Publ. RIMS, Kyoto Univ.
10 (1975), 545–562

On a Cartan Formula for the Algebraic Steenrod Operations Associated with a Pair of Hopf Algebras

By

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Introduction

S. Araki [2] and R. Vazquez [7] defined two types of Steenrod squaring operations in the spectral sequence mod 2 associated with a fibre space in the sense of Serre, by using the cubical singular cohomology theory. They computed the Cartan formula and Adams relation. H. Uehara [5] established an algebraic analogy to their work. He discovered and investigated the Steenrod operations in the Adams spectral sequence associated with a pair of Hopf algebras.

In Paragraph 1 of this paper, [5] is modified and reviewed. It is shown that the operations are independent of the higher homotopies under a certain filtration condition and the Cartan formula is obtained.

§ 1. Modification of [5]

In [5], a graded differential algebra with a decreasing filtration and cup-i-products was defined. Theorem 2 of [5] stated that in Adams filtered complex associated with a pair of Hopf algebras over Z_2, there exist Z_2-linear maps such that the Adams filtered complex is then a graded differential algebra with a decreasing filtration and cup-i-products. The proof of this theorem is not complete, in fact there is some question as to whether it can be proved using Definition 1 of [5].

Communicated by N. Shimada, April 26, 1974.
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In this chapter Definition 1 of [5] is modified and restated as Definition (*). With this definition the results of [5] are all true. If the change in definition affects the proof of propositions or theorems of [5], they are proved in this chapter. It should further be noted, if Definition 1 of [5] is satisfied then Definition (*) is satisfied but the converse may not be true.

Definition (*). By a graded differential algebra $G=\{C, \delta, F, \underset{i}{\cup}\}$ with a decreasing filtration F and with cup-i-products $\underset{i}{\cup}$, we mean
1) a graded cochain complex C over the field Z_2:

$$C: \ C^0 \longrightarrow C^1 \longrightarrow \dots \longrightarrow C^n \overset{\delta^n}{\longrightarrow} C^{n+1} \longrightarrow \dots,$$

where δ^n: $C^n \to C^{n+1}$ is a morphism of graded vector spaces over Z_2,
2) for each integer p, F^pC is a subcomplex of C such that
   i) $F^{p+1}C$ is a subcomplex of F^pC (in notation: $F^pC \supset F^{p+1}C$)
   ii) $F^pC=C$ if $p \leqq 0$, and
   iii) $F^pC^n=0$ if $p>n$,
3) for each integer i there exists a Z_2-linear map $\underset{i}{\cup}: C\otimes C \to C$ such that if $x\in F^pC^{m,s}$ and $y \in F^qC^{n,t}$, then $x\underset{i}{\cup}y\in F^{\alpha}C^{m+n-i,s+t}$ for

$$\alpha=\begin{cases} \operatorname{Max}\{p+q-i, p, q\} & \text{if } p=q \\ \operatorname{Max}\{p+q-i, \operatorname{Min}\{p, q\}+1\} & \text{if } p\neq q, \end{cases}$$

where $x\underset{i}{\cup}y=\underset{i}{\cup}(x\otimes y)$, $x\cup y=x\underset{0}{\cup}y$ in notations, and s, t stands for gradings. $\underset{i}{\cup}$ satisfies the following conditions:
   i) $\underset{i}{\cup}$ is trivial if $i<0$,
   ii) For $x\in F^qC^m$ and $y\in F^pC^n$, $x\underset{i}{\cup}y=0$ if $i>m$ or n,
   iii) $x\cup(y\cup z)=(x\cup y)\cup z$,
   iv) $1\cup x=x\cup 1$ for some $1\in C^{0,0}$, and
   v) $\delta(x\underset{i}{\cup}y)=x\underset{i-1}{\cup}y+y\underset{i-1}{\cup}x+\delta x\underset{i}{\cup}y+x\underset{i}{\cup}\delta y$:

Let $Z_r^{p,q}=\{x\in F^pC^{p+q} \mid \delta x\in F^{p+r}C^{p+q+1}\}$
$B_r^{p,q}=\{x\in F^pC^{p+q} \mid \text{there exists } y\in F^{p-r}C^{p+q-1} \text{ with } \delta y=x\}$,
$Z_\infty^{p,q}=\{x\in F^pC^{p+q} \mid \delta x=0\}$, and
$B_\infty^{p,q}=\{x\in F^pC^{p+q} \mid \text{there exists } y\in C^{p+q-1} \text{ with } \delta y=x\}$.

Define $$E_r^{p,q}=\frac{Z_r^{p,q}}{Z_{r-1}^{p+1,q-1}+B_{r-1}^{p,q}}\qquad \infty\geqq r\geqq 1$$

Thus, we have a spectral sequence $\{E_r, d_r | r\geqq 1\}$, where $d_r\colon E_r^{p,q}\to E_r^{p+r,q-r+1}$, is the composition

$$E_r^{p,q}=\frac{Z_r^{p,q}}{Z_{r-1}^{p+1,q-1}+B_{r-1}^{p,q}}\xrightarrow{\delta}\frac{Z_r^{p+r,q-r+1}}{B_{r-1}^{p+r,q-r+1}}$$
$$\xrightarrow{i}\frac{Z_r^{p+r,q-r+1}}{Z_{r-1}^{p+r+1,q-r}+B_{r-1}^{p+r,q-r+1}}=E_r^{p+r,q-r+1},$$

for more details see [4].

Let us define a map $\theta_i\colon C\to C$ by $\theta_i(x)=x\underset{i}{\cup}x+x\underset{i+1}{\cup}\delta x$.

Proposition 1. *θ_i induces Steenrod operations ${}_BSt_i$, ${}_FSt_i$ in the spectral sequence associated with the algebraic system G such that*

$${}_BSt_i\colon E_\gamma^{p,q}\longrightarrow E_{2\gamma-2}^{2p-i,2q}\quad\text{for}\quad \infty\geqq\gamma\geqq 2,$$

and

$${}_FSt_i\colon E_\gamma^{p,q}\longrightarrow E_\gamma^{p,2q+p-i}\quad\text{for}\quad \infty\geqq\gamma\geqq 1.$$

They are all Z_2-homomorphisms.

Proof. See [5].

Lemma 1. *Define $\tilde\theta_i\colon C\to C$ by $\tilde\theta_i(x)=x\underset{i}{\cup}x+\delta x\underset{i+1}{\cup}x$, then ${}_BSt_i=\widetilde{{}_BSt_i}$ and ${}_FSt_i=\widetilde{{}_FSt_i}$, where $\widetilde{{}_BSt_i}$, $\widetilde{{}_FSt_i}$ are the Steenrod operations induced by $\tilde\theta_i$.*

Proof. $\theta_i(x)+\tilde\theta_i(x)=x\underset{i}{\cup}x+x\underset{i+1}{\cup}\delta x+x\underset{i}{\cup}x+\delta x\underset{i+1}{\cup}x=x\underset{i+1}{\cup}\delta x+\delta x\underset{i+1}{\cup}x=\delta(x\underset{i+1}{\cup}x)$. Thus for $x\in Z_r^{p,q}$, we have $\delta(x\underset{i+1}{\cup}x)\in B_{r-1}^{p,2q+p-i}\cap B_{2r-3}^{2p-i,2q}$, and therefore

$${}_FSt_i=\widetilde{{}_FSt_i}\quad\text{and}\quad {}_BSt_i=\widetilde{{}_BSt_i}.$$

To prepare for the existence of a graded differential algebra $G=\{C, \delta, F, \underset{i}{\cup}\}$ with a decreasing filtration F and with cup-i-products $\underset{i}{\cup}$, we need the following proposition.

Proposition 2. *Let A be a cocommutative Hopf algebra over Z_2 and let $\Delta: A \rightarrow A \otimes A$ be the comultiplication. If $\in: \mathfrak{X} \rightarrow Z_2$ is a Z_2-split exact resolution of the A-module Z_2, then there exists a sequence of Δ-homomorphisms $h^i: \mathfrak{X} \rightarrow \mathfrak{X} \otimes \mathfrak{X}$ for $i \geqq 0$, such that*

1) *h^0 is a grade preserving Δ-chain map,*
2) *for $i>0$ h^i is a Δ-chain homotopy connecting h^{i-1} with ρh^{i-1} which raises the homological dimensions by i and preserves the gradings, where $\rho: \mathfrak{X} \otimes \mathfrak{X} \rightarrow \mathfrak{X} \otimes \mathfrak{X}$ is the twisting chain map. Moreover, if K^i for $i \geqq 0$ satisfies* 1) *and* 2), *then there exists a sequence of Δ-maps S^i for $i \geqq 0$ such that*
3) *$S^0=0$ and*
4) *For $i \geqq 0$, $S^{i+1}\delta + dS^{i+1} = h^i + k^i + S^i + \rho S^i$, where d, δ are boundary operators for $\mathfrak{X} \otimes \mathfrak{X}$ and $\mathfrak{X}$ respectively.*

Proof. See [5] and [6].

Consider the diagram

$$\begin{array}{ccc} \mathrm{Hom}_A(\mathfrak{X}, Z_2) \otimes \mathrm{Hom}_A(\mathfrak{X}, Z_2) & \dashrightarrow & \mathrm{Hom}_A(\mathfrak{X}, Z_2) \\ \chi \downarrow & \nearrow h^{i\#} & \\ \mathrm{Hom}_{A \otimes A}(\mathfrak{X} \otimes \mathfrak{X}, Z_2) & & \end{array}$$

where χ is the Z_2-chain map defined by

$$\chi(f \otimes g)(x \otimes y) = f(x) g(y) \quad \text{for} \quad f, g \in \mathrm{Hom}_A(\mathfrak{X}, Z_2)$$

and for $x, y \in \mathfrak{X}$.

Definition. The cup-i-product $\underset{i}{\cup}$ in the cochain complex $C = \mathrm{Hom}_A(\mathfrak{X}, Z_2)$ is denoted by $h^{i\#}\chi$.

Denoting $\mathrm{Hom}_A^s(X_p, Z_2)$ by $C^{p,s}$ for each homological dimension $p \geqq 0$ and the grading $s \geqq 0$, we have the cochain complex

$$C^{*s} = \{C^{p,s} \quad \text{for} \quad p=0, 1, \ldots, n, \ldots\}$$

such that $C = \{C^{*s} \mid s=0, 1, \ldots\}$. Then

$$f \underset{i}{\cup} g = \underset{i}{\cup}(f \otimes g) \in C^{p+q-i, s+t}$$

for $f \in C^{p,s}$ and $g \in C^{q,t}$. By the definition of $\underset{i}{\cup}$, we have,

Lemma 2. $\delta(f \underset{i}{\cup} g) = f \underset{i-1}{\cup} g + g \underset{i-1}{\cup} f + \delta f \underset{i}{\cup} g + f \underset{i}{\cup} \delta g.$

J. F. Adams [1] and A. Zachariou [8] computed explicitly a Δ-homomorphism h^i in case when $\mathcal{X}$ is the bar resolution $B(A)$. If $\Delta(a) = \sum a' \otimes a''$ for $a \in A$, then we have

$$h_n^0([a_1|a_2|\dots|a_n]) = 1 \otimes [a_1|\dots|a_n] + \sum_{1 \leq \rho \leq n} [a_1'|\dots|a_\rho'] \otimes a_1'' \dots a_\rho''[a_{\rho+1}|\dots|a_n]$$

for odd i,

$$\begin{aligned} h_n^i([a_1|\dots a_n]) &= \sum_{0 \leqq \rho_0 < \rho_1 < \dots < \rho_i \leqq n} [a_1'|\dots|a_{\rho_0}'|a_{\rho_0+1}' \dots a_{\rho_1}'|a_{\rho_1+1}'|\dots|a_{\rho_2}'|\dots \\ &\dots|a_{\rho_{i-1}+1}' \dots a_{\rho_i}'|a_{\rho_i+1}|\dots|a_n] \otimes a_1'' \dots a_{\rho_0}''[a_{\rho_0+1}''|\dots \\ &\dots|a_{\rho_1}'' a_{\rho_1+1}'' \dots a_{\rho_2}''|\dots|a_{\rho_{i-1}+1}''|\dots|a_{\rho_i}'']\end{aligned}$$

for even i,

$$\begin{aligned} h_n^i([a_1|\dots|a_n]) &= \sum_{0 \leqq \rho_0 < \rho_1 < \dots < \rho_i \leqq n} [a_1'|\dots|a_{\rho_0}'|a_{\rho_0+1}' \dots a_{\rho_1}'|\dots|a_{\rho_{i-1}+1}'|\dots|a_{\rho_i}'] \\ &\otimes a_1'' \dots a_{\rho_0}''[a_{\rho_0+1}''|\dots|a_{\rho_1}''|\dots|a_{\rho_{i-1}+1}'' \dots a_{\rho_i+1}''|\dots|a_n].\end{aligned}$$

The above h^i was computed in the following way. Let s be the contracting homotopy for $B(A)$, then $t = s \otimes 1 + \sigma\varepsilon \otimes s$ is a contracting homotopy for $B(A) \otimes B(A)$. Define h^i by the following inductive formulas

1) $h_o^0 = \Delta$
2) $h_n^0 s_{n-1} = t_{n-1} h_{n-1}^0$ for $n > 0$
3) $h_o^j = t_{j-i}(h_o^{j-1} + \rho h_o^{j-1})$ for $j \geqq 1$
4) $h_n^j s_{n-1} = t_{n+j-1}(h_n^{j-1} + \rho h_n^{j-1}) s_{n-1} + t_{n+j-1} h_{n-1}^j$ for $n > 0$ and $j \geqq 1$
5) $h_n^j(ax) = \Delta a\, h_n^j(x)$ for $j \geqq 0$, $a \in A$, $x \in I(A)^n$ and $n \geqq 1$.

Remark 1. $h_o^j = 0$ for $j > 0$.

Remark 2. $h_q^p = 0$ for $p > q$.

Remark 3. $h_0^j s_{n-1} = t_{n+j-1} h_n^{j-1} s_{n-1} + t_{n+j-1} h_{n-1}^j$ for $j \geqq 1$ and $n \geqq 1$.

Let (Γ, Λ) be a pair of connected locally finite cocommutative Hopf algebras over Z_2 such that, the subhopf algebra Λ is *central* in Γ, in the sense that

$$ab = ba \quad \text{if} \quad a \in \Lambda, \ b \in \Gamma.$$

We are going to associate with the pair (Γ, Λ) a graded differential algebra $G(\Gamma, \Lambda) = \{C, \delta, F, \underset{i}{\cup}\}$ with a decreasing filtration F and with cup-i-products $\underset{i}{\cup}$.

J. F. Adams [1] introduced a filtration in the bar construction $B(\Gamma)$, in the following way. For each integer p define a subcomplex $F_p B(\Gamma)$ of $B(\Gamma)$ such that $F_p B(\Gamma)_n$ is the Γ-submodule of $B(\Gamma)_n = \Gamma \otimes I(\Gamma)^n$ generated by elements of the form $\gamma[\gamma_1 | \ldots | \gamma_n]$ with the property $\gamma_s \in I(\Lambda)$ for at least $(n-p)$ values of s. Then it is immedeidate to see that F is the canonical increasing filtration in $B(\Gamma)$.

Define the product filtration $\overset{*}{F}$ in $B(\Gamma) \otimes B(\Gamma)$ by

$$\overset{*}{F}_p(B(\Gamma) \otimes B(\Gamma)) = \bigcup_{p \geqq s \geqq 0} F_{p-s} B(\Gamma) \otimes F_s B(\Gamma).$$

Then $(B(\Gamma) \otimes B(\Gamma), \overset{*}{F})$ is a resolution of the $\Gamma \otimes \Gamma$-module Z_2 with the increasing filtration $\overset{*}{F}$. Let $\Delta : \Gamma \to \Gamma \otimes \Gamma$ be the cocommutative diagonal and let ρ be the twisting chain map of $B(\Gamma) \otimes B(\Gamma)$. Then we have

Proposition 3. *There exists a sequence of Δ-homomorphisms h^i: $B(\Gamma) \to B(\Gamma) \otimes B(\Gamma)$ for $i \geqq 0$ such that*

1) *h^0 is a Δ-chain map which preserves grading and filtration,*
2) *h^i is a Δ-chain homotopy connecting h^{i-1} and ρh^{i-1} which preserves grading, raises homological dimension by i, and satisfies the filtration condition*

$$h^i(F_p B(\Gamma)) \subset \overset{*}{F}_\alpha(B(\Gamma) \otimes B(\Gamma)) \quad \textit{for} \quad \alpha = \mathrm{Min}\{2p, p+i\}.$$

Proof. The particular h_n^i defined previously satisfies the Proposition, [5].

Let (C, δ) be the cochain complex $\mathrm{Hom}_\Gamma(B(\Gamma), Z_2)$ over Z_2. For each integer p define a subcomplex $F^p(C)$ by the image of

$$\mathrm{Hom}_\Gamma\left(\frac{B(\Gamma)}{F_{p-1}B(\Gamma)}, Z_2\right)$$

under the dual of the projection

$$\pi: B(\Gamma) \longrightarrow \frac{B(\Gamma)}{F_{p-1}B(\Gamma)}.$$

Then (C, δ, F) is a cochain complex with a decreasing filtration. Let us call it Adams filtered complex associated with (Γ, Λ).

Proposition 4. *Let (C, δ, F) be the Adams filtered complex associated with a pair of Hopf algebras (Γ, Λ) over Z_2. Then there exist Z_2-linear maps $\underset{i}{\cup}: C\otimes C \to C$ such that $G(\Gamma, \Lambda)=\{C, \delta, F, \underset{i}{\cup}\}$ is a graded differential algebra with a decreasing filtration F and cup-i-products $\underset{i}{\cup}$, in the sense of Definition* (*).

Proof. Let $h^i: B(\Gamma) \to B(\Gamma)\otimes B(\Gamma)$ be the Δ-homomorphism in Proposition 3 and define $\underset{i}{\cup}: C\otimes C \to C$ by $h^{i*}\chi$. Since $\underset{i}{\cup}$ is the cup-i-product in $C=\mathrm{Hom}_\Gamma(B(\Gamma), Z_2)$, it is easy to see that $\underset{i}{\cup}$ satisfies all the necessary condition except the filtration condition. Consequently, it is sufficient to show that if $f\in F^pC^{m,s}$ and $g\in F^qC^{n,t}$, then $f\underset{i}{\cup} g\in F^\alpha C^{m+n-i,s+t}$ for

$$\alpha=\begin{cases} \mathrm{Max}\{p+q-i, p, q\} & \text{if } p=q \\ \mathrm{Max}\{p+q-i, \mathrm{Min}\{p, q\}+1\} & \text{if } p\neq q \end{cases}.$$

Consider first the case when $\alpha=p+q-i$. By Proposition 3

$$h^i(F_{\alpha-1}B(\Gamma))\subset \overset{*}{F}_{2(\alpha-1)}(B(\Gamma)\otimes B(\Gamma))\cap \overset{*}{F}_{(\alpha-1)+i}(B(\Gamma)\otimes B(\Gamma))$$

thus

$$\begin{aligned} h^i(F_{\alpha-1}B(\Gamma))&\subset \overset{*}{F}_{(\alpha-1)+i}(B(\Gamma)\otimes B(\Gamma)) \\ &=\overset{*}{F}_{p+q-1}(B(\Gamma)\otimes B(\Gamma)). \end{aligned}$$

Then

$$\begin{aligned}(f\underset{i}{\cup} g)(F_{\alpha-1}B(\Gamma))&\subset(f\otimes g)\overset{*}{F}_{p+q-1}(B(\Gamma)\otimes B(\Gamma))\\&=(f\otimes g)\sum F_{\xi}B(\Gamma)\otimes F_{\sigma}B(\Gamma)\end{aligned}$$

where $\xi+\sigma=p+q-1$ and thus

$$(f\underset{i}{\cup} g)(F_{\alpha-1}B(\Gamma))=0$$

because

$$\xi<p \quad \text{or} \quad \sigma<q.$$

If $\alpha=p$, then $p=q$. In this case also

$$\begin{aligned}(f\underset{i}{\cup} g)(F_{p-1}B(\Gamma))&\subset(f\otimes g)\overset{*}{F}_{2p-2}(B(\Gamma)\otimes B(\Gamma)\\&=(f\otimes g)\overset{*}{F}_{p+q-2}(B(\Gamma)\otimes B(\Gamma))\\&=0.\end{aligned}$$

If $\alpha=p+1$, then $p\leqq q-1$ and

$$\begin{aligned}(f\underset{i}{\cup} g)(F_{p}B(\Gamma))&\subset(f\otimes g)\overset{*}{F}_{2p}(B(\Gamma)\otimes B(\Gamma))\\&\subset(f\otimes g)\overset{*}{F}_{p+q-1}(B(\Gamma)\otimes B(\Gamma))\\&=0.\end{aligned}$$

Similarly when $\alpha=q+1$. Hence, the proof is completed.

From Propositions 4 and 1 we obtain,

Proposition 5. *Let (Γ, Λ) be a pair of connected locally finite cocommutative Hopf algebras over Z_2 such that Λ is central in Γ, and let $\{E_\gamma, d_\gamma\}$ be the Adams spectral sequence associated with the system $G(\Gamma, \Lambda)$. Then there exist algebraic Steenrod operations*

$$_{B}St_i:\ E_{\gamma}^{p,q}\longrightarrow E_{2\gamma-2}^{2p-i,2q} \quad \textit{for} \quad \infty\geqq\gamma\geqq 2$$

and

$$_{F}St_i:\ E_{\gamma}^{p,q}\longrightarrow E_{\gamma}^{p,2q+p-i} \quad \textit{for} \quad \infty\geqq\gamma\geqq 1.$$

Proposition 6. *Let (Γ, Λ) and (Γ', Λ') be pairs of Hopf algebras over Z_2 both of which satisfy the conditions stated before, and let E_γ and E'_γ be the Adams spectral sequence associated with $G(\Gamma, \Lambda)$ and $G(\Gamma', \Lambda')$ respectively. If $f\colon (\Gamma, \Lambda)\rightarrow(\Gamma', \Lambda')$ is a morphism of pairs of Hopf alge-*

bras, then f *induces a sequence of homomorphisms*

$$\varphi_\gamma : E'_\gamma \longrightarrow E_\gamma \quad \text{for} \quad \gamma \geqq 1$$

such that

$$\varphi_\gamma\, {}_FSt_i = {}_FSt_i \varphi_\gamma$$

and

$$\varphi_{2\gamma-2}\, {}_BSt_i = {}_BSt_i \varphi_\gamma \quad \text{for} \quad \gamma \geqq 2.$$

Proof. See [5].

Proposition 7. *Let* $k^i \colon B(\Gamma) \to B(\Gamma) \otimes B(\Gamma)$ *for* $i=0, 1, \dots, n, \dots,$ *be a sequence of* Δ*-homomorphisms such that*

1) k^0 *is a* Δ*-chain map which preserves grading and filtration.*
2) k^i *is a* Δ*-chain homotopy connecting* k^{i-1} *and* ρk^{i-1} *which preserves gradings, raises homological dimension by* i*, and satisfies the filtration condition*

$$k^i(F_p B(\Gamma)) \subset \overset{*}{F}_\alpha(B(\Gamma) \otimes B(\Gamma)) \quad \textit{for} \quad \alpha = \operatorname{Min}\{2p, p+i\}.$$

Then there exists a Δ*-homomorphism* $E^i \colon B(\Gamma) \to B(\Gamma) \otimes B(\Gamma)$ *such that*

i) $E^0 = 0$
ii) $h^i + k^i = E^i + \rho E^i + DE^{i+1} + E^{i+1} d.$

Moreover if $E^i(F_p B(\Gamma) \subset \overset{*}{F}_\alpha(B(\Gamma) \otimes B(\Gamma))$ *for* $\alpha = \operatorname{Min}\{2p, p+i\}$ *then the Steenrod operations* ${}_Bst_i$*,* ${}_Fst_i$ *induced by* k^i*, are equal to* ${}_BSt_i$*,* ${}_FSt_i$ *the Steenrod operations induced by* h^i *respectively.*

Proof. The first part is Proposition 2. For the rest of the proof let $\overline{\underset{i}{\cup}} = k^{i*}\chi$, and define

$$\psi_i \colon \operatorname{Hom}_\Gamma(B(\Gamma), Z_2) \longrightarrow \operatorname{Hom}_\Gamma(B(\Gamma), Z_2)$$

by $\psi_i(\xi) = \xi \overline{\underset{i}{\cup}} \xi + \xi \overline{\underset{i+1}{\cup}} \delta\xi$, then ψ_i induces

$${}_Bst_i \colon E_\gamma^{p,q} \longrightarrow E_{2\gamma-2}^{2p-i,2q} \quad \text{for} \quad \infty \geqq \gamma \geqq 2$$

and

$${}_Fst_i \colon E_\gamma^{p,q} \longrightarrow E_\gamma^{p,2q+p-i} \quad \text{for} \quad \infty \geqq \gamma \geqq 1.$$

Now $\theta_i(\xi)+\psi_i(\xi)=\xi \underset{i}{\cup} \xi+\xi \underset{i+1}{\cup} \delta\xi+\xi \underset{i}{\overline{\cup}} \xi+\xi \underset{i+1}{\overline{\cup}} \delta\xi$ and thus by the first part we can show that

$$\theta_i(\xi)+\psi_i(\xi)=D^*(E^{(i+1)*}\chi(\xi\otimes\xi)+E^{(i+2)*}\chi(\xi\otimes\delta\xi)) \\ +E^{(i+2)*}\chi(\delta\xi\otimes\delta\xi).$$

Since $E^iF_p\subset F_\alpha$ for $\alpha=\mathrm{Min}\{2p, p+i\}$, thus by Proposition 4 we have, for $\xi_1\in F^p\,\mathrm{Hom}_\Gamma(B(\Gamma), Z_2)$ and $\xi_2\in F^q\,\mathrm{Hom}_\Gamma(B(\Gamma), Z_2)$,

$$E^{j*}\chi(\xi_1\otimes\xi_2)\in F^\gamma\mathrm{Hom}_\Gamma(B(\Gamma), Z_2)$$

where

$$\gamma=\begin{cases}\mathrm{Max}(p+q-j, p, q) & \text{if } p=q\\ \mathrm{Max}(p+q-j, \mathrm{Min}(p, q)+1) & \text{if } p\neq q.\end{cases}$$

Therefore one can show that

$$\theta_i(\xi)+\psi_i(\xi)\in Z_{\gamma-1}^{p+1,2q+p-i-1}\cap Z_{2\gamma-3}^{2p-i+1,2q-1} \\ +B_{\gamma-1}^{p,2q+p-i}\cap B_{2\gamma-3}^{2p-i,2q}$$

which means that

$$_BSt_i={}_Bst_i \quad\text{and}\quad {}_FSt_i={}_Fst_i.$$

§ 2. Cartan Formula

Let Γ be a connected, locally finite, cocommutative Hopf algebra over the field Z_2 and let Λ be a Hopf subalgebra of Γ which is central in Γ.

Remark 4. $\Lambda\otimes\Lambda$ is central in $\Gamma\otimes\Gamma$ and

$$\Delta^*=(1\otimes t\otimes 1)(\Delta\otimes\Delta) : \Gamma\otimes\Gamma \longrightarrow \Gamma\otimes\Gamma\otimes\Gamma\otimes\Gamma$$

is commutative where $t(x\otimes y)=y\otimes x$. Therefore $(\Gamma\otimes\Gamma, \Lambda\otimes\Lambda)$ is a pair of Z_2-Hopf algebras with the properties of paragraph 1.

Consider the following diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow B(\Gamma\otimes\Gamma)_n \longrightarrow \cdots \longrightarrow & B(\Gamma\otimes\Gamma)_1 & \xrightarrow{d} & \Gamma\otimes\Gamma & & B(\Gamma\otimes\Gamma) \\
f_n\uparrow\ \downarrow g_n & f_1\uparrow\ \downarrow g_1 & & \Big| 1 & & f\uparrow\ \downarrow g \\
\cdots \longrightarrow (B(\Gamma)\otimes B(\Gamma))_n \longrightarrow \cdots & (B(\Gamma)\otimes B(\Gamma))_1 & \xrightarrow{d^*} & \Gamma\otimes\Gamma & & B(\Gamma)\otimes B(\Gamma)
\end{array}$$

V.K.A.M. Gugenheim [3], defines f and g as $\Gamma\otimes\Gamma$-chain maps by

$$g[a_1\otimes b_1|\ldots|a_n\otimes b_n]$$
$$=\sum_{0\le\rho\le n}\varepsilon\sigma(a_{\rho+1})\ldots\varepsilon\sigma(a_n)[a_1|\ldots|a_\rho]\otimes b_1\ldots b_\rho[b_{\rho+1}|\ldots|b_n]$$

and

$$f\{[a_1|\ldots|a_p]\otimes[b_1|\ldots|b_q]\}$$
$$=\sum[c_1|\ldots|c_{p+q}]$$

where $c_1, \ldots, c_{p+q}$ is a shuffle of $a_1\otimes 1, \ldots, a_p\otimes 1, 1\otimes b_1, \ldots, 1\otimes b_q$. Moreover he shows that $gf=I$. Define H^i: $B(\Gamma\otimes\Gamma)\to B(\Gamma\otimes\Gamma)\otimes B(\Gamma\otimes\Gamma)$ by

$$\begin{array}{ccc}
B(\Gamma\otimes\Gamma) & \xrightarrow{\qquad H^i \qquad} & B(\Gamma\otimes\Gamma)\otimes B(\Gamma\otimes\Gamma) \\
g\downarrow & & f\otimes f\uparrow \\
B(\Gamma)\otimes B(\Gamma) & \xrightarrow[(1\otimes t\otimes 1)(t\otimes 1\otimes 1)^i(\sum_{j=0}^{i} h^j\otimes h^{i-j})]{} & B(\Gamma)\otimes B(\Gamma)\otimes B(\Gamma)\otimes B(\Gamma)
\end{array}$$

Lemma 3. *H^i is a Δ^*-homomorphism.*

Proof. Construct the appropriate diagram and verify commutativity.

Lemma 4.

$$H^i d+DH^i=H^{i-1}+\rho H^{i-1}$$

Proof.

$$\begin{aligned}
H^i d &=(f\otimes f)(1\otimes t\otimes 1)(t\otimes 1\otimes 1)^i(\sum_{j=0}^{i} h^j\otimes h^{i-j})gd \\
&=(f\otimes f)(1\otimes t\otimes 1)(t\otimes 1\otimes 1)^i(\sum_{j=0}^{i} h^j\otimes h^{i-j})d^*g \\
&=(f\otimes f)(1\otimes t\otimes 1)(t\otimes 1\otimes 1)^i[h^0\otimes h^i+h^i\otimes h^0+\sum_{j=1}^{i-1} h^j\otimes h^{i-j}]d^*g \\
&=(f\otimes f)(1\otimes t\otimes 1)(t\otimes 1\otimes 1)^i[h^0 d\otimes h^i+h^0\otimes h^i d+h^i d\otimes h^0+h^i\otimes h^0 d \\
&\quad+\sum_{j=1}^{i-1} h^j d\otimes h^{i-j}+\sum_{j=1}^{i-1} h^j\otimes h^{i-j}d]g \\
&=(f\otimes f)(1\otimes t\otimes 1)(t\otimes 1\otimes 1)^i[\delta h^0\otimes h^i+h^0\otimes\delta h^i+h^0\otimes h^{i-1}
\end{aligned}$$

$$+h^o\otimes\rho h^{i-1}+\delta h^i\otimes h^o+h^{i-1}\otimes h^o+\rho h^{i-1}\otimes h^o+h^i\otimes\delta h^o$$
$$+\sum_{j=1}^{i-1}(\delta h^j+h^{j-1}+\rho h^{j-1})\otimes h^{i-j}+\sum_{j=1}^{i-1}h^j\otimes(\delta h^{i-j}+h^{i-j-1}+\rho h^{i-j-1})]g$$
$$=(f\otimes f)(1\otimes t\otimes 1)(t\otimes 1\otimes 1)^i(\delta\otimes 1+1\otimes\delta)(\sum_{j=0}^{i}h^j\otimes h^{i-j})g$$
$$+(f\otimes f)(1\otimes t\otimes 1)(t\otimes 1\otimes 1)^i(t\otimes 1\otimes 1+1\otimes 1\otimes t)(\sum_{j=0}^{i-1}h^j\otimes h^{i-j-1})g$$
$$=DH^i+(f\otimes f)(1\otimes t\otimes 1)(t\otimes 1\otimes 1)^{i-1}(\sum_{j=0}^{i-1}h^j\otimes h^{i-j-1})g$$
$$+(f\otimes f)(1\otimes t\otimes 1)(t\otimes 1\otimes 1)^i(1\otimes 1\otimes t)(\sum_{j=0}^{i-1}h^j\otimes h^{i-j-1})g$$
$$=DH^i+H^{i-1}+(f\otimes f)(1\otimes t\otimes 1)(t\otimes 1\otimes 1)^i(1\otimes 1\otimes t)(\sum_{j=0}^{i-1}h^j\otimes h^{i-j-1})g,$$

We only need to show that

$$(f\otimes f)(1\otimes t\otimes 1)(t\otimes 1\otimes 1)^i(1\otimes 1\otimes t)(\sum_{j=0}^{i-1}h^j\otimes h^{i-j-1})g=\rho H^{i-1},$$

Case 1. i is even, then

$$(1\otimes t\otimes 1)(t\otimes 1\otimes 1)^i(1\otimes 1\otimes t)=(1\otimes t\otimes 1)(1\otimes 1\otimes t)$$
$$=\rho(1\otimes t\otimes 1)(t\otimes 1\otimes 1)=\rho(1\otimes t\otimes 1)(t\otimes 1\otimes 1)^{i-1}$$

Case 2. i is odd, then

$$(1\otimes t\otimes 1)(t\otimes 1\otimes 1)^i(1\otimes 1\otimes t)=(1\otimes t\otimes 1)(t\otimes 1\otimes 1)(1\otimes 1\otimes t)$$
$$=\rho(1\otimes t\otimes 1)(t\otimes 1\otimes 1)^{i-1}$$

Thus

$$(f\otimes f)(1\otimes t\otimes 1)(t\otimes 1\otimes 1)^i(1\otimes 1\otimes t)(\sum_{j=0}^{i-1}h^j\otimes h^{i-j-1})g$$
$$=(f\otimes f)\rho(1\otimes t\otimes 1)(t\otimes 1\otimes 1)^{i-1}(\sum_{j=0}^{i-1}h^j\otimes h^{i-j-1})g$$
$$=\rho(f\otimes f)(1\otimes t\otimes 1)(t\otimes 1\otimes 1)^{i-1}(\sum_{j=0}^{i-1}h^j\otimes h^{i-j-1})g=\rho H^{i-1}.$$

This completes the proof.

Lemma 5.

$$H^iF'_pB(\Gamma\otimes\Gamma)\subset\overset{*}{F}{}'_\alpha(B(\Gamma\otimes\Gamma)\otimes B(\Gamma\otimes\Gamma)) \quad \text{for} \quad \alpha=\text{Min}\{2p, p+i\}.$$

Proof. It can be shown that g and f preserve the filtration.

Finally we need to show that

$$(h^j\otimes h^{i-j})(F_sB(\Gamma)\otimes F_{p-s}B(\Gamma))\subset\overset{**}{F}_\gamma(B(\Gamma)\otimes B(\Gamma)\otimes B(\Gamma)\otimes B(\Gamma))$$

where $\gamma=\text{Min}\{2p,\ p+i\}$.

Note that,

1) $h^j F_s \subset \overset{*}{F}_\alpha$ where $\alpha = \mathrm{Min}\{2s, s+j\}$ and

2) $h^{i-j} F_{p-s} \subset \overset{*}{F}_\beta$ where $\beta = \mathrm{Min}\{2(p-s), p-s+i-j\}$

Now $(h^j \otimes h^{i-j})(F_s \otimes F_{p-s}) \subset \overset{*}{F}_\alpha \otimes \overset{*}{F}_\beta \in \overset{**}{F}_{\alpha+\beta}$, so we have to show that $\alpha+\beta<\gamma$.

Case 1. $\alpha=2s$, $\beta=2(p-s)$ then $\alpha+\beta=2p$ and we need to show that $i \geqq p$.

$$p-s+i-j \geqq 2(p-2) \Rightarrow i \geqq p+j-s,$$

also

$$\alpha=2s \Rightarrow j \geqq s \Rightarrow j-s \geqq 0,$$

thus $i \geqq p$.

Case 2. $\alpha=s+j$, $\beta=p-s+i-j$ then $\alpha+\beta=p+i$ and we need to show that $i \leqq p$. But $\beta=p-s+i-j \Rightarrow p-s+i-j \leqq 2p-s \Rightarrow i-j \leqq p-s \Rightarrow i \leqq p-s+j$, also $\alpha=s+j \Rightarrow s+j \leqq 2s \Rightarrow j \leqq s$, thus $i \leqq p$.

Case 3. $\alpha=s+j$, $\beta=2p-2s$, then $\alpha+\beta=2p-s+j$. But $s+j \leqq 2s \Rightarrow j \leqq s \Rightarrow j-s \leqq 0 \Rightarrow \alpha+\beta \leqq 2p$ also $2p-2s \leqq p-s+i-j \Rightarrow p+s \leqq i-j \Rightarrow \alpha+\beta \leqq p+i$.

Therefore $\alpha+\beta \leqq \mathrm{Min}\{2p, p+i\} = \gamma$

Case 4. $\alpha=2s$, $\beta=p-s+i-j$ then $\alpha+\beta=p+s+i-j$.

But $s+j \geqq 2s \Rightarrow j \geqq s \Rightarrow -j \leqq -s \Rightarrow \alpha+\beta \leqq p+i$, also $p-s+i-j \leqq 2p-2s \Rightarrow i-j \leqq p-2 \Rightarrow \alpha+\beta \leqq 2p$, and thus $\alpha+\beta \leqq \mathrm{Min}\{2p, p+i\} = \gamma$.

Consider a diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\Gamma\otimes\Gamma}(B(\Gamma\otimes\Gamma), Z_2) \otimes \mathrm{Hom}_{\Gamma\otimes\Gamma}(B(\Gamma\otimes\Gamma), Z_2) & \xrightarrow[i]{\cup'} & \mathrm{Hom}_{\Gamma\otimes\Gamma}(B(\Gamma\otimes\Gamma), Z_2) \\
\downarrow \chi' & \nearrow H^{i\#} & \\
\mathrm{Hom}_{\Gamma\otimes\Gamma\otimes\Gamma\otimes\Gamma}(B(\Gamma\otimes\Gamma) \otimes B(\Gamma\otimes\Gamma), Z_2) & &
\end{array}$$

where $\chi'(f \otimes g)(x \otimes y) = f(x) g(y)$, then the cup-$i$-product $\underset{i}{\cup}'$ in the cochain complex $C' = \mathrm{Hom}_{\Gamma\otimes\Gamma}(B(\Gamma\otimes\Gamma), Z_2)$ is defined by $H^{i\#} \chi'$.

Lemma 6. *Let* $u \in C'^m$ *and* $v \in C'^n$, *then* $u \underset{i}{\cup}' v = 0$ *if* $i > m$ *or* $i > n$.

Proof. $u \underset{i}{\cup}' v = H^{i*}\chi'(u\otimes v)$
$= g^*(\sum h^{j*}\otimes h^{i-j*})(t^*\otimes 1\otimes 1)^i(1\otimes t^*\otimes 1)(f\otimes f)^*\chi'(u\otimes v)$
$= g^*(\sum h^{j*}\otimes h^{i-j*})(\chi\otimes\chi)(t^*\otimes 1\otimes 1)^i(1\otimes t^*\otimes 1)(\xi_1\otimes\xi_2\otimes\xi_3\otimes\xi_4)$

where

$$f(u)=\chi(\xi_1\otimes\xi_2) \quad \text{and} \quad f(v)=\chi(\xi_3\otimes\xi_4),$$

$\xi_i \in \mathrm{Hom}_\Gamma(B(\Gamma), Z_2)$ with $|\xi_1+\xi_2|=m$ and $|\xi_3+\xi_4|=n$.

$$= g^*(\sum h^{j*}\otimes h^{i-j*})(\chi\otimes\chi)(t^*\otimes 1\otimes 1)^i(\xi_1\otimes\xi_3\otimes\xi_2\otimes\xi_4)=0$$

if any of the following hold

1) $j>|\xi_1|$
2) $j>|\xi_3|$
3) $i-j>|\xi_2|$
4) $i-j>|\xi_4|$

Now $i>m=|\xi_1|+|\xi_2| \Rightarrow j+i-j>|\xi_1|+|\xi_2| \Rightarrow$ either $j>|\xi_1|$ or $i-j>|\xi_2|$ $\Rightarrow u\underset{i}{\cup}'v=0$ also $i>n=|\xi_3|+|\xi_4| \Rightarrow$ either $j>|\xi_3|$ or $i-j>|\xi_4| \Rightarrow u\underset{i}{\cup}'v=0$.

Lemma 7. $\delta'(x\underset{i}{\cup}'y)=x\underset{i-1}{\cup}'y+y\underset{i-1}{\cup}'x+\delta'x\underset{i}{\cup}'y+x\underset{i}{\cup}'\delta'y$.

Proof. Straightforward, since

$$H^i d+DH^i=H^{i-1}+\rho H^{i-1}.$$

Let (C', δ', F') be the Adams filtered complex associated with $(\Gamma\otimes\Gamma, \Lambda\otimes\Lambda)$, then by Lemmas 3, 4, 5, 6 and Proposition 3, we have that $G(\Gamma\otimes\Gamma, \Lambda\otimes\Lambda)=\{C', \delta', F', \underset{i}{\cup}'\}$ is a graded differential algebra with a decreasing filtration F' and with cup-i-products $\underset{i}{\cup}'$ in the sense of Definition (*).

Define $\theta'_i\colon C'\to C'$ by $\theta'_i(x)=x\underset{i}{\cup}'x+x\underset{i+1}{\cup}'\delta'x$, then by Proposition 1, θ'_1 induces Steenrod operations ${}_BST_i$, ${}_FST_i$ in the spectral sequence associated with $G(\Gamma\otimes\Gamma, \Lambda\otimes\Lambda)$ such that

$${}_BST_i\colon E_\gamma^{p,q}(\Gamma\otimes\Gamma)\longrightarrow E_{2\gamma-2}^{2p-i,2q}(\Gamma\otimes\Gamma) \quad \text{for} \quad \infty\geqq\gamma\geqq 2$$

and

$${}_FST_i\colon E_\gamma^{p,q}(\Gamma\otimes\Gamma)\longrightarrow E_\gamma^{p,2q+p-i}(\Gamma\otimes\Gamma) \quad \text{for} \quad \infty\geqq\gamma\geqq 1.$$

They are all Z_2-homomorphisms.

Proposition 8. *The following diagram is commutative*

$$\begin{array}{ccc} E_r^{s,t}(\Gamma)\otimes E_r^{s',t'}(\Gamma) & \xrightarrow{\xi_1} & E_r^{s,2t+s-j}(\Gamma)\otimes E_r^{s',2t'+s'-i+j}(\Gamma) \\ \overline{g^*\chi}\downarrow & & \downarrow\overline{g^*\chi} \\ E_r^{s+s',t+t'}(\Gamma\otimes\Gamma) & \xrightarrow{{}_FST_i} & E_r^{s+s',2t+2t'+s+s'-i}(\Gamma\otimes\Gamma) \end{array}$$

where $\xi_1=\sum_{j=0}^{i} {}_FST_j\otimes{}_FSt_{i-j}$ *for* $1\leqq r\leqq\infty$. *Similarly* ${}_BST_i$ *with* $2\leqq r\leqq\infty$.

Proof. Let $x=\bar{u}\in E_r^{s,t}(\Gamma)$ and $y=\bar{v}\in E_r^{s',t'}(\Gamma)$, then,

$$ST_i\overline{g^*\chi}(x\otimes y)=ST_i(\overline{g^*\chi(u\otimes v)})$$
$$=\overline{\theta'_i g^*(u\otimes v)}=\overline{g^*(u\otimes v)\underset{i}{\cup}{}'g^*(u\otimes v)+g^*(u\otimes v)\underset{i+1}{\cup}\delta g^*(u\otimes v)}.$$

Let $ST_i\overline{g^*}(x\otimes y)$ be represented by

$$z=g^*(u\otimes v)\underset{i}{\cup}{}'g^*(u\otimes v)+g^*(u\otimes v)\underset{i+1}{\cup}{}'\,\delta g^*(u\otimes v),$$

then

$$\begin{aligned} z=&g^*(\sum_{j=0}^{i} h^{j*}\otimes h^{i-j*})(t^*\otimes 1\otimes 1)^i(1\otimes t^*\otimes 1)(f^*\otimes f^*)(g^*\otimes g^*)\\ &(u\otimes v\otimes u\otimes v)+g^*(\sum_{\alpha=0}^{i+1} h^{\alpha*}\otimes h^{i+1-\alpha*})(t^*\otimes 1\otimes 1)^{i+1}\\ &(1\otimes t^*\otimes 1)(f^*\otimes f^*)(g^*\otimes g^*)(u\otimes v\otimes\delta u\otimes v)+u\otimes v\otimes u\otimes\delta v)\\ =&g^*(\sum_{j=0}^{i} h^{j*}(u\otimes u)\otimes h^{i-j*}(v\otimes v))+g^*(\sum_{\alpha=0}^{i+1} h^{\alpha*}(u\otimes u)\otimes h^{i+1-\alpha*}(v\otimes\delta v))\\ &+g^*(\sum_{\alpha=0}^{i+1} h^{\alpha*}\otimes h^{i+1-\alpha*})(t^*\otimes 1\otimes 1)^{i+1}(u\otimes\delta u\otimes v\otimes v) \end{aligned}$$

Case 1. i is odd, then

$$\begin{aligned} z=&g^*(\sum_{j=0}^{i} h^{j*}(u\otimes u)\otimes h^{i-j*}(v\otimes v)\\ &+\sum_{\alpha=0}^{i} h^{\alpha*}(u\otimes u)\otimes h^{i+1-\alpha*}(v\otimes\delta v))\\ &+g^*(\sum_{\alpha=1}^{i+1} h^{\alpha*}(u\otimes\delta u)\otimes h^{i+1-\alpha*}(v\otimes v))\\ &+g^*(h^{i+1*}(u\otimes u)\otimes h^{o*}(v\otimes\delta v))\\ &+g^*(h^{o*}(u\otimes\delta u)\otimes h^{i+1*}(v\otimes v))\\ =&g^*\{\sum_{j=0}^{i}(h^{j*}(u\otimes u)+h^{j+1*}(u\otimes\delta u))\otimes(h^{i-j*}(v\otimes v) \end{aligned}$$

$$+h^{i-j+1*}(v\otimes\delta v))\}+g^*(\sum_{j=0}^{i} h^{j+1*}(u\otimes\delta u)\otimes h^{i-j+1*}(v\otimes\delta v))+g^*(h^{i+1*}(u\otimes u)\otimes h^{0*}(v\otimes\delta v))+g^*(h^{0*}(u\otimes\delta u)\otimes h^{i+1*}(v\otimes v)).$$

On the other hand,

$$\overline{g^*\chi}\sum_{j=0}^{i}(St_j\otimes St_{i-j})(x\otimes y)$$

can be represented by

$$\begin{aligned} z' &= g^*\sum_{j=0}^{i}\theta_j(u)\otimes\theta_{i-j}(v) \\ &= g^*\{\sum_{j=0}^{i}(h^{j*}(u\otimes u)+h^{j+1*}(u\otimes\delta u))\otimes(h^{i-j*}(v\otimes v) \\ &\quad +h^{i-j+1*}(v\otimes\delta v))\} \end{aligned}$$

Now, we only have to show that $\bar{w}=0$, where

$$w=g^*(\sum_{j=0}^{i} h^{j+1*}(u\otimes\delta u)\otimes h^{i-j+1*}(v\otimes\delta v))+g^*(h^{0*}(u\otimes\delta u)\otimes h^{i+1*}(v\otimes v)+h^{i+1*}(u\otimes u)\otimes h^{0*}(v\otimes\delta v).$$

But

$$w'=u\cup\delta u\otimes v\underset{i-j+1}{\cup}\delta v+u\underset{i+1}{\cup}u\otimes v\underset{0}{\cup}\delta v+u\underset{0}{\cup}\delta u\otimes v\underset{i+1}{\cup}v \in F^{s+s'+1}\cap F^{2s+2s'-i+1}$$

and

$$\begin{aligned} \delta'w' &= (u\underset{j}{\cup}\delta u+\delta u\underset{j}{\cup}u+\delta u\underset{j+1}{\cup}\delta u)\otimes(v\underset{i-j+1}{\cup}\delta v) \\ &\quad +(u\underset{j+1}{\cup}\delta u)\otimes(v\underset{i-j}{\cup}\delta v+\delta v\underset{i-j}{\cup}v+\delta v\underset{i-j+1}{\cup}\delta v) \\ &\quad +(\delta u\underset{i+1}{\cup}u+u\underset{i+1}{\cup}\delta u)\otimes(v\cup\delta v)+(u\underset{i+1}{\cup}u)\otimes(\delta v\cup\delta v) \\ &\quad +(\delta u\cup\delta u)\otimes(v\underset{i+1}{\cup}v)+(u\cup\delta u)\otimes(\delta v\underset{i+1}{\cup}v+v\underset{i+1}{\cup}\delta v). \\ &\in F^{s+s'+r}C\cap F^{2s+ss'+2r-i-1}C\subset F^{s+s'+r}C\cap F^{2s+2s'+2r-i-2}C \\ &\Rightarrow\ w\in Z_{r-1}^{s+s'+1,2(t+t')+s+s'-i-1}(\Gamma\otimes\Gamma) \\ &\qquad \cap Z_{2r-3}^{2(s+s')-i+1,2(t+t')-1}(\Gamma\otimes\Gamma)\Rightarrow\bar{w}=0. \end{aligned}$$

Case 2. i is even, then

$$\begin{aligned} z &= g^*(\sum_{j=0}^{i} h^{j*}(u\otimes u)\otimes h^{i-j*}(v\otimes v)) \\ &\quad +g^*(\sum_{\alpha=0}^{i-1} h^{\alpha*}(u\otimes u)\otimes h^{i+1-\alpha*}(v\otimes\delta v)) \end{aligned}$$

$$+g^*\left(\sum_{\alpha=0}^{i-1}(h^{\alpha*}(\delta u\otimes u)\otimes h^{i+1-\alpha*}(v\otimes v))\right)$$

$$=g^*\sum_{j=0}^{i}\{(h^{j*}(u\otimes u)+h^{j-1*}(\delta u\otimes u))\otimes(h^{i-j*}(v\otimes v)$$

$$+h^{i-j+1*}(v\otimes\delta v))\}+g^*\left(\sum_{j=0}^{i}h^{j+1*}(\delta u\otimes u)\otimes\right.$$

$$\left.h^{i-j+1*}(v\otimes\delta u)\right)+g^*(h^{i+1*}(u\otimes u)\otimes h^{o*}(v\otimes\delta v))$$

$$+g^*(h^{o*}(\delta u\otimes u)\otimes h^{i+1*}(v\otimes v)).$$

Now Lemma 1 says that $\overline{u\cup_j u+u\cup_{j+1}\delta u}=\overline{u\cup_j u+\delta u\cup_{j+1}u}$, and the rest of the proof in this case will be the same as in Case 1. Hence the proof is completed.

Consider the following diagram

$$\begin{array}{ccc}
E_r^{s,t}(\Gamma\otimes\Gamma) & \xrightarrow{\ {}_F\widetilde{ST}_i\ } & E_r^{s,2t+s-i}(\Gamma\otimes\Gamma) \\
\downarrow{\scriptstyle \Delta^*} & & \downarrow{\scriptstyle \Delta^*} \\
E_r^{s,t}(\Gamma) & \xrightarrow{\ {}_FSt_i\ } & E_r^{s,2t+s-i}(\Gamma)
\end{array}$$

where ${}_F\widetilde{ST}_i$ induced by $\tilde{H}^i$: $B(\Gamma\otimes\Gamma)\to B(\Gamma\otimes\Gamma)^2$ and $\tilde{H}^i$ is defined in the same way h^i: $B(\Gamma)\to B(\Gamma)^2$ was defined previously. The above diagram is commutative (Proposition 6).

Now if $\tilde{H}^i$ and $H^i=(f\otimes f)(1\otimes t\otimes 1)(t\otimes 1\otimes 1)^i(\sum_{j=0}^{i}h^j\otimes h^{i-j})g$ satisfy Proposition 7, then

$${}_F\widetilde{ST}_i={}_FST_i \quad\text{and}\quad {}_B\widetilde{ST}_i={}_BST_i.$$

From [6], there exist homotopies E_i: $\tilde{H}_i\to H_i$. Now, if a family of homotopies E_i can be obtained satisfying the filtration conditions of Proposition 7, then we have the following (Cartan formula). For $\xi_1, \xi_2 \in E_r(\Gamma)$.

1) ${}_FSt_i(\xi_1\cdot\xi_2)=\sum_{j=0}^{i}{}_FSt_j\xi_1\cdot{}_FSt_{i-j}\xi_2$

2) ${}_BSt_i(\xi_1\cdot\xi_2)=\sum_{j=0}^{i}{}_BSt_j\xi_1\cdot{}_BSt_{i-j}\xi_2$

where $\xi_1\cdot\xi_2=\overline{\Delta^* g^*\chi}(\xi_1\otimes\xi_2)$.

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