

Non-linear Schrödinger Equation and Modified Korteweg-de Vries Equation; Construction of Solutions in Terms of Scattering Data.

By

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In [16], we have constructed the general solutions of the initial value problem for the Korteweg-de Vries (KdV) equation in terms of the scattering data of a Schrödinger operator. The purpose of the present paper is to extend the construction to the non-linear Schrödinger equation

$$(0.1) \quad iu_t + 2^{-1}u_{xx} + |u|^2u = 0$$

and to the the modified KdV equation

$$(0.2) \quad v_t + 6v^2v_x + v_{xxx} = 0.$$

Gardner, Greene, Kruskal and Miura (GGKM) [4] have discovered that the solution of the KdV equation can be described in terms of the scattering data of one-dimensional Schrödinger operator. They also proposed that the formalism of inverse scattering theory gives a certain explicit realization of solutions. If the reflection coefficient identically vanishes, the fundamental integral equation of the inverse problem reduces to linear algebraic equations and the potential is expressed in closed form in terms of exponentials. GGKM have also recognized that solutions of the KdV equation associated with reflectionless potentials play an important role in the study of asymptotic behavior of general solutions. See [13] for detailed discussion of reflectionless solutions of the KdV equation.

An analogue of the GGKM theory for (0.1) has been initiated by Zakharov and Shabat [19]. They found that solutions of (0.1) are de-

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scribed in terms of scattering data of the operator

$$(0.3) \quad L_u = i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} D - i \begin{bmatrix} 0 & u \\ u^* & 0 \end{bmatrix} \quad (D = d/dx)$$

with a complex potential u . Subsequently Wadati [17] and the present author [14] have associated (0.2) also with the operator L_u . In [19] some part of scattering theory of L_u has been described on formal basis. In particular a part of reflectionless potentials and solutions associated with them are constructed.

Scattering theory for L_u is more complicated than that of the one-dimensional Schrödinger operator given in Faddeev [3] in several points. To explain them let $a(\zeta)$ denote the wronskian of the right and left Jost solutions (See §§ 1 and 2 for detail). $a(\zeta)$ is defined and continuous for $\text{Im} \zeta \geq 0$, analytic in $\text{Im} \zeta > 0$ and tend to 1 as $|\zeta| \rightarrow \infty$. In the case of the Schrödinger operator, $a(\zeta)$ does not vanish on the real line, the number of its zeros are finite and all zeros are simple. These do not necessarily hold in our case.

In Shabat [10], [11], where scattering theory of L_u is discussed, potentials whose $a(\zeta)$ has no zeros in $\text{Im} \zeta \geq 0$ are considered in deriving the fundamental integral equation and consequently in solving the inverse problem. This assumption excludes all reflectionless potentials and is inconvenient for the study of equations (0.1) and (0.2).

We relax the condition and only assume that $a(\zeta)$ has no zeros on the real line. The number of its zeros are finite in consequence. Even under this simplifying assumption, the theory is rather complicated by the fact that $a(\zeta)$ has multiple zeros. The complication due to the presence of multiple zeros already occurs for the Schrödinger equation with a complex potential (Blashchak [2]).

In § 1 properties of the Jost solutions of L_u are discussed. Then in § 2 scattering data are defined and their properties are described. In § 3 we derive the fundamental integral equations. §§ 4 and 5 concern the inverse problem. In § 4 the solvability of the fundamental equation is discussed. Then in § 5 the inverse problem is formulated and solved.

In deriving the above results, methods developed for Schrödinger

equations ([1], [8] for radial equations; [3] for one-dimensional equations; [2] for equations with complex potentials) have been used in modified form.

The following two sections concern the construction of solutions of (0.1) (§ 6) and (0.2) (§ 7) in terms of scattering data of L_u . Several preliminary results have been published before in [14], [15].

Throughout the paper integrations are taken over $(-\infty, \infty)$ unless explicitly indicated. For a complex number c , c^* denotes its complex conjugate.

§ 1. Jost Solutions and Their Properties

For a complex-valued measurable function u , consider the operator L_u defined by (0.3) on the infinite interval $(-\infty, \infty)$.

For a two-dimensional vector $y = {}^t(y_1, y_2)$, put

$$y^* = {}^t(y_2^*, -y_1^*)$$

and

$$|y| = \max(|y_1|, |y_2|).$$

If $y = y(x)$ is a solution of

$$(1.1) \quad L_u y = \zeta y, \quad \zeta = \xi + i\eta,$$

then

$$L_u y^* = \zeta^* y^*$$

holds.

If $y(x)$ and $z(x)$ are solutions of (1.1), then the wronskian

$$[y, z] = y_1 z_2 - y_2 z_1$$

is constant.

Put

$$\psi_-^\circ(x, \zeta) = {}^t(1, 0) \exp(-i\zeta x),$$

$$\psi_+^\circ(x, \zeta) = {}^t(0, 1) \exp(i\zeta x).$$

They are solutions of (1.1) for $u \equiv 0$.

To obtain solutions with prescribed behavior as $x \rightarrow \pm\infty$, put the expression

$$\psi(x) = \alpha(x)\psi_+^{\circ}(x) + \beta(x)\psi_-^{\circ}(x)$$

into (1.1). Then we have

$$(1.2) \quad \begin{aligned} \alpha'(x) &= -u^*(x)\psi_1(x)\exp(-i\zeta x), \\ \beta'(x) &= u(x)\psi_2(x)\exp(i\psi x). \end{aligned}$$

We assume that the potential u is integrable and put

$$\sigma_{\pm}(x) = \pm \int_x^{\pm\infty} |u(y)| dy.$$

Theorem 1.1. *There exist unique solutions ψ_{\pm} of (1.1) for $\text{Im } \zeta \geq 0$ such that*

$$|\psi_{\pm}(x, \zeta) - \psi_{\pm}^{\circ}(x, \zeta)| \leq C\sigma_{\pm}(x)\exp(\mp\eta x)$$

as $x \rightarrow \pm\infty$. ψ_{\pm} are analytic in ζ in the upper half plane.

Proof. By (1.2) we have the integral equations for ψ_{\pm} :

$$\psi_{\pm}(x, \zeta) - \psi_{\pm}^{\circ}(x, \zeta) = \begin{bmatrix} - \int_x^{\pm\infty} u(y)\psi_{\pm 2}(y, \zeta)\exp(i\zeta(y-x))dy \\ \int_x^{\pm\infty} u^*(y)\psi_{\pm 1}(y, \zeta)\exp(i\zeta(x-y))dy \end{bmatrix}.$$

These integral equations can be solved by successive approximation leading to the existence of the solutions and their estimates. Q.E.D.

The solutions ψ_{\pm} will be called the Jost solutions.

We have also

Lemma 1.2. *There exist unique solutions $e_{\pm}(x, \zeta)$ of (1.1) for $\text{Im } \zeta > 0$ such that*

$$\exp(\pm i\zeta x)\{e_{\pm}(x, \zeta) - \psi_{\pm}^{\circ}(x, \zeta)\} \rightarrow 0, \quad x \rightarrow \pm\infty.$$

These solutions are analytic in ζ .

Proof. In this case we solve (1.2) by the requirement that $\alpha(a) = 0$

and $\beta(x) \rightarrow 1$ as $x \rightarrow \infty$. Then we have the integral equations for $e = e_+$:

$$e_1(x, \zeta) - \exp(-i\zeta x) = - \int_x^\infty u(y) e_2(y, \zeta) \exp(i\zeta(y-x)) dy,$$

$$e_2(x, \zeta) = - \int_a^x u^*(y) e_1(y, \zeta) \exp(i\zeta(x-y)) dy.$$

Putting

$$e(x, \zeta) = \exp(-i\zeta x) h(x, \zeta),$$

these are turned into integral equations for h which can be solved by successive approximation provided $\sigma_+(a) < 1$. It is easy to show that $h - t(1, 0) \rightarrow 0$ as $x \rightarrow \infty$. Q.E.D.

Put

$$(1.3) \quad \psi_\pm(x, \zeta) = \exp(\pm i\zeta x) h_\pm(x, \zeta).$$

Assuming the integral representations

$$(1.4+) \quad h_+(x, \zeta) = t(0, 1) + \int_0^\infty B_+(x, y) \exp(2i\zeta y) dy,$$

$$(1.4-) \quad h_-(x, \zeta) = t(1, 0) + \int_{-\infty}^0 B_-(x, y) \exp(-2i\zeta y) dy,$$

we put these into the integral equations for the Jost solutions. Then we have

$$(1.5) \quad B_{+1}(x, y) + \int_x^{x+y} u(z) B_{+2}(z, x+y-z) dz = -u(x+y),$$

$$(1.6) \quad B_{+2}(x, y) - \int_x^\infty u^*(z) B_{+1}(z, y) dz = 0$$

and

$$B_{-1}(x, y) - \int_{-\infty}^x u(z) B_{-2}(z, y) dz = 0,$$

$$B_{-2}(x, y) + \int_{x+y}^x u^*(z) B_{-1}(z, x+y-z) dz = -u^*(x+y).$$

The potential u is reproduced by

$$u(x) = -B_{+1}(x, 0) = -B_{-2}^*(x, 0).$$

Let u be bounded integrable and put

$$m_{\pm}(x) = \text{ess sup}_{\pm y \geq \pm x} |u(y)|.$$

Then the integral equations for the kernels B_{\pm} just obtained can be solved by successive approximation and the solutions are estimated as

$$(1.7) \quad |B_{\pm}(x, y)| \leq m_{\pm}(x+y) \exp(\sigma_{\pm}(x))$$

(see [10, pp. 167–168]).

Now we study how additional smoothness of u improves that of $B = B_{+}$.

Lemma 1.3. *Suppose that u is n -times differentiable with integrable derivatives. Then $B^{(j,k)} = \partial^{j+k} B / \partial x^j \partial y^k$ exist for $j+k \leq n$ and the estimate*

$$\begin{aligned} & |B_1^{(j,k)}(x, y) + u^{(j+k)}(x+y)| + |B_2^{(j,k)}(x, y)| \\ & \leq C \sum_{i=0}^{j+k-1} m_i(x+y), \quad 1 \leq j+k \leq n, \end{aligned}$$

where

$$m_i(x) = \sup_{y \geq x} |u^{(i)}(x)|$$

hold.

Proof. By (1.6), B_2 is differentiable and

$$(1.8) \quad B_{2x}(x, y) = -u^*(x)B_1(x, y)$$

holds. By (1.5) B_1 is differentiable with respect to x and estimate for B_x follows. Results for $B^{(j,0)}$ follow by induction on j using (1.5) and (1.8).

By (1.5) B_1 is differentiable with respect to y and

$$(1.9) \quad B_{1x}(x, y) - B_{1y}(x, y) = u(x)B_2(x, y)$$

holds. B_{2y} exists by (1.6) and the estimate for B_y follows.

For general $B^{(j,k)}$ the assertion follows from double induction on $j+k$ and k . Q.E.D.

If $m_{\pm}(\pm x) \in L^1(a, \infty)$, we can define ψ_{\pm} by (1.3) and (1.4). ψ_{\pm} coincide with the Jost solutions. So we have.

Theorem 1.4. *Suppose that u is bounded and $m_{\pm}(\pm x)$ are in $L^1(a, \infty)$*

for some a . Then there exist kernels B_{\pm} estimated as (1.7) and Jost solutions are represented as (1.3), (1.4).

§ 2. Scattering Data and Their Properties

Now let $\zeta = \xi$ be real. Then

$$[\psi_{\pm}, \psi_{\pm}^*] = -1.$$

As ψ_{+} and ψ_{+}^* are linearly independent solutions of (1.1) for $\zeta = \xi$, we can expand ψ_{-} uniquely as

$$(2.1) \quad \psi_{-} = a(\xi)\psi_{+}^* + b(\xi)\psi_{+}.$$

Similarly

$$(2.2) \quad \psi_{+} = c(\xi)\psi_{-}^* + d(\xi)\psi_{-}.$$

We have

$$(2.3) \quad a(\xi) = [\psi_{-}, \psi_{+}], \quad b(\xi) = -[\psi_{-}, \psi_{+}^*]$$

and

$$(2.4) \quad a(\xi) = -c(\xi), \quad b(\xi) = d(\xi)^*.$$

Putting (2.2) into (2.1) and using the relations (2.4), we have

$$|a(\xi)|^2 + |b(\xi)|^2 = 1.$$

Putting (1.3), (1.4) into (2.3), we have

$$(2.5) \quad a(\xi) = 1 + \int_0^{\infty} \Gamma(y) \exp(2i\xi y) dy,$$

where

$$(2.6) \quad \Gamma(y) = B_{-1}(x, -y) + B_{+2}(x, y) + \int_0^y [B_{-1}(x, -z)B_{+2}(x, y-z) - B_{-2}(x, -z)B_{+1}(x, y-z)] dz$$

and

$$b(\xi) = \int \Pi(y) \exp(-2i\xi y) dy,$$

where

$$\begin{aligned} \exp(2i\xi x)II(y) &= B_{+1}^*(x, y) + B_{-2}(x, y) \\ &+ \int B_{-1}(x, z)B_{+1}^*(x, y-z)dz + \int B_{-2}(x, z)B_{+2}^*(x, y-z)dz \end{aligned}$$

(we have extended $B_{\pm}(x, y)$ as 0 for $\pm y < 0$).

Putting $x=0$, for example, we have the estimates:

$$|II(y)| \leq Cm_{\pm}(\pm y), \quad \pm y > 0,$$

and

$$|\Gamma(y)| \leq C(m_{-}(-2^{-1}y) + m_{+}(2^{-1}y)), \quad y > 0.$$

Γ and II are bounded integrable functions.

There are other expressions for Γ and II which are sometimes more convenient.

Considering the limit for $x \rightarrow \infty$ in (2.1) and comparing the coefficients of $\exp(\pm i\xi x)$, we have the integral representations

$$\begin{aligned} a(\xi) &= 1 + \int u(y)\psi_{-2}(y, \xi) \exp(i\xi y)dy, \\ b(\xi) &= - \int u^*(y)\psi_{-1}(y, \xi) \exp(-i\xi y)dy. \end{aligned}$$

Putting (1.3), (1.4) into these expressions and performing a formal change of the order of integrations, we obtain the expressions:

$$(2.7) \quad II(y) = -u(y) - \int_y^{\infty} u^*(x)B_{-1}(x, y-x)dx,$$

$$(2.8) \quad \Gamma(y) = \int u(x)B_{-2}(x, -y)dx.$$

For functions defined by (2.7) and (2.8) we have the estimates

$$(2.9) \quad |II(y) + u(y)| \leq C\sigma_{+}(y)m_{-}(y)$$

and

$$(2.10) \quad |\Gamma(y)| \leq C(m_{-}(-2^{-1}y) + \sigma_{+}(2^{-1}y)).$$

Thus we have

Lemma 2.1. *If u satisfies the assumption of Theorem 1.4 and $(1+|x|)u$ is integrable, then expressions (2.7) and (2.8) hold.*

Lemma 2.2. *Let u be in S , the space of C^∞ -functions which are rapidly decreasing together with all derivatives. Then Π is in S . Γ is C^∞ and rapidly decreasing as $x \rightarrow \infty$ together with all derivatives.*

Proof. By repeated differentiation of (2.7), (2.8) and by Lemma 1.3, we obtain similar estimates to (2.9) and (2.10) for each derivative. Q.E.D.

Let ζ_0 be a zero of order m of $a(\zeta)$ in the upper half plane. Because ψ_\pm are linearly dependent at $\zeta = \zeta_0$, they are square integrable by their asymptotic property.

Theorem 2.3. *There exists a chain of complex numbers $\mathbf{c}^+ = \{c_0^+, \dots, c_{m-1}^+\}$ ($c_0 \neq 0$) such that*

$$(2.11) \quad j!^{-1}d^j\psi_-(x, \zeta_0)/d\zeta^j = \sum_{\mu=0}^j \mu!^{-1}c_{j-\mu}^+ d^\mu \psi_+(x, \zeta_0)/d\zeta^\mu, \quad 0 \leq j \leq m-1,$$

hold.

Proof. Let $e_+(x, \zeta)$ be the solution of (1.1) defined in Lemma 1.2. Noting that $[\psi_+, e_+] = -1$, we expand ψ_- as

$$(2.12) \quad \psi_- = c(\zeta)\psi_+ + a(\zeta)e_+$$

where $c(\zeta) = [e_+, \psi_-]$ is analytic in the upper half plane.

Comparing the j -th coefficient of the Taylor expansion of (2.12) at $\zeta = \zeta_0$, we have (2.11) where

$$c_{\bar{j}}^+ = \mu!^{-1}d^\mu c(\zeta_0)/d\zeta^\mu. \quad \text{Q.E.D.}$$

Obviously the chain of complex numbers $\mathbf{c}^- = \{c_0^-, \dots, c_{m-1}^-\}$ is uniquely determined such that

$$(2.13) \quad \begin{bmatrix} c_0^+ & 0 & \dots & 0 \\ c_1^+ & c_0^+ & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ c_{m-1}^+ & c_{m-2}^+ & \dots & \dots & c_0^+ \end{bmatrix} \begin{bmatrix} c_0^- & 0 & \dots & 0 \\ c_1^- & c_0^- & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ c_{m-1}^- & c_{m-2}^- & \dots & \dots & c_0^- \end{bmatrix} = I.$$

The chains \mathbf{c}^\pm are called the normalization chains associated with the zero

ζ_0 of $a(\zeta)$.

We now assume that

$$(2.14) \quad a(\xi) \neq 0$$

for any real ξ . As the integral representation (2.5) holds with the integrable kernel Γ , $a(\zeta) \rightarrow 1$ as $|\zeta| \rightarrow \infty$. By this property and (2.14), $a(\zeta)$ has only a finite number of zeros in the upper half plane. We denote them by ζ_j ($j=1, \dots, N$) and their multiplicity by $m(j)$. Denote by c_j^\pm the chain associated with the zero ζ_j by Theorem 2.3.

Put

$$r_+(\xi) = b(\xi)a(\xi)^{-1}, \quad r_-(\xi) = d(\xi)c(\xi)^{-1}.$$

$r_\pm(\xi)$ are called the reflection coefficients. The collections

$$s_\pm = \{r_\pm(\xi), \zeta_j, c_j^\pm\}$$

are called the scattering data of the operator L_u or the potential u .

An extension of the Wiener-Lévy theorem [6, Lemma 1.2] gives

Lemma 2.4. *Let $a(\zeta)$ be defined by*

$$a(\zeta) = 1 + \int_0^\infty \Gamma(y) \exp(2i\zeta y) dy, \quad \Gamma \in L^1(0, \infty),$$

and let $a(\xi) \neq 0$ for any real ξ . Let ζ_j ($j=1, \dots, N$) be the zeros of order $m(j)$ of $a(\zeta)$ in the upper half plane. Then $a(\zeta)$ has the representation

$$(2.15) \quad a(\zeta) = \prod_{j=1}^N (\zeta - \zeta_j / \zeta - \zeta_j^*)^{m(j)} \exp\left(\int_0^\infty \Gamma_1(t) \exp(2i\zeta t) dt\right),$$

where Γ_1 is in $L^1(0, \infty)$.

Putting $\zeta = \xi$ in (2.15), we have

$$(2.16) \quad \log |a(\xi)| = \operatorname{Re} \int_0^\infty \Gamma_1(t) \exp(2i\xi t) dt.$$

§ 3. Derivation of the Fundamental Integral Equations

In this section we derive integral equations which connect the kernels B_\pm with the scattering data.

As the coefficient $b(\xi)$ is the Fourier transform of a bounded L^1 -function, it is a bounded L^2 -function and tends to zero as $\xi \rightarrow \pm\infty$. So $r_{\pm}(\xi)$ are also bounded L^2 -functions and tend to zero as $\xi \rightarrow \pm\infty$. Put

$$F_{\pm}(x) = \pi^{-1} \int r_{\pm}(\xi) \exp(\pm 2i\xi x) dx$$

as L^2 -Fourier transforms.

Now on the identity

$$a^{-1}(\xi)h_{-}(x, \xi) - {}^t(1, 0) = \{h_{+}(x, \xi) - {}^t(0, 1)\}^* + r_{+}(\xi) \exp(2i\xi x)h_{+}(x, \xi)$$

multiply by $\pi^{-1} \exp(2i\xi y)$ and integrate over $(-\infty, \infty)$. By the convolution theorem, the right-hand side gives

$$(3.1) \quad B_{+}^{*}(x, y) + \left[\begin{array}{c} 0 \\ F_{+}(x+y) \end{array} \right] + \int_0^{\infty} F_{+}(x+y+z)B_{+}(x, z)dz.$$

By Jordan's lemma [12, § 60], we have

$$(3.2) \quad \pi^{-1} \int \{a^{-1}(\xi)h_{-}(x, \xi) - {}^t(1, 0)\} \exp(2i\xi y) d\xi \\ = 2i \sum_{j=1}^N R(\zeta_j)$$

where $R(\zeta_0)$ is the residue of

$$a^{-1}(\zeta)h_{-}(x, \zeta) \exp(2i\zeta y)$$

at a zero ζ_0 of $a(\zeta)$.

Next we express $R(\zeta_0)$ in terms of the kernel B_{+} . We have

$$R(\zeta_0) = (m-1)!^{-1} (d/d\zeta)^{m-1} a_0(\zeta)^{-1} h_{-}(x, \zeta) \exp(2i\zeta y) \quad (\zeta = \zeta_0)$$

where

$$a_0(\zeta) = a(\zeta)(\zeta - \zeta_0)^{-m}.$$

By the relation (2.11),

$$(3.3) \quad R(\zeta_0) = \sum_{\mu=0}^{m-1} \mu!^{-1} c_{m-\mu-1}^{+} (d/d\zeta)^{\mu} a_0(\zeta)^{-1} h_{+}(x, \zeta) \\ \times \exp(2i\zeta(x+y)) \quad (\zeta = \zeta_0)$$

Put

$$(3.4) \quad f_{\pm}(x) = \mp 2i \sum_{\mu=0}^{m-1} \mu!^{-1} c_{m-\mu-1}^{\pm} (d/d\xi)^{\mu} a_0(\xi)^{-1} \exp(\pm 2i\xi x) \quad (\xi = \zeta_0).$$

$f_{\pm}(x)$ are of the form

$$f_{\pm}(x) = p_{\pm}(x) \exp(\pm 2i\zeta_0 x),$$

where $p_{\pm}(x)$ are polynomials of order $m-1$.

$R(\zeta_0)$ is now expressed as

$$(3.5) \quad R(\zeta_0) = -(2i)^{-1} \left\{ \begin{matrix} 0 \\ f_+(x+y) \end{matrix} \right\} + \int_0^{\infty} f_+(x+y+z) B_+(x, z) dz \Big\}.$$

Coming back to the derivation of integral equations for the kernels B_{\pm} , let $f_{\pm j}(x)$ be functions defined by (3.4) at the zeros ζ_j of $a(\xi)$. Put

$$\Omega_{\pm}(x) = F_{\pm}(x) + \sum_{j=1}^N f_{\pm j}(x).$$

Putting (3.5) into the equality (3.1)=(3.2), we have

$$(3.6+) \quad B_+^*(x, y) + \int_0^{\infty} \Omega_+(x+y+z) B_+(x, z) dz + \left[\begin{matrix} 0 \\ \Omega_+(x+y) \end{matrix} \right] = 0.$$

Analogously

$$(3.6-) \quad B_-^*(x, y) + \int_{-\infty}^0 \Omega_-(x+y+z) B_-(x, z) dz + \left[\begin{matrix} \Omega_-(x+y) \\ 0 \end{matrix} \right] = 0$$

holds.

These analogues of the Marchenko equation are first introduced in [19] under the assumption that all zeros of $a(\xi)$ are simple.

§ 4. Solvability of the Fundamental Integral Equations

In this section we discuss the solvability of (3.6) as an integral equation for B_+ .

Given a function $\Omega(x)$ belonging to $L^1(a, \infty)$ for any a , we define the integral operator Ω_x by

$$\Omega_x \varphi(y) = \int_0^{\infty} \Omega(x+y+z) \varphi(z) dz$$

as an operator on $L^1(0, \infty)$. Ω_x is a completely continuous operator (see [1, pp. 70–72]).

We consider the solvability of

$$(4.1) \quad \varphi^* + \Omega_x \varphi = \psi$$

which is a slight generalization of (3.6+).

Eliminating ψ_1 or ψ_2 we have

$$(4.2) \quad (I + \Omega_x^* \Omega_x) \varphi_1 = -\psi_2^* - \Omega_x^* \psi_1,$$

$$(4.3) \quad (I + \Omega_x^* \Omega_x) \varphi_2 = \psi_1^* + \Omega_x^* \psi_2.$$

If one of these equations is solved, the other component can be determined from the original equation (4.1).

Lemma 4.1. *Suppose $\Omega(x)$ is bounded integrable in (a, ∞) for any a . Then*

- (a) *the operator $(I + \Omega_x^* \Omega_x)$ has bounded inverse,*
- (b) *$(I + \Omega_x^* \Omega_x)^{-1}$ is continuous in x and bounded as $x \rightarrow \infty$ in the operator norm (topology),*
- (c) *the system (4.1) is uniquely solvable for any ψ and the estimate*

$$\|\varphi\| \leq C(x) \|\psi\|$$

holds for the solution. $C(x)$ denotes a certain non-increasing function.

Proof. Suppose φ is a solution of

$$(4.4) \quad (I + \Omega_x^* \Omega_x) \varphi = 0$$

in $L^1(0, \infty)$. By the properties of Ω and (4.4), the boundedness of φ follows. So φ belongs to L^2 . In L^2 the operator Ω_x^* is equal to $(\Omega_x)^*$, the adjoint of Ω_x , and therefore φ should be zero. By the complete continuity of Ω_x , the assertion (a) follows.

In the operator norm Ω_x is continuous in x and tends to zero as $x \rightarrow \infty$. The assertion (b) follows. (c) is obtained by applying (b) to (4.2) and (4.3). Q.E.D.

Let $r(\xi)$ be a square-integrable function. Let ζ_1, \dots, ζ_N denote points

in the upper half plane. For each j , let $m(j)$ be natural numbers and $p_j(x)$ be polynomials of order $m(j)-1$. Put

$$f_j(x) = p_j(x) \exp(2i\zeta_j x).$$

Put

$$F(x) = \pi^{-1} \int r(\xi) \exp(2i\xi x) d\xi$$

and

$$\Omega(x) = F(x) + \sum_{j=1}^N f_j(x).$$

Put

$$\sigma_j(x) = \int_x^\infty |\Omega^{(j)}(y)| dy$$

if the right hand side exists.

By Lemma 4.1, we get

Theorem 4.2. *If $\Omega(x)$ is bounded integrable in (a, ∞) for any a , there exists a unique solution $B(x, y)$, integrable in y , of*

$$(4.5) \quad B^*(x, y) + \int_0^\infty \Omega(x+y+z) B(x, z) dz + \left[\begin{array}{c} 0 \\ \Omega(x+y) \end{array} \right] = 0.$$

The estimates

$$(4.6) \quad \int_0^\infty |B(x, y)| dy \leq C(x)$$

and

$$(4.7) \quad |B(x, y)| \leq C(x) \sup_{z \geq x+y} |\Omega(z)|$$

hold.

Returning to the general situation, we obtain

Lemma 4.3. *Suppose that Ω is absolutely continuous and Ω, Ω' are in $L^1(a, \infty)$ for any a . Let ψ_x be differentiable in x and let φ_x be the solution of (4.1) for $\psi = \psi_x$ i.e.*

$$(4.8) \quad \varphi_x^* + \Omega_x \varphi_x = \psi_x.$$

Then φ_x is differentiable in x and

$$(4.9) \quad \varphi'_x + \Omega_x \varphi'_x = \psi'_x - \Omega'_x \varphi_x$$

holds.

Proof. For the difference coefficient

$$\varphi_{x,h} = h^{-1}(\varphi_{x+h} - \varphi_x)$$

we have the equation

$$(4.10) \quad \varphi^*_{x,h} + \Omega_x \varphi_{x,h} = \psi_{x,h} - h^{-1}(\Omega_{x+h} - \Omega_x) \varphi_{x+h}.$$

The right hand side converges to

$$\psi'_x - \Omega'_x \varphi_x$$

as $h \rightarrow 0$. Eliminating each component of $\varphi_{x,h}$ from (4.10) and applying Lemma 4.1 (a), we conclude that $\varphi_{x,h}$ converges to φ'_x . Q.E.D.

Under the same assumption on Ω , B defined by (4.5) is differentiable in y and the estimates

$$\int_0^\infty |B_y(x, y)| dy \leq C(x)$$

and

$$(4.11) \quad \left| B_y(x, y) + \left[\begin{array}{c} 0 \\ \Omega'(x+y) \end{array} \right] \right| \leq C(x) \sigma_1(x+y)$$

hold. A similar result for the x -derivative can be obtained by Lemma 4.3. In fact $B(x, y)$ is differentiable in x in the L^1 sense and the estimate

$$\int_0^\infty |B_x(x, y)| dy \leq C(x)$$

holds. Then it is easy to see that the derivative

$$(d/dx) \int_0^\infty \Omega(x+y+z) B(x, z) dz$$

exists in the ordinary sense and we can change the order of integration and differentiation. B_x also satisfies the inequality (4.11).

Put

$$\psi(x, \zeta) = \exp(i\zeta x) \left\{ t(0, 1) + \int_0^\infty B(x, y) \exp(2i\zeta y) dy \right\}.$$

Then ψ is differentiable in x by the results just described.

Theorem 4.4. *If Ω is absolutely continuous and Ω, Ω' are in $L^1(a, \infty)$ for any a , then ψ defined above satisfies*

$$L_w \psi = \zeta \psi$$

for

$$u(x) = -B_1(x, 0).$$

Proof. Put

$$C_1(x, y) = -B_{2x}(x, y) - u^*(x)B_1(x, y),$$

$$C_2(x, y) = B_{1x}(x, y) - B_{1y}(x, y) - u(x)B_2(x, y).$$

Then by direct calculation we have

$$C^*(x, y) + \int_0^\infty \Omega(x+y+z)C(x, z)dz = 0.$$

So $C=0$ follows. Once $C=0$ is established, the assertion follows by straightforward calculation. Q.E.D.

By induction based on Lemma 4.3, we have

Lemma 4.5. *Suppose that $\Omega^{(j)}$, $j \leq n$ exist and belong to $L^1(a, \infty)$ for any a . Then $B^{(j,k)}(x, y)$, $j+k \leq n$, exist and the estimates*

$$\int_0^\infty |B^{(j,k)}(x, y)| dy \leq C(x)$$

and

$$\left| B^{(j,k)}(x, y) + \left[\begin{matrix} 0 \\ \Omega^{(j+k)}(x+y) \end{matrix} \right] \right| \leq C(x) \sum_{i=0}^{j+k} \sigma_i(x+y)$$

hold.

§ 5. Solution of the Inverse Problem

Let $r(\xi)$ be a square integrable continuous function. Assume that there exists $a(\xi)$ which has the properties assumed in Lemma 2.4 and is related to $r(\xi)$ by

$$|a(\xi)|^2 = (1 + |r(\xi)|^2)^{-1}.$$

By (2.15), $\Gamma_1(t)$ in Lemma 2.4 is uniquely determined by this relation.

Put $r_+(\xi) = r(\xi)$ and

$$r_-(\xi) = -a(\xi)^{-1} a^*(\xi) r^*(\xi).$$

Define $F_{\pm}(x)$ by the formula

$$(5.1) \quad F_{\pm}(x) = \pi^{-1} \int r_{\pm}(\xi) \exp(\pm 2i\xi x) d\xi.$$

Assume that $F_{\pm}(x)$ are absolutely integrable and $F_{\pm}(\pm x)$, $F'_{\pm}(\pm x)$ are in $L^1(a, \infty)$ for any a .

In the later sections, we will be concerned in the case where $r(\xi)$ is in S . Then $\log(1 + |r(\xi)|^2)$ is also in S . So (2.16) determines $\Gamma_1(t)$ as the restriction to $(0, \infty)$ of an element in S . $a(\xi)$ defined by (2.15) has the required properties. r_{\pm} are in S and so are F_{\pm} .

To each zero ζ_j of order $m(j)$ of $a(\xi)$, we associate the chains of complex numbers

$$c_j^{\pm} = \{c_{j,0}^{\pm}, c_{j,1}^{\pm}, \dots, c_{j,m(j)-1}^{\pm}\}, \quad c_{j,0}^{\pm} \neq 0,$$

each connected by the relation (2.13).

Define $f_{\pm j}(x)$ by the formulas (3.4) and put

$$(5.2) \quad \Omega_{\pm}(x) = F_{\pm}(x) + \sum_{j=1}^N f_{\pm j}(x).$$

Consider the fundamental integral equations (3.6) and let $B_{\pm}(\lambda, y)$ be their solutions. Define $h_{\pm}(x, \xi)$ by (1.4) and then $\psi_{\pm}(x, \xi)$ by (1.3). Put

$$u_+(x) = -B_{+1}(x, 0) \quad u_-(x) = -B_{-2}^*(x, 0).$$

Then, by Theorem 4.6, ψ_{\pm} satisfy (1.1) with $u = u_{\pm}$ respectively. For B_{\pm} , the estimates formulated in Theorem 4.2 hold.

We show that under certain additional conditions $u_{\pm}(x)$ coincide. It follows immediately, once the relation

$$(5.3) \quad a(\xi)^{-1}\psi_-(x, \xi) = \psi_+^*(x, \xi) + r_+(\xi)\psi_+(x, \xi)$$

is established.

To show that (5.3) holds, put

$$g(x, \xi) = h_+^*(x, \xi) + r_+(\xi) \exp(2i\xi x)h_+(x, \xi)$$

and

$$\begin{aligned} G(x, y) &= \pi^{-1} \int (g(x, \xi) - {}^t(1, 0)) \exp(2i\xi y) d\xi \\ &= B_+^*(x, y) + \int_0^\infty F_+(x+y+z)B_+(x, z)dz + \begin{bmatrix} 0 \\ F_+(x+y) \end{bmatrix}. \end{aligned}$$

Then by the fundamental equation, we have

$$G(x, y) = -\sum_{j=1}^N \left\{ \begin{bmatrix} 0 \\ f_{+j}(x+y) \end{bmatrix} + \int_0^\infty f_{+j}(x+y+z)B_+(x, z)dz \right\}$$

for $y > 0$.

$g(x, \xi)$ which are defined for real ξ can be analytically continued to the upper half plane. To show this we need

Lemma 5.1. *Put*

$$\begin{aligned} f(x) &= p(x) \exp(2i\xi_0 x) & x > 0, \\ 0 & & x < 0, \end{aligned}$$

where $p(x)$ is a polynomial of order $m-1$. Then we have, for $x+y > 0$,

$$(5.4) \quad \begin{bmatrix} 0 \\ f(x+y) \end{bmatrix} + \int_0^\infty f(x+y+z)B_+(x, z)dz = p(x, y) \exp(2i\xi_0 y),$$

where each component of $p(x, y)$ is defined for $x+y > 0$ and is a polynomial of order $m-1$ in y .

Proof. $f(x)$ is written as a Fourier transform:

$$f(x) = \pi^{-1} \int g(\xi) \exp(2i\xi x) d\xi$$

where $q(\xi)$ is a polynomial of order m in $(\xi - \zeta_0)^{-1}$ without constant term. Then for $x + y > 0$, the left hand side of (5.4) is equal to

$$\begin{aligned} & \pi^{-1} \int q(\xi) \psi_+(x, \xi) \exp(i\xi x + 2i\xi y) d\xi \\ &= 2i \operatorname{Res}_{\zeta=\zeta_0} [q(\zeta) \psi_+(x, \zeta) \exp(i\zeta x + 2i\zeta y)] \\ &= 2i(m-1)!^{-1} (d/d\zeta)^{m-1} q(\zeta) (\zeta - \zeta_0)^m \psi_+(x, \zeta) \exp(i\zeta x + 2i\zeta y) \\ & \hspace{15em} (\zeta = \zeta_0). \end{aligned}$$

The last term can be written as asserted. Q.E.D.

We have further

$$(5.5) \quad p(x, y) \exp(2i\zeta_0 y) = \pi^{-1} \int q(x, \xi) \exp(2i\xi y) d\xi, \quad y > 0,$$

where $q(x, \xi)$ is a polynomial of order m in $(\xi - \zeta_0)^{-1}$ without constant term.

Denote by $q_j(x, \zeta)$ the functions connected with $f_{+j}(x)$ by the above construction.

Put

$$g_1(x, \xi) = g(x, \xi) - {}^t(1, 0) + \sum_{j=1}^N q_j(x, \xi).$$

Then, for $x + y, y > 0$, we have

$$(5.6) \quad \int g_1(x, \xi) \exp(2i\xi y) d\xi = 0.$$

So $g_1(x, \xi)$ has analytical continuation as a holomorphic function into the upper half plane. $g(x, \xi)$ has analytical continuation as a meromorphic function into the upper half plane. Its poles are exhausted by ζ_1, \dots, ζ_N . ζ_j are poles of order $m(j)$.

Denote by R_j the residue of

$$g(x, \zeta) \exp(2i\zeta y), \quad y > 0,$$

at ζ_j , i.e. the residue of

$$-q_j(x, \zeta) \exp(2i\zeta y)$$

at ζ_j . It is expressed as

$$R_j = -(2\pi i)^{-1} \int g_j(x, \xi) \exp(2i\xi y) d\xi.$$

By (5.4) and (5.5), we have, for $x+y, y>0$,

$$R_j = -(2i)^{-1} \left\{ \left[\begin{matrix} 0 \\ f_j(x+y) \end{matrix} \right] + \int_0^\infty f_j(x+y+z) B_+(x, z) dz \right\}.$$

By the definition of f_j , we have

$$(5.7) \quad R_j = \sum_{\mu=0}^{m(j)-1} \mu!^{-1} c_{j, m(j)-1-\mu} (d/d\xi)^\mu a_j(\zeta) \psi_+(x, \zeta) \exp(i\zeta x + 2i\zeta y) \quad (\zeta = \zeta_j)$$

where

$$a_j(\zeta) = (\zeta - \zeta_j)^{-m(j)} a(\zeta).$$

Put

$$(5.8) \quad h(x, \zeta) = a(\zeta) g(x, \zeta)$$

and

$$(5.9) \quad \psi(x, \zeta) = h(x, \zeta) \exp(-i\zeta x).$$

Then $\psi(x, \zeta)$ is holomorphic in the upper half plane. R_j is also equal to the residue of

$$a(\zeta)^{-1} \psi(x, \zeta) \exp(i\zeta x + 2i\zeta y)$$

at ζ_j and is therefore expressed as

$$(5.10) \quad R_j = (m-1)!^{-1} (d/d\zeta)^{m-1} a_j(\zeta)^{-1} \psi(x, \zeta) \exp(i\zeta x + 2i\zeta y), \quad (\zeta = \zeta_j).$$

Comparison of (5.7), (5.10) and an elementary argument lead to

$$(5.11) \quad k!^{-1} (d/d\zeta)^k \psi(x, \zeta_j) = \sum_{\mu=0}^k \mu!^{-1} c_{j, k-\mu}^+ (d/d\zeta)^\mu \psi_+(x, \zeta_j),$$

$$0 \leq k \leq m(j) - 1.$$

Theorem 5.2. *Let $r(\xi) = O(\xi^{-1})$, $\xi \rightarrow \pm \infty$, and also let the conditions formulated in the beginning of this section be satisfied. Then $h(x, \zeta)$ defined by (5.8) is represented as*

$$(5.12) \quad h(x, \zeta) = {}^t(1, 0) + \int_{-\infty}^0 C(x, y) \exp(-2i\zeta y) dy, \quad x > 0,$$

$C(x, y)$ being square integrable in y . $C(x, y)$ ($x > 0$) is the solution of the fundamental equation

$$C(x, y) + \int_{-\infty}^0 \Omega_{-}(x+y+\varepsilon) C(x, z) dz + \left[\begin{matrix} \Omega_{-}(x+y) \\ 0 \end{matrix} \right] = 0.$$

Proof. As $B_y(x, y)$ is integrable in y , $\xi g_1(x, \xi)$ is bounded. By (5.6), We can apply the Phragmén-Lindelöf type argument (see [9, Problem 325]) and conclude that, for $x > 0$, $\zeta g_1(x, \zeta)$ is bounded in $\text{Im } \zeta \geq 0$. So $h(x, \zeta) - {}^t(1, 0) \rightarrow 0$ uniformly as $|\zeta| \rightarrow \infty, \text{Im } \zeta \geq 0$. By Jordan's lemma [12, §60],

$$\int \{h(x, \xi) - {}^t(1, 0)\} \exp(2i\xi y) d\xi = 0, \quad y > 0,$$

holds. The first assertion follows.

By direct calculation, it is easy to show that

$$-a(\xi)^{-1} h_+(x, \xi) = h^*(x, \xi) + r_-(\xi) h(x, \xi) \exp(-2i\xi x).$$

Taking into account (5.11) and repeating the argument of § 3, the second assertion follows. Q.E.D.

As the L^2 -solution of the fundamental equation is uniquely determined (see the proof of Lemma 4.2),

$$C(x, y) = B_-(x, y) \quad x > 0$$

i.e.

$$\psi(x, \zeta) = \psi_-(x, \zeta) \quad x > 0.$$

So we have (5.3) for $x > 0$ and therefore $u_{\pm}(x)$ coincide for $x > 0$.

By the estimate (4.7), we finally have

Theorem 5.3. *In addition to the assumption in the previous theorem, we assume that $m_{\pm}(\pm x)$ are in $L^1(a, \infty)$. Then*

$$\{r_{\pm}(\xi), \zeta_j, c_j^{\pm}\}$$

are the scattering data of an (uniquely determined) integrable potential which satisfies the assumptions of Theorem 1.4.

Application of Lemma 4.5 leads to

Corollary 5.4. *If $r_+(\xi)=r(\xi)$ belongs to S , then the potential belongs to S .*

§ 6. Non-linear Schrödinger Equation

We now turn to the construction of solutions for (0.1) in terms of scattering data of L_u .

Put

$$B_u = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} (D^2 + 2^{-1}|u|^2) - 2^{-1} \begin{bmatrix} 0 & u \\ u^* & 0 \end{bmatrix} D - 2^{-1} D \begin{bmatrix} 0 & u \\ u^* & 0 \end{bmatrix}.$$

Then the operator

$$[B_u, L_u] = B_u L_u - L_u B_u$$

is the multiplication by the matrix valued function:

$$(6.1) \quad [B_u, L_u] = \begin{bmatrix} 0 & -2^{-1}iuu_{xx} - i|u|^2u \\ 2^{-1}iu^*_{xx} + i|u|^2u^* & 0 \end{bmatrix}$$

So for $u=u(t)=u(x, t)$, (0.1) and the operator evolution equation

$$(6.2) \quad dL_u/dt = i[B_u, L_u]$$

are equivalent.

This relation is found in [15] and is different from the original relation between L_u and (0.1) first given in [19]. A similar relation for the KdV equation has been found by Lax [7].

We now describe a procedure to determine the time variation of the Jost solutions and scattering data of $L_u(t)$ on a formal basis. Relations found here will be used later to construct solutions.

Differentiating $L_u\psi_{\pm} = \zeta\psi_{\pm}$ with respect to t , we have

$$(6.3) \quad (dL_u/dt)\psi_{\pm} + L_u(d\psi_{\pm}/dt) = \zeta d\psi_{\pm}/dt.$$

If (6.2) holds, (6.3) can be written as

$$L_u(d\psi_{\pm}/dt - iB_u\psi_{\pm}) = \zeta(d\psi_{\pm}/dt - iB_u\psi_{\pm}).$$

Taking into account the asymptotic behavior and uniqueness of the Jost

solutions, we obtain

$$(6.4) \quad d\psi_{\pm}/dt - iB_w\psi_{\pm} = \mp i\zeta^2\psi_{\pm}.$$

We proceed to the determination of the time variation of the scattering data. Differentiating the relation (2.1) with respect to t and eliminating $d\psi_{\pm}/dt$ by (6.4), we obtain an identity

$$da(\xi)/dt\psi_+^* + (db(\xi)/dt - 2i\xi^2b(\xi))\psi_+ = 0.$$

So we have

$$a(\xi, t) = a(\xi, 0), \quad b(\xi, t) = b(\xi, 0)\exp(2i\xi^2t).$$

$a(\zeta, t)$ is independent of t and so are its zeros ζ_1, \dots, ζ_N in the upper half plane.

Differentiating (6.4) with respect to ζ , we get

$$(6.5) \quad \begin{aligned} d\psi_{\pm}^{(k)}/dt - iB_w\psi_{\pm}^{(k)} \\ = \mp i\zeta^2\psi_{\pm}^{(k)} \mp 2ik\zeta\psi_{\pm}^{(k-1)} \mp ik(k-1)\psi_{\pm}^{(k-2)} \quad (\zeta = \zeta_0) \end{aligned}$$

where $\psi^{(k)} = d^k\psi/d\zeta^k$. Then differentiate (2.11) in t and eliminate $d\psi_{\pm}^{(k)}/dt$ by (6.5). We have

$$\begin{aligned} \sum_{\mu=0}^k \mu!^{-1} (dc_{k-\mu}^{\pm}/dt \mp 2i\zeta_0^2c_{k-\mu}^{\pm} \mp 4i\zeta_0c_{k-1-\mu}^{\pm} \mp 2ic_{k-2-\mu}^{\pm})\psi^{(\mu)} = 0 \\ (c_{-1}^{\pm} \equiv c_{-2}^{\pm} \equiv 0) \end{aligned}$$

From these relations, by induction on k , we obtain the system of differential equations for $c_k^{\pm}(t)$:

$$(6.6) \quad dc_k^{\pm}/dt = \pm(2i\zeta_0^2c_k^{\pm} + 4i\zeta_0c_{k-1}^{\pm} + 2ic_{k-2}^{\pm}), \quad 0 \leq k \leq m-1.$$

Next we consider the properties of differential equations (6.6). If these differential equations are satisfied, then f_{\pm} defined by (3.4) satisfy

$$\partial f_{\pm}/\partial t = \mp 2^{-1}i\partial^2 f_{\pm}/\partial x^2.$$

It is easy to show by induction that the solution $c_k^{\pm}(t)$ are polynomials of order k in t multiplied by $\exp(\pm 2i\zeta_0^2t)$.

Lemma 6.1. *If the relation (2.13) for $c_k^\pm(t)$ is satisfied at $t=0$, then it is satisfied for all t .*

Proof. The (i, j) element of the product on the left hand side of (2.13) is expressed as

$$\sum_{k=j}^{m-i} c_{m-i-k}^+(t) c_{k-j}^-(t).$$

Their t -derivatives are seen to vanish by (6.6). Q.E.D.

Now we come to the construction of the solutions. Suppose $u(x)$ is an element of S . Then the coefficients $a(\xi)$, $b(\xi)$ and the scattering data $\{r_\pm(\xi), \zeta_j, c_j^\pm\}$ of L_u are determined. $a(\xi)$ is assumed to satisfy our basic assumption (2.14). Let $c_j^\pm(t)$ be the chain of complex numbers obtained as the solution of (6.6) for the initial conditions c_j^\pm . Put $a(\xi, t) = a(\xi)$ and let $f_{\pm j}(t)$ be defined by the chain $c_j^\pm(t)$ according to the formula (3.4) at the zero $\zeta = \zeta_j$ of $a(\xi)$. Put

$$r_\pm(\xi, t) = r_\pm(\xi) \exp(\pm 2i\xi^2 t).$$

As $r_\pm(\xi)$ are in S , so are $r_\pm(\xi, t)$ in ξ .

By Corollary 5.4, we have

Theorem 6.2. *The collection*

$$\{r_\pm(\xi, t), \zeta_j, c_j^\pm(t)\}$$

is the scattering data of $u(x, t)$ belonging to S for each t .

If we prove that one of the relations (6.4) holds with the differentiability of $u(x, t)$ in t , then $u(x, t)$ is a solution of (0.1). In fact put (6.4) into (6.3). Then

$$\langle dL_u/dt - i[B_u, L_u] \rangle \psi_\pm = 0$$

follows. By (6.1), $u(x, t)$ is a solution of (0.1). So we prove the equation (6.4+) for ψ_+ . We omit the suffix + in later discussion.

(6.4+) is equivalent to $dh/dt = ig$, $h = h_+(x, \zeta; t)$, where $g = g(x, \zeta; t)$ is written as

$$(6.7) \quad g = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \zeta^2 h + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} (2i\zeta h_x + h_{xx} + 2^{-1}|u|^2 h) - \begin{bmatrix} 0 & u \\ u^* & 0 \end{bmatrix} (i\zeta h + h_x) - 2^{-1} \begin{bmatrix} 0 & u_x \\ u_x^* & 0 \end{bmatrix} h.$$

Put

$$F(x, t) = \pi^{-1} \int r_+(\xi, t) \exp(2i\xi x) d\xi$$

and

$$\Omega(x, t) = F(x, t) + \sum_{j=1}^N f_j(x, t).$$

The differential equation

$$(6.8) \quad \Omega_t + 2^{-1}i\Omega_{xx} = 0$$

holds as F and f_j satisfy the equation of the same form.

Let $B = B_+(x, y; t)$ be the solution of the fundamental equation:

$$B^*(x, y; t) + \int_0^\infty \Omega(x+y+z; t) B(x, z; t) dz + \begin{bmatrix} 0 \\ \Omega(x+y; t) \end{bmatrix} = 0.$$

The parameter t will be omitted later for notational convenience.

By an argument analogous to Lemma 4.3 and the paragraph that follows, B is differentiable in t . The equation

$$(6.9) \quad B_t^*(x, y) + \int_0^\infty \Omega(x+y+z) B_t(x, z) dz + D(x, y) = 0,$$

where

$$D(x, y) = -2^{-1}i \int_0^\infty \Omega_{xx}(x+y+z) B(x, z) dz - \begin{bmatrix} 0 \\ 2^{-1}i\Omega_{xx}(x+y) \end{bmatrix},$$

hold by (6.8). Using the relations (1.8), (1.9) and integrating by part, we have the following integral representation for g defined by (6.7):

$$g(x, \zeta; t) = \int_0^\infty C(x, y; t) \exp(2i\zeta y) dy$$

where

$$C_1(x, y) = 2^{-1}B_{1xx}(x, y) - u(x)B_{2x}(x, y)$$

$$C_2(x, y) = -2^{-1}B_{2xx}(x, y) - u^*(x)B_{1x}(x, y).$$

By direct calculation we have

$$(6.10) \quad C^*(x, y) + \int_0^\infty \Omega(x+y+z)C(x, z)dz - iD(x, y) = 0.$$

Comparing (6.9) and (6.10), we get

$$B_t = iC,$$

because the homogeneous equation associated with the fundamental equation has only the trivial solution. The relation (6.4) has been established.

Thus we have proved

Theorem 6.3. *The function $u(x, t)$ in the previous theorem satisfies (0.1).*

§ 7. The Modified KdV Equation

In this section we indicate how the real solution for the modified KdV equation (0.2) can be constructed.

Put

$$A_v = -4D^3 + 3 \begin{bmatrix} -v^2 & iv_x \\ iv_x & -v^2 \end{bmatrix} D + 3D \begin{bmatrix} -v^2 & iv_x \\ iv_x & -v^2 \end{bmatrix}.$$

Then by direct calculation (0.2) is equivalent to

$$(7.1) \quad dL_{iv}/dt = [A_v, L_{iv}].$$

So it is necessary to find in terms of scattering data the condition that potential is purely-imaginary. Suppose that u is purely imaginary. Then the Jost solution have the symmetry property

$$\psi_\pm(x, -\xi^*) = t(\mp \psi_{\pm 1}^*(x, \xi), \pm \psi_{\pm 2}^*(x, \xi)).$$

Then by (2.3),

$$a(-\xi^*) = a^*(\xi), \quad b(-\xi) = -b^*(\xi).$$

So the reflection coefficients satisfy

$$(7.2) \quad r_{\pm}(-\xi) = -r_{\pm}^*(\xi).$$

Concerning the zeros ζ_j ($1 \leq j \leq N$) of $a(\zeta)$, there exists a permutation σ of order 2 of $\{1, \dots, N\}$ (we can change the numbering of the zeros so that $\sigma(2j-1) = 2j$, $\sigma(2j) = 2j-1$, $j \leq M$; $\sigma(j) = j$, $j > 2M$ for certain integer M , $0 \leq 2M \leq N$) such that

$$(7.3) \quad \zeta_{\sigma(j)} = -\zeta_j^*$$

and

$$(7.4) \quad m(\sigma(j)) = m(j)$$

hold. The chain associated with the zero ζ_j should satisfy

$$(7.5) \quad c_{\sigma(j),k}^{\pm} = (-1)^{k+1} c_{j,k}^{\pm,*} \quad 0 \leq k \leq m(j) - 1.$$

In fact, (7.5+) follows from

$$c^*(-\zeta^*) = -c(\zeta)$$

in the notation of Theorem 2.3.

Lemma 7.1. *If the basic assumption (2.14) is satisfied, conditions (7.2+), (7.3), (7.4) and (7.5+) are necessary and sufficient conditions for the potential to be purely imaginary valued.*

Proof. Necessity has been already shown. Suppose these conditions are satisfied. Then we have

$$\Omega_+^*(x) = -\Omega_+(x).$$

So by the uniqueness of the solution of the fundamental equation

$$B_{+i}^*(x, y) = (-1)^i B_{+i}(x, y) \quad i = 1, 2$$

follow. Therefore the potential is purely imaginary valued. Q.E.D.

As the remaining argument is completely analogous to that of § 6, we only indicate how the formulas should be modified.

If $v = v(t) = v(x, t)$ is a solution of (0.2) i.e. (7.1), then the Jost solutions

should satisfy

$$(7.6) \quad d\psi_{\pm}/dt - B_{iv}\psi_{\pm} = \pm 4i\xi^3\psi_{\pm}.$$

So we have

$$a(\xi, t) = a(\xi, 0), \quad b(\xi, t) = b(\xi, 0) \exp(8i\xi^3 t),$$

and therefore

$$r_{\pm}(\xi, t) = r_{\pm}(\xi, 0) \exp(\pm 8i\xi^3 t).$$

(6.6) is replaced by

$$dc_k^{\pm}/dt = \pm(8i\xi_0^3 c_k^{\pm} + 24i\xi_0^2 c_{k-1}^{\pm} + 24i\xi_0 c_{k-2}^{\pm} + 8ic_{k-3}^{\pm}), \quad 0 \leq k \leq m-1.$$

(7.6) is equivalent to $dh/dt = g$, $h = h_+$, where $g = g(x, \xi; t)$ is written as

$$\begin{aligned} & 12\xi^2 h_x - 12i\xi h_{xx} - 4h_{xxx} \\ & + 6 \begin{bmatrix} -v^2 & iv_x \\ iv_x & -v^2 \end{bmatrix} (i\xi h + h_x) + 3 \begin{bmatrix} -2vv_x & iv_{xx} \\ iv_{xx} & -2vv_x \end{bmatrix} h. \end{aligned}$$

We have the integral representation for g :

$$g(x, \xi; t) = \int_0^{\infty} C(x, y; t) \exp(2i\xi y) dy$$

where

$$C = -B_{xxx} + 3 \begin{bmatrix} -v^2 & iv_x \\ iv_x & -v^2 \end{bmatrix} B_x.$$

The remaining part of argument is completely analogous to that of § 6.

Our solutions contain all reflectionless solutions of (0.2). Previously a part of reflectionless solutions ($a(\xi)$ with simple zeros) have been studied (Hirota [5], Tanaka [14], [15] and Wadati [17], [18]).

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