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Approximation of Exterior Dirichlet Problems; Convergence of Wave and Scattering Operators

By

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§ 1. Introduction

In the present paper we show that the wave operator in the Dirichlet problem for an elliptic differential operator is the limit of those in the appropriate whole space problems.

The proof of this fact consists of three parts; the convergence of resolvents [Lemmas 1 and 2], the convergence of semi-groups [Lemma 3] and the convergence of wave operators [Theorem 1].

In proving the convergence of resolvents, we generalize the technique used by Ōeda [7]. As for the convergence of semi-groups, we apply a standard method (Kato [2]). And we prove the convergence of wave operators by means of the time-dependent scattering theory. The convergence of scattering operators is also considered [Corollary 1]. We give a comment on the Neumann problem for $-\Delta+q(x)$ [Remark 3].

§ 2. Statement of Results

Throughout the present paper we use the following notations: $D_j=-i\dfrac{\partial}{\partial x_j}$, $x=(x_1, \ldots, x_n)\in R^n$, a multi-index $\alpha=(\alpha_1, \ldots, \alpha_n)$, $|\alpha|=\alpha_1+\ldots+\alpha_n$ and $D^\alpha=D_1^{\alpha_1}\ldots D_n^{\alpha_n}$. We denote by $H^m(\Omega)$ the Sobolev space of order m for a domain Ω and by $\mathring{H}^m(\Omega)$ the closed subspace of $H^m(\Omega)$ which is the completion of $C_0^\infty(\Omega)$ by the Sobolev norm of order m, where $C_0^\infty(\Omega)$ is the set of all infinitely differentiable functions with compact support in Ω.

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We denote the inner product in $L^2(R^n)$ by $(\ ,\ )$. We denote the domain of a form $h[f, g]$ by $D[h]$ and the domain and the range of an operator A by $D(A)$ and $R(A)$ respectively. And we denote the adjoint of an operator A by A^*. We now consider the self-adjoint operators H and H° in $L^2(R^n)$ associated with closed elliptic Hermitian symmetric forms $h[f, g]$ and $h^\circ[f, g]$ respectively. The exact definitions and expressions of these operators and forms are given as follows. Let $h[f, g]$ and $h^\circ[f, g]$ be the sesqui-linear forms defined as

$$D[h]=D[h^\circ]=H^m(R^n),$$
$$h[f, g]=h^\circ[f, g]+\Big(\sum_{|\alpha|,|\beta|\leq m} a_{\alpha\beta}(x)D^\beta f,\ D^\alpha g\Big),\quad f, g\in D[h],$$

and

$$h^\circ[f, g]=\Big(\sum_{|\alpha|,|\beta|\leq m} \overset{\circ}{a}_{\alpha\beta}D^\beta f,\ D^\alpha g\Big),\quad f, g\in D[h^\circ].$$

On the coefficients $\overset{\circ}{a}_{\alpha\beta}$ and $a_{\alpha\beta}(x)$ we impose the following conditions:

$$\overset{\circ}{a}_{\alpha\beta}\ \textit{are constants},$$
$$\sum_{|\alpha|,|\beta|=m} \overset{\circ}{a}_{\alpha\beta}\xi^{\alpha+\beta}\geqq c|\xi|^{2m},\quad \overset{\circ}{a}_{\alpha\beta}=\overline{\overset{\circ}{a}_{\beta\alpha}},$$
$$\sum_{|\alpha|,|\beta|=m} \{\overset{\circ}{a}_{\alpha\beta}+a_{\alpha\beta}(x)\}\xi^{\alpha+\beta}\geqq c|\xi|^{2m},\quad a_{\alpha\beta}(x)=\overline{a_{\beta\alpha}(x)},$$
$$a_{\alpha\beta}(x)\ \textit{are bounded measurable functions and}$$
$$a_{\alpha\beta}(x)\ \textit{with}\ |\alpha|=|\beta|=m\ \textit{are uniformly continuous}.$$

Then, by virtue of the Gårding inequality, $h[f, g]$ and $h^\circ[f, g]$ are closed elliptic Hermitian symmetric forms bounded from below. We define the self-adjoint operators H and H° by these forms $h[f, g]$ and $h^\circ[f, g]$ respectively in the sense of Friedrichs. The operators H and H° are lower semi-bounded and in particular, for some real constant γ,

$$(Hf, f)\geqq(-\gamma+1)(f, f),\qquad f\in D(H).$$

We define the self-adjoint operator H_d in $L^2(\Omega)$ by the form $h_d[f, g]=h[f, g]$ with domain $\overset{\circ}{H}{}^m(\Omega)$ in the sense of Friedrichs. Here and hereafter we assume that Ω is the exterior of a compact set K with smooth boundary $\partial\Omega$. We define the sequence of self-adjoint operators $\{H_n\}$, which approximates H_d, as follows:

$$D(H_n)=D(H),$$
$$H_n f=Hf+n\chi_K f, \qquad f\in D(H_n),$$

where χ_K is the characteristic function of K.

Next, to state the main theorem, we need the following condition.

Condition A. *There exist a set $F\subset D(H^\circ)$ which is dense in $L^2(R^n)$, an infinitely differentiable function $\eta(x)$ and a constant R° such that: i) $\eta(x)=1$ on $\{x\,||x|>R^\circ+1\}$, $\eta(x)=0$ on $\{x\,||x|<R^\circ\}$ and $\{x\,||x|<R^\circ\}\supset K$, and ii) for any $u\in F$, $\eta(x)e^{-i\tau H^\circ}u\in D(H^\circ)\cap D(H)$ and the integral*

$$\int_{-\infty}^{\infty}\|(H\eta(x)-\eta(x)H^\circ)e^{-i\tau H^\circ}u\|\,d\tau$$

is finite.

Let J be the bounded operator from $L^2(R^n)$ to $L^2(\Omega)$ given by restriction of a function. Then we obtain the following theorem.

Theorem 1. *Under Condition A, the wave operators $W_\pm(H_n, H^\circ)$ and $W_\pm(H_d, H^\circ)$ given by*

$$W_\pm(H_n, H^\circ)=\operatorname*{s-lim}_{t\to\pm\infty} e^{itH_n}e^{-itH^\circ}$$

and

$$W_\pm(H_d, H^\circ)=\operatorname*{s-lim}_{t\to\pm\infty} e^{itH_d}Je^{-itH^\circ}$$

exist. And

$$W_\pm(H_d, H^\circ)=\operatorname*{s-lim}_{n\to\infty} JW_\pm(H_n, H^\circ).$$

Corollary 1. *If $W_\pm(H_n, H^\circ)$ and $W_\pm(H_d, H^\circ)$ are complete, that is,*

$$R(W_+(H_n, H^\circ))=R(W_-(H_n, H^\circ))$$

and

$$R(W_+(H_d, H^\circ))=R(W_-(H_d, H^\circ)),$$

then the scattering operator $S_d=W_+(H_d, H^\circ)^* W_-(H_d, H^\circ)$ for the Dirichlet problem is the strong limit of a sequence of operators $S_n=W_+(H_n, H^\circ)^* W_-(H_n, H^\circ)$ when n tends to infinity.

Remark 1. Condition A is satisfied if, for example, the Hessian of $P_{2m}(\xi)=\sum_{|\alpha|=|\beta|=m} a^{\circ}_{\alpha\beta}\xi^{\alpha+\beta}$ is not identically zero, $a_{\alpha\beta}(x)$ are differentiable to order $|\alpha|$ for $|x|>R>R^{\circ}$ and satisfy $\sum_{|\gamma|\leqq|\alpha|}|D^{\gamma}a_{\alpha\beta}(x)|\leqq\frac{c}{(1+|x|)^{1+\varepsilon}}$, $\varepsilon>0$, there. The proof of this fact is shown by the stationary phase method (Murata [6]).

Remark 2. The assumed completeness of wave operators in Corollary 1 is proved in Kuroda [3], [4] and [5] under the same condition as in Remark 1.

To prove the theorem, we need the following lemmas.

Lemma 1. *Let $R_n(\lambda)$ be the resolvent of H_n, then the strong limit*

$$G(\lambda)=\text{s-}\lim_{n\to\infty} R_n(\lambda)$$

exists.

Lemma 2. *If we denote by $R_d(\lambda)$ the resolvent of H_d, we have*

$$R_d(\lambda)=JG(\lambda)J^*=\text{s-}\lim_{n\to\infty} JR_n(\lambda)J^*$$

and

$$G(\lambda)=J^*R_d(\lambda)J, \qquad G(\lambda)J^*=J^*R_d(\lambda).$$

Lemma 3. *Let H_d and H_n be as above, then for any fixed t we have*

$$e^{itH_d}=\text{s-}\lim_{n\to\infty} Je^{itH_n}J^*.$$

Remark 3. The Neumann problem for $-\Delta+q(x)$: To approximate the Neumann problem for $-\Delta+q(x)$, we can use the following sequence of operators $\{H_n\}$. Let $h_n[f,g]$ be the closed Hermitian symmetric forms given by

$$h_n[f,g]=\int_{R^n}(1-\rho_n(x))\nabla f\cdot\overline{\nabla g}\,dx+\int_{\Omega}q(x)f\bar{g}\,dx$$

with domain $D[h_n]=H^1(R^n)$, where $\rho_n(x)$ are all in $C_0^{\infty}(K^{int})$ and monotone-increasing in n to 1 in K^{int}, the interior of K. The sequence of approximating operators $\{H_n\}$ is defined by these forms $\{h_n\}$ in the sense of Friedrichs. Then all the preceding results also hold in this situation. In fact, the sequence of forms $h_n[u,u]$ is monotone-decreasing and converges to a form $\mathring{h}_N[u,u]$, where $\mathring{h}_N[f,g]$ is the closed Hermitian symmetric form given by

$$\mathring{h}_N[f,g]=\int_\Omega \nabla f\cdot\overline{\nabla g}\,dx+\int_\Omega q(x)f\bar{g}\,dx$$

with domain $D[\mathring{h}_N]=H^1(\Omega)\oplus L^2(K)$, the direct sum of $H^1(\Omega)$ and $L^2(K)$. As this convergence is valid for the element belonging to $H^1(R^n)$ which is a core of $\mathring{h}_N[f,g]$, the sequence $\{H_n\}$ converges to $(H_N\oplus 0)$, the direct sum of operators H_N in $L^2(\Omega)$ and 0 in $L^2(K)$, in the generalized sense (see Kato [2] VIII § 3.2). Here H_N is $-\Delta+q(x)$ in Ω with the Neumann boundary condition. That is, the sequence of resolvents $R_n(-\gamma)=(H_n+\gamma)^{-1}$ converges to $((H_N\oplus 0)+\gamma)^{-1}=(H_N+\gamma)^{-1}\oplus\gamma^{-1}$. From this fact, the same result as in Lemma 1 follows immediately and the same results as in Lemmas 2, 3 and Theorem 1 are also valid.

§ 3. Proof of Results

i) *Proof of Lemma* 1: From the construction of an approximating series $\{H_n\}$,

$$(H_m+\gamma)\geqq(H_n+\gamma)\geqq(H+\gamma)\geqq 1, \qquad m\geqq n,$$

so that

$$1\geqq R_n(-\gamma)\geqq R_m(-\gamma)\geqq 0.$$

Accordingly, we obtain Lemma 1 by a well-known result (Kato [2] VIII § 3.1).

ii) *Proof of Lemma* 2: Put $(H_n+\gamma)^{-1}u=f_n$. Then

$$\begin{aligned}1&\geqq(R_n(-\gamma)u,u)\\&=((H_n+\gamma)f_n,f_n)\geqq h[f_n,f_n]+\gamma\|f_n\|^2\geqq c(h^\circ[f_n,f_n]+\gamma\|f_n\|^2).\end{aligned}$$

This shows $\{f_n\}$ is a bounded set in $H^m(R^n)=D[h^\circ]$. On the other hand, as $\{f_n\}$ converges to $G(-\gamma)u=g$ in $L^2(R^n)$ by Lemma 1, $g\in H^m(R^n)$ and

$\{f_n\}$ converges weakly to g in $H^m(R^n)$ (see, for example, Agmon [1] § 3. Theorem 3.12). Furthermore, as

$$1 \geqq ((H_n+\gamma)f_n, f_n) \geqq n(\chi_K f_n, f_n),$$

$\{\chi_K f_n\}$ converges to zero and hence $g|_K=0$. Then Jg, the restriction of $g \in H^m(R^n)$ to Ω, is in $\mathring{H}^m(\Omega)$. Consequently, $JG(-\gamma)J^* v \in \mathring{H}^m(\Omega)$ for $v \in L^2(\Omega)$. Now let $v \in L^2(\Omega)$ and $\phi \in \mathring{H}^m(\Omega)$, then

$$\begin{aligned}
(h_d+\gamma)[JG(-\gamma)J^*v, \phi] &= (h+\gamma)[G(-\gamma)J^*v, J^*\phi] \\
&= \lim_{n\to\infty} (h+\gamma)[(H_n+\gamma)^{-1}J^*v, J^*\phi] \\
&= (J^*v, J^*\phi) - \lim_{n\to\infty} n(\chi_K(H_n+\gamma)^{-1}J^*v, J^*\phi) = (JJ^*v, \phi) = (v, \phi).
\end{aligned}$$

In particular, for $\psi \in D(H_d) \subset \mathring{H}^m(\Omega)$ we have

$$(JG(-\gamma)J^*v, (H_d+\gamma)\psi) = (v, \psi) = ((H_d+\gamma)^{-1}v, (H_d+\gamma)\psi).$$

Then, as $\{(H_d+\gamma)\psi \mid \psi \in D(H_d)\} = L^2(\Omega)$, we have

$$JG(-\gamma)J^* = (H_d+\gamma)^{-1} = R_d(-\gamma).$$

This proves the first part of Lemma 2. To prove the second part, we multiply this equation by J and J^* from right and left:

$$J^*(H_d+\gamma)^{-1}J = J^*JG(-\gamma)J^*J.$$

On the other hand

$$G(-\gamma) - J^*JG(-\gamma)J^*J = G(-\gamma)(1-J^*J) + (1-J^*J)G(-\gamma)J^*J.$$

The second term of the righthand side of this equation is zero since $R(G(-\gamma)) \subset \{g \in L^2(R^n) \mid g|_K=0\}$. The first term is also zero, because for u with support in K and $f_n = (H_n+\gamma)^{-1}u$,

$$u = (H_n+\gamma)f_n = \chi_K(H_n+\gamma)f_n$$

so that

$$\|f_n\|^2 \leqq ((H_n+\gamma)f_n, f_n) = (u, \chi_K f_n) \leqq \|u\| \cdot \|\chi_K f_n\|$$

and, as $\lim_{n\to\infty} \|\chi_K f_n\|^{\frac{1}{2}} = 0$,

$$\|G(-\gamma)u\|=\lim_{n\to\infty}\|f_n\|\leqq\|u\|^{\frac{1}{2}}\cdot\lim_{n\to\infty}\|\chi_K f_n\|^{\frac{1}{2}}=0.$$

Hence

$$J^*(H_d+\gamma)^{-1}J=J^*JG(-\gamma)J^*J=G(-\gamma).$$

iii) *Proof of Lemma* 3: Using Lemma 2, we can show this lemma by a standard method of expressing the difference of two semi-groups by an integral formula (see Kato [2] IX § 2.4):

$$\begin{aligned}&R_n(-\gamma)(e^{itH_n}J^*-J^*e^{itH_d})R_d(-\gamma)u\\&=i\int_0^t e^{i(t-s)H_n}(J^*R_d(-\gamma)-R_n(-\gamma)J^*)e^{isH_d}uds.\end{aligned}$$

By Lemmas 1 and 2, the righthand side tends to zero as $n\to\pm\infty$. Therefore, using Lemma 2 and the uniform boundedness of $R_n(-\gamma)$ and e^{itH_n}, we obtain the desired result.

iv) *Proof of Theorem* 1: Put

$$f_u(t)=\|(H\eta(x)-H^\circ\eta(x)e^{itH^\circ})u\|,\quad\text{for}\quad u\in F.$$

Then, $f_u(t)$ is integrable and

$$\begin{aligned}(*)\qquad&\|(e^{itH_n}\eta(x)e^{-itH^\circ}-e^{isH_n}\eta(x)e^{-isH^\circ})u\|\\&=\left\|\int_s^t e^{i\tau H_n}(H_n\eta(x)-\eta(x)H^\circ)e^{-i\tau H^\circ}ud\tau\right\|\\&\leqq\int_s^t\|(H\eta(x)-\eta(x)H^\circ)e^{-i\tau H^\circ}u\|\,d\tau=\int_s^t f_u(\tau)\,d\tau.\end{aligned}$$

The existence of $W_\pm(H_n, H^\circ)$ follows immediately from the fact that

$$\operatorname*{s-lim}_{t\to\pm\infty}e^{itH_n}e^{-itH^\circ}=\operatorname*{s-lim}_{t\to\pm\infty}e^{itH_n}\eta(x)e^{-itH^\circ},$$

which follows from Rellich's compactness theorem. The same argument is valid for H_d and $W_\pm(H_d, H^\circ)$ exists. Now then, if we let t tend to $\pm\infty$ in (*), we have

$$\begin{aligned}(**)\qquad&\|(JW_\pm(H_n, H^\circ)-Je^{isH_n}J^*J\eta(x)e^{-isH^\circ})u\|\\&\leqq\|(W_\pm(H_n, H^\circ)-e^{isH_n}\eta(x)e^{-isH^\circ})u\|\leqq\int_s^\infty f_u(\tau)\,d\tau.\end{aligned}$$

On the other hand, for H_d we have

$$(***)\qquad \|(W_\pm(H_d, H^\circ)-e^{isH_d}J\eta(x)e^{-isH^\circ})u\|\leqq\int_s^\infty f_u(\tau)\,d\tau.$$

Since the righthand side of (**) is independent of n, we have, using Lemma 3 and the inequalities (**) and (***), that for any $\varepsilon>0$, there exists N such that

$$\begin{aligned}&\|W_\pm(H_d, H^\circ)u-JW_\pm(H_n, H^\circ)u\|\\ &\leqq\|(W_\pm(H_d, H^\circ)-e^{isH_d}J\eta(x)e^{-isH^\circ})u\|\\ &+\|(e^{isH_d}-Je^{isH_n}J^*)J\eta(x)e^{-isH^\circ}u\|\\ &+\|(JW_\pm(H_n, H^\circ)-Je^{isH_n}\eta(x)e^{-isH^\circ})u\|<\varepsilon,\quad\text{for}\quad n>N.\end{aligned}$$

Here we choose s sufficiently large so that the first and the third terms are less than $\varepsilon/3$.

v) *Proof of Corollary* 1: We have only to show the weak convergence since S_n and S_d are all unitary. The weak convergence is an immediate consequence of the strong convergence of wave operators.

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