we have

Lemma 2.8.

$$\left|K_m\left(x, t; \xi, \tau; \frac{\mu}{\rho}\right)\right| \leqq \left[\frac{\left\{2\left(2\pi\left|\frac{\mu}{\rho}\right|_T^{(0)}\right)^{1/2} C_{11}\Gamma(\alpha/2)T^{\alpha/2}\right\}^{m+1}}{2\left(2\pi\left|\frac{\mu}{\rho}\right|_T^{(0)}\right)^{1/2}\Gamma((m+1)\alpha/2)} T^{-\alpha/2}\right]_I \times$$

$$\times (t-\tau)^{-\frac{3-\alpha}{2}} \exp\left\{-(x-\xi)^2/32\left|\frac{\mu}{\rho}\right|_T^{(0)}(t-\tau)\right\}.$$

From Lemma 2.8, it follows that the series expansion of $\Phi(x, t; \xi, \tau; \frac{\mu}{\rho})$ is uniformly convergent for $T \geqq t \geqq 0$ and Φ is evaluated as follows:

Lemma 2.9. $|\Phi(x, t; \xi, \tau; \mu/\rho)| \leqq C_{12}(t-\tau)^{-\frac{3-\alpha}{2}} \times$

$$\times \exp\left\{-(x-\xi)^2/32\left|\frac{\mu}{\rho}\right|_T^{(0)}(t-\tau)\right\},$$

where $C_{12} = \sum_{m=0}^{\infty} [\cdots]_I$ *in Lemma* 2.8.

Thus, using Lemmas 2.4 and 2.9, we have

Lemma 2.10. $|D_x^m \Gamma(x, t; \xi, \tau; \mu/\rho)| \leqq C_{13}^{(m)}(t-\tau)^{-\frac{1+m}{2}} \times$

$$\times \exp\left\{-(x-\xi)^2/32\left|\frac{\mu}{\rho}\right|_T^{(0)}(t-\tau)\right\}.$$

In order to study Γ in more detail, we shall need the following lemmas.

Lemma 2.11. $|K(x, t; \xi, \tau; \mu/\rho) - K(x', t; \xi, \tau; \mu/\rho)| \leqq C_{14}(t-\tau)^{-3/2} \times$

$$\times |x-x'|^\alpha \exp\left\{-(x-\xi)^2/32\left|\frac{\mu}{\rho}\right|_T^{(0)}(t-\tau)\right\}.$$

Proof. From (2.20), it follows that the lemma holds, by using the Hölder continuity of $\frac{\mu}{\rho}$, Lemmas 2.4 and 2.5. Q.E.D.

By induction, we obtain

Lemma 2.12. $|\Phi(x, t; \xi, \tau; \mu/\rho) - \Phi(x', t; \xi, \tau; \mu/\rho)| \leqq C_{15}(t-\tau\rho)^{-\frac{3}{2}} \times$

$$\times |x-x'|^\alpha \exp\left\{-(x-\xi)^2/32\left|\frac{\mu}{\rho}\right|_T^{(0)}(t-\tau)\right\}.$$

As a result, by using Lemmas 2.4, 2.5, 2.6, 2.9 and 2.12, we have, after lengthy calculations,

Lemma 2.13.

(i) $$|\Gamma(x,t;\xi,\tau;\mu/\rho)-\Gamma(x,t';\xi,\tau;\mu/\rho)|\leqq C_{16}^{(0)}(t-t')(t'-\tau)^{-\frac{3}{2}}\times$$
$$\times\exp\{-(x-\xi)^2/32\left|\frac{\mu}{\rho}\right|_T^{(0)}(t-\tau)\},$$

(ii) $$|D_x^m\Gamma(x,t;\xi,\tau;\mu/\rho)-D_x^m\Gamma(x,t';\xi,\tau;\mu/\rho)|\leqq C_{16}^{(m)}\{(t-t')(t'-\tau)^{-\frac{3}{2}}+$$
$$+(t-t')^{\frac{2-m+\alpha}{2}}(t'-\tau)^{-\frac{3}{2}}\}\exp\{-(x-\xi)^2/32\left|\frac{\mu}{\rho}\right|_T^{(0)}(t-\tau)\}.$$

§3. The Existence of a Temporally Local Solution of (1.1), (2.1) and (2.2)

In the first place, we construct the sequence $\{v^n(x,t)\}$ such that

$$(3.1)\qquad \begin{cases} v^0(x,t)=v_0(x)\in H_T^{2+\alpha}, \\ v^n(x,t)=v_0(x)+\int_0^t d\tau\int_0^X \Gamma(x,t;\xi,\tau;\mu/\rho_{n-1})N_{n-1}(\xi,\tau)d\xi \end{cases}$$

$(0\leqq t\leqq T)$, where $\rho_{n-1}=\rho v_{n-1}$ and $N_{n-1}=\frac{\mu}{\rho_{n-1}}v_0''+v^{n-1}v_x^{n-1}$ (cf. (2.6)). We also assume (2.3) for $t=0$, i.e.,

$$v_0''(x)_{|x=0}=v_0''(x)_{|x=X}=0.$$

The functions v^n $(n=1,2,\dots)$ satisfy

$$(3.2)\qquad v_t^n=\frac{\mu}{\rho_{n-1}}v_{xx}^n-v^{n-1}v_x^{n-1}$$

and

$$(3.3)\qquad v^n(x,0)=v_0(x),\quad v^n(0,t)=v^n(X,t)=0.$$

By using the lemmas obtained in §2, especially Lemmas 2.10 and 2.13, we have

Lemma 3.1.

(i) $\|v^n\|_T^{(1)} \leqq C_{17,1}\left(T, \left|\!\left|\!\left|\frac{\mu}{\rho_{n-1}}\right|\!\right|\!\right|_T\right)|N_{n-1}|_T^{(0)} + \|v_0\|^{(1)}$

where $\left|\!\left|\!\left|\frac{\mu}{\rho_{n-1}}\right|\!\right|\!\right|_T = \left|\frac{\rho_{n-1}}{\mu}\right|_T^{(0)} + \left\|\frac{\mu}{\rho_{n-1}}\right\|_T^{(\alpha)}$,

(ii) $|v_{xx}^n|_T^{(0)} \leqq C_{17,2}\left(T, \left|\!\left|\!\left|\frac{\mu}{\rho_{n-1}}\right|\!\right|\!\right|_T\right)\|N_{n-1}\|_T^{(\alpha)} + |v_0''|^{(0)}$,

(iii) $|v^n|_{t,T}^{(\alpha/2)} \leqq C_{17,3}\left(T, \left|\!\left|\!\left|\frac{\mu}{\rho_{n-1}}\right|\!\right|\!\right|_T\right)|N_{n-1}|_T^{(0)}$,

(iv) $|v_x^n|_{x,T}^{(\alpha)} \leqq 2(|v_0''|^{(0)} + |v_0'|^{(0)}) + C_{17,4}\left(T, \left|\!\left|\!\left|\frac{\mu}{\rho_{n-1}}\right|\!\right|\!\right|_T\right)|N_{n-1}|^{(0)}$,

(v) $|v_x^n|_{t,T}^{(\alpha/2)} \leqq C_{17,5}\left(T, \left|\!\left|\!\left|\frac{\mu}{\rho_{n-1}}\right|\!\right|\!\right|_T\right)|N_{n-1}|_T^{(0)}$,

(vi) $|v_{xx}^n|_T^{(\alpha)} \leqq C_{17,6}\left(T, \left|\!\left|\!\left|\frac{\mu}{\rho_{n-1}}\right|\!\right|\!\right|_T\right)\|N_{n-1}\|_T^{(\alpha)} + |v_0''|^{(\alpha)}$.

Remark. The constants $C_{17,i}$ $(i=1, 2, 3, 4, 5)$ increase monotonically as each argument increases and $C_{17,i}\downarrow 0$ as $T\downarrow 0$.

It is easy to see that $v^{n-1}\in H_T^{2+\alpha}$ implies $\frac{\mu}{\rho_{n-1}}\in H_T^{\alpha}$ and $N_{n-1}\in H_T^{\alpha}$. Thus, by the above lemma, we see clearly that $v^n\in \hat{H}_T^{2+\alpha}$ and also $v^n\in H_T^{2+\alpha}$. Hence by induction we obtain

Lemma 3.2. $v^n(x, t)\in H_T^{2+\alpha}$.

Now, we take an arbitrary constant M_0 such that

$$\|v_0\|^{(2)} < M_0 < +\infty. \tag{3.4}$$

As for $\left|\!\left|\!\left|\frac{\mu}{\rho_{n-1}}\right|\!\right|\!\right|_T$, it holds that

$$\begin{aligned}\left|\!\left|\!\left|\frac{\mu}{\rho_{n-1}}\right|\!\right|\!\right|_T \leqq\ & \frac{\bar{\bar{\rho}}_0}{\mu}\exp\{T|v_x^{n-1}|_T^{(0)}\} + (\bar{\rho}_0)^{-1}\exp\{T|v_x^{n-1}|_T^{(0)}\} + \\ & + 2\mu[\{(\bar{\rho}_0)^{-2}|\rho_0'|^{(0)} + (\bar{\rho}_0)^{-1}T|v_{xx}^{n-1}|_T^{(0)}\exp\{2T|v_x^{n-1}|_T^{(0)}\}\} + \\ & + (\bar{\rho}_0)^{-1}\exp\{T|v_x^{n-1}|_T^{(0)}\}] + 2\mu[(\bar{\rho}_0)^{-1}(1+|v_x^{n-1}|_T^{(0)})\times \\ & \times\exp\{T|v_x^{n-1}|_T^{(0)}\} + |v_x^{n-1}|_T^{(0)}\{(\bar{\rho}_0)^{-2}|\rho_0'|^{(0)} +\end{aligned} \tag{3.5}$$

$$+(\bar{\rho}_0)^{-1}T|v_{xx}^{n-1}|_T^{(0)}\exp\{2T|v_x^{n-1}|_T^{(0)}\}\}] .$$

If we assume that $\|v^{n-1}\|_T^{(2)}<M_0$, then we have

$$\left|\left|\left|\frac{\mu}{\rho_{n-1}}\right|\right|\right|_T \leqq \left\{\frac{\bar{\bar{\rho}}_0}{\mu}+\mu(5+2M_0)(\bar{\rho}_0)^{-1}\right\}\exp\{M_0T\}+$$

$$+2\mu(1+M_0)\{(\bar{\rho}_0)^{-2}|\rho_0'|^{(0)}+(\bar{\rho}_0)^{-1}M_0Te^{2M_0T}\}\equiv A(T,M_0) .$$

By Lemma 3.1, we have $\|v^n\|_T^{(2)}\leqq\|v_0\|^{(2)}+(C_{17,1}+C_{17,2})\|N_{n-1}\|_T^{(\alpha)}$. Furthermore, $\|v^{n-1}\|_T^{(2)}<M_0$ implies

$$\|N_{n-1}\|_T^{(\alpha)}\leqq C_{18}\left(T,M_0,\left|\left|\left|\frac{\mu}{\rho_{n-1}}\right|\right|\right|_T\right), \tag{3.6}$$

where C_{18} is monotonically increasing in each argument and $C_{18}\downarrow$ ' a certain positive constant' as $T\downarrow 0$. Therefore, we have

$$\|v^n\|_T^{(2)}\leqq\|v_0\|_T^{(2)}+(C_{17,1}^*+C_{17,2}^*)C_{18}(T,M_0,A(T,M_0)),$$

where $C_{17,i}^*=C_{17,i}(T,A(T,M_0))$ $(i=1,2)$. Hence, for a sufficiently small $T_1\in(0,T]$

$$\|v^n\|_{T_1}^{(2)}\leqq M_0 . \tag{3.7}$$

By induction, for some $T_2\in(0,T]$

$$\|v^n\|_{T_2}^{(2)}\leqq M_0 \qquad (n=1,2,3,\ldots), \tag{3.8}$$

For simplicity we choose T from the beginning in such a way that $T=T_2$.

In the next place, by (3.2) the differences v^n-v^{n-1} satisfy the equation:

$$(v^n-v^{n-1})_t=\frac{\mu}{\rho_{n-1}}(v^n-v^{n-1})_{xx}+\tilde{N}_{n-1}, \qquad (n=1,2,3,\ldots), \tag{3.9}$$

where $\tilde{N}_{n-1}=\left(\frac{\mu}{\rho_{n-1}}-\frac{\mu}{\rho_{n-2}}\right)v_{xx}^{n-1}+v^{n-1}(v^{n-1}-v^{n-2})_x+(v^{n-1}-v^{n-2})v_x^{n-2}\in$ H_T^α $(n=2,3,4,\ldots)$, and by (3.3) it also satisfies the initial-boundary conditions:

(3.10) $$(v^n - v^{n-1})(x, 0) = 0, \quad (v^n - v^{n-1})(0, t) = (v^n - v^{n-1})(X, t) = 0.$$

In the same way as we did in §2, we can construct the fundamental solution of (3.9) and (3.10), and the solution of (3.9) and (3.10) is uniquely expressed by

(3.11) $$(v^n - v^{n-1})(x, t) = \int_0^t d\tau \int_0^X \Gamma(x, t; \xi, \tau; \mu/\rho_{n-1}) \tilde{N}_{n-1}(\xi, \tau) d\xi.$$

Similarly to Lemma 3.1, we have the following lemma.

Lemma 3.3. $\|v^n - v^{n-1}\|_T^{(1)} \leq C_{19}\left(T, \left\|\left|\frac{\mu}{\rho_{n-1}}\right|\right\|_T\right) |\tilde{N}_{n-1}|_T^{(0)}$

($n = 1, 2, 3, \ldots$), *where* C_{19} *has the same property as* $C_{17,1}$.

Directly by the above lemma, we have

(3.12) $$\|v^n - v^{n-1}\|_T^{(1)} \leqq C_{19}(T, A(T, M_0)) |\tilde{N}_{n-1}|_T^{(0)}.$$

Lemma 3.4.

$$|\tilde{N}_{n-1}|_T^{(0)} \leqq C_{20}\left(T, \left\|\left|\frac{\mu}{\rho_{n-1}}\right|\right\|_T + \left\|\left|\frac{\mu}{\rho_{n-2}}\right|\right\|_T\right) \|v^{n-1} - v^{n-2}\|_T^{(1)},$$

where C_{20} *has the same property as* C_{18}.

Proof. $$|\tilde{N}_{n-1}|_T^{(0)} \leqq \left|\frac{\mu}{\rho_{n-1}} - \frac{\mu}{\rho_{n-2}}\right|_T^{(0)} |v_{xx}^{n-1}|_T^{(0)} +$$
$$+ |v^{n-1}|_T^{(0)} |(v^{n-1} - v^{n-2})_x|_T^{(0)} + |v^{n-1} - v^{n-2}|_T^{(0)} |v_x^{n-2}|_T^{(0)}.$$

Using Lemma 2.2, we have

$$|\tilde{N}_{n-1}|_T^{(0)} \leqq C_{20}\left(T, \left\|\left|\frac{\mu}{\rho_{n-1}}\right|\right\|_T + \left\|\left|\frac{\mu}{\rho_{n-2}}\right|\right\|_T\right) \|v^{n-1} - v^{n-2}\|_T^{(1)}.$$

Q.E.D.

Hence we have

(3.13) $$|\tilde{N}_{n-1}|_T^{(0)} \leqq C_{20}(T, A(T, M_0)) \|v^{n-1} - v^{n-2}\|_T^{(1)}.$$

Combining (3.12) and (3.13), we obtain

$$\|v^n - v^{n-1}\|_T^{(1)} \leq C_{21} \|v^{n-1} - v^{n-2}\|_T^{(1)}, \tag{3.14}$$

where C_{21} *has the same property as* C_{19}.

By induction and (3.14), we have

$$\|v^n - v_n^{n-1}\|_T^{(1)} \leqq C_{21}^{n-1} \|v^1 - v^0\|_T^{(1)}. \tag{3.15}$$

Since $C_{21} \downarrow 0$ as $T \downarrow 0$, it holds, for some $T_0 \in (0, T]$, that $C_{21}(T_0, A(T_0, (M_0)) < 1$, whereas by Lemma 3.1

$$\|v^1 - v^0\|_{T_0}^{(1)} \leqq C_{17,1}(T_0, A(T_0, M_0))|N_0|_{T_0}^{(0)}$$

and

$$|N_0|_{T_0}^{(0)} \leqq (\bar{\rho}_0)^{-1} M_0 \exp\{M_0 T_0\} + M_0^2 < +\infty .$$

Thus

$$\sum_{n=1}^{\infty} C_{21}^{n-1} \|v^1 - v^0\|_{T_0}^{(1)} < +\infty . \tag{3.16}$$

Therefore, $\{v^n\}$ converges to an element v of $H_T^{2+\alpha}$ as $n \to \infty$. As is known the expression (2.5), $\{\rho_n\}$ converges to an element ρ_v of B_T^1 as $n \to \infty$. N_n also converges to $N = \frac{\mu}{\rho_v} v_0'' + v v_x$. Hence, by the formula (2.16), (2.18), (2.25), (2.26), (2.27), (2.28) and (2.29), $Z^0(x-\xi, t; \xi, \tau; \mu/\rho_{n-1})$, $Z(x-\xi, t; \xi, \tau; \mu/\rho_{n-1})$, $K_m(x, t; \xi, \tau; \mu/\rho_{n-1})$, $\Phi(x, t; \xi, \tau; \mu/\rho_{n-1})$, and $\Gamma(x, t; \xi, \tau; \mu/\rho_{n-1})$ converge to $Z^0(x-\xi, t; \xi, \tau; \mu/\rho_v)$, $Z(x-\xi, t; \xi, \tau; \mu/\rho_v)$, $K_m(x, t; \xi, \tau; \mu/\rho_v)$, $\Phi(x, t; \xi, \tau; \mu/\rho_v)$ and $\Gamma(x, t; \xi, \tau; \mu/\rho_v)$, respectively, as $n \to \infty$. Thus by (3.1), it holds, for $0 \leqq t \leqq T$, that

$$v(x, t) = v_0(x) + \int_0^t d\tau \int_0^X \Gamma(x, t; \xi, \tau; \mu/\rho_v) \left\{ \frac{\mu}{\rho_v} v_0''(\xi) - \right.$$

$$\left. - v v_\xi(\xi, \tau) \right\} d\xi .$$

As a result, we have

Theorem 3.1. *For some* $T \in (0, \infty)$, *there exists a solution of* (1.1), (2.1) *and* (2.2) *in* $H_T^{2+\alpha} \times B_T^1$.

Remark. For v, Lemma 3.1 also holds.

§4. The Uniqueness of the Solution of (1.1), (2.1) and (2.2) in $H_T^{4+\alpha}\times B_T^1$.

Now let us direct ourselves towards the problem of uniqueness concerning the system (1.1) of differential equations. (cf. [11]). We assume that there exist two solutions (v, ρ_v) and (w, ρ_w) of (1.1) in $H_T^{2+\alpha}\times B_T^1$ satisfying one and the same initial-boundary conditions (2.1) and (2.2). The difference $v-w$ satisfies the equation (3.9) and the initial-boundary condition (3.10) as v^n and v^{n-1} are replaced by v and w respectively. Then $v-w$ can be uniquely expressed in the form (3.11) as v^n and v^{n-1} are replaced by v and w respectively, i.e.,

$$(v-w)(x, t)=\int_0^t d\tau\int_0^X \Gamma(x, t; \xi, \tau; \mu/\rho_v)\tilde{N}(\xi, \tau)d\xi, \tag{4.1}$$

where $\tilde{N}(x, t)=\left(\frac{\mu}{\rho_v}-\frac{\mu}{\rho_w}\right)(x, t)-v(x, t)(v-w)_x(x, t)+(v-w)(x, t)w_x(x, t)$.
As for $v-w$, in a way analogous to that used in the preceding section for v^n-v^{n-1}, we obtain

Lemma 4.1.

(i) $$\|v-w\|_{T_0}^{(1)}\leqq C_{22}\left(T_0, \left|\left|\left|\frac{\mu}{\rho_v}\right|\right|\right|_T+\left|\left|\left|\frac{\mu}{\rho_w}\right|\right|\right|_T\right)|\tilde{N}|_{T_0}^{(0)},$$

(ii) $$|\tilde{N}|_{T_0}^{(0)}\leqq C_{23}\left(T_0, \left|\left|\left|\frac{\mu}{\rho_v}\right|\right|\right|_T+\left|\left|\left|\frac{\mu}{\rho_w}\right|\right|\right|_T\right)\|v-w\|_{T_0}^{(1)},$$

$(0<T_0\leqq T)$, *where* C_{22} *and* C_{23} *have the same property as* C_{19} *and* C_{20}, *respectively.*

Finally, we have an inequality similar to (3.14):

$$\|v-w\|_{T_0}^{(1)}\leqq C_{24}(T_0; v, w)\|v-w\|_{T_0}^{(1)}, \tag{4.2}$$

where $C_{24}(T_0;\cdots)$ *has the same property as* C_{22}.

Since $C_{24}(T_0;\cdots)\downarrow 0$ as $T_0\downarrow 0$, it holds for a sufficiently small $T_1\in(0, T]$, that

$$0 \leqq (1 - C_{24}(T_1)) \| v - w \|_{T_1}^{(1)} \leqq 0 . \tag{4.3}$$

Hence, we obtain $\| v - w \|_{T_1}^{(1)} = 0$, i.e., $v(x, t) = w(x, t)$ $(0 \leqq t \leqq T_1 \leqq T)$. According to the assumption, we can continue this procedure again by starting at $t = T_1$. After a finite number of repetitions of this procedure, it is shown in a conventional way that the following assertion holds.

Theorem 4.1. *If* (v, ρ) *and* $(w, \rho^*) \in H_T^{2+\alpha} \times B_T^1$ *satisfy* (1.1), (2.1) *and* (2.2), *then* $(v, \rho) = (w, \rho^*)$ $(\rho = \rho^* = \rho_v)$.

§5. An *a priori* Estimate for $|\rho|_T^{(0)}$

We begin with the following well known lemmas. (see, e.g., [5], [15]).

Lemma 5.1. *If* $u(x, t)$ *satisfies regularly the equation*:

$$\frac{\partial u}{\partial t} = a(x, t) \frac{\partial^2 u}{\partial x^2} + b(x, t) \frac{\partial u}{\partial x} + c(x, t) u, \qquad (0 < t \leqq T) \tag{5.1}$$

where $a(x, t)$, $b(x, t)$ *and* $c(x, t)$ *are continuous in* $\bar{Q}_T$ *and satisfy*

$$0 \leqq a(x, t) \leqq |a|_T^{(0)} < +\infty, \qquad c(x, t) \leqq 0, \tag{5.1$'$}$$

then it holds that

$$\max_{\bar{Q}_T} |u| \leqq \max_{\Gamma_T} |u| . \tag{5.2}$$

Lemma 5.2. *If* $u(x, t)$ *satisfies regularly the equation*:

$$\frac{\partial u}{\partial t} = a(x, t) \frac{\partial^2 u}{\partial x^2} + b(x, t) \frac{\partial u}{\partial x} + c(x, t) u + f(x, t) , \tag{5.3}$$

where $a(x, t) > 0$ *in* $\bar{Q}_T$ *and if*

$$D_t^k D_x^m a,\ D_t^k D_x^m b,\ D_t^k D_x^m c,\ D_t^k D_x^m f \qquad (0 \leqq \mathrm{m} + 2k \leqq p,\ k \leqq q)$$

belong to H_T^α, *then* $D_t^k D_x^m u$ $(0 \leqq m + 2k \leqq p + 2,\ k \leqq q + 1)$ *exist and are Hölder continuous (exponent* α*) in* $\bar{\Omega} \times [T', T]$ *for an arbitrary* $T' \in (0, T)$.

Directly by the above lemmas, we have

Lemma 5.3. *If* $(v, \rho)\in H_T^{2+\alpha}\times B_T^1$ *satisfies* (1.1), (2.1) *and* (2.2), *then it holds that*

$$|v|_T^{(0)}\leqq|v_0|^{(0)}. \tag{5.4}$$

Lemma 5.4. *If* $(v, \rho)\in H_T^{2+\alpha}\times B_T^1$ *satisfies* (1.1), (2.1), (2.2) *and an additional condition*:

$$\rho_0\in H^{1+\alpha}, \tag{5.5}$$

then $v\in H_{[T',T]}^{3+\alpha}$ *where the suffix* $[T', T]$ *denotes that* $\bar{Q}_T$ *in* (1.4) *and* (1.4)' *is replaced by* $\bar{\Omega}\times[T', T]$. [We note that (5.5) implies $\rho\in B_T^{1+\alpha}$.]

Lemma 5.5. *If* $(v, \rho)\in H_T^{2+\alpha}\times B_T^1$ *satisfies* (1.1), (2.1), (2.2) *and* (5.5), *then it holds that*

$$\bar{\rho}_0\exp\left\{-\frac{1}{\mu}\left|\rho_0v_0\right|^{(0)}X\right\}\leqq\rho(x, t)\leqq\bar{\bar{\rho}}_0\exp\left\{\frac{1}{\mu}\left|\rho_0v_0\right|^{(0)}X\right\}. \tag{5.6}$$

Proof. By (2.5), we need to estimate $\int_0^X\bar{v}_x(\tau; x, t)d\tau$. Since

$$\frac{\partial}{\partial x}\int_0^t\bar{v}_x(\tau; x, t)d\tau=\int_0^t\bar{v}_{xx}(\tau; x, t)x_x(\tau; x, t)d\tau=$$

$$=\frac{\rho(x, t)}{\mu}\{v(x, t)-v_0(x_0(x, t))\},$$

$$\int_0^t\bar{v}_x(\tau; x, t)d\tau=\left\{\frac{1}{\mu}\int_0^x\rho vdx+\int_0^t\bar{v}_x(\tau; 0, t)d\tau\right\}- \tag{5.7}$$

$$-\frac{1}{\mu}\int_0^x\rho(x, t)v_0(x_0(x, t))dx.$$

The second term of the right-hand side of (5.7) is transformed, by using (2.6), as follows:

$$\int_0^x\rho(x, t)v_0(x_0)dx=\int_0^{x_0}\rho_0(x_0')v_0(x_0')dx_0'.$$

We denote the first term of the right-hand side of (5.7) by $\psi(x, t)$. Then, we have

$$\psi_t(x,t)=\frac{1}{\mu}\int_0^x(\rho_t v+\rho v_t)dx+v_x(0,t)=\frac{1}{\mu}(-\rho v^2+\mu v_x).$$

On the other hand, $\psi_x(x,t)=\frac{1}{\mu}\rho v$, $\left(\frac{\mu}{\rho}\psi_x\right)_x=v_x$. Hence, we have

$$(5.8)\qquad \psi_t=\frac{\mu}{\rho}\psi_{xx}-\left(\frac{\mu\rho_x}{\rho^2}+v\right)\psi_x,$$

$$(5.9)\qquad \psi(x,0)=\frac{1}{\mu}\int_0^x\rho_0 v_0 dx,\ \psi_x(0,t)=\psi_x(X,t)=0\qquad (T\geqq t\geqq 0).$$

Therefore, $\psi(x,t)$ is to be expressed by utilizing the fundamental solution of the linear parabolic equation (5.8) in the following way:

$$(5.10)\qquad \psi(x,t)=\int_0^X\tilde{\Gamma}(x,t;\xi,0)\psi(\xi,0)d\xi=\frac{1}{\mu}\int_0^X\tilde{\Gamma}(x,t;\xi,0)\left(\int_0^\xi\rho_0 v_0 dy\right)d\xi$$

where $\tilde{\Gamma}$ is the fundamental solution of (5.8).
Hence, we have $\left|\psi-\frac{1}{\mu}\int_0^x\rho(x,t)v_0(x_0(x,t))dx\right|_T^{(0)}\leqq\frac{1}{\mu}|\rho_0v_0|^{(0)}X$.
As a result, we obtain (5.6). Q.E.D.

If $(v,\rho)\in H_T^{2+\alpha}\times B_T^1$ satisfies (1.1), (2.1), (2.2), and (5.5), then by Lemmas 2.1 and 5.2, we have

$$(5.11)\qquad \left\|\frac{\mu}{\rho}\right\|_T^{(\alpha)}\leqq 5\left|\frac{\mu}{\rho}\right|_T^{(0)}+2\left|\left(\frac{\mu}{\rho}\right)_x\right|_T^{(0)}+2\left|\left(\frac{\mu}{\rho}\right)_t\right|_T^{(0)}$$
$$\leqq(5+2|v_x|_T^{(0)})\mu(\bar{\rho}_0)^{-1}\exp\{|\int_0^t\bar{v}_x(\tau;x,t)d\tau|_T^{(0)}\}+$$
$$+4|v_0|^{(0)}(1+|v_0|^{(0)})+2\mu(\bar{\rho}_0)^{-2}|\rho_0'|^{(0)}(1+|v_0|^{(0)}),$$

since
$$\begin{cases}\left(\dfrac{\mu}{\rho}\right)_x=-\dfrac{\mu\rho_x}{\rho^2}=-\dfrac{\rho_0'}{\rho_0^2}+v(x,t)-v_0(x_0(x,t)),\\[2mm] \left(\dfrac{\mu}{\rho}\right)_t=-\dfrac{\mu\rho_t}{\rho^2}=\mu\left\{\dfrac{v_x}{\rho}-v\left(\dfrac{1}{\rho}\right)_x\right\}.\end{cases}$$

By (5.11), we know that, in order to have an *a priori* estimate for $\left\|\frac{\mu}{\rho}\right\|_T^{(\alpha)}$, we have to obtain beforehand one for $|v_x|_T^{(0)}$. Hereafter in §6, we shall endeavor to have an *a priori* estimate for $|v_x|_T^{(0)}$.

§6. An *a priori* Estimate for $|v_x|_T^{(0)}$

Lemma 6.1. *Under the initial-boundary conditions* (2.1), (2.2) *and* (5.5), $|v_x|_T^{(0)}$ *is bounded by a constant depending only on the quantities appearing in* (2.1), (2.2) *and* (5.5) *but independent of T.*

Proof. The procedure of the demonstration is divided into three steps.

((1-st step)). First of all, we note that (5.6) holds by Lemma 5.5. Now we define $v_\lambda(x, t)$ by

$$v_\lambda(x,t)\equiv v_x(x,t)^2+\lambda v(x,t)^2\,, \tag{6.1}$$

where λ is a constant to be determined later.

Since $v\in H^{3+\alpha}_{[T',T]}$ by Lemma 5.4, v_x satisfies the equation:

$$(v_x)_t=\frac{\mu}{\rho}(v_x)_{xx}+\left\{\left(\frac{\mu}{\rho}\right)_x-v\right\}(v_x)_x-(v_x)^2. \tag{6.2}$$

Let $\mathscr{L}$ be defined by

$$\mathscr{L}\equiv\frac{\partial}{\partial t}-\frac{\mu}{\rho}\frac{\partial^2}{\partial x^2}+v\frac{\partial}{\partial x}\,. \tag{6.3}$$

Then, we have for any $\varepsilon>0$

$$\begin{aligned}\mathscr{L}v_\lambda&=2\left(\frac{\mu}{\rho}\right)_x v_xv_{xx}-2v_x^3-2\frac{\mu}{\rho}v_{xx}^2-2\frac{\mu\lambda}{\rho}v_x^2\\&\leqq\left[2\{(\bar\rho_0)^{-2}|\rho_0'|^{(0)}+2|v_0|^{(0)}\}\varepsilon-\frac{2\mu}{|\rho|_T^{(0)}}\right]v_{xx}^2+\\&\quad+\left[\frac{1}{2\varepsilon}\{(\bar\rho_0)^{-2}|\rho_0'|^{(0)}+2|v_0|^{(0)}\}+2|v_x|_T^{(0)}-\right.\\&\quad\left.-\frac{2\mu\lambda}{|\rho|_T^{(0)}}\right]v_x^2\,.\end{aligned} \tag{6.4}$$

We choose $\varepsilon=\varepsilon_0$ in such a way that

$$(6.5)\qquad \varepsilon_0 \equiv \begin{cases} 1 & (\text{if } |\rho_0'|^{(0)} = |v_0|^{(0)} = 0) \\ \dfrac{\mu}{|\rho|_T^{(0)}} \{(\bar{\rho}_0)^{-2}|\rho_0'|^{(0)} + 2|v_0|^{(0)}\}^{-1} & \\ & (\text{otherwise}). \end{cases}$$

For such a fixed number $\varepsilon_0(>0)$, it holds that

$$(6.6)\qquad \mathscr{L} v_\lambda \leqq \Big[\frac{1}{2\varepsilon_0} \{(\bar{\rho}_0)^{-2}|\rho_0'|^{(0)} + 2|v_0|^{(0)}\} + 2|v_x|_T^{(0)} - \\ - \frac{2\mu\lambda}{|\rho|_T^{(0)}} \Big] v_x^2 .$$

If we take $\lambda = \lambda_0 \equiv \dfrac{|\rho|_T^{(0)}}{2\mu} \Big[\dfrac{1}{2\varepsilon_0} \{(\bar{\rho}_0)^{-2}|\rho_0'|^{(0)} + 2|v_0|^{(0)}\} + 2|v_x|_T^{(0)} \Big]$, then we have an inequality

$$(6.7)\qquad \mathscr{L} v_{\lambda_0} \leqq 0 .$$

By (6.7) and the maximum principle, it holds that

$$(6.8)\qquad \max_{\bar{Q}_T} v_{\lambda_0} \leqq \max_{\Gamma_T} (v_x^2 + \lambda_0 v^2) \\ \leqq (|v_0'|^{(0)})^2 + \lambda_0 (|v_0|^{(0)})^2 + \max_{S_T} v_x^2 .$$

((2-nd step)). To evaluate the last term of (6.8), it is clear the case $|v_0|^{(0)} = 0$. Then, suppose $|v_0|^{(0)} \neq 0$ and consider $v = \phi(w)$, where ϕ is a smooth function to be determined later. Thus, we get

$$(6.9)\qquad \mathscr{L} v = \phi' \Big[w_t - \frac{\mu}{\rho} w_{xx} - \frac{\mu}{\rho} \frac{\phi''}{\phi'} w_x^2 + \frac{1}{\phi'} v v_x \Big] = 0 .$$

If $\phi' > 0$ and $\phi'' < 0$, then it follows that

$$(6.10)\qquad w_t - \frac{\mu}{\rho} w_{xx} = \frac{\phi''}{\phi'} \frac{\mu}{\rho} w_x^2 - \frac{1}{\phi'} v v_x \\ \leqq \frac{\mu}{|\rho|_T^{(0)}} \Big\{ \Big[\frac{\phi''}{\phi'} + \frac{|\rho|_T^{(0)} |v_0|^{(0)}}{2\mu} \phi' \Big] w_x^2 + \\ + \frac{|\rho|_T^{(0)} |v_0|^{(0)}}{2\mu} \frac{1}{\phi'} \Big\}$$

Furthermore, we choose ϕ in such a way that

$$\frac{\phi''}{\phi'}+\frac{|\rho|_T^{(0)}|v_0|^{(0)}}{2\mu}\phi'\leqq 0 \quad \text{and} \quad \phi(0)=0\,,$$

that is to say, for example,

$$\phi(w)=\frac{2\mu}{|\rho|_T^{(0)}|v_0|^{(0)}}\log(1+w)\,, \tag{6.11}$$

or

$$w=-1+\exp\left\{\frac{2\mu}{|\rho|_T^{(0)}|v_0|^{(0)}}v\right\}.$$

By (6.11), it is clear that $w_{|S_T}=0$.
For such a function ϕ, it follows from (6.10) that

$$w_t-\frac{\mu}{\rho}w_{xx}\leqq\frac{|\rho|_T^{(0)}(|v_0|^{(0)})^2}{4\mu}\exp\left\{\frac{2\mu}{|\rho|_T^{(0)}}\right\}\equiv C_{25}. \tag{6.12}$$

Differentiating both sides of (6.11) once in x and putting $t=0$, we get

$$\max_{\bar{\Omega}}|w_x(x,0)|\leqq\frac{2\mu}{|\rho|_T^{(0)}|v_0|^{(0)}}|v_0'|^{(0)}\exp\left\{\frac{2\mu}{|\rho|_T^{(0)}}\right\}. \tag{6.13}$$

Define the constant C_{26} by $\max\left\{\frac{2\mu|v_0'|^{(0)}}{|\rho|_T^{(0)}|v_0|^{(0)}}\exp\left\{\frac{2\mu}{|\rho|_T^{(0)}}\right\},1\right\}$, then it follows that

$$\max_{\bar{\Omega}}|w_x(x,0)|\leqq C_{26}. \tag{6.14}$$

Now, consider the function $w(x,t)+ve^{-x}$. For $v\geqq C_{26}e^X$, $w(x,t)+ve^{-x}\geqq 0$ and

$$\max_{\Gamma_T}\{w+ve^{-x}\}\leqq\max_{S_T^0}\{w+ve^{-x}\}=v\,, \tag{6.15}$$

since $\frac{\partial}{\partial x}\{w(x,0)+ve^{-x}\}_{|x\in\bar{\Omega}}\leqq C_{26}-ve^{-X}\leqq 0$.

Next, if we take $v=v_0\equiv e^X\max\left\{C_{26},\frac{|\rho|_T^{(0)}}{\mu}C_{25}\right\}$, then

$$\frac{\partial}{\partial t}\{w+v_0e^{-x}\}-\frac{\mu}{\rho}\frac{\partial^2}{\partial x^2}\{w+v_0e^{-x}\}\leqq 0\,, \tag{6.16}$$

because it holds that

$$w_t-\frac{\mu}{\rho}\{w+v_0e^{-x}\}_{xx}\leqq C_{25}-\frac{\mu v_0}{|\rho|_T^{(0)}}e^{-X}\leqq 0 .$$

Hence, by (6.15) and (6.16), the maximum of $w+v_0e^{-x}$ in $\bar{Q}_T$ is attained at all points of S_T^0. Thus

$$\frac{\partial w}{\partial x}|_{S_T^0}\leqq v_0 . \tag{6.17}$$

Directly, from (6.17), we have

$$\frac{\partial v}{\partial x}|_{S_T^0}\leqq\frac{2\mu v_0}{|\rho|_T^{(0)}|v_0|^{(0)}} . \tag{6.18}$$

In order to obtain the estimate for $\frac{\partial v}{\partial x}|_{S_T^0}$ from below, it is sufficient to apply the above one to the solution $-v(x, t)$ of the equation:

$$(-v)_t-\frac{\mu}{\rho}(-v)_{xx}-vv_x=0 .$$

As a result, we have

$$\left|\frac{\partial v}{\partial x}|_{S_T^0}\right|\leqq\frac{2\mu v_0}{|\rho|_T^{(0)}|v_0|^{(0)}} . \tag{6.19}$$

As for $\frac{\partial v}{\partial x}|_{S_T^X}$, consider the function $\tilde{v}(x, t)=v(X-x, t)$, and repeat the same argument for $\tilde{v}_x$ on S_T^0 as for v_x on S_T^0. Finally we get

$$\max_{S_T}|v_x|\leqq\frac{2\mu v_0}{|\rho|_T^{(0)}|v_0|^{(0)}} . \tag{6.20}$$

((3-rd step)). By (6.8) and (6.20), we have

$$\begin{aligned}(|v_x|_T^{(0)})^2\leqq&(|v_0'|^{(0)})^2+\lambda_0(|v_0|^{(0)})^2+\\&+\left(\frac{2\mu v_0}{|\rho|_T^{(0)}|v_0|^{(0)}}\right)^2\\\leqq&(|v_0'|^{(0)})^2+(|v_0|^{(0)})^2\frac{|\rho|_T^{(0)}}{2\mu}\left[\frac{1}{2\varepsilon_0}\{(\bar{\rho}_0)^{-2}|\rho_0'|^{(0)}+\right.\end{aligned} \tag{6.21}$$

$$+2|v_0|^{(0)}\}+2|v_x|_T^{(0)}\Big]+\left(\frac{2\mu v_0}{|\rho|_T^{(0)}|v_0|^{(0)}}\right)^2$$

$$\equiv a_0+b_0|v_x|_T^{(0)}.$$

where
$$\begin{cases} a_0=(|v_0'|^{(0)})^2+\dfrac{|\rho|_T^{(0)}}{2\mu}\dfrac{1}{2\varepsilon_0}\{(\bar{\rho}_0)^{-2}|\rho_0'|^{(0)}+2|v_0|^{(0)}\}\times \\ \qquad \times(|v_0|^{(0)})^2+\left(\dfrac{2\mu v_0}{|\rho|_T^{(0)}|v_0|^{(0)}}\right)^2, \\ b_0=\dfrac{|\rho|_T^{(0)}(|v_0|^{(0)})^2}{\mu}. \end{cases}$$

Thus, it holds that

$$|v_x|_T^{(0)}\leqq\frac{b_0+(b_0^2+4a_0)^{1/2}}{2}. \tag{6.22}$$ Q. E. D.

From (5.11), it follows that

$$\left\|\frac{\mu}{\rho}\right\|_T^{(\alpha)}\leqq C_{27}(v_0,\rho_0). \tag{6.23}$$

§7. An *a priori* Estimate for $\|v\|_T^{(2+\alpha)}$ and the Main Theorem

By Lemma 3.1 for v instead of v'', we have

Lemma 7.1. $\|v\|_T^{(1+\alpha)}\leqq C_{28}(T;v_0,\rho_0),$

where $C_{28}(T;\cdots)$ increases monotonically as T increases.

Hence, from this, it follows that

$$\|N\|_T^{(\alpha)}\leqq C_{29}(T;v_0,\rho_0),$$

where $C_{29}(T;\cdots)$ has the same property as C_{28}. Thus, we have

Lemma 7.2. $\|v_{xx}\|_T^{(\alpha)}\leqq C_{30}(T;v_0,\rho_0).$

From the discussions made thus far follows:

Lemma 7.3. *Under the initial-boundary conditions* (2.1), (2.2) *and* (5.5), *if there exists a solution* $(v, \rho)\in H_T^{2+\alpha}\times B_T^1$ *of* (1.1), *then* $\|v\|_T^{(2+\alpha)}+[\rho]_T^{(1)}$ *has a priori bounds in* T, *where*

$$[\rho]_T^{(n)}\equiv \sum_{r+s=0}^{n} |D_t^r D_x^s \rho|_T^{(0)} \quad (r \textit{ and } s, \textit{ integers} \geqq 0).$$

Proof. We have only to note that

$$\|v\|_T^{(2+\alpha)} \leqq \|v\|_T^{(1+\alpha)} + \left(1+\left\|\frac{\mu}{\rho}\right\|_T^{(\alpha)}\right)\|v_{xx}\|_T^{(\alpha)} + $$

$$+\|vv_x\|_T^{(\alpha)}.$$

The results obtained since §5 guarantee that each term of the right-hand side of the above inequality has *a priori* bounds in T.

Q.E.D.

Combining Theorems 3.1, 4.1 and Lemma 7.3, we have the following main theorem on the existence of a temporally global solution of (1.1), (2.1), (2.2) and (5.5).

Theorem 7.1. *Under the initial-boundary conditions* (2.1), (2.2) *and* (5.5), *there uniquely exists a regular temporally global solution and it holds that*

$$(7.1)\quad \begin{cases} |v(x, t)| \leqq |v_0|^{(0)}, \\ 0 \leqq \bar{\rho}_0 \exp\left\{-\frac{1}{\mu}|\rho_0 v_0|^{(0)} X\right\} \leqq \rho(x, t) \leqq \\ \qquad \leqq \bar{\bar{\rho}}_0 \exp\left\{\frac{1}{\mu}|\rho_0 v_0|^{(0)} X\right\}, \\ |v_x(x, t)| \leqq K\left(\|v_0\|^{(1)}, \|\rho_0\|^{(1)}, \frac{1}{\bar{\rho}_0}\right) < +\infty, \end{cases}$$

where K *increases as each argument increases.*

Remark. (i) The word "regular" means, exactly speaking, regular up to the boundary.
(ii) If there exists a regular solution (v, ρ) defined in $[0, X]\times[0, \infty)$, then $(v, \rho)\in H_T^{2+\alpha}\times B_T^{1+\alpha}$ for an arbitrary $T\in(0, \infty)$.

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