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# Singular Supports of Solutions of Partial Differential Equations in a Slab Domain

By

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## § 1. Introduction

In [6], F. John proved that for a differential operator, non-solvability to the non-characteristic Canchy problem for any initial data with compact support is equivalent to rather stringent non-hyperbolicity.[1)] In the present paper, we shall study an analogous question where we shall be interested not in the support but in the singular support of solution. To state more precisely, let $D$ denote the imaginary gradient $-i\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)$ and $P(D)$ be a differential operator with constant coefficients obtained from a polynomial $P(\xi)$ of $n$ variables $\xi=(\xi_1, \ldots, \xi_n)$. Our main result is the following

**Theorem 1.** *Assume that the polynomial* $P(\xi)$ *has the form*

$$P(\xi)=a\xi_1^l+\sum_{\alpha_1<l} a_\alpha \xi^\alpha \tag{1.1}$$

*with* $a\neq 0$*, and that the zeros of the polynomial have the property*;

(1.2) $\quad \operatorname{Im} \zeta_1 \to \infty$ *when* $\xi' \to \infty$ *in* $\boldsymbol{R}^{n-1}$ *and* $P(\zeta_1, \xi')=0$.

*Under these assumptions, if the equation*

$$P(D)u=f,$$

*in a slab domain* $\Omega=\{x\in \boldsymbol{R}^n;\ c_1<x_1<c_2\}$*, admits a solution* $u\in \mathcal{D}'(\Omega)$ *with bounded singular support for a given* $f\in C^\infty(\Omega)$*, then we have that*

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1) Cf. Matsuura [8] for the extension to general systems.

$u \in C^\infty(\Omega)$.

The proof of Theorem 1 will be given in section 2. In section 3, we shall state and prove an analogous theorem for Gevrey classes. In section 4 we shall consider partial converses of these theorems by constructing suitable fundamental solutions, inspired by an idea due to Kashiwara [7].

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## § 2. Proof of Theorem 1

We shall first consider to what extent the regularity of solutions $u$ can be deduced from the condition (1.1).

**Lemma 1.** *A polynomial $P(\xi)$ can be written in the form* (1.1) *if and only if there exist positive constants $\varepsilon$, $\rho$, $C$ with $0<\rho\leqq 1$ such that*

$$|P^{(\alpha)}(\xi)P(\xi)|\leqq C(1+|\xi|)^{-\alpha_1-\rho|\alpha'|} \tag{2.1}$$

*for all multi-indices $\alpha=(\alpha_1, \alpha')$ and $\xi\in\Gamma$, where $\Gamma=\{\xi\in R^n;\ |\xi'|<\varepsilon|\xi_1|^\rho\}$.*

*Proof.* Suppose that $P$ has the form $P(\xi)=a\xi_1^l(1+\sum_{\alpha_1<l} a_\alpha \xi_1^{\alpha_1-l}\xi'^{\alpha'})$ for $\xi_1\neq 0$. If $\rho=\inf((l-\alpha_1)/|\alpha'|, 1)$, the infimum being taken over all $\alpha$ with $a_\alpha \neq 0$, and if $\varepsilon>0$ is sufficiently small, then we have

$$\frac{1}{2}|a\xi_1^l|\leqq|P(\xi)|\leqq\frac{3}{2}|a\xi_1^l|,\ \xi\in\Gamma$$

$$|P^{(\beta)}(\xi)|\leqq C|\xi_1|^{l-\beta_1-\rho|\beta'|},\ \xi\in\Gamma$$

which proves the estimate (2.1). Conversely if the highest order tern of $P(\xi)$ with respect to $\xi_1$ is dependent on the other variables, there exists a multi-index $\alpha\neq 0$ such that $P^{(\alpha)}(\xi)$ has the same order with respect to $\xi_1$ as $P(\xi)$ and such that the term is independent of the other variables. Thus $|P^{(\alpha)}(\xi)/P(\xi)|$ does not converge to zero when $\xi_1\to\infty$ and $\xi'$ bounded, which proves the lemma.

**Lemma 2.** *Assume that there is a subset $\Gamma\subset \boldsymbol{R}^n$ and a constant*

$\rho>0$ *such that*

$$|P^{(\alpha)}(\xi)/P(\xi)|\leqq C(1+|\xi|)^{-\rho|\alpha|},\ \xi\in\Gamma. \tag{2.2}$$

*If* $P(D)u=f\in C^{\infty}(U)$ *where* $U$ *is an open subset in* $\mathbf{R}^n$*, it follows that for every* $\varphi\in C_0^{\infty}(U)$ *there exist constants* $C_N$*, such that*

$$|\widehat{\varphi u}(\xi)|\leqq C_N(1+|\xi|)^{-N},\ N=1, 2, \cdots, \xi\in\Gamma.$$

*Proof.* Let us consider a solution $v(x, \xi)$ of the equation

$${}^tP(D)v=\varphi e^{-i<x,\xi>} \tag{2.3}$$

where ${}^tP(D)=P(-D)$ is the formal adjoint of $P$. If we set $v(x, \xi)=e^{-i<x,\xi>}\ w(x, \xi)/P(\xi)$ when $\xi\in\Gamma$, Leibniz formula gives

$${}^tP(D)v(x, \xi)=(w(x, \xi)+\sum_{\alpha\neq 0}\frac{P^{(\alpha)}(\xi)}{P(\xi)}\frac{D^{\alpha}}{\alpha!}w(x, \xi))e^{-i<x,\xi>}.$$

Hence (2.3) is equivalent to the equation

$$w-R(\xi, D)w=\varphi,\ \xi\in\Gamma, \tag{2.4}$$

where $R(\xi, D)=-\sum_{\alpha\neq 0}(P^{\alpha}(\xi)D^{\alpha}/\alpha!P(\xi))$. Suggested by the formal solution $w=\sum_0^{\infty}R(\xi, D)^k\varphi$ to (2.4), we consider the following approximate solutions

$$w_N=\sum_0^N R(\xi, D)^k\varphi,$$
$$v_N=e^{-i<x,\xi>}w_N/P(\xi).$$

The same calculation gives

$$\begin{aligned}{}^tP(D)v_N(x, \xi)&=(w_N-R(\xi, D)w_N)e^{-i<x,\xi>}\\&=(\varphi(x)-R(\xi, D)^{N+1}\varphi(x))e^{-i<x,\xi>}.\end{aligned}$$

Then it follows that

$$\begin{aligned}\widehat{\varphi u}(\xi)&=\int u(x)\cdot\varphi(x)e^{-i<x,\xi>}dx\\&=\int u(x)(R(\xi, D)^{N+1}\varphi)e^{-i<x,\xi>}dx\\&\quad+\int f(x)\frac{w_N(x, \xi)}{P(\xi)}e^{-i<x,\xi>}dx,\ \xi\in\Gamma.\end{aligned} \tag{2.5}$$

Since the order of distributions $(D^{\alpha}\varphi)u$ is independent of $\alpha$, using (2.2) we can estimate the first term on the right-hand side of (2.5) by

$$C_N(1+|\xi|)^{-\rho(N+1)+M},\ N=1, 2, \ldots, \qquad \xi\in\Gamma$$

with some constants $C_N$ and $M$. The second term also decreases rapidly in $\Gamma$ for $w_N f\in C_0^{\infty}$ and every derivative $D_x^{\alpha}(w_N(x,\xi)/P(\xi))$ is bounded when $\xi\in\Gamma$. The proof is now complete.

*Proof of Theorem* 1. Since the preceding lemmas show that for each $\varphi\in C_0^{\infty}(\Omega)$ $\widehat{\varphi u}(\xi)$ is rapidly decreasing when $|\xi'|<\varepsilon|\xi_1|^{\rho}$, our aim now is to prove the same fact also when $|\xi'|\geqq\varepsilon|\xi_1|^{\rho}$. In doing so we assume in what follows without loss of generality that $\Omega=\{x;\ -2<x_1<2\}$, $a=1$ in (1.1) and that bounded is supp $u$ as well as sing supp $u$. Moreover we can assume that $u\in C^0$ when $-1\leqq x_1\leqq 1$. In fact, if we take a test function $\phi(x_1)\in C_0^{\infty}((-2, 2))$ which is equal to 1 in a neighborhood of the interval $\left[-\frac{3}{2}, \frac{3}{2}\right]$ and if we replace $p(D)$ by $P(D)(1-\Delta)^k$ and $u$ by $(1-\Delta)^{-k}\phi u$ in the statement of the theorem, then to prove the theorem is equivalent to do it in this case at least when $-\frac{3}{2}<x_1<\frac{3}{2}$, and the fact that $(1-\Delta)^{-k}\phi u\in C^0$ when $k$ is large assures the claim. The partial Fourier transforms of $u$ and $f$,

$$\hat{u}_1(x_1,\xi')=\int u(x)e^{-i\langle x',\xi'\rangle}dx', \qquad -1\leqq x_1\leqq 1,$$

$$\hat{f}_1(x_1, \xi')=\int f(x)e^{-i\langle x',\xi'\rangle}dx', \qquad -1\leqq x_1\leqq 1,$$

are then analytic functions of $\xi'$ for fixed $x_1$. Since $P(D)u=f$, it follows that

$$P(D_1, \xi')\hat{u}_1(x_1, \xi')=\hat{f}_1(x_1, \xi'), \qquad -1\leqq x_1\leqq 1. \tag{2.6}$$

We shall now prove that there exist constants $C$, $C_N$ and $M$ which are independent of $x_1$, $-1\leqq x_1\leqq 1$, such that

$$|D_1^k\hat{u}_1(x_1, \xi')|\leqq C(1+|\xi'|)^M, \qquad 1\leqq k\leqq l,\ -1\leqq x_1\leqq 1, \tag{2.7}$$

$$|\hat{f}_1(x_1, \xi')|\leqq C_N(1+|\xi'|)^{-N}, \qquad N=1, 2, \ldots . \tag{2.8}$$

To do so we choose a function $\phi\in C_0^\infty((-2, 2))$ such that $\phi=1$ when $-1\leqq x_1\leqq 1$ and set $v=\phi u$. Thus we have when $-1\leqq x_1\leqq 1$

$$
\begin{aligned}
D_1^k\hat{u}_1(x_1, \xi')&=D_1^k\hat{v}_1(x_1, \xi') \\
&=(2\pi)^{-1}\int_{-\infty}^{\infty}\xi_1^k\hat{v}(\xi)e^{ix_1\xi_1}d\xi_1.
\end{aligned} \tag{2.9}
$$

Since $v\in\mathcal{E}'(\Omega)$, we obtain for some constants

$$|\hat{v}(\xi)|\leqq C(1+|\xi|)^M.$$

Moreover Lemma 1 and 2 give that when $|\xi'|<\varepsilon|\xi_1|^\rho$

$$|\hat{v}(\xi)|\leqq C_N(1+|\xi|)^{-N}, \quad N=1, 2, \ldots.$$

Applying these estimates to (2.9) we obtain

$$
\begin{aligned}
|D_1^k\hat{u}_1(x_1, \xi')|\leqq&|(2\pi)^{-1}C\int_{|\xi_1|\leqq\varepsilon^{-1/\rho}|\xi'|}\xi_1^k(1+|\xi|)^M d\xi_1| \\
&+|(2\pi)^{-1}C_N\int_{|\xi_1|\geqq\varepsilon^{-1/\rho}|\xi'|}\xi_1^k(1+|\xi|)^{-N}d\xi_1|.
\end{aligned} \tag{2.10}
$$

Since the second integral in (2.10) is absolutely convergent when $N>l+1$, (2.7) follows with other constants $C$ and $M$. (2.8) also follows similary.

Now decomposing the ordinary differential equation (2.6), we obtain

$$(D_1-\sigma_1(\xi'))\ldots(D_1-\sigma_l(\xi'))\hat{u}_1(x_1, \xi')=\hat{f}_1(x_1, \xi')$$

where $\sigma_1(\xi'), \ldots, \sigma_l(\xi')$ are the roots of the polynomial $P(\xi_1, \xi')$ of $\xi_1$, which satisfy the following estimates with positive constants $C_1$, $C_2$, $\delta_1$ and $\delta_2$

$$C_1|\xi'|^{\delta_1}\leqq\operatorname{Im}\sigma_k(\xi')|\leqq|\sigma_k(\xi')|\leqq C_2|\xi'|^{\delta_2}, \quad k=1, 2, \ldots l, \tag{2.11}$$

when $\xi'$ is large. For the first inequality follows from the hypothesis (1.2) (see Lemma 2.1 in Appendix in Hörmander [4]) and the last from the usual estimate of the roots of a polynomial by its coefficients. Now we introduce the functions $w_k(x_1, \xi')$, $0\leqq k\leqq l$, by the equations

$$(D_1-\sigma_{k+1}(\xi'))w_{k+1}(x_1, \xi')=w_k(x_1, \xi'), \quad 0\leqq k\leqq l-1, \tag{2.12}$$

where $w_0(x_1, \xi')=\hat{f}_1(x_1, \xi')$ and $w_l(x_1, \xi')=\hat{u}_1(x_1, \xi')$. In view of (2.7) and (2.11) it follows that there are constants $C$ and $M$ such that

(2.13) $$|w_k(x_1, \xi')| \leqq C(1+|\xi|)^M, \quad -1 \leqq x_1 \leqq 1.$$

Solving the equation when $k=0$ in (2.2), we have

$$w_1(x_1, \xi') = e^{-i\sigma_1(t-x_1)} w_1(t, \xi') - i\int_{x_1}^{t} e^{-i\sigma_1(s-x_1)} \hat{f}_1(s, \xi') ds$$

where $-1 \leqq x_1, t \leqq 1$. If we restrict the range of $x_1$ to the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$, we can always choose $t$ so that $\mathrm{Im}\ \sigma_1(\xi')(t-x_1) = -\frac{1}{2}|\mathrm{Im}\ \sigma_1(\xi')|$. Thus we have

(2.14) $$|w_1(x_1, \xi')| \leqq |w_1(x_1 \pm \tfrac{1}{2}, \xi')| e^{-|\mathrm{Im}\sigma_1|/2} + \sup_{-1 \leqq x_1 \leqq 1} |\hat{f}(x_1, \xi')|$$

In view of (2.8), (2.11) and (2.13), this means that there exist constants $C_N$ such that

$$|w_1(x_1, \xi')| \leqq C_N(1+|\xi'|)^{-N} \quad N=1, 2, \ldots, \quad -\frac{1}{2} \leqq x_1 \leqq \frac{1}{2}.$$

Repeating the same arguments to the rest equations in (2.12), we obtain finally

(2.15) $$|\hat{u}_1(x_1, \xi')| \leqq C_N(1+|\xi'|)^{-N}, \quad N=1, 2, \ldots,$$

when $x_1$ is in a neiborhood of 0. Since this condition, however, plays no essential role, we may assume that (2.15) is valid also when $-1 \leqq x_1 \leqq 1$. Since in the region defined by the inequality, $|\xi'| \geqq \varepsilon |\xi_1|^\rho$,

$$1+|\xi| \leqq C(1+|\xi'|)^{1/\rho}$$

with a constant $C$, for each $\varphi \in C_0^\infty(R')$ such that supp $\varphi \subset [-1, 1]$ we have according to (2.15)

(2.16) $$\begin{aligned} |\widehat{\varphi u}(\xi)| &= \left|\int_{-\infty}^{\infty} \varphi(x_1) \hat{u}_1(x_1, \xi') e^{-ix_1\xi_1} dx_1\right| \\ &\leqq 2C_N \sup |\varphi| \cdot C^{\rho N}(1+|\xi|)^{-\rho N}, \quad N=1, 2, \ldots, \end{aligned}$$

when $|\xi'| \geqq \varepsilon |\xi_1|^\rho$. Adding the result obtained from the lemmas we conclude that $u \in C^\infty$ under the condition, $-1 < x_1 < 1$, which we can of course remove. The proof of the theorem is complete.

**Examples.** The hypoelliptic operators satisfy both the conditions (1.1) and (1.2), although the theorem in this case is a consequence of the well known regularity theorem. On the other hand, the polynomial $P(\xi_1, \xi_2, \xi_3)=\xi_1^2+\xi_2^2+\xi_2^2\xi_3^2+\xi_3^2$, clearly satisfying the conditions, is not hypoelliptic. In fact the highest order term with respect to $\xi_2$ is dependent on $\xi_3$.

## § 3. Gevrey case

We shall consider here the case when in Theorem 1 $f$ will belong to a Gevrey class which will be introduced as follows.

**Definition 1.** *Let $\Omega$ be an open set in $\boldsymbol{R}^n$. We denote by $\Gamma^\rho(\Omega)$, $0<\rho\leqq 1$, the set of functions $f$ in $C^\infty(\Omega)$ such that for every compact set $K\subset\Omega$ there is a constant $C$ for which the inequality*

$$(3.1)\qquad |D^\alpha f(x)|\leqq C(C|\alpha|)^{|\alpha|/\rho},\ x\in K,$$

*is valid for every multi-index $\alpha$. We slao set $\Gamma_0^\rho=\Gamma^\rho\cap C_0^\infty$ and define the notation $\Gamma^\rho$* sing supp *$u$ for a distribution $u$ in $\Omega$ as the smallest subset outside whicn $u\in\Gamma^\rho$.*

The definition above is a special case of Definition 4.4.2. in Hörmander [4]. The following lemma is an easy consequence of (3.1).

**Lemma 3.** *When $\rho<1$, $u\in\mathcal{D}'(\Omega)$ belongs to $\Gamma^\rho(\Omega)$ if and only if for each $\varphi\in\Gamma_0^\rho$ there exists a constant $C$ which is independent of $N$ such that*

$$(3.2)\qquad |\widehat{\varphi u}(\xi)|\leqq C(CN)^N(1+|\xi|)^{-\rho N},\quad N=1, 2, \dots,\quad \xi\in\boldsymbol{R}^n,$$

*or equivalently that*

$$(3.3)\qquad |\widehat{\varphi u}(\xi)|\leqq C(CN)^{N/\rho}(1+|\xi|)^{-N},\quad N=1, 2, \dots,\quad \xi\in\boldsymbol{R}^n,$$

*which follows from* (3.2) *replacing $N$ by $N/\rho$.*

*Proof.* If $u\in\Gamma^\rho(\Omega)$, we have for each $\varphi\in\Gamma_0^\rho(\Omega)$ and each multi-index $\alpha$

$$|\xi^\alpha \widehat{\varphi u}(\xi)| = \int_{\mathrm{supp}\,\varphi} e^{-i<x,\xi>} D^\alpha(\varphi u) dx$$

which means from (3.1) when $|\alpha| \leqq N$ that

$$|\widehat{\varphi u}(\xi)| \leqq C(CN)^{N/\rho}(1+|\xi|)^{-N}.$$

Conversely let $u \in \mathscr{D}'(\Omega)$ satisfy the condition (3.2). If we note that for every compact set $K \subset \Omega$ we can shoose a function $\varphi \in \Gamma_0^\rho(\Omega)$ such that $\varphi = 1$ near $K$ (see e.g. Lemma 5.7.1. in [4]), we have for such $\varphi$

$$\begin{aligned} D^\alpha u(x) &= D^\alpha(\varphi u)(x) \\ &= (2\pi)^{-n} \int e^{i<x,\xi>} \xi^\alpha \hat{\varphi u}(\xi)\, d\xi, \qquad x \in K. \end{aligned}$$

If we choose $N = n+1+|\alpha|$ in (3.3), we obtain the estimate,

$$|D^\alpha u(x)| \leqq C(C(|\alpha|+n+1))^{(|\alpha|+n+1)/\rho}, \quad x \in K. \tag{3.4}$$

If we note that when $(N+K)^{k/N} < 2$ and $N \geqq K$

$$(N+k)^{N+k} \leqq (4N)^N, \tag{3.5}$$

(3.4) implies that (3.1) is valid with another constant when $x \in K$, which complete the proof.

Now we shall have instead of Theorem 1 the following

**Theorem 2.** *Let the notations and the hypotheses be as in Theorem* 1. *In addition suppose the following conditions*;

(*i*) $|\mathrm{Im}\, \xi_1| \geqq C|\xi'|^{\delta_1}$, $0 < C$, *when* $P(\xi_1, \xi') = 0$ *and real* $\xi'$ *is large,*

(*ii*) $\Gamma^{\delta_2}$ sing supp $u$ *is bounded in* $\Omega$.

(*iii*) $f \in \Gamma^{\delta_3}(\Omega)$,

*then if* $0 < \delta = \min(\delta_1, \delta_2, \delta_3) < 1$ *it follows that* $u \in \Gamma^{\rho\delta}(\Omega)$ *where* $\rho$ *is the number introduced in Lemma* 1.

The verification of the present theorem is a routine repetition of that of Theorem 1 except for some estimates concerning with Lemma 3. At first Lemma 2 will be replaced by the following

**Lemma 4.** *Besides the hypotheses in Lemma* 2 *if* $f \in \Gamma^{\rho_0}(U)$ *it follows that for each* $\varphi \in \Gamma_0^{\rho_1}$, $\rho_1 = \min(\rho, \rho_0)$,

$$(3.6)\qquad |\widehat{\varphi u}(\xi)| \leqq C(CN)^N(1+|\xi|)^{-\rho_1 N}, \qquad N=1, 2, \ldots, \qquad \xi \in \Gamma.$$

*Proof*. Although the proof will be carried on parallel to that of Lemma 2, we must define $w_N$ more carefully, that is, we set

$$(3.7)\quad w_N = \sum_{|\alpha_1|+\cdots+|\alpha_k| \leqq N-m} (-1)^k (P^{(\alpha_1)}(\xi)\ldots P^{(\alpha_k)}(\xi) D^{\alpha_1+\cdots+\alpha_k}/\alpha_1!\ldots\alpha_k! P(\xi)^k)\varphi$$

$$k=1, 2, \ldots,$$

where $m=\deg P$. Thus we have

$$w_N - R(\xi, D)w_N$$
$$= \varphi - \sum'_{N-m \leqq |\alpha_1|+\cdots+|\alpha_k| \leqq N} (-1)^k (P^{(\alpha_1)}(\xi)\ldots P^{(\alpha_k)}(\xi) D^{\alpha_1+\cdots+\alpha_k}/\alpha_1!\ldots\alpha_k! P(\xi)^k)\varphi$$

where $\sum'$ means a partial summation in the range noted below. Since each term in the summation $\sum'$ is bounded by $C(CN)^N(1+|\xi|)^{-\rho(N-m)}$ and the number of the terms in $\sum'$ does not exceed $A^N$ with a constant $A$ independent of $N$, the summation $\sum'$ can be estimated when $\xi \in \Gamma$ by $C(CN)^N(1+|\xi|)^{-\rho(N-m)}$, $N=1, 2, \ldots$, with $C$ independent of $N$. If we note that the coefficient of each term in $w_N/P(\xi)$ is bounded when $\xi \in \Gamma$ and that the number of its terms does not exceed $A^N$, using (2.5) and (3.5) we obtain (3.6).

*Proof of Theorem* 2. We may of course assume that $\delta=\delta_1=\delta_2=\delta_3$. We shall first replace (2.8) by

$$(3.8)\qquad |\hat{f}_1(x_1, \xi')| \leqq C(CN)^N(1+|\xi'|)^{-\delta N}, \qquad N=1, 2, \ldots,$$

following the hypothesis that $f \in \Gamma^\delta(\Omega)$. When we utilize the estimate (2.14) we note the elementary fact such that for any positive constants $\varepsilon$, $M$ there is a constant $C$ independent of $N$ such that

$$(3.9)\qquad (1+\tau)^M \exp(-\varepsilon\tau^\delta) \leqq C(CN)^N(1+\tau)^{-\delta N}, \quad \tau>0, \quad N=1, 2, \ldots,$$

obtained by calculating the value $\tau_0$ which makes the left-hand side of (3.9) maximum. Then we have instead of (2.15)

$$|\hat{u}_1(x_1, \xi')| \leqq C(CN)^N(1+|\xi'|)^{-\delta N}, \qquad N=1, 2, \ldots,$$

Thus in view of (2.16) we have proved Theorem 2.

## § 4. Fundamental Solutions with Singular Supports in a Proper Cone

In this section we shall show roughly speaking that if contrary to the condition (1.2) of Theorem 1 imaginary parts of some roots are bounded when $\xi' \in \boldsymbol{R}^{n-1}$ while those of the other roots go to infinity, we can construct a solution which has actually a bounded singular support in $\Omega$.

**Theorem 3.** *When* $0<\rho<1$, *there exists a fundamental solution* $E \in \mathcal{D}'(\Omega)$ *of* $P(D)$ *such that* $\Gamma^{\rho}$ sing supp $E$ *is contained in a* $\vartheta$*-proper cone, that is, in a cone contained in* $\{x; <x, \vartheta> >0\} \cup \{0\}$ *if and only if for some positive constants* $\varepsilon$, $\tau_0$, $t_0$ *and a neighborhood* $U$ *of* $\vartheta$

$$(4.1) \qquad P(\xi+it\eta) \neq 0, \quad -\varepsilon|\xi|^{\rho}<t<-t_0, \quad |\xi| \geqq \tau_0,$$

*when* $\eta \in U$.

*Proof.* Assume that such a solution $E$ exists. Choosing a function $\psi \in \Gamma_0^{\rho}$ such that $\psi=1$ near the origin, we have

$$(4.2) \qquad P(D)f(x)=\delta(x)+g(x)$$

where $f=\psi E$ and $g=P(D)$ $((\psi-1)E)$. Since $f, g \in \mathcal{E}'(\boldsymbol{R}^n)$, taking the Fourier-Laplace transforms of (4.2), we have

$$(4.3) \qquad P(\zeta)\hat{f}(\zeta)=1+g(\zeta), \quad \zeta \in \boldsymbol{C}^n.$$

we now note that $\Gamma^{\rho}$ sing supp $g \subset \Gamma^{\rho}$ sing supp $E$ and does not intersect a neighborhood of the origin, which makes it possible to choose a compact neighborhood $K$ of $\Gamma^{\rho}$ sing supp $g$ and a neighborhood $U$ of $\vartheta$ such that $h(-\eta) <$ const. $<0$ when $\eta \in U$, where $h(\xi)$ is the support function of $K$; $h(\xi)=\sup_{x \in K} \langle x, \xi \rangle$.

Writing that $g=g_1+g_2$ where $g_1 \in \mathcal{E}'$, supp $g_1 \subset K$ and $g_2 \in \Gamma_0^{\rho}$, we have in view of the Paley-Wiener theorem and the proof of Lemma 5.7.2. in Hörmander [4]

$$(4.4) \qquad \begin{aligned} &|\hat{g}_1(\zeta)| \leqq C(1+|\zeta|)^M e^{h(\operatorname{Im}\zeta)} \\ &|\hat{g}_2(\zeta)| \leqq C \exp(A|\operatorname{Im}\zeta|-B|R_e\zeta|^{\rho}) \end{aligned}$$

for some positive constants $C$, $A$, $B$ and $M$. Since (4.3) implies that when $P(\zeta)=0$, $|\hat{g}_1(\zeta)|\geqq\frac{1}{2}$ or $|\hat{g}_2(\zeta)|\geqq\frac{1}{2}$ it follows that

$$0\leqq\log(2c)-th(-\eta)+M\log(1+|\xi|+|t\eta|)$$
$$\text{or}\quad 0\leqq\log(2c)+A|t\eta|-B|\xi|^{\rho}$$

when $P(\xi+it\eta)=0$ and $\xi$, $\eta$ and $t>0$ are real. Replacing the constants suitably we have $P(\xi+it\eta)\neq0$ if

$$-\varepsilon|\xi|^{\rho}<t<-C_1-C_2\log(1+|\xi|+|t|). \tag{4.5}$$

when $\eta\in U$ and $\xi$ is large. From the Tarski-Seidenberg theorem we can eliminate the logarithmic term in (4.5) (see the proof of Lemma 2.1 of Appendix in [4] ), which proves (4.1).

Conversely assume that (4.1) is valid. We now define a distribution $E=E_{\tau_0}$ by the identity

$$E_{\tau_0}(x)=(2\pi)^{-n}\int_{|\xi|\geqq\tau_0}\frac{e^{i<x,\xi-it_1\vartheta>}}{p(\xi-it_1\vartheta)}d\xi$$

with $t_0<t_1<\varepsilon\tau_0^{\rho}$. This means of course the distribution;

$$C_0^{\infty}\ni\check{\varphi}\longmapsto\langle E_{\tau_0},\check{\varphi}\rangle$$
$$=(2\pi)^{-n}\int_{|\xi|\geqq\tau_0}\frac{\hat{\varphi}(\xi-it_1\vartheta)}{p(\xi-it_1\vartheta)}d\xi.$$

The convergence of the integral follows from the Paley-Wiener theorem. Note that $P(D)\,E_{\tau_0}-\delta$ is real analytic and moreover entire holomorphic in $\mathbf{C}^n$. Hence, by Ehrenpreis theorem there exists an analytic function $f$ such that $P(D)\,(E_{\tau_0}+f)=\delta$. What we have to do is thus to show that $\Gamma^{\rho}$ sing supp $E_{\tau_0}$ is contained in a $\vartheta$-proper cone. In doing so, let $H=R_+U$ and $\Gamma$ be its dual cone, namely

$$\Gamma=\{x;\ \langle x,\xi\rangle\geqq0\text{ for all }\xi\in H\}$$

which is a $\vartheta$-proper cone. For a given $x_0\notin\Gamma$, let us choose a so small neighborhood $V$ of $x_0$ that $V\cap\Gamma=\phi$ and that there is a vector $\eta\in U$ such that with $a>0$

$$\langle x,\eta\rangle<-a<0,\qquad x\in V. \tag{4.6}$$

We shall now estimate $\widehat{\varphi E}(\xi)$, $\xi \in \boldsymbol{R}^n$, for each $\varphi \in \Gamma_0^\rho(V)$. In doing so we may assume that $U$ is connected, by shrinking it if necessary.
Stokes' formula now gives

$$\widehat{\varphi E}(\xi) = (2\pi)^{-n} \int_{|\xi'| \geqq \tau_0} \frac{\hat{\varphi}(\xi - \xi' + it_1\vartheta)}{P(\xi' - it_1\vartheta)} d\xi'$$
$$= (2\pi)^{-n} \int_{\gamma_1} \frac{\hat{\varphi}(\xi - \zeta)}{P(\zeta)} d\zeta$$
$$+ (2\pi)^{-n} \int_{|\xi'| \geqq \tau_0} \frac{\hat{\varphi}(\xi - \xi' + it_1\eta)}{P(\xi' - it_1\eta)} d\xi'$$

where $\gamma_1$ is a compact chain in $\boldsymbol{C}^n$. If we note that with positive constants $C$, $B$ and $a$ introduced in (4.6),

$$|\hat{\varphi}(\xi - \xi' + is\eta)| \leqq C \exp(-B|\xi - \xi'|^\rho - as) \tag{4.7}$$

when $\xi$, $\xi'$ and $s$ are real, another application of Stokes' formula gives with $\tau_1 > 0$

$$\widehat{\varphi E}(\xi) = (2\pi)^{-n} \int_{\gamma_2} \frac{\hat{\varphi}(\xi - \zeta)}{P(\zeta)} d\zeta + (2\pi)^{-n} \int_{|\xi'| \geqq \tau_1} \frac{\hat{\varphi}(\xi - \xi' + iv(\xi'))}{P(\xi' - iv(\xi'))} d\xi' \tag{4.8}$$

where $v(\xi') = \varepsilon |\xi'|^\rho \eta/2$ and $\gamma_2$ another compact chain. Using (4.7) the first term on the right-hand side of (4.8) can be easily estimated by a constant times $\exp(-B|\xi|^\rho)$. To do the second term, observe that

$$|P(\xi - iv(\xi))^{-1}| \leqq C(1 + |\xi|)^M,$$

which follows also from the Tarski-Seidenberg theorem. Furthermore we have

$$|\hat{\varphi}(\xi - \xi' + iv(\xi'))| \leqq C \exp(-B|\xi - \xi'|^\rho - \frac{a\varepsilon}{2}|\xi'|^\rho)$$
$$\leqq C \exp(-C_1|\xi'|^\rho) \exp(-C_2|\xi|^\rho)$$

where all the constants are positive and we have used the inequality: max $(|\xi - \xi'|^\rho, |\xi'|^\rho) \geqq (|\xi|^\rho/4) + (|\xi'|^\rho/2)$. Thus the second term can be estimated also by a constant times $\exp(-C_2|\xi|^\rho)$, which means in view of (3.9) and Lemma 3 that $E \in \Gamma^\rho$ in $V$, in other words that $\Gamma^\rho$ sing supp $E \subset \Gamma$. The proof of Theorem 3 is now complete.

When we need not consider the Gevrey class we have in like manner the following theorem, already studied in somewhat different forms by Shirota [9] and Hörmander [5].

**Theorem 4.** *There exists a fundamental solution $E$ of $P(D)$ such that* sing supp $E$ *is contained in a $\vartheta$-proper cone if and only if there exist positive constants $\varepsilon$, $\tau_0$, $t_0$ and $\rho$ and a neighborhood $U$ of $\vartheta$ such that*

$$p(\xi+it\eta)\neq 0, \quad -\varepsilon|\xi|^{\rho}<t<-t_0, \quad |\xi|\geqq\tau_0, \quad \textit{when} \quad \eta\in U. \tag{4.9}$$

If $P$ satisfies (4.9) and in addition is not hypoelliptic then we have a fundamental solution $E$ of $P(D)$ whose singular support is contained in a $\vartheta$-proper cone and not equal to the origin. Thus choosing a slab $\Omega=\{x;\ a<\langle x, \vartheta\rangle<b\}$ suitably we have $E\in\mathcal{D}'(\Omega)$ such that $P(D)E\in C^{\infty}(\Omega)$ and sing supp $E$ is bounded and actually exists. The requirement, $\eta\in U$, in (4.9) however makes it slightly difficult to obtain examples which satisfy the condition (4.9). But when $P$ is homogeneous, the condition becomes much simpler.

**Proposition 1.** *When $P$ is a homogeneous polynomial, the condition* (4.9) *is equivalent to the following; there exists a constant $\varepsilon>0$ such that*

$$P(\xi+it\vartheta)\neq 0 \quad \text{if} \quad 0<t<\varepsilon|\xi|. \tag{4.10}$$

*Proof.* We first show the equivalence of (4.9) to the following;

$$P(\xi+it\eta)\neq 0 \ \text{if} \ 0<t<\varepsilon|\xi| \qquad \eta\in U \tag{4.11}$$

for $\varepsilon>0$ and a neighborhood $U$ of $\vartheta$. To do so assume that (4.9) is valid for some $\rho$ which may be chosen equal to 1 by the homogeneity of $P$. Since $P(\xi+it\eta)=\tau^m P(\tau^{-1}\xi+i\tau^{-1}t\eta)$, $P(\xi+it\eta)\neq 0$ if $-\varepsilon\tau^{-1}|\xi|<\tau^{-1}t<-t_0$, which implies $-\varepsilon|\xi|<t<0$ by letting $\tau\to 0$. Since that (4.11) implies (4.9) is trivial, our claim is now verified. In order to prove that (4.10) means (4.11) we introduce the localization $P_\xi$ of $P$ at $\xi\in\boldsymbol{R}^n\backslash 0$, following Atiyah-Bott-Gårding [2] and Andersson [1], which is defined as the coefficient of the lowest order term in $\tau$ when we develop $P(\xi+\tau\zeta)$ around $\xi\in\boldsymbol{R}^n\backslash 0$;

$$P(\xi+\tau\zeta)=\tau^{p}P_{\xi}(\zeta)+\text{terms of higher order in } \tau$$

where $p$ is called the multiplicity of $P$ at $\xi$. If $P$ satisfies (4.10) and $P_{\xi_0}(\xi+s\vartheta)$, $\xi\in\boldsymbol{R}^n$, has a root $s_0$ with $\operatorname{Im} s_0\neq 0$, in view of the identity $\tau^{-p}P(\xi_0+\tau(\xi+s\vartheta))=P_{\xi_0}(\xi+s\vartheta)+0(|\tau|)$, $P(\xi_0+\tau\xi+\tau s\vartheta)$ also has a root $s_1$ near $s_0$, in particular with $\operatorname{Im} s_1\neq 0$, for sufficiently small $\tau>0$, which contradicts the condition (4.10). It thus follows that $P_{\xi_0}(\xi+s\vartheta)\neq 0$ when $\xi\in\boldsymbol{R}^n$ and $\operatorname{Im} s\neq 0$, which means that $P_{\xi_0}$ is hyperbolic with respect to $\vartheta$. Now that $P(\xi+\cdot)$ is locally hyperbolic following the terminology of Gårding [3], we have (4.11) from Main Lemma in [3]. However, for the convenience of the reader, we shall here copy the proof from [3]. We consider the function

$$f(s,\ t,\ u,\ \eta)=\tau^{-p}P(\xi_0+\tau(\xi+s\vartheta+t\eta))$$

where $|\xi_0|=1$, $u=(\tau,\xi)$, $p$ the multiplicity of $P$ at $\xi_0$ and $\eta$ belongs to a compact convex neighborhood $K$ of $\vartheta$ where $P_{\xi_0}$ does not vanish. Since $f(0,\ t,\ 0,\ \eta)=t^p P_{\xi_0}(\eta)$, we have when $s$, $t$, $u$ are small

$$f(s,t,u,\eta)=P_{\xi_0}(\eta)\prod_{1}^{p}(t+\lambda_k(s,u,\eta))F(s,t,u,\eta) \tag{4.12}$$

where $\lambda_1,\ \dots,\ \lambda_p$, $F$ are continuous and $F(0,0,0,\eta)=1$ Because the hyperbolicity of $P_{\xi_0}$ implies that $P_{\xi_0}(s\vartheta+t\eta)\neq 0$ when $0<\operatorname{Im}(s+t)$ and because of (4.10), we have $\operatorname{Im}\lambda_k(s,0,\eta)>0$ and $\operatorname{Im}\lambda_k(s,u,\eta)\neq 0$ when $\operatorname{Im} s>0$ and real $u$, $s$, $t$ are small. Thus we have $\operatorname{Im}\lambda_k(s,u,\eta)\geqq 0$ when $\operatorname{Im} s\geqq 0$ and real $u$, $s$, $t$ are small. Putting $s=0$ in (4.12), we obtain with $\varepsilon_0>0$,

$$P(\xi_0+\tau\xi+t\eta)\neq 0,\quad 0<\operatorname{Im} t<\varepsilon_0,$$

when $\eta\in K$ and real $\tau$, $|t|$, $\xi$ are small. Moving $\xi_0$ with $|\xi_0|=1$, we have (4.11) by the compactness of the sphere.

**Examples.** All the polynomials written in products of hypoelliptic and hyperbolic polynomials satisfy the condition (4.9). $P(\xi_1,\xi_2,\xi_3)=\xi_1^4-\xi_2^4-\xi_3^4$ which is irreducible and neither hyperbolic nor hypoelliptic satisfies the condition (4.9) when $\vartheta=(1,0,\dots,0)$ in view of Proposition 1.

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