Publ. RIMS, Kyoto Univ.
10 (1974), 293-328

On Resolutions of Cyclic Quotient Singularities

By

Akira FUJIKI

Introduction

Let G be a finite cyclic group with a fixed generator g, acting on a complex affine m-space $\boldsymbol{C}^m = \boldsymbol{C}^m(z_1,\dots, z_m)$ by the formula;

$$(1) \qquad g: (z_1,\dots, z_m) \longrightarrow (e_n^{p_1} z_1,\dots, e_n^{p_m} z_m),$$

where $n, p_i, 1 \leqq i \leqq m$, are integers satisfying $0 \leqq p_i < n$ and $e_n^{p_i} = \exp(2\pi\sqrt{-1}p_i/n)$. Then the quotient space $X = \boldsymbol{C}^m/G$ has the natural structure of a normal affine algebraic variety such that the quotient map $\pi: \boldsymbol{C}^m \to X$ is a morphism of algebraic varieties [10] [13]. We call this X a cyclic quotient singularity and denote it often by $N_{n,p_1,\dots,p_m}$ according to the particular expression of the generator g as above.

Now the main purpose of this paper is to show the existence of certain natural ways of resolution of these cyclic quotient singularities, which have some good properties. (For the precise statement, see Theorem 1.) Such resolutions were first constructed by Hirzebruch in [3] when $m=2$ and then, by Ueno [12], when $m=3$, $p_1=1$ and $p_2=p_3$ in (1). On the other hand, the author recently learned that Mumford has found the equivariant resolutions of toroidal singularities, which contain cyclic quotient singularities as special cases [6]. However, our method is different from his and is connected more closely with the above expression of g. So it may be of some interest to compare the resolutions obtained here with those in [6].

In §1 we prove Theorem 1 and then, in §2 we apply this theorem to obtain resolutions of the general isolated quotient singularities and the isolated singularity with $\boldsymbol{C}^*$ action in the case where the dimension

Received January 31, 1974.

of the spaces is 3. This is done by methods similar to those used in [1] for the former and in [8] for the latter respectively in the 2-dimensional cases. The unpleasant restriction on the dimension comes from the fact that we cannot prove the statement corresponding to Lemma 7 in higher dimensional cases.

In this paper the variety means an irreducible algebraic variety defined over $\boldsymbol{C}$ in the sense of FAC. Further we adopt the following notational conventions; for positive integers $n_1,\dots, n_l$, $(n_1,\dots, n_l)$ denotes the greatest common divisor of them. If T is an automorphism of $\boldsymbol{C}^m$ defined by $T(z_1,\dots, z_m)=(\tau_1 z_1,\dots, \tau_m z_m)$ with $\tau_i \in \boldsymbol{C}^*$, then we abbreviate it to $T=(\tau_1,\dots, \tau_m)$. Moreover for $0 \leqq p < n$, $e_n^p = \exp(2\pi\sqrt{-1}\, p/n)$. $N_{n,p_1,\dots,p_m} = \boldsymbol{C}^m/\{g\}$ if $g=(e_n^{p_1},\dots, e_n^{p_m})$ in the above notation. We also use the notation $N_{n,p,q}$ to express $N_{n,1,p,q}$ in 1.4.2.

In concluding the introduction, the author wishes to express his hearty thanks to Prof. S. Nakano and Dr. K. Ueno for their kind encouragement during the preparation of this paper.

§1. Resolutions of Cyclic Quotient Singularities

1.1. Let the notations be the same as in the introduction, except that in the sequel we assume $(n, p_1,\dots, p_m)=1$ in the expression (1) of g. In this section we study the singular locus S of X. For this purpose, let F be the fixed point set of G in $\boldsymbol{C}^m$, namely, the set of those points whose stabilizers are nontrivial with respect to the action of G. This F, with reduced structure, is in general a subvariety of $\boldsymbol{C}^m$. But to describe F more closely, we put in general, for any nonempty subset $M=\{m_1,\dots, m_k\} \subseteq \{1,\dots, m\}$, $H(M)=\{(z_1)\in \boldsymbol{C}^m;\ z_{m_1}=z_{m_2}=\cdots=0\}$ and call it a coordinate subspace of $\boldsymbol{C}^m$ defined by M. Then we have

Lemma 1. *Let $F(i)$ be the fixed point set of g^i for $1 \leqq i < n$. Then each $F(i)$ coincides with some coordinate subspace $H(M)$ of $\boldsymbol{C}^m$. Conversely, a coordinate subspace $H(M)$ of $\boldsymbol{C}^m$ coincides with $F(i)$ for some i if and only if the following condition is satisfied; if we put $d=d(M)=(n, p_{j_1},\dots, p_{j_{m-k}})$, then $d \nmid (n, p_{m_t})$ for any t, where $M=\{m_1,\dots, m_k\}$ and $\{j_1,\dots, j_{m-k}\}$ is the complementary set of M in $\{1,\dots, m\}$.*

Proof. Suppose $g^i=(e_n^{p_1 i}, .., e_n^{p_m i})$ has $t=t(i)$ 1's as its component, say, on the $i_1,\dots, i_t$-th places. Then a point $z=(z_1, .., z_m)$ is fixed by g^i if and only if $z_j=0$ for any $j\neq i_\alpha$, $1\leqq\alpha\leqq t$. Thus if we take M to be the complementary set of $\{i_1,\dots, i_t\}$, then $F(i)=H(M)$ and the first assertion is proved. Next, for the given M, let $n_1=n/d(M)$. Then n_1 is easily seen to be the least integer μ for which g^μ has 1 on the i-th component for every $i\notin M$. Thus we have only to show that $H(M)=F(n_1)$ if and only if the above condition holds. But $H(M)=F(n_1)$ is equivalent to the fact that for any $j\in M$, the j-th component of g^{n_1} is different from 1, or equivalently, $(e_n^{p_j})^{n_1}=e_{n'_j}^{p'_j n_1}\neq 1$, where $n'_j=n/(n, p_j)$ and $p'_j=p_j/(n, p_j)$. Then, since $(n'_j, p'_j)=1$, this in turn is equivalent to the fact that $n'_j\nmid n_1$, namely, $(n/d)/(n/(n, p_j))=(n, p_j)/d$ is not an integer for $j\in M$. This proves the second assertion. Q.E.D.

In particular, if $(n, p_t)=1$ for some t, hyperplane $z_t=0$ contains every $F(i)$.

Now let $G(i)$ be the stabilizer of the coordinate hyperplane $H\{i\}$, $i=1,\dots, m$, $g(i)$ a generator of $G(i)$ and G_0 be the subgroup of G generated by $\{G(i)\}_{i=1,\dots,m}$. Each $g(i)$ has the form $(1,\dots, e_{n_i}, . , 1)$ with e_{n_i} on the i-th component and $n_i|n$. From this, we see that $\boldsymbol{C}^m(z)/G_0\cong\boldsymbol{C}^m(w)$ and in fact the quotient map $h_0: \boldsymbol{C}^m(z)\to\boldsymbol{C}^m(w)$ is defined by $(w)=h_0(z)=(z_1^{n_1},\dots, z_m^{n_m})$. Further the quotient group $\bar{G}=G/G_0$ acts naturally on $\boldsymbol{C}^m(w)$ by $\bar{g}=(e_n^{p_1 n_1},\dots, e_n^{p_m n_m})$ so that $X=\boldsymbol{C}^m(w)/\bar{G}$, where $\bar{g}$ is the natural image of g in $\bar{G}$. Then $\bar{G}$ has no element whose fixed point set is a coordinate hyperplane of $\boldsymbol{C}^m(w)$. For otherwise, there exists an element $\bar{g}^k\in\bar{G}$, the fixed point set of which is, say, $H\{1\}$ of $\boldsymbol{C}^m(w)$. Then this has the form $\bar{g}^k=(e_t, 1, . , 1)$ and we infer from this that g^k has the form $(e_n^{p_1 k}, e_{n_2}^{t_2},\dots, e_{n_m}^{t_m})$ and hence $g^k g(2)^{-t_2}\dots g(m)^{-t_m}=(e_n^{p_1 k}, 1,\dots, 1)\in G(1)$, $g^k\in G_0$. This implies $\bar{g}^k=e$, the unit element of G, which is a contradiction. Now we say that a cyclic group acting on $\boldsymbol{C}^m$ as above is *small*, if it has no element whose fixed point set coincides with some coordinate hyperplane (c.f. [9]). Thus we have shown that in the expression $X=\boldsymbol{C}^m/G$ of the cyclic quotient singularities, we may always take G to be small.

The following lemma is a special case of the fundamental result of Prill [9, Theorem 2].

Lemma 2. *Two cyclic quotient singularities* $N_{n,p_1,\dots p_m}=\boldsymbol{C}^m/G$ *and* $N_{n,p'_1,\dots,p'_m}=\boldsymbol{C}^m/G'$, *with* G *and* G' *small, are isomorphic if and only if* $n=n'$, *and there exist a permutation* τ *of* $\{1,\dots,m\}$ *and an integer* k, $0<k<m$, *such that* $(k, n)=1$ *and* $p_ik\equiv p'_{\pi(i)}$ *mod* n.

In fact, Prill has proved more precisely that $\boldsymbol{C}^m/G$ and $\boldsymbol{C}^m/G'$ are analytically isomorphic as germs of analytic spaces at $\pi(0)$ and $\pi'(0)$ respectively, if and only if G and G' are conjugate in $GL(m, \boldsymbol{C})$, where π and π' are quotient maps.

From this we get the following.

Corollary. *If* G *is small, then the singular locus* S *of* X *coincides with* $\pi(F)$, *where* F *is the fixed point set of* G.

Proof. It is clear that $S\subseteq\pi(F)$. So let $\mathfrak{P}\in\pi(F)$ and take a point $\mathfrak{Q}\in F$ such that $\pi(\mathfrak{Q})=\mathfrak{P}$. Then there exists a neighborhood U of $\mathfrak{Q}$ in $\boldsymbol{C}^m$ such that if $G_{\mathfrak{Q}}$ is the stabilizer of $\mathfrak{Q}$, then U is $G_{\mathfrak{Q}}$ invariant and the quotient $U/G_{\mathfrak{Q}}$ is analytically isomorphic to some neighborhood V of $\mathfrak{P}$. On the other hand, as we have already seen, the fixed point set of $g_{\mathfrak{Q}}$, a generator of $G_{\mathfrak{Q}}$, is some coordinate subspace $H(M)$ of $\boldsymbol{C}^m$. After a suitable renumbering of coordinates if necessary, we may assume that $M=\{1,\dots, k\}$, where $\dim H(M)=m-k$ and that $g_{\mathfrak{Q}}$ acts on $\boldsymbol{C}^k$ by $g_{\mathfrak{Q}}=(e_{n_0}^{s_1},\dots, e_{n_0}^{s_k})$, where s_i are some integers. Let $\mathfrak{Q}'=(0, z')\in \boldsymbol{C}^k/G_{\mathfrak{Q}}\times\boldsymbol{C}^{m-k}$ be the image of $\mathfrak{Q}$ by the quotient map $\boldsymbol{C}^m\to\boldsymbol{C}^k/G_{\mathfrak{Q}}\times \boldsymbol{C}^{m-k}$ composed with the above isomorphism. Then $(U/G_{\mathfrak{Q}}, \pi(\mathfrak{Q}))$ and $(\boldsymbol{C}^k/G_{\mathfrak{Q}}\times\boldsymbol{C}^{m-k},, \mathfrak{Q}')$ must be analytically isomorphic as germs of analytic spaces. Then if V is nonsingular, we conclude that $\boldsymbol{C}^k/G_{\mathfrak{Q}}\times\boldsymbol{C}^{m-k}$ and hence $\boldsymbol{C}^k/G_{\mathfrak{Q}}$ is nonsingular. But since it is easy to see that $G_{\mathfrak{Q}}$, considered naturally as a subgroup of $GL(k, C)$, is small, we have $G_{\mathfrak{Q}}=\{e\}$ by the lemma. This contradicts the assumption that $\mathfrak{Q}\in F$.

Q.E.D.

Remark 1. By Lemma 1 we have in particular

a) G is small if and only if $(n, p_1,\dots, \hat{p}_i,\dots, p_m)=1$ for every i, where $\hat{p}_i$ means that p_i is omitted, and

b) if G is small, then $\dim S=0$ if and only if $(n, p_i)=1$ for every i.

In fact, a) and the sufficiency of the condition in b) are immediate. So suppose $\dim S=0$, and $(n, p_i)>1$ for some i. Let $\{j_1,\dots, j_k\}$ be the subset of $\{1, \dots, m\}$ consisting of those elements for which (n, p_{j_α}) is divisible by (n, p_i) for every α, $1\leqq\alpha\leqq k$, and let M be the complementary set of $\{j_1,\dots, j_k\}$ in $\{1,\dots, m\}$. Then by Lemma 1, $H(M)$ coincides with $F(i)$ for some i. But since $F(i)=\{0\}$ by the assumption, M must coincide with $\{1,\dots, m\}$. This is a contradiction because $i\notin M$.

1.2. The next procedure is the key step in the course of our resolution of X. Namely, we propose to show the following

Lemma 3. *Suppose $X=N_{n,p_1,\dots,p_m}$ is a cyclic quotient singularity and assume that the group G is small. Then there exist a variety X_1, a finite affine open covering $\mathfrak{U}=\{U_1,\dots, U_l\}$ of X_1 for an integer l, $1\leqq l\leqq m$, and a proper birational morphism $f: X_1\to X$ such that for each i there are isomorphisms $\varphi_i: U_i\cong N_{p'_i,q^i_1,\dots,q^i_m}$, where the integers p'_i and q^i_α are determined by the following formula:*

$$(2)\qquad \begin{cases} p'_i=p_i/d, \quad \text{with} \quad d=(p_1, \dots, p_m), \\ q^i_\alpha\equiv p'_\alpha \quad \mod p'_i \ \text{ if } \alpha\neq i, \text{ and } 0\leqq q^i_\alpha<p'_i, \\ q^i_i+n\equiv 0 \quad \mod p'_i. \end{cases}$$

Proof. The construction we have in mind when $p_i>0$ for every i is roughly as follows; first, we take an abelian covering $h: \boldsymbol{C}^m(t)\to \boldsymbol{C}^m(z)$ with the covering transformation group H and then, define an action of G on $\boldsymbol{C}^m(t)$ compatible with h so that $X=\boldsymbol{C}^m(t)/G\oplus H$. Next we perform a monoidal transformation $\sigma: W_0\to \boldsymbol{C}^m(t)$ at the origin and observe that the actions of G and H extend naturally onto W_0. Finally, we put $X_1=W_0/G\oplus H$ and define f to be the morphism induced by σ. Then we see that this X_1 and f have the desired properties.

Now we shall see these more closely. First we consider the case where $p_i>0$ for each i. Then the abelian covering $h: \boldsymbol{C}^m(t)\to\boldsymbol{C}^m(z)$ is defined by

$$h(t_1,\dots, t_m)=(t_1^{p_1},\dots, t_m^{p_m}).$$

The covering transformation group H of h is isomorphic to the direct sum $G_1 \oplus \cdots \oplus G_m$, where G_i is the cyclic group of order p_i, a generator g_i of which acts on $\boldsymbol{C}^m(t)$ by $g_i=(1,\dots, e_{p_i},\dots, 1)$ with e_{p_i} on the i-th place. Moreover, we define the action of g on $\boldsymbol{C}^m(t)$ by

$$g=(e_n,\dots, e_n).$$

Then it is easy to see that this is compatible with h, namely, the equality $g\cdot h=h\cdot g$ holds. Now let $\sigma\colon W_0 \to \boldsymbol{C}^m(t)$ be the monoidal transformation at the origin. Then W_0 has the natural structure of a line bundle over a projective $(m-1)$-space $\boldsymbol{P}^{m-1}$ with homogeneous coordinates $(\xi_1:\cdots:\xi_m)$. Indeed, if we define $V_i=\{(\xi)\in \boldsymbol{P}^{m-1};\ \xi_i\neq 0\}$, then W_0 is expressed as $W_0=\bigcup_{i-1}^{m}(V_i\times \boldsymbol{C})$, where $(p, \zeta_i)\in V_i\times \boldsymbol{C}$ and $(q, \zeta_j)\in V_j\times \boldsymbol{C}$ are identified if and only if $p=q$ and $\zeta_i=(\xi_i/\xi_j)\zeta_j$. Then the map σ has the following form on each $W_i^0=V_i\times \boldsymbol{C}$;

$$(3) \qquad \begin{cases} t_j=(\xi_j/\xi_i)\zeta_i & \text{if } \ j\neq i \\ t_i=\zeta_i . \end{cases}$$

Note that W_i^0 is isomorphic to $\boldsymbol{C}^m$ with the coordinates $(\xi_1/\xi_i,\dots, \xi_m/\xi_i, \zeta_i)$. By (3) we see that the actions of g and g_i, $i=1,\dots, m$, extend onto W_0 as follows; on W_i^0,

$$\begin{cases} g_j=(1,\dots, 1, e_{p_j}, 1,\dots, 1) \text{ with } e_{p_j} \text{ on the } j\text{-th place, if } j\neq i \\ g=(1,\dots, 1, e_n) \\ g_i=(e_{p_i}^{-1},\dots, e_{p_i}^{-1}, e_{p_i}) . \end{cases}$$

Then, as in 1.1 if we set $W_i^1=W_i^0/\check{G}_i$ with $\check{G}_i=G\oplus G_1\oplus \cdots\cdots \oplus \hat{G}_i \oplus \cdots\cdots \oplus G_m$, then $W_i^1\cong \boldsymbol{C}^m(w_1^{i1},\dots, w_m^{i1})$ and if we identify W_i^1 with $\boldsymbol{C}^m(w^{i1})$ by this isomorphism, then the quotient map is defined by

$$\rho_i\colon (\xi_1/\xi_i,\dots, \xi_m/\xi_i, \zeta_i) \longrightarrow ((\xi_1/\xi_i)^{p_1},\dots, \zeta_i^n,\dots, (\xi_m/\xi_i)^{p_m})$$

with ζ_i^n on the i-th place. Moreover the action of G_i induced on W_i^1 takes the form;

$$g_i=(e_{p_i}^{-p_1},\dots, e_{p_i}^n,\dots, e_{p_i}^{-p_m}) .$$

However, G_i may not be small with respect to the action on W_i^1. In fact, by Remark 1a), G_i is not small if and only if $d=(p_1,\dots,p_m)>1$, since G is small. In any case we put $W_i=W_i^1/\{g_i^d\}$. Then again as in 1.1, $W_i\cong \boldsymbol{C}^m(w_1^i,\dots,w_m^i)$, the quotient map $\bar{\rho}_i\colon W_i^1\to W_i$ is defined by

$$\bar{\rho}_i(w_1^{i1},\dots,w_m^{i1})=(w_1^{im},\dots,(w_i^{i1})^d,\dots,w_m^{i1}),$$

and the induced action of $\bar{G}_i=G_i/\{g_i^d\}$ has the form

$$\bar{g}_i=(e^{-p_i'p_1'},\dots,e^{n'}_{p_i'},\dots,e^{-p_m'}_{p_i'}),$$

where $p_i'=p_i/d$, $\bar{g}_i$ is the natural image of g_i in $\bar{G}_i$, and $\bar{G}_i$ is small. Now we set $X_1=W_0/G\oplus H$ and $U_i=\pi(W_i^0)$, $1\leqq i\leqq m$, where $\pi\colon W_0\to X_1$ is the quotient map. Then by the above description we have $U_i=W_i/\bar{G}_i$, and hence $\mathfrak{U}=\{U_1,\dots,U_m\}$ makes a finite affine open covering of X_1 such that each member U_i is isomorphic to a cyclic quotient singularity $N_{p_i',q_1^i,\dots,q_m^i}$ of order p_i'. Here the integers q_α^i are defined by (2), taking $\bar{g}_i^{-1}$ as a generator of $\bar{G}_i$. Finally, if we recall that $X=\boldsymbol{C}^m(z)/G\oplus H$, and that the action of G and H on W_0 and X commute with σ, then we see that σ induces a birational morphism $f\colon X_1\to X$.

Next, we consider the case when $p_i=0$ for some i. After a suitable permutation of w_i, we may assume that $p_{l+1}=\cdots=p_m=0$ and $p_i>0$ for $i\leqq l$ for some $l>0$. Then $g=(e_n^{p_1},\dots,e_n^{p_l},1,\dots,1)$ and X is naturally isomorphic to $(\boldsymbol{C}^l/G)\times\boldsymbol{C}^{m-l}$, where the action of G on $\boldsymbol{C}^l$ is defined by $g=(e_n^{p_1},\dots,e_n^{p_l})$. Put $Y=\boldsymbol{C}^l/G$. Then we may apply the above considerations to Y instead of to X. Suppose Y_1, $\mathfrak{V}=\{V_1,\dots,V_l\}$, and $g\colon Y_1\to Y$ correspond in the above consideration to X_1, $\mathfrak{U}$, and $f\colon X_1\to X$ respectively. Then we put $X_1=Y\times\boldsymbol{C}^{m-l}$, $\mathfrak{U}=\{U_i;\ U_i=V_i\times\boldsymbol{C},\ 1\leqq i\leqq l\}$, and $f=g\times id\colon X_1\to X$, where id is the identity map of $\boldsymbol{C}^{m-l}$. Correspondingly, we get the groups $\bar{G}_i$ and the isomorphisms $\varphi_i\colon U_i\simeq N_{p_i',q_1,\dots,q_m}$ with $q_i=0$ for $i\geqq l+1$. The relations in (2) are obvious, and hence the proof of the lemma is completed.

Now we summarize in the following lemma the properties of the covering $\mathfrak{U}$, and the morphism f, thus obtained.

Lemma 4. i) *If we denote by $\pi_i\colon W_i\to U_i$, $1\leqq i\leqq l$, the quotient maps, then $W_{ij}=\pi_i^{-1}(U_i\cap U_j)$ is an open subset of W_i defined by $w_j^i\neq 0$.*

ii) *The multivalued map $\pi_{ij}=\pi_i^{-1}\pi_j\colon \pi_j^{-1}(U_i\cap U_j)\to\pi_i^{-1}(U_i\cap U_j)$ is de-*

fined by

$$\begin{cases} w^i_k = w^i_k(w^i_j)^{-p'_k/p'_i} & \textit{if } k \neq i, \neq j\,, \\ w^i_i = w^j_j(w^j_i)^{n/p'_i}\,, & \\ w^i_j = (w^j_i)^{-p'_j/p'_i}\,, & \end{cases}$$

where $(w^j_i)^{i/p'_i}$ *denotes any* p'_i*-th root of* w^j_i.

iii) *The multivalued map* $\pi_{0i} = \pi^{-1}\pi_i\colon W_i \to \boldsymbol{C}^m(z)$ *is defined by*

$$\begin{cases} z_k = w^i_k(w^i_i)^{p'_k/n} & \textit{if } k \neq i \\ z_i = (w^i_i)^{p'_i/n}\,. & \end{cases}$$

iv) *Let* T *be the automorphism of* X *induced by an automorphism* T' *of* $\boldsymbol{C}^m(z)$ *of the form* $T' = (\tau_1, \dots, \tau_m)$, *then* T *extends uniquely onto* X *so that it leaves each* U_i *invariant and* $T|U_i$ *is induced by the automorphism* T_i *of* $W_i = \boldsymbol{C}^m(w^i)$ *of the form*

$$T_i = (\tau_1\tau_i^{-p_1/p_i}, \dots, \tau_i^{n/p_1}, \dots, \tau_m\tau_i^{-p_m/p_i})\,.$$

v) *Let* F_0 *be the fixed point set of* g, $S_0 = \pi(F_0)$, *and* $S_1 = f^{-1}(S_0)$, *then* $\pi_i^{-1}(U_i \cap S_1)$ *is defined by* $w^i_i = 0$ *in* W_i.

The proofs are all straightforward and we omit them. We only note that the construction of X_1 and f_1 depends crucially on the choice of a generator g of G as in (1), or equivalently, on the choice of an isomorphism $\varphi\colon X \simeq N_{n, p_1, \dots, p_m}$ (see an example in 1.4).

1.3. For resolutions of the cyclic quotient singularities, we have to deal with a little more general situation. So let X be a variety and suppose there exists a finite affine open covering $\mathfrak{U} = \{U_1, \dots, U_s\}$ of X such that there exists for each i an isomorphism $\varphi_i\colon U_i \cong N_{n_i p_1^{(i)} \dots p_m^{(i)}}$, where each $N_{n_i p_1^{(i)} \dots p_m^{(i)}} = \boldsymbol{C}^m(u^i_1, \dots, u^i_m)/G_i$ is a cyclic quotient singularity defined in the introduction. In this case we call n_i the order of U_i. We denote such X and $\mathfrak{U}$ simply by the pair $(X, \mathfrak{U})$. Hence, when we speak of a pair $(X, \mathfrak{U})$, it is supposed that we are given a variety X and a finite affine open covering $\mathfrak{U}$ of X such that to each U_i there are assoicated an m-dimensional affine space C^m with the coordinate system $(u_1^i,\dots, u_m^i)$, a cyclic group G_i with a fixed generator g_i, and an action of G_i of the form $g_i=(e_{ni}^{p_1^{(i)}},\dots, e_{n_i}^{p_m^{(i)}})$ on C^m together with an isomorphism $\varphi_i\colon U_i\simeq C^m/G_i$. Now let $(X, \mathfrak{U})$ be a pair. We say that $U_i\in\mathfrak{U}$ is *adjacent* to $U_j\in\mathfrak{U}$ if $\pi_i^{-1}(U_i\cap U_j)$ is defined by the equation $u_k^i\neq 0$ for some $k=k(i,j)$. Note that in this case U_j is also adjacent to U_i.

Definition 1. A pair $(X, \mathfrak{U})$ as above is said to be *admissible* if the following three conditions are satisfied:

(α) Any two members U_i and U_j of $\mathfrak{U}$ can be connected by a finite sequence of adjacent ones, namely, there exists a finite sequence $U_1,\dots, U_d$ with $U_\alpha\in\mathfrak{U}$ such that $U_1=U_i$, $U_d=U_j$ and U_{t-1} is adjacent to U_t for $0<t\leqq d$.

(β) Suppose U_i is adjacent to U_j and $\pi_i^{-1}(U_i\cap U_j)$ (resp. $\pi_j^{-1}(U_i\subset U_j)$) is defined by $u_k^i\neq 0$ (resp. $u_{k'}^j\neq 0$) in $C^m(u^i)$ (resp. in $C^m(u^j)$), where $\pi_i\colon C^m\to U_i$ (resp. $\pi_j\colon C^m\to U_j$) is the quotient map $\rho_i\colon C^m\to C^m/G_i$ (resp. $\rho_j\colon C^m\to C^m/G_j$) composed with the isomorphism φ_i^{-1} (resp. φ_j^{-1}). Then the multivalued map $\pi_{ij}=\pi_i^{-1}\pi_j\colon \pi_j^{-1}(U_i\cap U_j)\to\pi_i^{-1}(U_i\cap U_j)$ has the following form;

$$\begin{cases} u_k^i=(u_l^j)^{-n_j/n_i} \\ u_s^i=(u_l^j)^{a(s)}u_{\pi(s)}^j \qquad \text{if } s\neq k\,, \end{cases}$$

where $n_i=\operatorname{ord} U_i$, $n_j=\operatorname{ord} U_j$, $a(s)$, $1\leqq s\leqq m$, $s\neq k$, are certain rational numbers, and finally $\pi\colon\{1,\dots,\hat{k},\dots, m\}\to\{1,\dots,\hat{l},\dots, m\}$ is some bijective map.

(γ) The groups G_i are all small.

Remark 2. a) For any point $P\in C^m(u^i)$ $\pi_i^{-1}(\pi(P))$ consists of at most n_i distinct points. From this, we see that the denominators of $a(s)$ do not exceed n_i in their irreducible expressions.

b) Let U_i and U_j be as in (β) and if π_{ij} has the form stated in (β), then so does π_{ji} as is seen by solving the equation with respect to u^j.

Now let $(X, \mathfrak{U})$ and $(Y, \mathfrak{V})$ be admissible pairs and $f\colon X\to Y$ be a proper birational morphism. We say that f is compatible with the coverings $\mathfrak{U}$ and $\mathfrak{V}$ if for any $V_\alpha \in \mathfrak{V}$, $\mathfrak{U}_\alpha = \{U_i \in \mathfrak{U} \mid U_i \subseteq f^{-1}(V_\alpha)\}$ makes an affine open covering of $f^{-1}(V_\alpha)$. Then the pair $(f^{-1}(V_\alpha), \mathfrak{U}_\alpha)$ with groups G_i, coordinates $\{u^i\}$, and isomorphisms φ_i: $U_i \cong \boldsymbol{C}^m(u^i)/G_i$ induced by those of the original U_i is admissible. We denote this also by $f^{-1}((V_\alpha, \{V_\alpha\}))$.

Definition 2. Suppose $(X, \mathfrak{U})$, $(Y, \mathfrak{V})$, and f are as above. We say that f is *admissible* as a morphism of admissible pairs $(X, \mathfrak{U})$ and $(Y, \mathfrak{V})$ if f satisfies the following two conditions:

(α) f is compatible with the coverings $\mathfrak{U}$ and $\mathfrak{V}$.

(β) Suppose $f(U_i) \subseteq V_\alpha$, then the multivalued map $\pi_\alpha^{-1} f \pi_i$: $\boldsymbol{C}^m(u^i) \to \boldsymbol{C}^m(v^\alpha)$ takes the following form;

$$v_s^\alpha = (u_1^i)^{b(1,s)} \cdots (u_m^i)^{b(m,s)}, \qquad 1 \leqq s \leqq m,$$

where $b(i, s)$ are positive and rational numbers. Moreover an automorphism T of X is said to be *admissible* if $T(U_i) = U_i$ for every $U_i \in \mathfrak{U}$ and $T|U_i$ is induced by an automorphism $\tilde{T}_i$ of $\boldsymbol{C}^m(u^i)$ of the form $\tilde{T}_i = (\tau_1, \dots, \tau_m)$, $\tau_i \in \boldsymbol{C}^*$, where $U_i \cong \boldsymbol{C}^m(u^i)/G_i$.

For example, let $X = \boldsymbol{C}^m/G$ be a cyclic quotient singularity. Then $(X, \{X\})$ can be trivially regarded as an admissible pair. Next let X_1, $\mathfrak{U}$, and f: $X_1 \to X$ be as in Lemma 3. Then, by Lemma 4 $(X_1, \mathfrak{U})$ and f are admissible, and any admissible automorphism of X extends uniquely to that of $(X_1, \mathfrak{U})$.

On the other hand, if X is nonsingular in an admissible pair $(X, \mathfrak{U})$, then since G_i are small, we must have $n_i = 1$ and U_i are isomorphic to $\boldsymbol{C}^m(u^i)$. Then the multivalued maps π_{ij} in (β) are nothing but the trandition functions with respect to this covering. Rewriting these, we have

$$\begin{cases} u_k^i = (u_l^j)^{-1} \\ u_s^i = (u_l^j)^{a(s)} u_{\pi(s)}^i, & s \neq k, \end{cases}$$

where $a(s)$ are now integers.

Now recall that a resolution of a variety X is a pair $(\tilde{X}, f)$ consisting of a variety $\tilde{X}$ and a proper birational morphism f: $\tilde{X} \to X$ such that f is isomorphic outside $f^{-1}(S)$, S being the singular locus of X. Then we are able to state our main theorem.

Theorem 1. *Suppose* $(X, \mathfrak{U})$ *is an admissible pair. Then there exists a resolution* $(\tilde{X}, f)$ *of* X *and a finite affine open covering* $\tilde{\mathfrak{U}}$ *of* $\tilde{X}$ *such that the following conditions are satisfied;*

1) $(\tilde{X}, \tilde{\mathfrak{U}})$ *is admissible,*

2) f *is admissible as a morphism of admissible pairs* $(\tilde{X}, \tilde{\mathfrak{U}})$ *and* $(X, \mathfrak{U})$,

3) *any admissible automorphism* T *of* U_i *extends uniquely to that of* $f^{-1}((U_i, \{U_i\}))$, *and*

4) $f^{-1}(S) \cap \tilde{U}_\alpha$ *is defined by* $\tilde{u}^\alpha_{k_1} \cdots \tilde{u}^\alpha_{k_t} = 0$ *in* $\tilde{U}_\alpha \cong \boldsymbol{C}^m(\tilde{u}^\alpha)$ *for some* $k_1, \ldots, k_t$, *if it is not empty.*

From 4) we derive easily the following.

Corollary. *Let* $E = f^{-1}(S)$. *Then* E *has only normal crossings in* $\tilde{X}$ *and every irreducible component* E_i *of* E *is nonsingular and rational. Further* E_i *is covered by finite affine open subsets, each of which is isomorphic to* $\boldsymbol{C}^{m-1}$.

Proof of Theorem 1. First we consider the set Φ of maps $\varphi: N \to N \cup \{0\}$ such that $\varphi(N) \neq 0$ and $\varphi(n) = 0$ for all but a finite number of n's, where N is the set of natural numbers. We shall introduce an order on the set Φ in the following manner. Let $\varphi_1, \varphi_2 \in \Phi$ and n_0 be the largest integer for which $\varphi_1(n_0) \neq \varphi_2(n_0)$. Then we define $\varphi_1 < \varphi_2$ by the inequality $\varphi_1(n_0) < \varphi_2(n_0)$. By this, Φ becomes a totally ordered set with a minimal element φ_0, which is defined by $\varphi_0(1) = 1$ and $\varphi_0(n) = 0$ for $n \geqq 2$. We associate then to each admissible pair $(X, \mathfrak{U})$ an element of Φ, which we denote simply by φ_X since no confusion may arise, by $\varphi_X(n) = \#\{U_i \in \mathfrak{U}; \operatorname{ord} U_i = n\}$ for $n \in N$, where $\#$ means the number of elements of the corresponding set. Then by the above remark we may try to prove the theorem by induction on φ_X. If $\varphi_X = \varphi_0$, or more generally, if $\varphi_X(n) = 0$ for $n \geqq 2$, then it is sufficient to define $(\tilde{X}, \tilde{\mathfrak{U}}) = (X, \mathfrak{U})$ and $f = id_X$, the identity map of X, because then X is non-singular. Thus we may assume that $\varphi_X(n) \geqq 1$ for some $n \geqq 2$ and that the theorem has already been proved for admissible pairs $(Y, \mathfrak{B})$ for which $\varphi_Y < \varphi_X$.

We define an integer $l(i)$ for each U_i by $l(i) = \#\{p^{(i)}_k;\ p^{(i)}_k \neq 0,\ 1 \leqq$

$k \leqq m\}$, where U_i is isomorphic to $N_{n_i, p_1^{(i)}, \dots, p_m^{(i)}}$ by the isomorphism φ_i. Clearly this is equal to the codimension of the fixed point set of G_i, and hence does not depend on the particular choice of the generator. Set $l = \max_i l(i)$ and take and fix one $U_{i_0} \in U$ for which $l = l(i_0)$. Next, let F_0 be the fixed point set of g_{i0}, $S_{i0} = \pi_{i0}(F_0)$ and S_0 the closure of S_{i0} in X, where $\pi_{i0}\colon \boldsymbol{C}^m \to U_{i0}$ is the quotient map. Then our purpose is to construct an admissible pair $(X_1, \mathfrak{U}_1)$ and a birational morphism $f_1\colon X_1 \to X$, such that f_1 is admissible for $(X_1, \mathfrak{U}_1)$ and $(X, \mathfrak{U})$, that f_1 is isomorphic outside $f^{-1}(S_0)$, and that $\varphi_{X_1} < \varphi_X$. For this, first we construct for each U_i an admissible pair $(W_i, \mathfrak{V}_i)$ and a birational morphism $\psi_i\colon W_i \to U_i$ admissible for $(W_i, \mathfrak{V}_i)$ and $(U_i, \{U_i\})$, such that ψ_i is isomorphic outside $\psi_i^{-1}(S_0 \cap U_i)$. And then we show that these W_i and f_i are patched together and form the desired variety X_1 and the morphism f_1.

First, put $W_i = U_i$, $V_i = \{U_i\}$, and $\psi_i = id_{\mathfrak{U}_i}$ if $U_i \cap S_0 = \phi$.

Suppose then $U_i \in \mathfrak{U}$ is such that $U_i \cap S_0 \neq \phi$, then, since $F_i = \pi_i^{-1}(S_0)$ is coordinate subspace in $\boldsymbol{C}^m(u^i)$ of codimension l, we may assume that F_i is defined by the equations $u_1^i = \cdots = u_l^i = 0$ in $\boldsymbol{C}^m(u^i)$. Then g_i must be of the form $g_i = (e_{n_i}^{p_1^{(i)}}, \dots, e_{n_i}^{p_m^{(i)}})$ with $p_k^{(i)} \neq 0$ for $1 \leqq k \leqq l$, but then by the maximality of l, we have $p_k^{(i)} = 0$ for $k > l$, namely, $g_i = (e_{n_i}^{p_1^{(i)}}, \dots, e_{n_i}^{p_l^{(i)}}, 1, \dots, 1)$.

Now suppose further that U_j is adjacent to U_i and $U_j \cap S \neq \phi$. Then if $\pi_i^{-1}(U_i \cap U_j)$ is defined by $u_k^i \neq 0$ in $\boldsymbol{C}^m(u^i)$, then we must have $k > l$, for otherwise F_i would be contained in the hyperplane $u_k^i = 0$ and hence $S_0 \cap U_i \cap U_j = \phi$, which contradicts the assumption. Thus we may assume that $k = m$ renumbering the coordinate if necessary. Considering analogously with U_j, we may also assume that $F_j = \pi_j^{-1}(S_0)$ (resp. $\pi_j^{-1}(U_i \cap U_j)$) is defined by $u_1^j = \cdots = u_l^j = 0$ (resp. $u_m^j \neq 0$) in $\boldsymbol{C}_m(u^j)$ and $g_j = (e_{n_j}^{p_1^{(j)}}, \dots, e_{n_j}^{p_l^{(j)}}, 1, \dots, 1)$ on $\boldsymbol{C}^m(u^j)$. Then we get the multivalued map π_{ij} in he following form;

$$u_k^i = u_k^j (u_m^j)^{a_k} \qquad \text{if } k \neq i$$

$$u_m^i = (u_m^j)^{-n_j/n_i}.$$

Assertion 1. i) *For the given generator* g_i *of* G_i, *we can take a generator* g_j *of* G_j *so that* $n_i=n_j$ *and* $p_\alpha^{(i)}=p_\alpha^{(j)}$ *for all* α.

ii) *The multivalued map* π_{ij} *above has in fact the following form*;

$$(4)\qquad \begin{cases} u_k^i=u_k^j(u_m^j)^{a_k} & \text{if } k\neq i \text{ and } k\leqq l \\ u_k^i=u_k^j(u_m^j)^{a_k}, & \textit{with } a_k \textit{ integers} \quad \textit{if } k>l \\ u_m^i=(u_m^j)^{-1}. & \end{cases}$$

Proof. Let G_i act on $\boldsymbol{C}^l(u_1^i,\dots,u_l^i)$ by $g=(e_{n_i}^{p_1^{(i)}},\dots,e_{n_i}^{p_l^{(i)}})$. Then corresponding to the decomposition $\boldsymbol{C}^m(u^i)=\boldsymbol{C}^l(u_1^i,\dots,u_l^i)\times\boldsymbol{C}^{m-l}(u_{l+1}^i,\dots,u_m^i)$, we have an isomorphism $\boldsymbol{C}^m/G_i\cong\boldsymbol{C}^l/G_i\times\boldsymbol{C}^{m-l}$ such that the quotient map $\rho_i\colon\boldsymbol{C}^m\to\boldsymbol{C}^m/G_i$ corresponds to the product of $p_i'\colon\boldsymbol{C}^l\to\boldsymbol{C}^l/G_i$ and the identity map of $\boldsymbol{C}^{m-l}$. Identify $\boldsymbol{C}^m/G_i$ with $\boldsymbol{C}^l/G_i\times\boldsymbol{C}^{m-l}$ by this isomorphism. Then we have the commutative diagram

$$\begin{array}{ccc} \boldsymbol{C}^l\times\boldsymbol{C}^{m-l} & & \\ \Big\downarrow{\scriptstyle\rho_i} & \searrow{\scriptstyle\pi_i} & \\ \boldsymbol{C}^l/G_i\times\boldsymbol{C}^{m-l} & \xleftarrow[\varphi_i]{\sim} & U. \end{array}$$

Proceeding analogously with U_j, we have the similar diagram for j. Thus we get $\rho_i^{-1}\cdot\varphi_i^{-1}\cdot\varphi_j\cdot\rho_j=\pi_i^{-1}\cdot\pi_j$ on $\pi_j^{-1}(U_i\cap U_j)$. From this, we obtain the following commutative diagram

$$\begin{array}{ccc} \boldsymbol{C}^l/G_i\times(\boldsymbol{C}^{m-l}-\{u_m^i=0\}) & \xrightarrow{\varphi_{ij}} & \boldsymbol{C}^l/G_j\times(\boldsymbol{C}^{m-l}-\{u_m^i=0\}) \\ \Big\downarrow{\scriptstyle\tilde{\omega}_i} & & \Big\downarrow{\scriptstyle\tilde{\omega}_j} \\ \boldsymbol{C}^{m-l}-\{u_m^i=0\} & \xleftarrow{\quad\bar{\pi}_{ij}\quad} & \boldsymbol{C}^{m-l}-\{u_m^j=0\}, \end{array}$$

where $\varphi_{ij}=\varphi_i\varphi_j^{-1}|\varphi_{j(U_i\cap U_j)}^{-1}$, $\tilde{\omega}_i$ and $\tilde{\omega}_j$ are the projections to the second factors and $\bar{\pi}_{ij}$ is the multivalued map defined by $u_k^i=u_k^j(u_m^j)^{a_k}$, $l+1\leqq k\leqq m-1$, and $u_m^i=(u_n^j)^{-n_j/n_i}$. But since φ_{ij} is an isomorphism, so must be $\bar{\pi}_{ij}$. In particular it is single-valued. Hence, we conclude that $n_i=n_j$ and a_k, $l+1\leqq k\leqq m-1$, are all integers. This proves ii). On the other hand, the linear isomorphism $\tilde{\varphi}_{ij}(P)\colon\boldsymbol{C}^l\to\boldsymbol{C}^l$ defined by $u_k^i=u_k^j(\bar{u}_n^j)^{a_k}$, $l\geqq k\geqq 1$, for each fixed point $P=(\bar{u}_{l+1}^j,\dots,\bar{u}_m^j)\in\boldsymbol{C}^{m-l}-\{u_m^j=0\}$, induces the isomorphism $\varphi_{ij}|\tilde{\omega}_j^{-1}(P)\colon\boldsymbol{C}^l/G_j=\boldsymbol{C}^l/G_i$. Hence, by $\tilde{\varphi}_{ij}(P)^{-1}$,

G_i must be mapped isomorphically onto G_j by Lemma 2, since both G_i and G_j are small. But g_i is mapped to $g'_j=(e_{n_i}^{p_1^{(i)}},\dots, e_{n_i}^{p_l^{(i)}})$ by the definition of $\tilde{\varphi}_{ij}(p)$, so we may take this to be the generator of G_j.

Q.E.D.

Now fix once and for all a generator $g_{i_0}\in G_{i_0}$. Then by the admissibility of $(X, \mathfrak{U})$ and by the above assertion, we can successively define a distinguished generator g'_i of G_i for every i for which $U_i\cap S_0\neq\phi$, with the following properties; if U_i is adjacent to U_j, and is related to U_j by the multivalued map π_{ij} as above after a suitable renumbering of coordinates depending on i and j, then $n_i=n_j$ $(=n_{i_0})$ and $p_k^{(i)}=p_k^{(j)}$, $1\leqq k\leqq l$. We leave the precise argument to the reader. But for the proof it makes no difference if we change the generator g_i to g'_i. So we assume that this change has already been done, and we denote the new generators also by the same letters g_i. Then by Lemma 4 3) and 5), we see that ψ_i is isomorphic outside $\psi_i^{-1}(S_0\cap U_i)$.

We define now for each U_i, $U_i\cap S_0\neq\phi$, W_i, $\mathfrak{V}_i=\{V'_1,\dots, V_l^i\}$ and ψ_i: $W_i\to U_i$ as those constructed in Lemma 3, taking $U_i=X$, $G_i=G$ and so on.

The next step is to show

Assertion 2. *The birational map $\psi_{ij}=\psi_i^{-1}\psi_j$ gives an isomorphism of $\psi_j^{-1}(U_i\cap U_j)$ and $\psi_i^{-1}(U_i\cap U_j)$.*

Proof. If either $U_i\cap S_0=\phi$ or $U_j\cap S=\phi$, then this is obvious, because then both ψ_i^{-1} and ψ_j^{-1} are isomorphic on $U_i\cap U_j$. So we assume that $U_i\cap U_j\cap S_0\neq\phi$ (note that this is equivalent to $U_i\cap S_0\neq\phi$ and $U_i\cap S_0\neq\phi$). If we show that ψ_{ij} is holomorphic at each point of $\psi_j^{-1}(U_i\cap U_j)$ as a map of analytic spaces, then by Z.M.T., we see that this is a morphism of varieties, and since this holds also for ψ_{ij}, we get that ψ_{ij} is an isomorphism. To prove that ψ_{ij} is holomorphic, we go back to the situation and the notations given in the proof of the Assertion 1. There, we defined for each point $P=(\bar{u}_{l+1}^j,\dots, \bar{u}_m^j)\in \boldsymbol{C}^{m-l}-\{u_m^j=0\}$, the linear map $\tilde{\varphi}_{ij}(P)$: $\boldsymbol{C}^l\to\boldsymbol{C}^l$. But if we take a simply connected subdomain D in $\boldsymbol{C}^{m-l}-\{u_m^j=0\}$ and choose a suitable branch for each $(\bar{u}_m^j)^{a_k}$, the same formula with $\tilde{\varphi}_{ij}(P)$ defines the isomorphism $\tilde{\varphi}_{ij}(D)$:

$C^l \to C^l$ depending holomorphically on D: Further this fits into the following commutative diagram;

$$\begin{array}{ccc} C^l \times \bar{\pi}_{ij}(D) & \xleftarrow{\tilde{\varphi}_{ij}(D) \times \bar{\pi}_{ij}} & C^l \times D \\ \downarrow & & \downarrow \\ C^l/G_i \times \bar{\pi}_{ij}(D) & \xleftarrow{\varphi_{ij}} & C^l/G_j \times D \\ \downarrow & & \downarrow \\ \bar{\pi}_{ij}(D) & \xleftarrow{\bar{\pi}_{ij}} & D. \end{array}$$

From this we infer readily that ψ_{ij} is holomorphic on $\psi_j^{-1}(D)$ by the definition of ψ_j. Q.E.D.

Now we define X_1 to be the union $X_1 = \bigcup_{i=1}^{s} W_i$, where if U_i is adjacent to U_j, then $(w^i) \in W_i$ and $(w^j) \in W_j$ are identified if and only if $(w^i) \in \psi^{-1}(U_i \cap U_j)$ (resp. $(w^j) \in \psi_j^{-1}(U_i \cap U_j)$) and $\psi_{ji}(w^i) = (w^j)$. Then define $f_1: X_1 \to X$ by the condition that $f_1 | W_i = \psi_i$. It is easy to see that by the definition, X_1 becomes a variety and f_1 a morphism of varieties. Moreover we shall prove

Assertion 3. *The pair* $(X_1, \mathfrak{U}_1)$ *is admissible, where* $\mathfrak{U}_1$ *is the union of the ocverings* $\mathfrak{V}_i$.

Proof. First we fix U_i and U_j which are mutually adjacent and make some preliminary considerations. We distinguish three cases; 1) neither U_i nor U_j intersects with S_0, 2) either U_i or U_j intersects with S_0 but not both, and 3) both U_i and U_j intersects with S_0.

In case 1), since W_i (resp. W_j) $= U_i$ (resp. U_j) by construction, W_i is adjacent to W_j and the corresponding multivalued map satisfies the condition (β).

Next we consider the case 3). We may assume that π_{ij} is given by (4). Then W_i (resp. W_j) is covered by l affine open subsets $V_{i1},\dots, V_{il}$ (resp. $V_{j1},\dots, V_{jl}$) and the multivalued map $\pi_i^{-1}\psi_i\pi_{is}$: $C^m(v^{is}) \to C^m(u^i)$, $1 \leqq s \leqq l$, has the form;

$$(5)\qquad \begin{cases} u_s^i=(V_s^{is})^{n_{is}/n_i} \\ u_k^i=(V_s^{is})^{b_k}v_k^{is} & \text{for } 1\leqq k\leqq l \text{ and } k\neq s \\ u_k^i=v_k^{is} & \text{for } m\geqq k\geqq l+1, \end{cases}$$

where π_{is}: $\boldsymbol{C}^m(v^{is})\to V_{is}$ is the quotient map, and b_k are certain rational numbers. Here we write down also the inverse to the above map for convenience.

$$(6)\qquad \begin{cases} v_s^{is}=(u_s^i)^{n_i/n_{is}} \\ v_k^{is}=u_k^i(u_s^i)^{-n_ib_k/n_{is}} & \text{for } 1\leqq k\leqq l \text{ and } k\neq s \\ v_k^{is}=u_k^i & \text{for } m\geqq k\geqq l+1\,. \end{cases}$$

These follow from Lemma 4. From this, it is easy to see that V_{is} and V_{js} are adjacent to each other, and no other two V_{ik} and $V_{jk'}$ are adjacent. Moreover since $\pi_{ij}=\pi_i^{-1}\pi_j=(\pi_i\psi_i\pi_{ik})^{-1}(\pi_i^{-1}\pi_j)(\pi_j^{-1}\psi_j\pi_{js})$, using (4) and (5) we infer that this has the following form;

$$(7)\qquad \begin{cases} v_s^{is}=(v_m^{js})^{\frac{n_ia_s}{n_{is}}}v_s^{js} \\ v_k^{is}=(v_m^{js})^{\frac{n_{is}a_k-a_sn_ib_k}{n_{is}}}v_k^{js},\ k\neq s \\ v_m^{is}=(v_m^{js})^{-1}\,. \end{cases}$$

Hence the condition (β) is satisfied for V_{is} and V_{js}.

Finally, we deal with the case 2), say, when $U_j\cap S_0\neq\phi$. The alternative case can be treated analogously. Then if $\pi_i^{-1}(U_i\cap U_j)$ is defined by $u_s^i\neq 0$, then $1\leqq s\leqq l$. For, otherwise, $U_i\cap S_0\neq\phi$. Hence we may assume that the multivalued map π_{ij} has the form $u_k^i=(u_s^j)^{a_k}u_k^j$ for $k\neq s$ and $u_s^i=(u_s^i)^{n_j/n_i}$. Then we see readily that V_{is} is the only element which is adjacent to W_j and that $\pi_i^{-1}\pi_{js}=(\pi_i^{-1}\pi_j)(\pi_j^{-1}\psi_j\pi_{js})$ takes the following form;

$$(8)\qquad \begin{cases} u_k^i=(v_s^{js})^{\frac{n_{js}a_n+b_kn_j}{n_j}}v_k^{js},\ k\neq s \\ u_s^i=(v_s^{js})^{-n_{js}/n_i}\,. \end{cases}$$

This gives the condition (β) for W_i and V_{js}.

Using these results we shall now prove the condition (α) for $(X_1, \mathfrak{U}_1)$. So suppose V_i^α is adjacent to V_j^β. Then, by the admissibility of $(X, \mathfrak{U})$, there exists a finite sequence $U_0, \dots, U_t$ such that $U_0 = U_i$, $U_t = U_j$, and U_μ is adjacent to $U_{\mu+1}$ for $0 \leqq \mu \leqq t-1$. Then, by the previous considerations, there exist for each successive pair (U_k, U_{k+1}), $0 \leqq k < t$, $V_{\alpha(k)} \in \mathfrak{V}_k$ and $V_{\beta(k)} \in \mathfrak{V}_{k+1}$ such that $V_{\alpha(k)}$ is adjacent to $V_{\beta(k)}$. But by virtue of Lemma 4 $V_{\alpha(k+1)}$ is adjacent to $V_{\beta(k)}$. Hence, V_α, $V_{\alpha(0)}$, $V_{\beta(0)}, \dots, V_{\beta(t-1)}$, V_β is the sequence of adjacent members of $\mathfrak{U}_1$, which connects V_α and V_β. This proves (α).

Next, in order to see the condition (β), again by the previous consideration, we have only to show that if V_α is adjacent to V_β and if $V_\alpha \in \mathfrak{V}_i$ and $V_\beta \in \mathfrak{V}_j$, then U_i is adjacent to U_j. But this is easy to check by virtue of Lemma 4, iii).

Finally, the groups G_i are small by construction. This is (γ), thus completing the proof of the assertion.

In order to apply the induction hypothesis to the admissible pair $(X_1, \mathfrak{U}_1)$, we have to see that $\varphi_{X_1} < \varphi_X$. But by construction we see that for any $V_\alpha \in \mathfrak{V}_i$ if $U_i \cap S_0 \neq \phi$, then $\operatorname{ord} V_\alpha < \operatorname{ord} U_i$ and if $U_i \cap S_0 = \phi$, $\operatorname{ord} V_\alpha = \operatorname{ord} U_i$, where $\operatorname{ord} V_\alpha$ (resp. $\operatorname{ord} U_i$) denotes the order of V_α (resp. of U_i). Hence $\varphi_{X_1} < \varphi_X$.

Hence by the induction hypothesis there exist a resolution $(\tilde{X}, f_2)$ of X_1 and a finite affine open covering $\tilde{U}$ of $\tilde{X}$ satisfying the properties stated in the theorem. Hence $2)_1$ f_2 is admissible as a morphism of admissible pairs $(\tilde{X}, \tilde{\mathfrak{U}})$ and $(X_1, \mathfrak{U}_1)$, $3)_1$ any admissible automorphism of $(X_1, \mathfrak{U}_1)$ extends uniquely onto that of $(\tilde{X}, \tilde{\mathfrak{U}})$, and $4)_1$ $f_2^{-1}(S_1) \cap \tilde{U}_s$ is defined by $\tilde{u}_{k_1}^s \dots \tilde{u}_{k_\mu}^s = 0$ for some $k_1, \dots, k_\mu$ if it is not empty, where S_1 is the singular locus of X_1. Set $f = f_2 \cdot f_1$, then since f_1 is isomorphic outside $f_1^{-1}(S_0)$ and $S_0 \subseteq S$, we see that $(\tilde{X}, f)$ gives a resolution of X. We show that these (X, f) and $(\tilde{X}, \tilde{\mathfrak{U}})$ satisfy the requirement of the theorem. For this, we have to check the conditions 2), 3) and 4) of the theorem. Note first that f_1 is admissible since so is each ψ_i by Lemma 4, 3). Then it is clear that f is admissible as a composition of two admissible morphisms f_1 and f_2. This checks 2). Next, let T be an admissible automorphism of U_i, then by Lemma 4 this extends uniquely onto an admissible automorphism of $(W_i, \mathfrak{V}_i)$ and this in turn has the unique extension to an admissible automorphism $f^{-1}((U_i, \{U_i\}))$ by $3)_1$. This is 3). Finally we deal with 4). Note first that $f^{-1}(S)= f_2^{-1}(S_1) \cup f_2^{-1}(f_1^{-1}[S_0])$, where $f_1^{-1}[S_0]$ is the proper transform of S_0 in X_1. By virtue of $4)_1$ above it is enough to see that $f_2^{-1}(f_1^{-1}[S_0]) \cap \tilde{U}_s$ is defined by the equation $\tilde{u}_{k_1}^s \tilde{u}_{k_2}^s \ldots \tilde{u}_{k_\mu}^s = 0$ on $\tilde{U}_s$ if it is not empty. But this follows from 5) of Lemma 4 and the admissibility of f. This completes the proof of the theorem.

Remark 3. *a*) From our construction the resolution map f is seen to be naturally decomposed into $f=f_1\ldots f_d$, such that f_i: $X_i \to X_{i-1}$, $1 \leqq i \leqq d$, are proper birational morphisms with $X_0=X$ and $X_d=\tilde{X}$ whose exceptional loci E_i are irreducible. Moreover if we put $s(i)=\dim f_i(E_i)$, then the function $s(i)$ is nondecreasing for $1 \leqq i \leqq d$. Further if $s(i)$ is constant for $a \leqq i \leqq b$ and if we put $f_1 \ldots f_i(E_i)=E_a^i$, then these E_a^i, $a \leqq i \leqq b$, are disjoint. Hence we have $f_i f_{i+1} = f_{i+1} f_i$ for $a \leqq i$, $i+1 \leqq b$ in an obvious sense. For example let $X=N_{n,p_1,\ldots,p_m}$ be a cyclic quotient singularity. Then $s(i)=0$ for all i if and only if $n, p_1, \ldots, p_m$ are all relatively prime as follows from Lemma 4 and Remark 1*b*).

b) In constructing a resolution of an admissible pair $(X, \mathfrak{U})$ according to the inductive method described in the proof of the theorem, we have a finite number of choices in taking generators of G_i, or taking isomorphism φ_i: $U_i \cong N_{n_i,p_1^{(i)},\ldots,p_m^{(i)}}$ at each step. In general different choices of generators lead to different resolutions, as was indicated in the remark at the end of 1.2 (see an example in 1.4).

c) We say a resolution f: $\tilde{X} \to X$ is *special* if in each step f_i: $X_i \to X_{i-1}$ of the resolution, the isomorphisms φ_i: $U_i \cong N_{n_i,p_1^{(i)},\ldots,p_m^{(i)}}$ are taken so that $p_k^{(i)}=1$ for some k. If $X=N_{n,p_1,\ldots,p_q}$ and $(p_i, n)=1$ for some i, then a special resolution of X exists as follows from Lemma 3. For the motivation for this definition, we refer to Remark 6 after Proposition 1 in the next section.

1.4. By way of illustration we apply the above method when $m=2$, and next, examine the case when $m=3$.

1.4.1. Let $X=\boldsymbol{C}^2/G$ be the cyclic quotient singularity of dimension 2.

If the group G is small, then it is easy to see that there exists a unique generator $g \in G$ such that $g=(e_n, e_n^p)$ with $0 \leqq p<n$ and $(p, n)=1$ with respect to the coordinate (z_1, z_2) of $\boldsymbol{C}^2$. Then taking g as a generator of G we construct X_1, $\mathfrak{U}=\{U_1, U_2\}$ and $f: X_1 \to X$ as in 1.2. Then by Lemma 3 $U_1 \cong \boldsymbol{C}^2(u_1, v_1)$ and $U_2 \cong N_{p,1,p_1}$, where p_1 is defined by the following formula; $0 \leqq p_1 < p$, $n=b_1 p - p_1$, $b_1 \geqq 2$. Next applying the same process to $U_2 = N_{p,1,p_1}$, we have a variety X_2 with an affine open covering $\{U_1, U_{21}, U_{22}\}$ such that $U_{21} \cong \boldsymbol{C}^2(u_2, v_2)$ and $U_{22} \cong N_{p_1,1,p_2}$, where this time p_2 is defined by $0 \leqq p_2 < p_1$, $p=b_2 p_1 - p_2$, $b_2 \geqq 2$. Then using 2) and 3) of Lemma 4 the transition functions between U_1 and U_{21} are calculated as follows;

$$\begin{cases} u_1 = (u_2)^{b_1} v_2 \\ v_1 = (u_2)^{-1}. \end{cases}$$

Continueing analogously we are finally led to the minimal resolution f: $\tilde{X} \to X$ of X first constructed by Hirzebruch in [3] (see also [12]): Define positive integers λ_k, μ_k $0 \leqq k < s+1$, and b_k, $1 \leqq k \leqq s$ by the formula;

$$(9) \quad \begin{cases} \lambda_0 = n \ \ \lambda_1 = p \ \ \lambda_{k+1} = b_k \lambda_k - \lambda_{k-1}, \ b_k \geqq 2, \ 0 \leqq \lambda_{k-1} < \lambda_k \ \ \lambda_s = 1 \ \ \lambda_{s+1} = 0 \\ \mu_0 = 0 \ \ \mu_1 = 1 \ \ \mu_{k+1} = b_k \mu_k - \mu_{k-1} \end{cases}$$

Then $\tilde{X}$ is covered by $s+1$ copies W_k, $0 \leqq k \leqq s$, of complex affine place $\boldsymbol{C}^2$ with the coordinate (u^k, v^k) and the transition functions between W_{k-1} and W_k are given by the following formula;

$$\begin{cases} u_k = 1/v_{k-1} \\ v_k = (v_{k-1})^{b_k} u_{k-1}. \qquad k=1, \dots, s. \end{cases}$$

The rational map $T: \boldsymbol{C}^2(z_1, z_2) \to \tilde{X}$ is given by

$$(10) \qquad T_k: (z_1, z_2) \longrightarrow (z_1^{\lambda_k} z_2^{-\mu_k}, z_1^{-\lambda_{k+1}} z_2^{\mu_{k+1}}) \in W_k.$$

Further if we define nonsingular subvarieties θ_k, $1 \leqq k \leqq s$, on $\tilde{X}$ by

$$\begin{cases} u_{s-k} = 0 \quad \text{on} \quad W_{s-k} \\ v_{s-k+1} = 0 \quad \text{on} \quad W_{s-k+1}, \end{cases}$$

then f is isomorphic outside $\theta=\bigcup_{i=1}^{k}\theta_i$, $\theta_k\cong \boldsymbol{P}^1$ and the selfintersection number $\theta_k\cdot\theta_k=-b_k$. Hence in particular there are no exceptional curves of the first kind in $f^{-1}(\mathfrak{P})$, where $\mathfrak{P}$ is the singular point of X. But if we make no normalization of the generator as above, then the resolution is in general not minimal. For instance, let $X=\boldsymbol{C}^2/G$ and $G=\{g\}$ with g acting on $\boldsymbol{C}^2$ by $g=(e_5^3, e_5^4)$. Further let $f\colon \tilde{X}\to X$ be the resolution of X obtained in the theorem and let $\mathfrak{P}\in X$ the singular point of X. Then $f^{-1}(\mathfrak{P})$ consists of nonsingular rational curves C_1, C_2, and C_3 with intersection numbers $(C_1)^2=-3$, $(C_2)^2=-1$, $(C_2)^2=-4$, $C_1\cdot C_2=1$, $C_2\cdot C_3=1$ and $C_1\cdot C_3=0$.

1.4.2. So we may consider some kind of normalizations also in the higher dimensional cases, and expect a certain minimality condition for the resolutions. But in the following we shall restrict ourselves half for simplicity to the case when $m=\dim X=3$, and assume that $(n, p_1)=1$. In this case we can take the canonical generator $g\in G$ by the condition that $g=(e_n, e_n^p, e_n^q)$, where $0\leqq p$, $q<n$ and $(n, p, q)=1$. Here we have assumed that the group G is small (c.f. Remark 1). Then we often write $X=N_{n;p.k}$ instead of $X=N_{n,1,p,q}$.

Now suppose X_1, $\mathfrak{U}_1=\{U_1, U_2, U_3\}$ and $f_1\colon X_1\to X$ are as in Lemma 3. Then since $q_2^1=q_3^1=1$, we have the canonical isomorphisms

$$U_1\cong \boldsymbol{C}^3,\ U_2\cong N_{p;p_2p_3} \quad\text{and}\quad U_3\cong N_{q;q_2q_3},$$

where p_2, p_3, q_2 and q_3 are determined by the following formulas;

$$\text{(11)}\qquad p_2+n\equiv 0 \mod p,\quad p_3\equiv q \mod p \quad\text{and}\quad 0\leqq p_i<p$$
$$q_2\equiv p \mod q \quad q_3+n\equiv 0 \mod q \quad\text{and}\quad 0\leqq q_i<q\,.$$

By Lemma 4 the transition functions π_{ij} and the rational map $X_1\to \boldsymbol{C}^3(z)$ are given respectively by

$$\text{(12)}\quad \begin{cases} w_1^2=(w_2^1)^{-1/p} \\ w_2^2=w_1^1(w_2^1)^{n/p} \\ w_3^2=w_3^1(w_2^1)^{-q/p} \end{cases} \qquad \begin{cases} w_1^3=(w_3^1)^{-1/q} \\ w_2^3=w_2^1(w_3^1)^{-p/q} \\ w_3^3=w_1^1(w_3^1)^{n/q} \end{cases} \qquad \begin{cases} w_1^2=w_1^3(w_2^3)^{-1/p} \\ w_2^2=w_3^3(w_2^3)^{n/p} \\ w_3^2=(w_2^3)^{-q/p} \end{cases}$$

$$(13)\quad \begin{cases} z_1=(w_1^1)^{1/n} \\ z_2=w_2^1(w_1^1)^{p/n} \\ z_3=w_3^1(w_1^1)^{q/n} \end{cases} \quad \begin{cases} z_1=w_1^2(w_2^2)^{1/n} \\ z_2=(w_2^2)^{p/n} \\ z_3=w_1^3(w_2^2)^{q/n} \end{cases} \quad \begin{cases} z_1=w_1^3(w_3^3)^{1/n} \\ z_2=w_2^3(w_3^3)^{p/n} \\ z_3=(w_3^3)^{q/n}\,. \end{cases}$$

From this we see that we can define the canonical way of resolution for such X. To state this precisely, we shall consider some preliminary cases. So suppose $X=N_{n,1,p}\times \boldsymbol{C}$ and $h_0\colon \tilde{N}_{n,1,p}\to N_{n,1,p}$ is the minimal resolution. Let $h=h_0\times id_{\boldsymbol{C}}\colon \tilde{N}_{n,1,p}\times \boldsymbol{C}\to N_{n,1,p}\times \boldsymbol{C}$ be the resolution of X obtained as the product of h_0 and the identity of $\boldsymbol{C}$. Then we say that the resolution h of X is minimal. Next suppose $(X,\mathfrak{U})$ with $\mathfrak{U}=\{U_1, U_2\}$ is an admissible pair. Assume that $U_i=\boldsymbol{C}^3(u^i, v^i, w^i)/\{g_i\}$ and $g_i=(e_n, e_n^p, 1)$, $i=1, 2$. Obviously each U_i is isomorphic to $N_{n,1,p}\times \boldsymbol{C}$. Then the minimal resolutions of U_i coincides on the intersection $U_1\cap U_2$ and gives a resolution $f\colon \tilde{X}\to X$ of X. We call this *the minimal resolution* of X. Further in this case if the transition functions between U_1 and U_2 are given by the formula

$$\begin{cases} u^i=u^i(w^j)^{a_1} \\ v^i=v^j(w^j)^{a_2} \\ w^i=(w^j)^{-1} \end{cases}$$

with a_i some rational numbers, then X is covered by $2s$ copies v_k^i, $1\leqq k\leqq s$, $i=1, 2$, of $\boldsymbol{C}^3$ with the coordinates (u_k^i, v_k^i, w_k^i) such that the transition functions between them are given by the following formula;

$$(14)\quad \begin{cases} u_k^i=(v_{k-1}^i)^{-1} \\ v_k^i=(v_{k-1}^i)^{b_k}u_{k-1}^i \\ w_k^i=w_{k-1}^i \end{cases} \qquad \begin{array}{l} u_k^1=u_k^2(w_k^2)^{a_1\lambda_k-a_2\mu_k} \\ v_k^1=v_k^2(w_k^2)^{-a_1\lambda_{k+1}+a_2\mu_{k+1}} \\ w_k^1=(w_k^2)^{-1}\,, \end{array}$$

where b_k and λ_k, μ_k are defined by (9). This can be proved easily if one uses (10). Note that the minimal resolution is unique.

Now suppose $X=N_{n;p,q}$ as before and $f\colon \tilde{X}\to X$ is one of the resolutions obtained in the theorem. Decompose f into $f=f_1\dots f_d$ as in the

Remark 3 a). Let E_i be the exceptional locus of $f_i\colon X_i \to X_{i-1}$ and set $D_i = f_i(E_i)$. D_i is either a single point or isomorphic to a projective line. We may assume that D_i is a point for $i \leqq b$ and is a line if $i > b$ (c.f. Remark 3 a)). Each X_i has a natural affine open covering $\mathfrak{U}_i$ such that $(X_i, \mathfrak{U}_i)$ is an admissible pair. If D_i is a point, then there exists a unique member $U_\alpha^{(i-1)} \in \mathfrak{U}_{i-1}$ such that $D_i \in U_\alpha^{(i-1)}$ and $f_i | f_i^{-1}(U_\alpha^{(i-1)})$: $f_i^{-1}(U_\alpha^{(i-1)}) \to U_\alpha^{(i-1)}$ is the map which replaces $U_\alpha^{(i-1)}$ by three affine open sets $U_{\alpha 1}^{(i)}$, $U_{\alpha 2}^{(i)}$, and $U_{\alpha 3}^{(i)}$ according to the method of 1.2. We say that $f_1 \dots f_i$, $1 \leqq i \leqq b$, is canonically defined if inductively 1) $f_1 \dots f_{i-1}$ is canonically defined and 2) $f_i | f_i^{-1}(U_\alpha^{(i-1)})$ is with respect to the isomorphisms

$$U_{\alpha 1}^{(i)} \cong \boldsymbol{C}^3,\ U_{\alpha 2}^{(i)} = N_{n_{\alpha 2}^{(i)}; p_{\alpha 2}^{(i)} q_{\alpha 2}^{(i)}} \quad \text{and} \quad U_{\alpha 3}^{(i)} \cong N_{n_{\alpha 3}^{(i)}, p_{\alpha 3}^{(i)} q_{\alpha 3}^{(i)}}$$

prescribed in Lemma 3 as explained above for suitable integers $n_{\alpha k}^{(i)}$, $p_{\alpha k}^{(i)}$, $q_{\alpha k}^{(i)}$. Next, consider the admissible pair $(X_b, \mathfrak{U}_b)$. The singular locus of X_b is the disjoint union of nonsingular curves C_i, $i = 1, \dots, l$, each isomorphic to a projective line. For each C_i there exists a unique pair $(U_\alpha^{(b)}, U_\beta^{(b)})$ of the members of $\mathfrak{U}_b$ such that $(U_\alpha^{(b)} \cup U_\beta^{(b)}, \{U_\alpha^{(b)}, U_\beta^{(b)}\})$ is an admissible pair of the type considered before. So we may speak of the minimal resolution of X along each C_i. Then

Definition 3. We say that the resolution f *is canonical*, or f *is the canonical resolution of* X, if i) $f_1 \cdot \dots \cdot f_b$ is canonically defined and ii) $f_{b+1} \dots f_d$ defines the minimal resolution of each C_i.

The canonical resolution is one of the special resolutions defined in Remark 3 c).

To describe the minimality condition, we make the following definition after Moishezon.

Definition 4. Suppose $\tilde{X}$ is a complex manifold of dimension 3 and S is a connected submanifold of X of codimension 1. Let $N_{S/X}$ be the normal bundle of S in X. We say that S is *the exceptional surface of the first kind*, if either S is isomorphic to $\boldsymbol{P}^2$ and $N_{S/\tilde{X}} \cong -H_{\boldsymbol{P}^2}$, or S is isomorphic to a $\boldsymbol{P}^1$ bundle over a manifold of dimension 1 and $N_{S/\tilde{X}} |_F \cong -H_F$, where H is the hyperplane bundle of the corresponding projective space, and F is the general fiber of the fibering of S.

Remark 4. If an exceptional surface of the first kind S is compact and rational, then either $S \cong \boldsymbol{P}^2$ or $S \cong \Sigma_m$, the Hirzebruch surface of degree m. Σ_m is a $\boldsymbol{P}^1$ bundle over $\boldsymbol{P}^1$ obtained by adding an ∞-section to a line bundle of degree $-m$ on $\boldsymbol{P}^1$. We call the 0-section of this line bundle also the 0-section of Σ_m. In particular, $\Sigma_0 = \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and it has two different fiberings of $\boldsymbol{P}^1$ associated to the projections to the first and to the second factors. Conversely, it is known that Σ_0 is the only one among Σ_m which has two structures of $\boldsymbol{P}^1$ bundles.

Now let $f: \tilde{X} \to X$ be the canonical resolution of $X = N_{n.p,q}$. Let S be the singular locus of X, $\theta = f^{-1}(S)$ and $\theta_1, \dots, \theta_c$ be the irreducible components of θ. Let $f = f_1, \dots, f_d$ be the decomposition of f as in Remark 3. Suppose some $\theta_\alpha \cong \Sigma_0$ and is the proper transform of the exceptional locus of $f_i: X_i \to X_{i-1}$. Then either of its fiberings $\mu: \theta_\alpha \to \boldsymbol{P}^1$ is said to be *incompatible with* f if $f_i, \dots, d_d(\theta_\alpha)$ is a curve C_α and $f_i, \dots, f_d$ sends each fiber of μ onto C_α. Then we can prove

Proposition 1. *Suppose $f: \tilde{X} \to X$ is the canonical resolution of $X = N_{n;p,q}$ and S, θ and θ_i are as above. Then none of θ_α are exceptional surfaces of the first kind except when $\theta_\alpha \cong \Sigma_0$ and $N_{\theta_\alpha/X}|_F \cong -H_F$, where F is the general fiber of the fibering of θ_α incompatible with f.*

Proof. Let $f = f_1, \dots, f_d$ be the decomposition of f as in Remark 3 and E_i the exceptional locus of $f_i: X_i \to X_{i-1}$. Let $\tilde{E}_i$ be the proper transform of E_i in $\tilde{X}$. We have to show that $\tilde{E}_i$ are not the exceptional surface of the first kind unless it comes under the above exceptional case. Set $D_i = f_i(E_i)$. First we consider the case when $\dim D_i = 0$. Then E_i is isomorphic to a projective plane divided by a cyclic group. Indeed, by Lemma 3 and 5) of Lemma 4 we see that E_i is covered by 3 affine open subsets V_1, V_2, V_3, each isomorphic to $\boldsymbol{C}^2$, $N_{p',1,p'_3}$, and $N_{q',q'_2,1}$ respectively, where we assumed that there exists $U_\alpha \in \mathfrak{U}^{(i-1)}$ with the canonical isomorphism with $N_{n';p'q'}$ such that $D_i \in \mathfrak{U}_\alpha$ and where p'_3 and q'_2 are determined from p' and q' by the formula corresponding to (11). But since $\tilde{E}_i$ are nonsingular by the theorem, the induced map $\tilde{E}_i \to E_i$ gives the resolution of the singularity of E_i. Now since Σ_m (resp. $\boldsymbol{P}^2$) have 2 (resp. 1) as the second betti number, from this, we can readily infer that $\tilde{E}_i$ is isomorphic to neither of them unless in the following three cases; i) $(p'_3, q'_2)=(0, 0)$, ii) $(p'_3, q'_2)=(1, 0)$ (iii) $(p'_3, q'_2)=(0, 1)$. But i) is equivalent to $p=q$ ii) to $p=1$ and iii) to $q=1$. These cases are dealt with in the following two lemmas.

Lemma 5. (Ueno [12]) *Suppose $f\colon \tilde{X}\to X$ is the canonical resolution of $X=N_{n;p,p}$. Then $\tilde{X}$ is covered by $2s+1$ copies V_k, $1\leqq k\leqq s$, $i=1, 2$, and V_0, of $\boldsymbol{C}^3$ with the coordinates (u_k^i, v_k^i, w_k^i) and (u^0, v^0, w^0) respectively and they are connected by the system of transition functions as follows;*

$$(15)\quad \begin{cases} u_k^i=(v_{k-1}^i)^{-1} \\ v_k^i=u_{k-1}^i(v_{k-1}^i)^{b_k} \\ w_k^i=w_{k-1}^i \end{cases} \qquad \begin{cases} u_k^1=u_k^2(w_k^2)^{-\mu_k} \\ v_k^1=v_k^2(w_k^2)^{\mu_{k+1}} \\ w_k^1=(w_k^2)^{-1} \end{cases} \qquad \begin{cases} u_1^i=(u^0)^{-1} \\ v_1^i=v^0(u^0)^{b_1} \\ w_1^i=w^0(u^0)^{-1}, \end{cases}$$

where λ_k, μ_k and b_k are given by (9).

Proof. Set $h=f_2,\cdots,f_d\colon \tilde{X}\to X_1$ and $W_i=h^{-1}(U_i)$ for $i=1, 2, 3$. Then $h|_{W_2\cup W_3}$ defines the minimal resolution of $U_2\cup U_3$. Hence $W_2\cup W_3$ are covered by $2s$ copies V_k^i, $1\leqq k\leqq s$, $i=1, 2$ of $\boldsymbol{C}^3$ with the coordinates (u_k^i, v_k^i, w_k^i) and the transition functions among them are given by (14), namely,

$$\begin{cases} u_k^i=(v_{k-1}^i)^{-1} \\ v_k^i=(v_{k-1}^i)^{b'_k}u_{k-1}^i \\ w_k^i=w_{k-1}^i \end{cases} \qquad \begin{cases} u_k^1=u_k^2(w_k^2)^{-(\lambda'_k+n\mu'_k)/p} \\ v_k^1=v_k^2(w_k^2)^{(\lambda'_{k+1}+n\mu'_{k+1})/p} \\ w_k^1=(w_k^2)^{-1} \end{cases}$$

where λ'_k, μ'_k, b'_k are determined by the euclidian algorithm of (9) putting this time $\lambda_0=p$ and $\lambda_1=p_2$. In fact, in this case we can take $a_1=-1/p$ and $a_2=n/p$ in (14), as is seen from (12). But if λ_k and μ_k are the integers defined by (9) from n and p, then we can show inductively the following relations;

$$\lambda'_k=\lambda_{k+1},\ b'_k=b_{k+1}\quad (0\leqq k\leqq s-1), \qquad \lambda_{k+1}+n\mu'_n=p\mu_{k+1}\quad (k\geqq 1).$$

Hence we have obtained the first two relations. The last one is easily deduced from (12), (13) and (8). Q.E.D.

Lemma 6. *Suppose* $f\colon \tilde{X}\to X$ *and* $f'\colon \tilde{X}'\to X$ *be the two canonical resolutions of* X, *according to the isomorphisms* $X\cong N_{n;p,p}$ *and* $X\cong N_{n;1,p'}$ *respectively, where* p *and* p' *are related by* $pp'\equiv 1 \bmod n$. *Then there exists an isomorphism* $h\colon X\to X'$ *such that* $f'h=f$.

Proof. We shall only indicate the method of proof and leave the explicit computations to the readers. It suffices to prove that $\tilde{X}'$ is covered by $2s+1$ copies of $\boldsymbol{C}^3$ and that the transition functions between them are given by (15). Decompose f into $f=f_1\cdots f_d$ as in Remark 3. We prove the lemma by induction on d. In fact, in this case we have $U_1\cong U_2\cong \boldsymbol{C}^3$ and $U_3=N_{p';1,p'_3}$ by (11). Thus we may apply the induction hypothesis to the canonical resolution $h\colon h^{-1}(U_3)\to U_3$, where $h=f_2\cdots f_d|(f_d\cdots f_2)^{-1}(U_3)$. Hence if we set $W_3=h^{-1}(U_3)$, then W_3 is isomorphic to the canonical resolution $\tilde{N}_{p';p''p''}$ of $N_{p';p''p''}$, where p'' is defined by the formula $p''p'\equiv 1 \pmod{p'}$ and $0\leqq p''<p'$. Then we have to show that $\tilde{N}_{p';p''p''}$ is isomorphic to $\bigcup_{k=1}^{s-1} V_k^i \cup V_0$ in the notation of Lemma 5. But this corresponds to the fact that if $n/p=b_1\_1/b_2\_1/b_3\_\cdots\_1/b_s$, $b_i\geqq 2$, is the expansion of n/p into the continued fraction, then that of p'/p'' is given by $p'/p''=b_1\_1/b_2\_\cdots\_1/b_{s-1}$. This can be derived from $n/p'=b_s\_1/b_{s-1}\_\cdots\_1/b_1$. Now it remains to see that the transition functions with respect to V_1^i and U_0. But these can be calculated using (12) and (8) to coincide with the last relations of (15). Q.E.D.

Now we define nonsingular subvarieties θ_k, $1\leqq k\leqq s$, of $\tilde{X}$ by the following formula;

$$\theta_1:\ u^0=0 \text{ in } V_0,\ v_1^1=0 \text{ in } V_1^1,\ v_1^2=0 \text{ in } V_1^2$$

$$\theta_k:\ \begin{cases} u_{k-1}^i=0 & \text{in } V_{k-1}^i \qquad i=2,3 \text{ and } k=2,\dots,s. \\ v_k^i=0 & \text{in } V_k^i \end{cases}$$

Then $f^{-1}(P)=\bigcup_{k=1}^{s}\theta_k$, where P is the singular point of X and θ_k is the proper transform of the exceptional locus of f_k, when we decompose

f into $f=f_1\cdots f_d$ as usual. Note on the other hand that if $f'=f'_1\cdots f'_d$ is the decomposition of f', then θ_k is the proper transform the exceptional locus of f_{d-k}. Roughly speaking, f and f' are the resolutions of X from the opposite sides. Now from Lemma 5 we can see immediately

Corollary to Lemma 6 [12]. i) *$\theta_1 \cong \boldsymbol{P}^2$ and $N_{\theta_1/\tilde{X}} \cong -b_1H$.* ii) *$\theta_k$, $k=2,\cdots,s$, is isomorphic to Σ_{μ_k}. If F is the general fibering of θ_k, then $N_{\theta_k/\tilde{X}}|_F \cong -b_kH$.*

For the precise proof we refer the reader to Lemma 4.3–4.6 of [12]. We only note that since $\mu_k \geqq \mu_2 \geqq 2$ for $k \geqq 2$, we conclude that $\theta_k \not\cong \Sigma_0$, $\not\cong \Sigma_1$ for any k. Hence in particular there exists no exceptional surface of the first kind at all among θ_k.

Now we continue the proof of the proposition. By virtue of the above results we may assume now that $\dim D_i=1$. Then $\tilde{E}_i$ coincides with an irreducible component of $(f_b\cdots f_d)^{-1}(C_j)$ for some C_j. Set $f^b=f_b\cdots f_d$. Then $f^b|\tilde{E}_i$: $\tilde{E}_i \to C_j$ gives the natural structure of a $\boldsymbol{P}^1$ bunddle on $\tilde{E}_i$. Then by [14] we see that $N_{\tilde{E}_i/\tilde{X}}|_F \cong -bH_F$ with $b \geqq 2$, where F is the general fiber of $f^p|\tilde{E}_i$. Thus $\tilde{E}_i$ can possibly be an exceptional surface of the first kind only along the incompatible fibering of $\tilde{E}_i$. This proves the proposition.

Remark 5. If n, p, q satisfy the following condition, then in the canonical resolution f: $\tilde{X} \to X$, the exceptional case of the proposition occurs:

Set $(p, q)=d$ and define integers p_i, q_j, n_i^p, n_j^q, $0 \leqq i \leqq s+1$ $0 \leqq j \leqq t \leqq +1$, by the formulas;

$$q=b_0p+p_1,\ 0 \leqq p_1<p,\ p=c_0q+q_1,\ 0 \leqq q_1<q$$

$$p_{i-1}=b_ip_i-p_{i+1},\ 0 \leqq p_{i+1}<p_i,\ q_{j-1}c_jq_j-q_{j+1},\ 0 \leqq q_{j+1}<q_j$$

$$p_0=p,\ p_s=d,\ p_{s+1}=0 \qquad q_0=q,\ q_t=d,\ q_{p+1}=0$$

$$n=d_0p-n_1^p,\ 0 \leqq n_1^p<p \qquad n=e_0q-n_1^q,\ 0 \leqq n_1^q<q$$

$$n_{i-1}^p=d_ip_i+n_i^p,\ 0 \leqq n_i^p<p_i, \qquad n_{j-1}^q=e_jq_j+n_j^q,\ 0 \leqq n_q<q_j.$$

Then $d_1 = n^{qp}_{s+1} = n^q_{t+1}$ is the integer defined by $d_1 + n \equiv 0 \pmod d$ and $0 \leqq d_1 < d$. Now let $p'_i = p_i/d$, $q'_j = q_j/d$ and set

$$P' = 1/p'_0 p'_1 + \cdots + 1/p'_{s-1} p'_s, \qquad Q' = 1/q'_0 q'_1 + \cdots + 1/q'_{t-1} q'_t$$

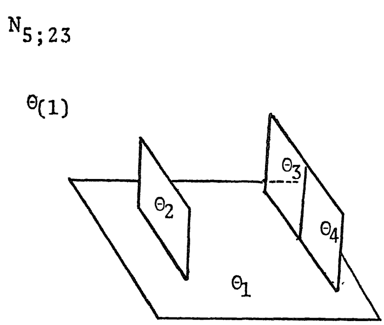
$$N'_p = n^p_1/p'_0 p'_1 + \cdots + n^p_s/p'_{s-1} p'_s, \quad N'_q = n^q_1/q'_0 q'_1 + \cdots + n^q_t/q'_{t-1} q'_t .$$

Moreover let λ_k, μ_k, $1 \leqq k \leqq b$, be defined by the algorithm of (9) putting $\lambda_0 = d$ and $\lambda_1 = d_1$ there. Now our condition is stated as follows; there exists k, $1 \leqq k \leqq b$, such that

$$P' + Q' - 1/p'q' = \mu_k \quad \text{and} \quad N'_p + N'_q + 1/p'q' = \lambda_k .$$

Remark 6. Lemma 6 and Proposition 1 would certainly be true for any special resolution.

As an example of the explicit resolutions we take $X = \boldsymbol{C}^3/G$, where $G = \{g\}$ and g acts on $\boldsymbol{C}^3$ by $g = (e_5, e_5^2, e_5^3)$. Then according to whether we take g, g^2, or g^3 as a generator of G, we have the isomorphisms $X \cong N_{5;2,3}$, $X \cong N_{5;2,4}$, and $X \cong N_{5;3,4}$ respectively. Let f_i: $\tilde{X} \to X$, $1 \leqq i \leqq 3$, be the corresponding canonical resolutions of X. Then $f_i^{-1}(p) = \theta_{(i)}$ look as in the following figure, where P is the singular point of X.

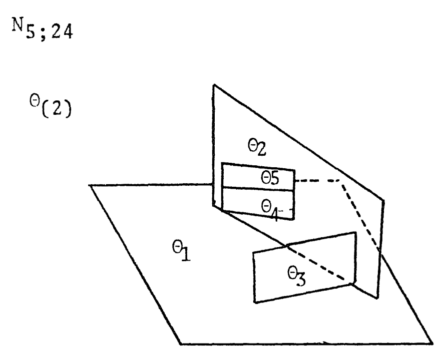


θ_1: rational

$\theta_2 \cong \boldsymbol{P}^2$ $\quad N_{\theta_2/\tilde{X}_1} \cong -2H$

$\theta_3 \cong \Sigma_2$ $\quad N_{\theta_3/\tilde{X}_1}|_F \cong -2H_F$

$\theta_4 \cong \boldsymbol{P}^2$ $\quad N_{\theta_4/\tilde{X}_1} \cong -2H$



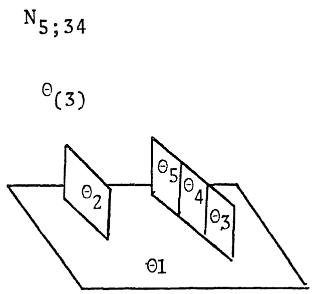
$\theta_1 \cong \Sigma_2$ $\quad N_{\theta_1/\tilde{X}_2}|_F \cong -3H_F$

θ_2: rational

$\theta_3 \cong \Sigma_2$ $\quad N_{\theta_3/\tilde{X}_2}|_F \cong -2H_F$

$\theta_4 \cong \boldsymbol{P}^2$ $\quad N_{\theta_4/\tilde{X}_2} \cong -2H$

$\theta_5 \cong \Sigma_2$ $\quad N_{\theta_5/\tilde{X}_2}|_F \cong -2H_E$



θ_1: rational

$\theta_2 \cong \boldsymbol{P}^2$ $\quad N_{\theta_2/\tilde{X}_3} \cong -3H$

$\theta_3 \cong \Sigma_2$ $\quad N_{\theta_3/\tilde{X}_3}|_F \cong -2H_F$

$\theta_4 \cong \Sigma_3$ $\quad N_{\theta_4/\tilde{X}_3}|_F \cong -2H_F$

$\theta_5 \cong \boldsymbol{P}^2$ $\quad N_{\theta_5/\tilde{X}_3} \cong -2H$

§2. Resolutions of General Isolated Quotient Singularities and Isolated Singularities with $\boldsymbol{C}^*$-action of Dimension 3

2.1. Let X be a complex space of dimension 3 and $\mathfrak{P}$ be a point of X. Suppose there exists a neighborhood U of $\mathfrak{P}$ in X such that U is siomorphic to $D \times Y$, where D is a unit disc $\{t; 0 \leqq |t| < 1\}$ and Y is some neighborhood of the singular point $\mathfrak{Q}$ of the cyclic quotient singularity $N_{n,1,p}$. Let φ: $U \cong D \times Y$ be the isomorphism, and h: $\tilde{Y} \to Y$ be the minimal resolution of Y. Then $f = (\varphi)^{-1}(id_D \times h)$: $D \times \tilde{Y} \to U$ defines a resolution of U. Now let U' be another neighborhood of $\mathfrak{P}$ with an isomorphism φ': $U' \cong D \times Y'$ and let f'; $D \times \tilde{Y}' \to U'$ be the resolution obtained analogously using the minimal resolution h': $\tilde{Y}' \to Y'$ of Y', where Y' is again some neighborhood of the singular point $\mathfrak{Q}'$ of a cyclic quotient singularity $N_{n',1,p'}$. Then we have necessarily that $n = n'$ and $p = p'$[1)].

Lemma 7. *The above f and f' coincide on $U \cap U'$, or more precisely, there exists an isomorphism ψ: $f^{-1}(U \cap U') = f'^{-1}(U \cap U')$ such that $f'\psi = f$.*

Proof. First, recall that a quotient singularity is ratinoal [1, Satz 1.7]. Then, by [7, Chap. I], [11, Theorem 1] any of its resolutions is obtained by succession of a finite number of quadratic transformations. Namely, let Y_0 be a quotient singularity and h_0: $\tilde{Y}_0 \to Y_0$ a resolution.

1) This is in fact the consequence of the following lemma 7, which makes no use of this fact, together with the uniqueness of the minimal resolution of a normal singularity of a surface and the rigidity of quotient singularities [1].

Then h_0 can be written as $h_0=h_d^0\dots h_1^0$, where $h_i^0\colon Z_i\to Z_{i-1}$, $1\leqq i\leqq d$, is the moniodal transformation with center the singular locus S_i of Z_i with its reduced structure and $Z_0=Y_0$ and $Z_d=\tilde{Y}_0$. In fact, Z_i are all normal and S_i consists only of finite number of points [7, Prop. 8.1], [11, Prop. 1.2]. Now we define a resolution $f_0\colon \tilde{U}''\to U''$ of $U''=U\cap U'$ inductively as a succession of monoidal transformations with nonsingular center as follows; first let $f_1^0\colon U_1\to U''$ be the moniodal transformation with center the singular locus S'' of U'' (S'' coincides with $U'\cap\varphi(D\times\mathfrak{Q})$ and hence is non-singular). Assume now that $f_i^0\colon U_i\to U_{k-1}$, $1\leqq i\leqq s$, have already be defined. Then we define $f_{s+1}^0\colon U_{s+1}\to U_s$ as a monoidal transformation with center the singular locus of U_s. Then by the definition of f_0, f_i^0 is naturally isomorphic to both $f_i=id_D\times h_i\colon D\times Z_i\to D\times Z_{i-1}$, and $f'_1=id_D\times h'_i\colon D\times Z'_i\to D\times Z'_{i-1}$, where $h=h_d\dots h_1$ with $h_i\colon Z_i\to Z_{i-1}$ (resp. $h'=h'_{d'}\dots h'_1$ with $h'_i\colon Z'_i\to Z'_{i-1}$) is the decomposition of h (resp. h') into the quadratic transformations as above. Hence we see that $d=d', f^0=f_d^0..f_1^0$ gives a resolution of $U\cap U'$, and finally both f and f' coincide with f^0 on $U\cap U'$. This completes the proof.

We call the resolution obtained in the lemma the minimal resolution of X at $\mathfrak{P}$. Moreover, suppose X_1 is a complex space of $\dim X_1=3$ and each point $\mathfrak{P}_1\in X_1$ admits a neighborhood U_1 which is isomorphic to $D\times Y_1$ with Y_1 a neighborhood of the singular point of some cyclic quotient singularity. Then a resolution $f\colon \tilde{X}_1\to X_1$ is said to be minimal if it gives the minimal resolution at each singular point of X_1.

2.2, Let $G\subseteqq GL(3, \boldsymbol{C})$ be a finite subgroup. Then G acts naturally on $\boldsymbol{C}^3$ and the quotient space $X=\boldsymbol{C}^3/G$ has the natural structure of a normal affine algebraic variety [10, Prop. 18]. Let S be the singular locus of X. In this section we shall prove

Theorem 2. *Suppose X has only an isolated singularity at the point $\mathfrak{P}_0$ corresponding to the origin. Then there exists a resolution $f\colon \tilde{X}\to X$ of X with the following propoerties; if we denote by $\theta_1,\dots,\theta_s$ the irreducible components of $\theta=f^{-1}(S)$, then*

1) *θ has only normal crossings in X,*
2) *each θ_i is a nonsingular raitonal surface,*
3) *$\theta_i\cap\theta_j$, $i\neq j$, is isomprohic to $\boldsymbol{P}^1$ if it is not empty, and*

4) $\theta_i \cap \theta_j \cap \theta_k$, $i \neq j \neq k \neq i$ *consists of a single point if it is not empty.*

Proof. We follow after the proof of Satz 2.11 of [1]. We may assume that G is small, namely, no elements of G have 1 as its eigenvalues with multiplicity exactly 2 [9. Def. 2]. Let $\sigma: W \to \boldsymbol{C}^3$ be the monoidal transformation at the origin. Then $E = \sigma^{-1}(0)$ is isomorphic to $\boldsymbol{C}^3 - \{0\}/\boldsymbol{C}^*$, and hence to $\boldsymbol{P}^2$, where $\boldsymbol{C}^*$ acts naturally on $\boldsymbol{C}^3$. Further W has the natural structure of a line bundle over $\boldsymbol{P}^2$. Let $\omega: W \to \boldsymbol{P}^2$ be the projection. The action of G exetnds naturally onto W so that it leaves E invariant. In more detail, let $g \in G$ be an arbitrary element. There exists a linear change of coordinate of $\boldsymbol{C}^3$ such that with respect to this new coordinate g has the diagonal form with eigenvalues a, b and c in this order. Let (w_1, w_2, w_3) be this new coordinate. We may also consider $(w_1: w_2: w_3)$ as a homogeneous coordinate of $E \cong \boldsymbol{P}^2$. Now set $V_i = \{w^i \neq 0; (w) \in \boldsymbol{P}^2\}$. Then as usual W is described as the union $W = \bigcup_{i=1}^{3} (V_i \times \boldsymbol{C})$, where $(p, \zeta_i) \in V_i \times \boldsymbol{C}$ is identified with $(q, \zeta_j) \in V_j \times \boldsymbol{C}$ if and only if $\zeta_i = w_i/w_j \zeta_j$. Set $U_i = V_i \times \boldsymbol{C}$. Then U_i is naturally isomorphic to $\boldsymbol{C}^3$ and the extended action of g on W is given with respect to this coordinate of U_i by

$$g = (b/a, c/a, a) \text{ on } U_1, \ = (a/b, c/b, b) \text{ on } U_2, \text{ and}$$
$$= (a/c, b/c, c) \text{ on } U_3.$$

From this, we infer that the fixed point sets on W of the elements of G are classified according to the eigenvalues of the elements as follows; if the eigenvalues of an element $g \in G$ are as a set (i) $\{1, a, a\}$, (ii) $\{1, a, b\}$, (iii) $\{a, a, a\}$, (iv) $(a, a, b\}$ or (v) $\{a, b, c\}$, then the fixed point set of g is (i) union of a fiber of ω and a line in E, (ii) union of a fiber and a point on E, (iii) E, (iv) uion of a line and a point on E, or (v) three distinct points in E, respectively, where a, b and c are roots of unity which are mutually distinct and different from 1. Now let $X_1 = W/G$, $\pi_1: W \to X_1$ the quotient map, and $f_1: X_1 \to X$ be the morphism induced by σ. Note that f_1 is isomorphic outisde $f_1^{-1}(\mathfrak{P}^0)$ so that the singular locus S_1 of X is contained in $f_1^{-1}(\mathfrak{P}^0)$. From this, we infer that no elements of G can fix a fiber of ω. Thus the classes (i) and (ii) above are empty. Now let $\mathfrak{P} \in E$ be an arbitray point, and $G_{\mathfrak{P}}$ be the stabilizer of $\mathfrak{P}$. We show that $G_{\mathfrak{P}}$ is cyclic. For this, we fix an open neighborhood U of $\mathfrak{P}$ in E such that $\omega^{-1}(U)$ is $G_{\mathfrak{P}}$ invariant. Such a U exists because G acts fiber-preservingly on $\omega: W \to \boldsymbol{P}^2$. Moreover, taking U sufficiently small we may assume that $W|U$ is trivial i.e. there exists an isomorphism $\varphi_U: W|U \cong U \times \boldsymbol{C}$. Identify $W|U$ with $U \times \boldsymbol{C}$ by this isomorphism. Let ζ be the coordinate of $\boldsymbol{C}$. Then the action of an element g of $G_{\mathfrak{P}}$ on the ζ-component is of the following form $\zeta = h(u)\zeta$, where h is a regular function on U. But since g has a finite order, $h(u)$ is a root of unity and is a constant $a(g)$. Now define a map $\mu: G_{\mathfrak{P}} \to \boldsymbol{C}^*$ by $\mu(g) = a(g)$, then it is clear that μ is a homomorphism of the groups. It is easy to see that this $a(g)$ is independent of the trivialization φ_U and coincides with the corresponding eigenvalue of g. Hence by the above classification of the elements of G, μ defines an isomorphism of $G_{\mathfrak{P}}$ with a subgroup of $\boldsymbol{C}^*$. Thus $G_{\mathfrak{P}}$ is cyclic. Now taking a generator g of G and a covering of W by 3 open subsets U_1, U_2 and U_3 corresponding to g as in the beginning of the proof, we see that X_1 is isomorphic at $\pi_1(\mathfrak{P})$ as a germ of a variety to a cyclic quotient singularity. Hence by Theorem 1 there exist a neighborhood V of $\mathfrak{Q} = \pi_1(\mathfrak{P})$ in X_1 (in the usual topology), and a resolution $(\tilde{V}, f_{\mathfrak{P}})$ of V satifsying the properties stated in the Theorem. Moreover by the consideration in 1.4 f is minimal on $V - \mathfrak{Q}$. Finally, we take a finite number of points $\mathfrak{Q}_1, \dots, \mathfrak{Q}_s$ of X_1, neighborhoods U_i of $\mathfrak{Q}_i$ in X_1, and resolutions $f_i: \tilde{U}_i \to U_i$, such that U_i cover the singular locus of X_1. Then by Lemma 7 we can see that these f_i coincide on the intersections, and thus patch together to give a resolution $f_2: \tilde{X} \to X_1$ of X_1. Then $f = f_1 \cdot f_2: \tilde{X} \to X$ is the desired resolution. In fact, since any irreducible component of S_1 is either isomorphic to $\boldsymbol{P}^1$ or a point, we deduce that each θ_i is nonsingular and rational except the proper transform, say θ_1, of $f_1^{-1}(\mathfrak{P}^0)$ in $\tilde{X}$. But the latter is obtained by resolving the singularity of $f_1^{-1}(\mathfrak{P}^0)$, which in turn is a projective plane E divided by G, considering G as a subgroup of $PGL(3, \boldsymbol{C})$. Hence by the theorem of Castelnuovo (See e.g. Šavarevič, Alg. Surfaces Steklov Institute of Math. 1965), $f_1^{-1}(\mathfrak{P}_0)$ and thus θ_1 is rational. The other statements can be treated analogously and we do not repeat it.

Remark 7. The above proof shows that also in the higher dimensional cases the stabilizer $G_{\mathfrak{P}}$ at each point $\mathfrak{P}\in E$ is cyclic if X has only an isolated singularity. But if we allow X to have the singularity of positive dimensions, then the map μ above has necessarily a kernel and $G_{\mathfrak{P}}$ is in general not cyclic.

2.3. Suppose X is a normal affine algebraic variety embedded in $\boldsymbol{C}^n(z_1,\dots, z_n)$ and there exists a $\boldsymbol{C}^*$ action μ on $\boldsymbol{C}^n$ which leaves X invariant of the form;

$$\mu(t, (z_1, \ldots, z_n))=(t^{q_1}z_1,\dots, t^{q_n}z_n), \qquad t\in \boldsymbol{C}^*,$$

where q_i are positive integers satisfying $(q_1,\dots, q_n)=1$ [cf. 8]. This means in particular that the action is effective. We assume further that X is not contained in any linear subspace of $\boldsymbol{C}^n$. Noe suppose that X has an isolated singularity at the origin. We call such a variety X an isolated singularity with $\boldsymbol{C}^*$ action. For such an X we have the canonical way of inserting a '0-section' at the singular point, due to Orlik and Wagreich [8, 1.2]. More precisely, let $X'=X-\{0\}$, $Z=X'/\boldsymbol{C}^*$, π': $X'\to Z$ be the projection and $\Gamma'\subseteq X'\times Z$ be the graph of π'. It is known that Z is a projective variety [8].

Let Γ be the closure of Γ' in $X\times Z$ and $\beta_1: \Gamma\to X$ (resp $\pi: \Gamma\to Z$) be induced by the natural projection $p_1: X\times Z\to X$ (resp $p_2: X\times Z\to Z$) to the first (resp. to the second) factor. We have then the canonical section $i: Z\to \Gamma$ defined by $i(Z)=(0, z)\in X\times Z$. In the sequel we identify Z with $i(Z)$. Then we show

Lemma 8. *Γ has only the cyclic quotient singularities: For each point $\overline{\mathfrak{P}}\in\Gamma$, Γ is isomorphisc as a germ of an analytic space at $\overline{\mathfrak{P}}$ to a cyclic quotient singularity.*

Proof. We consider Γ and Z as analytic spaces. First, by the fundamental result of Holmann [5], for each point $\mathfrak{P}\in\Gamma'$ there exists a neighborhood $U_{\mathfrak{P}}$ of $\mathfrak{P}$ in Γ' and a 1-codimensional closed submanifold $A_{\mathfrak{P}}$ of $U_{\mathfrak{P}}$ such that (i) both $U_{\mathfrak{P}}$ and $A_{\mathfrak{P}}$ are $G_{\mathfrak{P}}$ invariant and (ii) $\pi'|A_{\mathfrak{P}}$ induces an isomorphism between $A_{\mathfrak{P}}/G_{\mathfrak{P}}$ and some neighborhood $V_{\overline{\mathfrak{P}}}$ of $\overline{\mathfrak{P}}=\pi'(\mathfrak{P})$, where $G_{\mathfrak{P}}$ is as usual the stabilizer of $\mathfrak{P}$ (refer [2] for the arguments). Now we fix a point $\overline{\mathfrak{P}} \in Z$ and then a point $\mathfrak{P} \in \pi^{-1}(\overline{\mathfrak{P}})$ and take $U_{\mathfrak{P}}$, $A_{\mathfrak{P}}$ and $V_{\overline{\mathfrak{P}}}$ as above. Let $Y = \pi'^{-1}(V_{\overline{\mathfrak{P}}})$. Define a holomorphic retraction $r\colon Y \to \pi'^{-1}(\overline{\mathfrak{P}})$, as follows; let $\mathfrak{Q} \in Y$ be any point. Then there exists a $t \in \boldsymbol{C}^*$ such that $\mu(t, \mathfrak{Q}) \in A_{\mathfrak{P}}$. Then we define $r(\mathfrak{Q}) = \mu(t, \mathfrak{P})$. This is easily seen to be independent of the choice of such a t, and the map is well-defined. Note that $\pi'^{-1}(\overline{\mathfrak{P}})$ is naturally isomorphic to $\boldsymbol{C}^*/G_{\mathfrak{P}}$. Then we set $\tilde{Y} = Y \times_{\boldsymbol{C}^*/G_{\mathfrak{P}}} \boldsymbol{C}^*$, the fiber product of Y and $\boldsymbol{C}^*$ over $\pi'^{-1}(\overline{\mathfrak{P}}) = \boldsymbol{C}^*/G_{\mathfrak{P}}$, where $\boldsymbol{C}^* \to \boldsymbol{C}^*/G_{\mathfrak{P}}$, which we denote by α, is the natural projection. Then there exists an isomorphism $\varphi\colon \tilde{Y} \to \boldsymbol{C}^* \times A_{\mathfrak{P}}$ and the following diagram is commutative;

$$
\begin{array}{ccc}
Y & \xleftarrow{\ \nu\ } & \tilde{Y} \underset{\varphi}{\cong} \boldsymbol{C}^* \times A_{\mathfrak{P}} \\
\downarrow & & {\scriptstyle \rho_1}\searrow\ \swarrow{\scriptstyle \rho} \\
\boldsymbol{C}^*/G & \xleftarrow[\ \alpha\ ]{} & \boldsymbol{C}^*
\end{array}
$$

where ν and ρ are natural projections and ρ_1 the projections to the first factor. Indeed, φ is explicitly defined by $\varphi^{-1}(t, a) = (t, \mu(t, a))$, with $t \in \boldsymbol{C}^*$ and $a \in A_{\mathfrak{P}}$.

Further, the natural actions of $G_{\mathfrak{P}}$ on $\tilde{Y}$ and $\boldsymbol{C}^* \times A_{\mathfrak{P}}$ commute with φ and thus $Y = \tilde{Y}/G_{\mathfrak{P}}$ is isomorphic to $(\boldsymbol{C}^* \times A_{\mathfrak{P}})/G_{\mathfrak{P}}$. On the other hand $G_{\mathfrak{P}}$ acts naturally on $\boldsymbol{C} \times A_{\mathfrak{P}}$ and $\boldsymbol{C}$, extending those on $\boldsymbol{C}^* \times A_{\mathfrak{P}}$ and $\boldsymbol{C}^*$ respectively. From this we infer that $\pi^{-1}(V_{\overline{\mathfrak{P}}}) \cong (\boldsymbol{C} \times A_{\mathfrak{P}})/G_{\mathfrak{P}}$ with respect to this action of $G_{\mathfrak{P}}$ on $\boldsymbol{C} \times A_{\mathfrak{P}}$. This proves the lemma, since $G_{\mathfrak{P}}$ is cyclic.

Remark 8. Acutually, we have the following commutative diagram extending the above one;

$$
\begin{array}{ccc}
\pi^{-1}(V_{\overline{\mathfrak{P}}}) & \xleftarrow{\ \bar{\nu}\ } & \boldsymbol{C} \times A_{\mathfrak{P}} \\
\downarrow{\scriptstyle r} & & \downarrow{\scriptstyle \bar{\rho}_1} \\
\boldsymbol{C} & \xleftarrow[\ \alpha\ ]{} & \boldsymbol{C}
\end{array}
$$

such that the map $\bar{\nu}$ is equavariant, where $\boldsymbol{C}^*$ acts on $\boldsymbol{C} \times A_{\mathfrak{P}}$ in a natural manner.

Combining Lemma 7 and Lemma 8 with Theorem 1 we have the following theorem when dimension $X = 3$.

Theorem 3. *Suppose X is an isolated singularity with C^* action and* $\dim X=3$. *Then there exists an equivariant resolution $f\colon \tilde{X}\to X$ of X with the following properites; if we denote by $\theta_0, \dots, \theta_b$, the irreducible components of $\theta=f^{-1}(\mathfrak{P})$ with $\mathfrak{P}_0$ the singular point of X, then*

1) *θ has only normal crossings in $\tilde{X}$.*

2) *each θ_i is a nonsingular ruled surfce (i.e. it is birationally equivalent to the product of a nonsingular curve C and projective line P^1) except one, say θ_0, which is the proper transform of $i(Z)$. θ_0 also is nonsingular.*

3) *$\theta_i \cap \theta_j$, $i\neq j$ is a nonsingular curve.*

We only note that the equivariance can be deduced from the above remark.

It may be conjectured that the same kind of resolution could be obtained also in higher dimensional cases.

As an example we shall examine the resolutions of Brieskorn varieties. Let X_a be a hypersurface in C^4 defined by the equation

$$z_1^{a_1}+z_2^{a_2}+z_3^{a_3}+z_4^{a_4}=0\,,$$

where a_i are integers $\geqq 2$. Put $a=l.c.d.$ of a_i and $q_i=a/a_i$. Then the map $h\colon C^4(t)\to C^4(z)$ with $f(t_1, t_2, t_3, t_4)=(t_1^{q_1}, t_2^{q_2}, t_3^{q_3}, t_4^{q_4})$ defines an abelian covering X_1 of X_a in $C^4(t)$. In fact X_1 is defined by

$$t_1^a+t_2^a+t_3^a+t_4^a=0 \tag{16}$$

Let $f\colon L\to X_1$ be the monoidal transformation at the origin. L had naturally the structure of a line bundle over a nonsingular surface S defined by (16), considering this time $(t_1 : t_2 ; t_3 : t_4)$ as a homogeneous coordinates of P^3. The action of G extends onto L and the diagram

$$\begin{array}{ccc} \Gamma & \xleftarrow{\bar{\mu}} & L \\ \downarrow & & \downarrow \\ Z & \xleftarrow{\bar{\nu}_0} & X \end{array}$$

commutes and compatible with the natural C^* action on both L and Γ, where Γ and Z are as above. Now let H_i and l_{ij} be the hyperplanes and lines in P^2 defined by $t_i=0$, and $t_i=t_j=0$ respectively. Then

$C_i=H_i\cap X$ is a nonsingular curve and $l_{ij}\cap X$ consists of n distinct points $\mathfrak{P}_{ij}^{(1)},\dots,\mathfrak{P}_{ij}^{(\mu)}$. Hence we identified X with the 0-section of L. Then it is readily seen that Γ has a singular point at $\nu(\mathfrak{P}_{ij}^{(\alpha)})$, isomorphic as germs of anayltic spaces to the cyclic quotient singularity $\boldsymbol{C}^3/\{g_{ij}\}$, where g_{ij} acts on $\boldsymbol{C}^3$ by

$$g_{ij}=(e_{c_{kl}},\ e_{c_{kl}}^{-q_i},\ e_{c_{kl}}^{-q_j}),$$

where $c_{kl}=(q_k, q_l)$ and i, j, k, l are all distinct.

Here are some examples:

1) $(a_i, a_j)=1,\ i\neq j$. Then $g_{ij}=(e_{a_ia_j},\ e_{a_j}^{-a_ka_l},\ e_{a_i}^{-a_ka_l})$. In this case Z is a projective plane [2] and $\nu(C_i)$ form 4 lines in general position. Then using Lemma 3 we could resolve the singularity of Γ rather easily. Note that then the irreducible components θ_i are all rational.

2) $a_1=a_2=b$ and $a_3=a_4=c$. Put $(b, c)=d$, $b'=b/d$ and $c'=c/d$. In this case Γ has b singular points each isomorphic to $N_{c';b'_1,b'_1}$ and c singular points each isomorphic to $N_{b';c'_1,c'_1}$ where b'_1 and c'_1 are determined by the following formula; $0\leqq b'_1<c'$, $b'_1+b'\equiv 0(c)$ $0\leqq c'_1<c'$, $c'_1+c'\equiv 0 \pmod{b'}$. The resolution of each singular point is obtained according to Lemma 5.

3) $a_1=b,\ a_2=a_3=a_4=c$. Let d, b' and c' be as above. Γ has the singular locus along the curve $\nu(C_1)$, of the type $N_{b';c_1 1}$. Let $f_1\colon\tilde{X}\to\Gamma$ the minimal resolution of Γ, θ_0 the proper transform of Z in $\tilde{X}$ and $\theta_1,\dots,\theta_s$ be the irreducible components of $f_1^{-1}(\nu(c_1))$ such that $\theta_0\cap\theta_1=\bar{C}\neq\phi$. Then θ_0 is isomorphic to the cyclic covering of $\boldsymbol{P}^2$ of degree d, with branch locus C defined by $z_2^c+z_3^c+z_4^c=0$, considering $(z_2\colon z_3\colon z_4)$ as the homogeneous coordinates of $\boldsymbol{P}^2$. Thus θ_0 is isomorphic to $\boldsymbol{P}^2$ if $d=1$ and isomorphic to $\boldsymbol{P}^1\times\boldsymbol{P}^1$ if $d=2$. θ_i, $i\geqq 1$, are isomorphic to the $\boldsymbol{P}^1$ bundles over C. Further we can see that the normal bundle $N_{\theta_0/\tilde{X}}$ of θ_0 in $\tilde{X}$ is given by

$$N_{\theta_0/\tilde{X}}=-(c'p'+1)/b'c'[\bar{C}]$$

with $0\leqq p'<b'$ and $-(c'p'+1)\equiv 0 \mod b'$ (cf. [4]). From this, we infer that θ_0 is the exceptional surface of the first kind if and only if either

$$l=1 \text{ and } c|b-1, \text{ or } c=2 \text{ and } b=2m \text{ is even.}$$

In the latter case we have a birational morphism $h: \tilde{X} \to X_0$ such that X_0 is nonsingular, h is isomorphic outside θ and $C_0=h(\theta)$ is isomorphic to $\boldsymbol{P}^1$ and the normal bundle N_{C_0/X_0} of C_0 in X_0 is isomorphic to $(-2H_{C_0})\oplus 1_{C_0}$, where H_{C_0} and 1_{C_0} denotes the hyperplane bundle of C_0 and the trivial bundle respectively.

References

[1] Brieskorn, E., Rationale Singuiaritäten komplexer flächen, *Invent. Math.* **4** (1968), 336–358.

[2] Brieskorn, E. and Van de Ven, A., Some complex structures on products of homotopy spheres, *Topology* **7** (1968), 389–393.

[3] Hirzebruch, F., Über vierdimensionale Riemannsche Flächen mehrdetiger analytischer Funktionen von zwei komplexen Veräderlichen, *Math. Ann.* **126** (1953), 1–22.

[4] Hirzebruch, F. and Jänig, K., Involutions and singularities, *Proceedings of the Bombay Colloq. on Alg. Geometry*, Oxford (1969), 220–240.

[5] Holmann, H., Quotiennteräume komplexer Mannigfaltigkeiten nach komplexen Lieschen Automorphismengruppen, *Math. Ann.* **139** (1960), 383–402.

[6] Kempf, G., Knudsen, F., Mumford, D., and Saint-Donat, B., Toroidal embeddings I, *Lecture Notes in Math.*, **339** Springer, (1973).

[7] Lipman, J., Rational singularities, with applications to algebraic surfaces and unique factorization, *Publ. Math. IHES*, **36** (1969), 195–279.

[8] Orlik, P. and Wagreich, P., Isolated singularities of algebraic surfaces with $\boldsymbol{C}^*$ action, *Ann. of Math.* **93** (1971), 205-228.

[9] Prill, D., Local classification of quotients of complex manifolds by discontinuous groups, *Duke Math. J.* **34** (1967), 375–386.

[10] Serre, J-P., *Groupes algébriques et corps de classes*, Hermann, Paris, (1959).

[11] Tyurina, G. N., Absolute isolatedness of rational singularities and triple rational points, *Functional analysis and its applications*, **2** (1968), 324–333.

[12] Ueno, K., On fiber spaces of normally polarized abelian varieties of dim 2, I, *J. Fac. Sci. Univ. of Tokyo, Sec. IA*, **18** (1971), 37–95.

[13] Cartan, H., Quotient d'un espace analytique par un groupe d'automorphismes, in "algebraic geometry and topology", Princeton, (1957), 90–102.