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On the Construction of Invariant Measure over the Orthogonal Group on the Hilbert Space by the Method of Cayley Transformation

By

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The purpose of this paper is to construct some invariant measures over the infinite dimensional rotation group, analogously to the Haar measure in the finite dimensional case. In this direction there are some results making use of Haar measure on compact groups or Gaussian measure on Hilbert spaces. See, [5], [6], [9] and [10]. But it seems to me that the treatment of the whole group is complicated and difficult. On the other hand, the Cayley transformation in the finite dimensional Euclid space gives a correspondence between the special orthogonal group and the set of skew-symmetric operators, and still may be useful for the infinite dimensional case. Thus, we restrict our consideration to a subgroup which is included in the domain of Cayley transformation. Then the problem is transformed as follows. *To the rotationally invariant measure on this subgroup what measure corresponds on a suitable class of infinite dimensional skew-symmetric operators?* In order to solve it we first consider the Cayley image of Haar measure in the n-dimensional case and second construct a finitely additive measure as the limit of $n\to\infty$. Lastly we discuss the countably additive extension of so obtained measure.
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§ 1. Some properties of Cayley transformation in the finite dimensional case.

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§ 1. Some Properties of Cayley Transformation in the Infinite Dimensional Case

Let H be a real separable Hilbert space, $O(H)$ be the orthogonal group on it and I be the identity operator. We start from the following subgroup of $O(H)$. $O_0(H)=\{T\in O(H);\ I-T$ is of finite rank$\}$. Various topologies are considerable, but here we give two special norms on this group for our later discussion. One is the operator norm and the other is Hilbert-Shmidt norm. Perhaps the lack of completeness is inconvenient, so we extend $O_0(H)$ to larger classes $O_c(H)$ and $O_h(H)$, where $O_c(H)=\{T\in O(H);\ I-T$ is a compact operator$\}$, $O_h(H)=\{T\in O(H);\ I-T$ is a Hilbert-Shmidt operator$\}$. The metric d_c and d_h are defined as follows. $d_c(T, S)=\|T-S\|_{op}$ and $d_h(T, S)=\|T-S\|_{HS}$.

Theorem 1.1. *$(O_c(H), d_c)$ and $(O_h(H), d_h)$ are complete separable metric groups and these two metrics are invariant under the action of $O(H)$ from left and right.*

The assertion of the theorem is easily checked, so that we omit the proof.

Lemma 1.1. *Let T belong to $O_c(H)$. If T has not eigen value -1, then $(I+T)$ has a bounded inverse.*

Proof is omitted. See [11].

The subset of $O_c(H)$ appeared in the above lemma will be denoted by G_c and $G_c\cap O_h(H)$ will be denoted by G_h.

Lemma 1.2. *G_c and G_h are open sets in (O_c, d_c) and (O_h, d_h) respectively.*

Proof. The natural mapping from (O_h, d_h) to (O_c, d_c) is trivially continuous and $G_h = G_c \cap O_h$, so that we have only to prove in the case of compact one. Supposing $T \in G_c$, we shall prove that if $\|S\|_{op} < \frac{1}{\|(I+T)^{-1}\|_{op}}$, $T+S$ has not eigen value -1. The relation $(T+S)x = -x$ implies $Sx = -(I+T)x$, hence $(I+T)^{-1}Sx = -x$, so that we have $1 \leqq \|(I+T)^{-1}S\|_{op} \leqq \|(I+T)^{-1}\|_{op}\|S\|_{op}$, which is a contradiction. Q.E.D.

Let $S_c(H)$, $S_h(H)$, $S_n(H)$ be the sets of skew-symmetric compact, skew-symmetric Hilbert-Shmidt, and skew-symmetric nuclear operators on the Hilbert space H, respectively. We equip the operator norm, the Hilbert-Shmidt norm and the trace norm in these spaces, respectively.

Theorem 1.2. *The infinite dimensional Cayley transformation* $C_\infty(U) = (I-U)(I+U)^{-1}$ *is an onto homeomorphism from* G_c *and* G_h *to* $S_c(H)$ *and* $S_h(H)$ *respectively and the inverse mapping has the same form as* C_∞.

Proof. The algebraic assertion and onto properties are the same as for the finite dimensional case, so that we only check the continuity property. Before the proof we state several lemmas.

Lemma 1.3. *Let* X *be a bounded operator. If* $\|X\|_{op} < 1$, $(I-X)$ *has a bounded inverse and* $\|I-(I-X)^{-1}\|_{op} \leqq \frac{\|X\|_{op}}{1-\|X\|_{op}}$.

Lemma 1.4. *Let* X *be a bounded operator. If* X *has a bounded inverse, the mapping* $X \rightarrow X^{-1}$ *is continuous with respect to the operator norm.*

The above two lemmas are well known. See [12].

Proof of theorem 1.2. Suppose $U_n \rightarrow U_0$ in G_α, where $\|\cdot\|_\alpha$ is the operator norm or Hilbert-Shmidt norm. Then we have

$$\begin{aligned}
&\|(I+U_n)^{-1}(I-U_n) - (I+U_0)^{-1}(I-U_0)\|_\alpha \\
&\leqq \|(I+U_n)^{-1}(I-U_n) - (I+U_n)^{-1}(I-U_0)\|_\alpha \\
&+ \|(I+U_n)^{-1}(I-U_0) - (I+U_0)^{-1}(I-U_0)\|_\alpha \\
&\leqq \|(I+U_n)^{-1}\|_{op}\|U_n - U_0\|_\alpha + \|(I+U_n)^{-1} - (I+U_0)^{-1}\|_{op}\|I-U_0\|_\alpha
\end{aligned}$$

Thus the assertion will be proved by the above inequality together with Lemma 1.4. The same argument holds for the inverse mapping. Q.E.D.

Remark. (1) The set G_α (where α is c or h) is invariant under the operation of group inverse and the action of $T(\cdot)\,T^{-1}$ for all $T\in O(H)$.

(2) $C_\infty(I)=0$, $C_\infty(U^{-1})=-C_\infty(U)$ and $C_\infty(TUT^{-1})=TC_\infty(U)T^{-1}$ for all $T\in O(H)$, $U\in G_\alpha$.

§2. The Cayley Image of Haar Measure and Its Continuity in the Finite Dimensional Case.

Let $SO(n)$ be the special orthogonal group on the n-dimensional Euclid space, μ_n be the normalized Haar measure on it and C_n be the Cayley transformation from $SO(n)$ to $S(n)$, where $S(n)$ is the set of n-dimensional skew-symmetric matrices. The domain of C_n is the set of operators which have not eigen value -1 and will be denoted by G_n.

Lemma 2.1. *$\mu_n(G_n)=1$ and G_n is a open dense set in $SO(n)$ in the natural topology.*

The assertion is easily checked, so that we omit it.

Naturally $S(n)$ may be considerable as $\frac{1}{2}n(n-1)$-dimensional Euclid space, regarding the entries of the upper part of the diagonal as independent variables. Symbolically we shall write $dA=da_{1,2}da_{1,3}\dots da_{n-1,n}$ for the volume element of the Lebesgue measure in this space.

Theorem 2.1. *Let ν_n be the image of μ_n under the map C_n. Then it has the following explicit form.* $\nu_n(E)=\gamma_n\int_E \frac{dA}{\det(I+A)^{n-1}}$; *where γ_n is a normalizing constant, and E is a Borel set.*

Proof. If X, X_1, and X_2 are sufficiently near to 0, then

$$(I-X)(I+X)^{-1}=I-2X+2X^2-\dots, \text{ and}$$
$$(I-X_1)(I+X_1)^{-1}(I-X_2)(I+X_2)^{-1}=I-2(X_1+X_2)+\dots.$$

Thus, an infinitesimal rotation on $SO(n)$ corresponds to an infinitesimal translation on $S(n)$, therefore the invariance property of μ_n implies that ν_n is approximately Lebesgue at origin up to a constant factor; $d\nu_n=\gamma_n dX$ as denoted symbolically. Let $A_0=(I-U_0)(I+U_0)^{-1}$ be an arbitrary element in $S(n)$, where $U_0\in SO(n)$. The transformation $W(X)=C_n(C_n^{-1}(X)\cdot U_0)$ is a measure preserving map on $S(n)$ and $W(0)=A_0$, so that $d\nu_n(0)=d(W\nu_n)(0)=d\nu_n(A_0)$. If we neglect more than first order terms, then $A=W(X)\sim A_0+(I+A_0)X(I-A_0)$ and $dA=\frac{\partial(A)}{\partial(X)}dX=\frac{1}{\gamma_n}\frac{\partial(A)}{\partial(X)}d\nu_n(A_0)$, where $\frac{\partial(A)}{\partial(X)}=\frac{\partial(a_{1,2}a_{1,3}\dots a_{n-1,n})}{\partial(x_{1,2}x_{1,3}\dots x_{n-1,n})}$ is a Jacobian. In order to complete the proof, we have only to check the following lemma.

Lemma 2.2. *If $A=(A_0+I)X(I-A_0)$, then $\frac{\partial(A)}{\partial(X)}=\det(I+A_0)^{n-1}$, where all matrices are elements of $S(n)$.*

This is an elementary calculation and can be proved by induction for n. We omit it.

Remark. (1) For the case $n=2$, ν_2 is one dimensional Cauchy distribution and its Fourier transformation becomes $\exp(-|x|)$.

(2) $\nu_n(TET^{-1})=\nu_n(E)$ for all $T\in O(n)$ and Borel sets.

We put $\langle A, B\rangle_n=\frac{1}{2}\operatorname{tr}(B^*A)=-\frac{1}{2}\operatorname{tr}(BA)$ for $A, B\in S(n)$. It is a scalar product on $S(n)$, and $\|A\|_n^2=\langle A, A\rangle_n$ is the Hilbert-Shmidt norm.

We define a linear mapping $\pi_{n,m}$ from $S(n)$ to $S(m)$ $(n\geqq m)$.

$$\pi_{n,m}:\ \left(\begin{array}{c|c} A_m & * \\ \hline -* & ** \end{array}\right) \longrightarrow A_m$$

Theorem 2.2. *The sequence $\{\nu_n\}$ is consistent relative to the maps $\pi_{n,m}$ and the normalizing constant γ_n can be calculated by the following recurrent formula.*

$$\frac{\gamma_n}{\gamma_{n+1}}=\int_{R^n}\frac{dy_1dy_2\dots dy_n}{(1+\|y\|^2)^n}$$

Proof. Let $\chi_n(X)$ be the Fourier transformation of ν_n, namely

$\chi_n(X)=\int_{S(n)} e^{i\langle A,X\rangle_n} d\nu_n(A)$. If we write $A=\begin{pmatrix} A_1 & a \\ -{}^t a & 0 \end{pmatrix}$, $X=\begin{pmatrix} X_1 & x \\ -{}^t x & 0 \end{pmatrix}$, where A_1, X_1 are $(n-1)$-dimensional matrices and a, x are $(n-1)$-dimensional vectors, then, since $\det(I+A)=\det(I+A_1)\{1+\langle (I-A_1^2)^{-1}a, a\rangle\}$, we have

$$\chi_n(X)=\gamma_n \int \frac{e^{i\langle A_1,X_1\rangle_{n-1}} e^{i\langle a,x\rangle} da dA_1}{\det(I+A_1)^{n-1}\{1+\langle (I-A_1^2)^{-1}a, a\rangle\}^{n-1}}$$

$$=\gamma_n \int \frac{e^{i\langle A_1,X_1\rangle_{n-1}} e^{i\|(I+A_1)x\|y_1}}{\det(I+A_1)^{n-2}(1+\|y\|^2)^{n-1}} dA_1 dy_1 \dots dy_{n-1}$$

If $x=0$, then $\chi_n(X)=\chi_{n-1}(X_1)$ with the requested result for $\frac{\gamma_n}{\gamma_{n-1}}$. In terms of the measure this shows $\pi_{n,n-1}\nu_n=\nu_{n-1}$. On the other hand, $\pi_{n,k}=\pi_{m,k}\pi_{n,m}$ $(k\leqq m\leqq n)$ holds trivially. Combining these facts, the assertion is proved. Q.E.D.

Lemma 2.3. *In the notation of Theorem* 2,2, $|\chi_n(X)-\chi_{n-1}(X_1)| \leqq \|x\|$.

Proof. Put $X'=\begin{pmatrix} X_1, & -x \\ {}^t x, & 0 \end{pmatrix}$, then we have $\chi_n(X)=\chi_n(X')$, because $\det(I+A)$ is a even function of a. Therefore we have

$$\chi_{n-1}(X_1)-\chi_n(X)=\chi_{n-1}(X_1)-\frac{1}{2}(\chi_n(X)+\chi_n(X'))$$

$$=\gamma_n \int \frac{e^{i\langle A_1,X_1\rangle_{n-1}}(1-\cos\langle a, x\rangle)}{\det(I+A)^{n-1}} dA,$$

so that

$$|\chi_{n-1}(X_1)-\chi_n(X)|\leqq|1-\chi_n(\hat{X})|, \text{ where } \hat{X}=\begin{pmatrix} 0 & x \\ -{}^t x & 0 \end{pmatrix}.$$

Since $\chi_n(\hat{X})=\chi_n(T\hat{X}T^{-1})$ for all T in $SO(n)$ and $T\hat{X}T^{-1}=\begin{pmatrix} 0 & \|x\| & 0 \\ -\|x\| & 0 & \\ 0 & & \ddots\; 0 \end{pmatrix}$ for some $T\in SO(n)$, by the remark after Lemma 2.2 $\chi_n(\hat{X})=\exp(-\|x\|)$, and $|\chi_{n-1}(X_1)-\chi_n(X)|\leqq 1-\exp(-\|x\|)\leqq\|x\|$. Q.E.D.

Theorem 2.3. *Again we use the notation of Theorem* 2.2. *Then,*

$|1-\chi_n(X)|\leqq\frac{1}{2}\|X\|_{\mathrm{tr}}$, *where* $\|\cdot\|_{\mathrm{tr}}$ *is the trace norm.*

Proof. By the remark (2) after Lemma 2.2, we can assume X to be in the canonical form without loss of generality.

$$X=\begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & 0 \\ & & \ddots & & \\ 0 & & & 0 & \lambda_\nu \\ & & & -\lambda_\nu & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \lambda_1 & & & & \\ -\lambda_1 & 0 & & & 0 & \\ & & \ddots & & & \\ 0 & & & 0 & \lambda_\nu & \\ & & & -\lambda_\nu & 0 & \\ & & & & & 0 \end{pmatrix}$$

(n: even) (n: odd)

Making use of Lemma 2.3 repeatedly, in both cases we get the following inequality. $|1-\chi_n(X)|\leqq\sum_{j=1}|\lambda_j|=\frac{1}{2}\|X\|_{\mathrm{tr}}$. Q.E.D.

We remark that the above estimate does not depend on the dimension.

§ 3. The Cylindrical Measure on Some Class of Skew-Symmetric Operators and Its Countably Additive Extension

In this section we shall denote the collection of finite dimensional linear subspaces of H by L^f. If M is the element of L^f, $SO(M)$, $S(M)$, μ_M, ν_M and χ_M have the same meaning as the the preceding section. We define the mappings $\pi_{N,M}$ from $S(N)$ to $S(M)$ as follows ($N\supseteq M$); $\pi_{N,M}(A)=P_{N,M}A|M$, where $P_{N,M}$ is the projection from N to M and $\cdot|M$ means the restriction to M. Then the discussion of section 2 tells us that $\{\nu_M : M\in L^f\}$ is a consistent sequence relative to the maps $\pi_{N,M}$. In order to obtain σ-additive extension of $\{\nu_M\}$ we shall briefly discuss the projective limit method. First of all we take a sequence of operators $\{A_M\}$ which has the following two properties.

(1) $A_M\in S(M)$. (2) $\pi_{N,M}(A_N)=A_M$ ($N\supseteq M$).

Then we can define a bilinear functional B on $H\times H$ such that $B(x,y)=\langle A_Mx,y\rangle$, where M includes both x and y. It is well defined independent from the choice of M by the property (2) and satisfies $B(x,y)=-B(y,x)$. Conversely if a bilinear skew-symmetric functional B is given, $B(x, y)=\langle A_M x, y\rangle$ $x, y\in M$ determines the operator $A_M\in S(M)$ and the sequence $\{A_M\}$ satisfies (2). Evidently this correspondence is one to one. Thus,

Theorem 3.1. *The projective limit space of* $\{S(M), \pi_{N,M}\}$ *is realized as the set of bilinear skew-symmetric functionals on* $H\times H$.

Theorem 3.2. *A* σ*-additive extension of* $\{\nu_M\}$ *exists uniquely on the projective limit space of* $\{S(M), \pi_{N,M}\}$.

The proof is carried out by Bochner's famous theorem. We omit it.

Though it can be shown more directly, Theorem 3.2 tells us that we can define a positive definite function χ on the set $S_0(H)$ of all finite rank skew-symmetric operators on H.

More exactly, χ has the following two properties.

(1) χ *is a positive definite function and* $\chi(0)=1$.

(2) $\chi(A)=\chi_M(A)$ *for any* $M\in L^f$ *such that* $AH\subseteq M$.

The last theorem in Section 2, together with the fact that $S_0(H)$ is dense in $S_n(H)$ enables us to extend χ to $S_n(H)$ uniquely. To say repeatedly, $S_n(H)$ is the set composed of the skew-symmetric trace class operators which is equipped with the trace norm.

Theorem 3.3. *The dual of the space of compact linear operators on* H *equipped with the operator norm is identified with the space of nuclear operators on* H *equipped with the trace norm. More exactly, for the dual element* F*, there exists one and only one nuclear operator* T*, such that* $F(A)=\mathrm{tr}(T^*A)$ *and* $\|F\|=\|T\|_{\mathrm{tr}}$.

This is a well known result. We omit it. See [7].

Collorary. *The dual of* $S_c(H)$ *is identified with* $S_n(H)$ *in the above sense.*

Proof. Using Hahn-Banach theorem, we can easily reduce it to the above theorem, so that we omit it.

Theorem 3.4. *A unique cylindrical measure (weak distribution) exists on $S_c(H)$ such that its finite dimensional projection is given by the Cayley image of Haar measure.*

Proof. Since a weak distribution corresponds uniquely to a continuous positive definite functional on its dual space, we have only to check the continuity of χ. But $|1-\chi(X)|\leqq\frac{1}{2}\|X\|_{tr}$ always holds, so that we complete the proof. In details, see [3].

(1) Consider the spectral decomposition of $T\in S_n(H)$

$T(X)=\sum\lambda_j\{\langle x, e_{2j}\rangle e_{2j-1}-\langle x, e_{2j-1}\rangle e_{2j}\}$. Then

$$\chi(T)=\int_{S_c(H)}\exp\left\{i\frac{1}{2}\mathrm{tr}(T^*A)\right\}d\nu(A)$$

$$=\lim_n\gamma_{2n}\int_{S(2n)}\frac{\exp\{i\sum_{j=1}^n\lambda_j a_{2j-1,2j}\}}{\det(I+A)^{2n-1}}dA$$

(2) Let e and f be the elements in H. The function $\xi_{e,f}(A)=\langle Ae, f\rangle$ is of course measurable and its distribution is Cauchy. For real y,

$$\int_{S_c(H)}e^{i\langle Ae,f\rangle y}d\nu(A)=\exp(-|y|\sqrt{\|e\|^2\|f\|^2-\langle e,f\rangle^2})$$

(3) $\nu(E)=\nu(-E)$ and $\nu(TET^{-1})=\nu(E)$ for all $T\in O(H)$ and measurable set E.

Our final problem in this paper is the σ-additive extension of this weak distribution. Apparently its characteristic functional tells us the lack of σ-additiveness on $S_c(H)$. So that we must extend $S_c(H)$ to a larger space which is so called a nuclear extension. As the projective limit in Theorem 3.1 is considered to be the largest extension, we have only to check the support of the measure in Theorem 3.2. But on the other hand, since we wish to obtain a measure on $O(H)$, the extension on $S_c(H)$ is desirable. Then the problem is set up as follows. *Let T_1 and T_2 be bounded operators on H and consider the map, $A\to T_1AT_2$, where A belong to $S_c(H)$. Then whether the image of ν by this transformation has* *a σ-extension on $S_c(H)$ or not?* Before answering it, we prove the following lemma.[1)]

Lemma 3.1. *Let T_1 and T_2 be bounded operators on H and assume that $T_1AT_2 \in S_c(H)$ for all $A \in S_c(H)$. Then, (a) $T_1^* = cT_2$ for some real constant, or (b) $T_1AT_2 = 0$ for all $A \in S_c(H)$.*

Proof. If the rank of T_2 is less than 1, then the rank of T_1AT_2 is also less than 1, but such a skew-symmetric operator must be zero. This is the case (b).

The skew-symmetricity of T_1AT_2 implies that for any $A \in S_c(H)$ and $x \in H$, we have $\langle AT_2x, T_1^*x \rangle = 0$. This means that for some real number C_x, $T_1^*x = C_xT_2x$ if $T_2x \neq 0$. Thus, we have only to prove (1) $C_x = C_y$ for $x \neq y$, and (2) $T_2x = 0$ implies $T_1^*x = 0$. First we shall prove (2). If $T_2x = 0$ and $T_2y \neq 0$, we have $T_1^*x = T_1^*(x+y) - T_1^*y = (C_{x+y} - C_y)T_2y$. Thus, T_1^*x is linearly dependent with T_2y. If the rank of T_2 is more than 2, this means $T_1^*x = 0$. Next we shall prove (1). If $T_2x \neq 0$ and $T_2y \neq 0$, we have $T_1^*(x+y) = T_1^*x + T_1^*y = C_xT_2x + C_yT_2y = C_{x+y}T_2(x+y) = C_{x+y}T_2x + C_{x+y}T_2y$. If T_2x and T_2y are linearly independent, comparing the coefficients we get $C_{x+y} = C_x = C_y$. If T_2x and T_2y are linearly dependent, $T_2x = \alpha T_2y$ for some α. Then, in virtue of (2) we have $T_1^*x = \alpha T_1^*y$, namely $C_xT_2x = \alpha C_yT_2y = C_yT_2x$, hence $C_x = C_y$.

Q.E.D.

Theorem 3.5. *Let T be a bounded operator on H and put $\tau(A) = T^*AT$. Then τ is a bounded transformation on $S_c(H)$. In order that $\tau\nu(E) = \nu(\tau^{-1}(E))$ has a σ-additive extension on $S_c(H)$, it is sufficient that T is a nuclear operator. Further under the above condition the support of $\tau\nu$ is always contained in $S_h(H)$.*

Proof. We assume that T is a nuclear operator. Then the range of τ is included in $S_h(H)$. Thus, it will be sufficient to show that $\tau\nu$ has a σ-additive extension on $S_c(H)$. We shall observe its Fourier transformation: $\widehat{\tau\nu}(B) = \int_{X \in S_h(H)} e^{i\,\mathrm{tr}\,(X^*B)} d\tau\nu(X)$, where B is an element in $S_h(H)$.

1) This lemma was improved by Y. Yamasaki.

Then by the continuity of ν

$$1-\widehat{\tau\nu}(B)=\int_{Sc(H)} 1-e^{-i\,\mathrm{tr}\,(T^*ATB)}d\nu(A)=\sup_{\|A\|\leqq 1}|\mathrm{tr}(T^*ATB)|=\|TBT^*\|_{\mathrm{tr}}.$$

We put $P_T(B)=\sup_{\|A\|\leqq 1}|\mathrm{tr}(T^*ATB)|$, then it is a semi norm. Naturally the space $S_h(H)$ becomes a Hilbert space with the Hilbert-Schmidt norm: $\langle B_1, B_2\rangle=\frac{1}{2}\sum_{j=1}^{\infty}\langle B_1e_j, B_2e_j\rangle$, where $e_1, e_2, \ldots, e_n, \ldots$ is a complete orthonormal system (c.o.n.s) in H. Put $q_{i,j}(x)=\langle x, e_i\rangle e_j-\langle x, e_j\rangle e_i$, then the sequence of operators $\{q_{i,j}\}\in S_h(H): i, j=1\,2\ldots, i<j$ forms a c.o.n.s. in $S_h(H)$. After some caluculations we can show that $P_T(q_{i,j})=2\|Te_i\|\|Te_j\|$, therefore we have $\sum_{i<j}P_T(q_{i,j})=\sum_{i<j}2\|Te_i\|\|Te_j\|<\infty$. We define an operator W on $S_h(H)$ as follows : $W(B)=\sum_{i<j}\langle B, q_{i,j}\rangle P_T(q_{i,j})q_{i,j}$. Then W is a trace class operator on $S_h(H)$ and $P_T(B)\leqq\sum_{i<j}|\langle B, q_{i,j}\rangle|P_T(q_{i,j})$ $\leqq(\sum_{i<j}\langle B, q_{i,j}\rangle^2P_T(q_{i,j}))^{\frac{1}{2}}(\sum_{i<j}P_T(q_{i,j}))^{\frac{1}{2}}\leqq\langle W(B), B\rangle^{\frac{1}{2}}\|W\|_{\mathrm{tr}}^{\frac{1}{2}}$. Therefore $\widehat{\tau\nu}$ is continuous with respect to the norm $\|B\|_W=\langle W(B), B\rangle^{\frac{1}{2}}$, and since W is trace class, $\tau\nu$ is a σ-additive measure on $S_h(H)$. See [4]. Q.E.D.

Theorem 3.6. *We use the notation of Theorem* 3.5. *In order that $\tau\nu$ has a σ-additive extension on $S_B(H)$, it is necessary that T is a Hilbert-Shmidt operator, where $S_B(H)$ is the set of all bounded skew-symmetric operators.*

Proof. Suppose that $\tau\nu$ has a σ-additive extension on $S_B(H)$. If T vanishes identically, the assertion is trivial, so that we take an element e such that $Te\neq 0$, $\|e\|=1$. Extending it in a suitable way, we form a c.o.n.s $e_1, e_2, \ldots, e_n, \ldots$ such that $e_1=e$. The mapping ξ which sends $A\in S_B(H)$ to $Ae\in H$ is clearly a measurable mapping. Again we observe a Fourier transformation of a measure η, which is the image of $\tau\nu$ by the map ξ. As η is a σ-additive measure on H, for any positive ε, there exist $\delta>0$ and a Hilbert-Schmidt operator S on H such that $|1-\hat{\eta}(y)|<\varepsilon$ for all $\|Sy\|\leqq\delta$. However, since

$$\hat{\eta}(y)=\int_{S_c(H)} e^{i\langle Ty, ATe\rangle}d\nu(A)=\exp(-\sqrt{\|Te\|^2\|Ty\|^2-\langle Te, Ty\rangle^2}),$$

the inequality $|1-\hat{\eta}(y)|<\varepsilon$ is equivalent with $\|Te\|^2\|Ty\|^2-\langle Te, Ty\rangle^2$

$<(\mathrm{Log}(1-\varepsilon))^2$. Next we take an arbitrary element y from H. If λ is small enough to satisfy $\|\lambda S y\| \leqq \delta$, then $\|Te\|^2\|Ty\|^2 - \langle Te, Ty\rangle^2 \leqq \frac{1}{\delta^2}(\mathrm{Log}(1-\varepsilon))^2$. Letting $\frac{1}{\lambda}$ to infimum under the above condition, we get $\|Te\|^2\|Ty\|^2 - \langle Te, Ty\rangle^2 \leqq \frac{1}{\delta^2}(\mathrm{Log}(1-\varepsilon))^2\|Sy\|^2$. Substituting e_j for y and summing up over j, we have $\|Te\|^2 \sum_j \|Te_j\|^2 - \|T^*Te\|^2 \leqq \frac{1}{\delta^2}(\mathrm{Log}(1-\varepsilon))^2\|S\|_{HS}^2$. Therefore $\sum_j \|Te_j\|^2 < \infty$, namely T is a Hilbert-Schmidt operator. Q.E.D.

Making use of the infinite dimensional Cayley inverse transformation, we can obtain some invariant measures on the group $O_c(H)$. The author wishes to discuss various properties of these measures in the subsequent paper.

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