The Products in the Steenrod Rings of the Complex and Sympletic Cobordism Theories

By

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§1. Introduction

This paper is concerned with the Hopf algebra structure of a certain subalgebra S of the Steenrod ring A^G of stable cohomology operations in the complex cobordism theory and the sympletic cobordism theory $MG^*(\ )$, where $G=U$ or Sp, the infinite dimensional unitary or sympletic group, respectively.

The Hopf algebra structure and its applications of Steenrod algebra with coefficients Z_p, for a prime p, have long been studied by many topologists (Steenrod-Epstein [6]). Novikov [3] investigated the Steenrod rings of generalized cohomology theories. Landweber [1] also studied general properties of A^G as Hopf algebra.

The main purpose of this paper is to determine the explicit product formula in S (Theorem 3.1) and the indecomposable quotient $\bar{S}/\bar{S}^2$ (Theorem 4.1), where $\bar{S}$ denotes the kernel of the augmentation $S \to Z$.

We use the following notations. Let Z be the ring of integers and $Z_m = Z/mZ$. According to Landweber [1], A^G can be expressed as $A^G = \Lambda \hat{\otimes} S$, with the coefficient $\Lambda = \Omega_G^* \cong MG^*(point)$. In case $G = U$, $\Lambda = Z[x_1, x_2,...]$, $\deg(x_i) = -2i$. In case $G = Sp$, Λ has not been determined completely. The subalgebra S is a Hopf algebra over Z and has a Z-free basis $\{S_I\}$, with $\deg(S_I) = d\sum_r r i_r$, where $d=2$ or 4 according as $G=U$ or Sp and $I=(i_1, i_2,...)$ is a sequence of non-negative integers such that all but a finite number of i_r are zero. For two Z-graded modules $M = \sum_{i=-\infty}^{\infty} M_i$ and $N = \sum_{i=-\infty}^{\infty} N_i$, the completed tensor product

$M \hat{\otimes} N$ is defined by

$$(M \hat{\otimes} N)_n = \prod_{i+j=n} M_i \otimes N_j.$$

We remark that S_I in this paper and that in Adams [2] are the same as S^I in Landweber [1]. S_I in Landweber [1] and Novikov [3] are different from S_I in this paper. (For the detail on the relationship between two notations S_I and S^I by Landweber, see Landweber [1, Page 101].) While revising the manuscript, the author was informed that the main theorem of the present paper was obtained independently by I. Kojima.

The author expresses his hearty thanks to Dr. K. Shibata for many improvements of the contents (see Remark after Theorem 4.1) and also to Professor M. Mimura for reading the manuscript.

§2. Notations

Let $I = (i_1, i_2, ...)$ be a sequence such that each i_r is a non-negative integer and all but a finite number of i_r are zero. We define

$$|I| = \sum_r i_r \qquad \text{and} \qquad \|I\| = \sum_r r i_r.$$

We order sequences such that

$$I < J \quad \text{if} \quad i_{r_0} > j_{r_0} \quad \text{for some } r_0 \text{ and } i_r = j_r \text{ for all } r > r_0;$$

$$I \leqq J \quad \text{if} \quad i_r \leqq j_r \quad \text{for all} \quad r \geqq 1.$$

Then "$<$" is a total order and "$\leq$" is a partial order. We denote by 0 the sequence $(0, 0, ...)$; by Δ_n the sequence $(i_1, i_2, ...)$ with $i_r = 0$ for $r \neq n$ and $i_n = 1$; by nI the sequence with each component in I multiplied by n. Sometimes we abbreviate $I = (i_1, i_2, ..., i_n, 0, 0, ...)$ by $(i_1, i_2, ..., i_n)$. $I + J$ denotes a componentwise sum. $I - J$ is defined componentwise if $I \geqq J$. A small letter indexed by n denotes the n-th component of the sequence expressed by the corresponding capital letter.

Let M be a module, $m_1, m_2, ..., m_r$ in M and R a ring with unit. Then we denote by $R\{m_1, ..., m_r\}$ a free R-module generated by $m_1, ..., m_r$.

Let R be a ring, m_1 and m_2 in R. Then we denote $[m_1, m_2] = m_1 m_2 - m_2 m_1$.

Let X be an indexed set of non-negative integers:

(2.1) $\qquad X = (k_{m,n,s}; \ m, n \geq 0, \ (m, n) \neq (0, 0), \ 1 \leq s \leq s(m, n))$.

we consider another indexed set

$$(i_{r,m,n,s}; \ m > 0, \ n \geq 0, \ 0 \leq r \leq m, \ 1 \leq s \leq s(m, n))$$

such that

$$\sum_{r=0}^{m} i_{r,m,n,s} = n + 1, \quad \sum_{r=1}^{m} r i_{r,m,n,s} = m$$

For fixed m and n we order the set of

$$(i_{r,m,n,s}; \ 0 \leq r \leq m); \quad 1 \leq s \leq s(m, n)$$

such that

$s < s'$ implies $(i_{r,m,n,s}; 0 \leq r \leq m) < (i_{r,m,n,s'}; 0 \leq r \leq m)$, where $s(m, n)$ is the number given by

$$s(m, n) = \#\{(i_1, \ldots, i_m); \ \sum_{r=1}^{m} i_r \leq n + 1, \ \sum_{r=1}^{m} r i_r$$

$$= m, \ 0 \leq i_r \in Z\}.$$

The set of such $(i_1, \ldots, i_m)$ is denoted by $I(m, n)$. Then $s(m, n)$ has the following properties:

$$s(m+n, n) = s(m+n, n-1) + s(m-1, n), \ n \geq 0$$

$$s(m, 0) = 1 = s(0, m) = s(1, m),$$

$$s(m, 1) = [m/2] + 1, \ ([\ \ ] \text{ denotes a Gauss integer.}),$$

$$s(m, 2) = \begin{cases} 3j^2 + (k+3)j + k, & 0 < k \leq 5, \ m = 6j + k, \\ 3j^2 + 3j + 1, & m = 6j. \end{cases}$$

We use the following notations:

$$I(X) = (i_1(X), \ i_2(X), \ldots),$$

$$J(X) = (j_1(X), \ j_2(X), \ldots),$$

$$K(X) = (k_1(X),\ k_2(X),\dots),$$

$$i_r(X) = \sum_{n \geq 0, m \geq r, 1 \leq s \leq s(m,n)} i_{r,m,n,s} k_{m,n,s},$$

$$j_n(X) = \sum_{m \geq 0} k_{m,n},$$

$$k_m(X) = \sum_{0 \leq n \leq m} k_{m-n,n},$$

$$k_{m,n} = \sum_{1 \leq s \leq s(m,n)} k_{m,n,s},$$

$$a(X) = \prod_{m \geq 1} \binom{k_m(X)}{k_{m,0},\dots,\ k_{0,m}}$$

$$\times \prod_{m,n \geq 1} \binom{k_{m,n}}{k_{m,n,1},\dots,\ k_{m,n,s(m,n)}}$$

$$\times \prod_{m,n \geq 1, 1 \leq s \leq s(m,n)} \binom{n+1}{i_{0,m,n,s},\dots,\ i_{m,m,n,s}}^{k_{m,n,s}}$$

We define polynomial coefficients for $I = (i_1,\dots,i_r)$ with $|I| \leq i$ as follows:

$$\binom{i}{I} = \binom{i}{i_1,\dots,\ i_r} = \frac{i!}{(i-|I|)! i_1! i_2! \dots i_r!}$$

Remark. The definition of $i_r(X)$ is equivalent to the following;

$$i_r(X) = k_{r,0} + \sum_{m \geq r, n < 0, 1 \leq s \leq s(m,n)} i_{r,m,n,s} k_{m,n,s}$$

§3. Product Formula

Theorem 3.1.

$$S_I S_J = \sum_{I(X)=I, J(X)=J} a(X) S_{K(X)}.$$

Remark. The sum in Theorem 3.1 is a finite sum. $\sum$ in Theorem 3.1 means that if two sequences I and J are given, then the sum runs over all indexed sets X as in (2.1) such that $I(X)=I$ and $J(X)=J$. The product formula in Theorem 3.1 corresponds to the product formula for the Milnor basis in the mod p Steenrod algebra of ordinary cohomology theory with coefficients Z_p (for the Milnor basis, see Milnor [5]; Theorem 4b).

Proof. The proof is complicated and we will give it in the last part of this section.

Corollary 3.2.

$$S_{i\Delta_r}S_{j\Delta_n} = \sum \binom{k+k_0}{k}^{\delta_{r,n}} \prod_{1 \leq c \leq n} \binom{n+1}{c}^{k_c} S_K,$$

where the sum runs over all $(k, k_0, ..., k_{n+1})$ such that

$$\sum_{c=0}^{n+1} k_c = j, \qquad k + \sum_{c=1}^{n+1} ck_c = i,$$

$\delta_{r,n}$ is a Kronecker's delta and

$$K = k\Delta_r + \sum_{c=0}^{n+1} k_c \Delta_{n+rc}.$$

Corollary 3.3.

$$S_{\Delta_r}S_{\Delta_n} = (n+1)S_{\Delta_{r+n}} + (1+\delta_{r,n})S_{\Delta_r+\Delta_n}.$$

Corollary 3.4.

$$S_I S_{\Delta_n} = \sum (i_{m+n}+1)\binom{n+1}{J} S_{I-J+\Delta_{m+n}},$$

and in particular if $i_r = 0$ for all $r \geq n$, then

$$S_I S_{\Delta_n} = \sum \binom{n+1}{J} S_{I-J+\Delta_{m+n}},$$

where the two sums run over all pairs (m, J) such that

$$m \geq 0, \quad J \in I(m, n), \quad 0 \leq J \leq I.$$

For the next corollary define $h(I) = \max\{r; i_r > 0\}$ for each sequence $I = (i_1, i_2, ...)$.

Corollary 3.5. *If $h(I) < n$, then for any $j \geq 1$,*

$$S_I S_{j\Delta_n} = S_{I+j\Delta_n} + \sum_{h(K)>n} a_K S_K, \qquad a_K \in Z.$$

Proof of Theorem 3.1.
Let φ and ψ be the product and the coproduct of the Hopf algebra

S, respectively.   Then

$$\psi(S_K) = \sum_{I+J=K} S_I \otimes S_J.$$

Let S^* be the Hopf algebra dual to S and b_i the element dual to S_{A_i}. Then S^* is a polynomial algebra on generators $b_1, b_2, b_3,\ldots$. Set $b = \sum_{i=0}^{\infty} b_i$, where $b_0 = 1$.   Then Theorem 6.3 in Adams [2] says that the coproduct in S^* is as follows:

$$\varphi^*(b) = \sum_{n \geq 0} b^{n+1} \otimes b_n.$$

Therefore

$$\varphi^*(b) = \sum_{n \geq 0} \sum \binom{n+1}{i_0, i_1,\ldots, i_{n+1}} b_1^{i_1} b_2^{i_2} \ldots b_{n+1}^{i_{n+1}} \otimes b_n,$$

where the second sum runs over all $(i_0, i_1,\ldots, i_{n+1})$ such that $\sum i_r = n+1$. Then the homogeneous component is

$$\varphi^*(b_m) = \sum_{n=0}^{m} B_{m,n} \otimes b_n,$$

where

$$B_{m,n} = \sum \binom{n+1}{i_0, i_1,\ldots, i_{m-n}} b_1^{i_1} b_2^{i_2} \ldots b_{m-n}^{i_{m-n}}$$

(the sum runs over all $(i_0, i_1,\ldots, i_{m-n})$ such that $\sum_{r=0}^{m-n} i_r = n+1$, $\sum_{r=1}^{m-n} r i_r = m-n$, $i_r \geq 0$.)
In other words, using the notation in §2, we have

$$B_{m,n} = \sum_{s=1}^{s(m-n,n)} \binom{n+1}{i_{0,m-n,n,s},\ldots, i_{m-n,m-n,n,s}}$$

$$\times b_1^{i_{1,m-n,n,s}} \ldots b_{m-n}^{i_{m-n,m-n,n,s}}.$$

Since φ^* is a homomorphism of algebras,

$$\varphi^*(b_m^{k_m}) = (\varphi^*(b_m))^{k_m} = \sum_{(1)} \binom{k_m}{k_{m,0}, k_{m-1,1},\ldots, k_{0,m}}$$

$$\times B_{m,0}^{k_m,0} B_{m,1}^{k_{m-1},1} \ldots B_{m,m}^{k_0,m} \otimes b_1^{k_{m-1},1} b_2^{k_{m-2},2} \ldots b_m^{k_0,m},$$

where $\sum_{(1)}$ runs over all $(k_{m,0},\ldots, k_{0,m})$ such that $\sum_{n=0}^{m} k_{m-n,n} = k_m$,

$k_{m-n,n} \geq 0 \ (m \geq n \geq 0)$. From now on we have a convention that two sums $\sum_{(n)}$ with the same index number n run over the same set.

$$B_{m,n}^{k_{m-n,n}} = \sum_{(2)} \binom{k_{m-n,n}}{k_{m-n,n,1}, \ldots, k_{m-n,n,s(m-n,n)}}$$

$$\times \prod_{1 \leq s \leq s(m-n,n)} \binom{n+1}{i_{0,m-n,n,s}, \ldots, i_{m-n,m-n,n,s}}^{k_{m-n,n,s}}$$

$$\times b_1^{\sum_{(3)} i_{1,m-n,n,s} k_{m-n,n,s}} \ldots b_{m-n}^{\sum_{(3)} i_{m-n,m-n,n,s} k_{m-n,n,s}}$$

where $\sum_{(2)}$ runs over all $(k_{m-n,n,1}, \ldots, k_{m-n,n,s(m-n,n)})$ such that $\sum_{(3)} k_{m-n,n,s} = k_{m-n,n}, \ k_{m-n,n,s} \geq 0 \ (1 \leq s \leq s(m-n, n))$ and $\sum_{(3)} = \sum_{1 \leq s \leq s(m-n,n)}$.

$$\varphi^*(b_m^{k_m}) = \sum_{(1)} \binom{k_m}{k_{m,0}, \ldots, k_{0,m}} \sum_{(2)} \prod_{0 \leq n \leq m} \binom{k_{m-n,n}}{k_{m-n,n,1}, \ldots, k_{m-n,n,s(m-n,n)}}$$

$$\times \prod_{\substack{0 \leq n \leq m \\ 1 \leq s \leq s(m-n,n)}} \binom{n+1}{i_{0,m-n,n,s}, \ldots, i_{m-n,m-n,n,s}}^{k_{m-n,n,s}}$$

$$\times \prod_{r \geq 1} b_r^{\sum_{(4)} i_{r,m-n,n,s} k_{m-n,n,s}} \otimes b_1^{k_{m-1,1}} \ldots b_n^{k_{m-n,n}},$$

where the sum $\sum_{(4)}$ runs over all pairs (n, s) such that $m \geq n \geq 0$, $m - n \geq r$, $1 \leq s \leq s(m-n, n)$.

Let $K = (k_1, k_2, \ldots)$ be a sequence of non-negative integers such that all but a finite number of k_m are zero. We denote $b^K = b_1^{k_1} b_2^{k_2} \ldots b_m^{k_m} \ldots$. Then we have

$$\varphi^*(b^K) = \sum_{(5)} \prod_{m \geq 1} \binom{k_m}{k_{m,0}, \ldots, k_{0,m}}$$

$$\times \prod_{\substack{m \geq 1 \\ m \geq n \geq 0}} \binom{k_{m-n,n}}{k_{m-n,n,1}, \ldots, k_{m-n,n,s(m-n,n)}}$$

$$\times \prod_{\substack{m \geq 1 \\ m \geq n \geq 0 \\ 1 \leq s \leq s(m-n,n)}} \binom{n+1}{i_{0,m-n,n,s}, \ldots, i_{m-n,m-n,n,s}}^{k_{m-n,n,s}}$$

$$\times \prod_{r \geq 1} b_r^{\sum_{(6)} i_{r,m-n,n,s} k_{m-n,n,s}} \otimes \prod_{n \geq 1} b_n^{\sum_{m \geq n} k_{m-n,n}},$$

where $\sum_{(5)}$ runs over all sequences $(k_{m,n,s})$ such that $\sum_{n=0}^{m} k_{m-n,n} = k_m$, $\sum_{s=1}^{s(m-n,n)} k_{m-n,n,s} = k_{m-n,n}$, $k_{m-n,n} \geq 0$, $k_{m-n,n,s} \geq 0$ for all m, n, s; $\sum_{(6)}$ runs

over all triples (m, n, s) such that $m \geq 1$, $n \geq 0$, $m - n \geq r$, $1 \leq s \leq s(m - n, n)$. Replacing $m - n$ with m, we have

$$\varphi^*(b^K) = \sum_{(7)} \prod_{m \geq 1} \binom{k_m}{k_{m,0}, \ldots, k_{0,m}} \prod_{\substack{m, n \geq 0 \\ (m,n) \neq (0,0)}} \binom{k_{m,n}}{k_{m,n,1}, \ldots, k_{m,n,s(m,n)}}$$

$$\times \prod_{\substack{m, n \geq 0 \\ (m,n) \neq (0,0) \\ 1 \leq s \leq s(m,n)}} \binom{n+1}{i_{0,m,n,s}, \ldots, i_{m,m,n,s}}^{k_{m,n,s}}$$

$$\times \prod_{r \geq 1} b_r^{\sum_{(8)} i_{r,m,n,s} k_{m,n,s}} \otimes \prod_{n \geq 1} b_r^{\sum_{m \geq 0} k_{m,n}}$$

where $\sum_{(7)}$ runs over all sequences $(k_{m,n,s})$ such that $\sum_{n=0}^{m} k_{m-n,n} = k_m$, $\sum_{s=1}^{s(m,n)} k_{m,n,s} = k_{m,n}$, $k_{m,n} \geq 0$, $k_{m,n,s} \geq 0$, for all m, n, s; $\sum_{(8)}$ runs over all triples (m, n, s) such that $n \geq 0$, $(m, n) \neq (0, 0)$, $r \leq m$, $1 \leq s \leq s(m, n)$. Since $s(m, 0) = s(0, m) = 1$, we have for $m = 0$ or $n = 0$

$$\binom{k_{m,n}}{k_{m,n,1}, \ldots, k_{m,n,s(m,n)}} = 1, \quad \binom{n+1}{i_{0,m,n,s}, \ldots, i_{m,m,n,s}} = 1 .$$

Thus the coproduct formula in S^* is

$$\varphi^*(b^K) = \sum_{K(X) = K} a(X) b^{I(X)} \otimes b^{J(X)} .$$

Thus we can prove the theorem by using the following lemma.

Lemma 3.6. Let R be a principal ideal domain, A an R-free Hopf algebra over R with a basis $\{a_i\}$ and a product φ, A^* the Hopf algebra dual to A, a^i the element dual to a_i, φ^* the coproduct in A^*. If

$$\varphi^*(a^i) = \sum_{j,k} c^i_{j,k} a^j \otimes a^k, \quad c^i_{j,k} \in R,$$

then

$$\varphi(a_j \otimes a_k) = \sum_i (-1)^e c^i_{j,k} a_i,$$

where $e = (\deg a_j) \times (\deg a_k)$.

Proof. The proof is essentially a part of the proof of Theorem 4b in Milnor [5; Page 164].

§4. The Indecomposable Quotient

In this section we determine the indecomposable quotient $\bar{S}/\bar{S}^2$. Throughout this section "$=$", "$\equiv$" and "$\equiv_p$" denote the equality in S, the congruence in $\bar{S}/\bar{S}^2$ and $(\bar{S}/\bar{S}^2)\otimes Z_p$, respectively. All S_I are called monomials. $I_{p,a}$ denotes $2^a \Delta_2$ for $p=2$ and denotes $2p^a \Delta_1$ or $p^a \Delta_2$ for p an odd prime in the next theorem.

Theorem 4.1. (1) S_I is indecomposable in S if and only if

$$I = p^a \Delta_e \quad or \quad 2p^a \Delta_1, \quad for \ some \ prime \quad p, \ a \geq 0, \ e = 1, 2.$$

(2) *Let V be a set of monomials. Then V is a minimal set of generators of S if and only if $V = \{S_{p^a \Delta_1}, S_{I_{p,a}}; p: prime, a \geq 0\}$.*

(3) $$\bar{S}/\bar{S}^2 = Z\{S_{\Delta_1}\} \oplus Z\{S_{\Delta_2}, S_{2\Delta_1}\}/2Z\{S_{\Delta_2} + S_{2\Delta_1}\}$$

$$\oplus \sum_{p:\,Prime} Z_p\{S_{p^a \Delta_e}; \ a \geq 1, \ e = 1, 2, \ (p, a, e) \neq (2, 1, 1)\}$$

The only relations in $\bar{S}/\bar{S}^2$ are:

$$S_{2p^a \Delta_1} \equiv - S_{p^a \Delta_2} \qquad (p: odd \ prime, \ a \geq 1)$$

Suppose that p is any prime in (4), (5) and (6) below:
(4) S_I *is indecomposable in $S \otimes Z_p$ if and only if*

$$I = p^a \Delta_e \ or \ 2p^a \Delta_1; \ for \ a \geq 0, \ e = 1, 2 \ (p: odd \ prime),$$

$$I = 2^a \Delta_e \qquad \qquad ; \ for \ a \geq 0, \ e = 1, 2 \ (p = 2).$$

(5) *Let V be a set of monomials. Then V is a set of generators of $(\bar{S}/\bar{S}^2) \otimes Z_p$ if and only if $V = \{S_{p^a \Delta_1}, S_{I_{p,a}}; a \geq 0, e = 1, 2\}$.*

(6) $$\overline{S \otimes Z_p}/\overline{S \otimes Z_p}^2 = Z_p\{S_{p^a \Delta_e}; \ a \geq 0, \ e = 1, 2\}.$$

Among representatives there are relations:

$$S_{2p^a \Delta_1} \equiv_p - S_{p^a \Delta_2} \qquad (p: odd, \ a \geq 1).$$

Proof. The remainder of this section, except for Corollary 4.14, is

devoted to the proof of Theorem 4.1.

Remark. Of course the order of $S_{2p^a\Delta_1}$ is p. Landweber [1] has proved (4), (5) and (6) except for the relations. The original proof of Proposition 4.13 is improved by Dr. K. Shibata as mentioned below in this section. Corollary 3.5 and Lemma 4.12 are preparations for Proposition 4.13 and are due to Dr. K. Shibata.

Proposition 4.2.

$$S_I S_J = \prod_{m \geq 1} \binom{i_m + j_m}{i_m, j_m} S_{I+J} + \sum_{I(X)=I, J(X)=J, K(X) < I+J} a(X) S_{K(X)}.$$

Proof. It suffices to show the following two statement.

Statement A: There is a unique sequence X such that $I(X)=I$, $J(X)=J$ and $K(X)=I+J$. Moreover the sequence X satisfies $a(X)=\prod\binom{i_m+j_m}{i_m, j_m}$.

Statement B: If $I(X)=I$, $J(X)=J$ and $K(X)\neq I+J$, then $K(X) < I+J$.

First we show Statement A. There is an integer m_0 such that $i_m = j_m = 0$ for all $m > m_0$. Then $\sum_{n=0}^{m} k_{m-n,n} = k_m(X) = i_m + j_m = 0$, for all $m > m_0$. Thus $k_{m,n} = 0$ for all $m, n \geq 0$, $m+n > m_0$

$$k_{m,m_0} = \begin{cases} j_{m_0}, & m=0 \\ 0, & m>0. \end{cases}$$

Since $k_{m,n,s} = 0$ for all $m \geq m_0$, $n > 0$, we have $i_{m_0} = k_{m_0,0}$. Thus

$$\sum_{n=0}^{m_0} k_{m_0-n,n} = k_{m_0}(X) = i_{m_0} + j_{m_0} = k_{m_0,0} + k_{0,m_0}.$$

Therefore for $m+n \geq m_0$,

$$k_{m,n,s} = \begin{cases} i_{m_0}, & (m, n, s) = (m_0, 0, 1) \\ j_{m_0}, & (m, n, s) = (0, m_0, 1) \\ 0, & \text{otherwise.} \end{cases}$$

This implies

$$k_{m,m_0-1} = \begin{cases} j_{m_0-1}, & m=0 \\ 0, & m>0 \end{cases}$$

and $i_{m_0-1}=k_{m_0-1,0}$. Thus Statement A is proved by repeating the similar process.

Secondly we show Statement B. This is equivalent to the statement that if $k_{m_1}(X)\neq i_{m_1}+j_{m_1}$, for some $m_1\geq 1$, and $k_m(X)=i_m+j_m$, for all $m>m_1$, then $k_{m_1}(X)>i_{m_1}+j_{m_1}$.

Similarly to the proof of the first part, we have

$$k_{m,n}=\begin{cases} 0, & m>0,\ n>0,\ m+n>m_1 \\ j_{m_1}, & (m,n)=(0,m_1) \\ i_{m_1}, & (m,n)=(m_1,0) \end{cases}$$

Thus

$$k_{m_1}=\sum_{0<n<m_1}k_{m_1-n,n}+i_{m_1}+j_{m_1}\geq i_{m_1}+j_{m_1}.$$

Thus Statement B is proved.

For the next corollary we define the length of a sequence I as follows:

$$l(I)=\#\{r;\ i_r>0\}.$$

Corollary 4.3. (1) $l(K)\geq 2$ implies

$$S_K\in\bar{S}^2+Z\{S_I;\ I<K,\ \|I\|=\|K\|,\ l(I)=1\}.$$

(2) If $K=k\Delta_m$, $2\leq k$ and k is not a prime power, then

$$S_K\in\bar{S}^2+Z\{S_{q^b\Delta_n};\ m<n,\ q^bn=km,\ q:\ prime\}.$$

(3) If $K=p^a\Delta_m$, $m\geq 1$, p is a prime and $a\geq 1$, then

$$pS_K\in\bar{S}^2+Z\{S_{q^b\Delta_n};\ m<n,\ q^bn=p^am,\ q:\ prime\}.$$

Proof. (1): Let I and J be such that $I\neq 0,\ J\neq 0,\ I+J=K$ and $i_rj_r=0$ for all r. Then

$$S_IS_J=S_K+\sum_{I(X)=I,J(X)=J,K(X)<I+J}a(X)S_{K(X)}.$$

Repeating this process, we find that all S_K, $l(K)\geq 2$, are generated by S_I, $l(I)=1$, modulo $\bar{S}^2$. (2), (3): Let $K=k\Delta_m$ and $0<k'<k$. Then

$$S_{k'\Delta_m}S_{(k-k')\Delta_m}=\binom{k}{k'}S_{k\Delta_m}+(\text{terms of lower order}).$$

Thus (2) and (3) follow from the second and first equalities in the next lemma, respectively.

Lemma 4.4.

$$g.c.d.\left\{\binom{k}{i};\ 0<i<k\right\}=\begin{cases} p, & k=p^j, j\geqq 1,\ p:\ prime \\ 1, & otherwise. \end{cases}$$

Proposition 4.5. *The components of degree d and $2d$ of $\bar{S}/\bar{S}^2$ are $Z\{S_{\Delta_1}\}$ and $Z\{S_{\Delta_2}, S_{2\Delta_1}\}/2Z\{S_{\Delta_2}+S_{2\Delta_1}\}$, respectively.*

Proof. $S_{\Delta_1}S_{\Delta_1}=2S_{\Delta_2}+2S_{2\Delta_1}$.

Proposition 4.6. *If $n\geqq 3$, then S_{Δ_n} is in $\bar{S}^2$.*

Proof. Case(1): $n\equiv 1\ (\mathrm{mod}\,2)$. It follows from the following relation:

$$[S_{\Delta_m}, S_{\Delta_{m+1}}]=S_{\Delta_{2m+1}}, m\geqq 1.$$

Case(2): $n\equiv 0\ (\mathrm{mod}\,4)$. By Corollary 3.2 we have

$$S_{n\Delta_1}S_{\Delta_n}=\sum_{i=0}^{n}\binom{n+1}{i}\ T_i\quad \text{and}$$

$$S_{\Delta_{n+i}}S_{(n-i)\Delta_1}=T_i+2T_{i+1}, 0\leqq i<n,$$

where $T_i=S_{(n-i)\Delta_1+\Delta_{n+i}}$.

We define inductively as follows:

$$c_0=1,\quad c_i=\binom{n+1}{i}-2c_{i-1},\quad 0<i\leqq n.$$

Then we have

$$c_k=\sum_{i=0}^{k}(-2)^{k-i}\binom{n+1}{i},$$

$$(-1)^{n+1}=(-2+1)^{n+1}=-2c_n+1.$$

Thus $c_n=1$, (n: even), $=0$, (n: odd). Then we have

$$S_{n\Delta_1}S_{\Delta_n} - \sum_{i=0}^{n-1} c_i S_{\Delta_{n+i}} S_{(n-i)\Delta_1} = \begin{cases} S_{\Delta_{2n}}, & n: \text{even} \\ 0, & n: \text{odd}. \end{cases}$$

Case (3): $n \equiv 2 \pmod 4$. By Corollary 3.2, we have

$$[S_{2\Delta_{2m}}, S_{\Delta_2}] + (2m-2)S_{\Delta_{2m+2}}S_{\Delta_{2m}} - (2m^2 - m)[S_{\Delta_{2m}}, S_{\Delta_{2m+4}}]$$

$$= S_{\Delta_{4m+2}}.$$

Thus the proof is completed.

We can consider $S \otimes Z_p$ as a tensor product of algebras S and Z_p.

Theorem 4.7. (Landweber [1]) *For any prime p, $S \otimes Z_p$ has*

$$S_{p^a \Delta_e}; \quad a \geq 0, \ e = 1, 2$$

as a minimal set of generators.

Remark. (Landweber [1]). If p is odd, then another minimal set of generators is $\{S_{e p^a \Delta_1} : a \geq 0, e = 1, 2\}$. If $p=2$, no other monomials are decomposable.

Proposition 4.8. *Under the correspondence of the same representative S_I we have an isomorphism*

$$\overline{S \otimes Z_p} / \overline{S \otimes Z_p}^2 \cong (\bar{S}/\bar{S}^2) \otimes Z_p.$$

Proof. The short exact sequence

$$0 \longrightarrow \bar{S}^2 \longrightarrow \bar{S} \longrightarrow \bar{S}/\bar{S}^2 \longrightarrow 0$$

and an isomorphism $\bar{S} \otimes Z_p \to \overline{S \otimes Z_p}$ induced by the middle linear function (for the definition, see MacLane [4; Page 138]) $\bar{S} \times Z_p \to \overline{S \otimes Z_p}$ imply that the following diagram is commutative:

$$\begin{array}{ccccccc}
\bar{S}^2 \otimes Z_p & \longrightarrow & \bar{S} \otimes Z_p & \longrightarrow & (\bar{S}/\bar{S}^2) \otimes Z_p & \longrightarrow & 0 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\
\overline{S \otimes Z_p}^2 & \longrightarrow & \overline{S \otimes Z_p} & \longrightarrow & \overline{S \otimes Z_p}/\overline{S \otimes Z_p}^2 & \longrightarrow & 0
\end{array}$$

Since f_1 and f_2 are isomorphisms, it follows from the five Lemma that f_3 is an isomorphism. Thus the proof is completed.

Proposition 4.9. $(\bar{S}/\bar{S}^2)_n = 0$, unless $n = dep^a$, for some p: prime, $a \geq 0$, $e = 1, 2$.

Proof. Since $\bar{S}$ is locally finitely generated as a Z-module, so is $\bar{S}/\bar{S}^2$, that is, $(\bar{S}/\bar{S}^2)_n$ is finitely generated for all n. Thus $(\bar{S}/\bar{S}^2)_n$ is decomposed into a direct sum of m copies of Z and m_q copies of Z_{q^b}, $b \geq 1$, for primes q.

Thus $(\bar{S}/\bar{S}^2)_n \otimes Z_p$ is a direct sum of $m + m_p$ copies of Z_p. Theorem 4.7 yields $(\bar{S}/\bar{S}^2)_n \otimes Z_p = 0$, for all primes p, under the assumption of this proposition. Thus in this case, $(\bar{S}/\bar{S}^2)_n = 0$.

Lemma 4.10. *Let p be a prime, $b = \sum_{i=1}^m a_i$,*

$$b = \sum_{j \geq 0} b_j p^j, \ 0 \leq b_j < p \quad \text{and} \quad a_i = \sum_{j \geq 0} a_{i,j} p^j, \ 0 \leq a_{i,j} < p.$$

Then we have

$$\binom{b}{a_1, \ldots, a_m} \equiv \prod_{j \geq 0} \binom{b_j}{a_{1,j}, \ldots, a_{m,j}} \pmod{p},$$

where if $b_j < \sum_i a_{i,j}$, then we define $\binom{b_j}{a_{1,j}, \ldots, a_{m,j}} = 0$.

Proof. Well known.

For the next proposition, we give some preparations. Lemma 5.6 in Landweber [1] says that

$$MG^*(FP^\infty) = \Lambda \hat{\otimes} Z[x], \quad \deg(x) = d,$$

where FP^∞ is the infinite dimensional complex or quaternion projective space according as $d = 2$ or 4, respectively, and

$$S_I x = \begin{cases} x^{n+1}, & I = \Delta_n, \\ 0, & \text{otherwise}. \end{cases}$$

Proposition 4.11.

$$S_I x^n = \begin{cases} \binom{n}{I} x^{n+\|I\|}, & |I| \leq n, \\ 0, & |I| > n. \end{cases}$$

Proof. We prove by induction on n. Use the Cartan formula repeatedly.

Lemma 4.12. *If $m, n \geq 1$ and $mn \geq 3$, then $mS_{m\Delta_n}$ is in $\bar{S}^2$.*

Proof. By Corollary 3.2, we have for $i \geq 1$

$$S_{\Delta_{in}}S_{(m-i)\Delta_n} = (n+1)S_{\Delta_{(i+1)n}+(m-i-1)\Delta_n} + S_{\Delta_{in}+(m-i)\Delta_n}.$$

Thus by Proposition 4.6,

$$mS_{m\Delta_n} \equiv (-1)^m(n+1)^{m-1}S_{\Delta_{mn}} \equiv 0.$$

Proposition 4.13. *If p is a prime, $a \geq 1$, $n \geq 3$, then $S_{p^a\Delta_n} \in \bar{S}^2$.*

Proof. It suffices to prove the following assertions $Q(i)$ for each $i \geq 1$.

Assertion $Q(i)$: $S_{p^a\Delta_n} \equiv 0$ for each p, a, n such that p is a prime, $a \geq 0$, $n \geq 3$ and $1 \leq p^a \leq i$.

We prove Assertion $Q(i)$ by induction on i. The assertion $Q(1)$ is just Proposition 4.6.

Suppose that $Q(i-1)$ holds for $i \geq 2$ and that $p^a = i$.

Case (1): $n \not\equiv 2 \pmod p$. By Corollary 3.2,

$$S_{p^a\Delta_{n-1}}S_{p^a\Delta_1} = \sum_K 2^{k_1}S_K$$

with $K = k_0\Delta_1 + k\Delta_{n-1} + k_1\Delta_n + k_2\Delta_{2n-1}$, $k_0 + k_1 + k_2 = p^a$ and $k + k_1 + 2k_2 = p^a$. By Corollary 4.3 and $Q(i-1)$, we have

$$S_{p^a\Delta_{n-1}}S_{p^a\Delta_1} \equiv \sum_{k, k_0, k_1} 2^{k_1}S_{k_0\Delta_1 + k\Delta_{n-1} + k_1\Delta_n}$$

with $k_0 + k_1 = k + k_1 = p^a$. Unless $k_1 = 0$ or p^a, Corollary 3.5 yields that

$$S_{k_0\Delta_1 + k\Delta_{n-1}}S_{k_1\Delta_n} = S_{k_0\Delta_1 + k\Delta_{n-1} + k_1\Delta_n} + \sum_{h(K)>n} a_K S_K.$$

So again by Corollary 3.2 and $Q(i-1)$,

$$S_{k_0\Delta_1 + k\Delta_{n-1} + k_1\Delta_n} \equiv 0.$$

Therefore

$$S_{p^a \Delta_{n-1}} S_{p^a \Delta_1} \equiv S_{p^a \Delta_{n-1} + p^a \Delta_1} + 2^{p^a} S_{p^a \Delta_n} .$$

On the other hand, Corollary 3.2, Corollary 4.3 and $Q(i-1)$ yield

$$S_{p^a \Delta_1} S_{p^a \Delta_{n-1}} \equiv S_{p^a \Delta_1 + p^a \Delta_{n-1}} + n^{p^a} S_{p^a \Delta_n} .$$

Thus $(n^{p^a} - 2^{p^a}) S_{p^a \Delta_n} \equiv [S_{p^a \Delta_1}, S_{p^a \Delta_{n-1}}] \equiv 0 .$

But $p^a S_{p^a \Delta_n} \equiv 0$ by Lemma 4.12 and $(n^{p^a} - 2^{p^a}, p^a) = 1$ by our hypothesis on n, and so $S_{p^a \Delta_n} \equiv 0$.

Case (2): $n = mp + 2$ $(m \geq 1)$, p: an odd prime.
As in the preceding case, Corollaries 3.2, 3.5, 4.3 and $Q(i-1)$ yield

$$[S_{p^a \Delta_2}, S_{p^a \Delta_{mp}}] \equiv ((mp+1)^{p^a} - 3^{p^a}) S_{p^a \Delta_n} .$$

$((mp+1)^{p^a} - 3^{p^a}, p^a) = 1$ yields $S_{p^a \Delta_n} \equiv 0$.

Case (3): $p = 2$, $n = 2m$ $(m \geq 2)$.
As in the preceding cases, Corollaries 3.2, 3.5, 4.3 and $Q(i-1)$ yields

$$(4.1) \qquad S_{2^{a+1} \Delta_{m-1}} S_{2^a \Delta_2} \equiv 3^{2^a} S_{2^a \Delta_{2m}}$$

$$+ \sum \binom{k+k_0}{k}^{\delta_{m,3}} 3^{k_1} S_{k \Delta_2 + k_0 \Delta_{m-1} + k_1 \Delta_{m+1}}$$

with $k_0 + k_1 = 2^{a+1}$ and $k + k_1 = 2^a$.

Now the proof of $Q(i)$ will be continued after we prove the following assertion $R(j)$, $2 \leq j \leq [m/2] + 1$, under the hypothesis $Q(i-1)$;

Assertion $R(j)$: If $\sum_{c=0}^{j} k_c = 2^{a+1}$, $k + \sum_{c=1}^{j} c k_c = 2^a$, $k_j \neq 0$ and $K_j = k \Delta_2 + \sum_{c=0}^{j} k_c \Delta_{m-1+2c}$, then $S_{K_j} \equiv 0$.

Proof. $R([m/2]+1)$ follows from Corollaries 3.5, 4.3 and $Q(i-1)$. Suppose that $R([m/2]+1), \ldots, R(j+1)$ hold for $[m/2] \geq j \geq 2$. If $k_j > 0$, then $k + k_0 + \cdots + k_{j-1} > 0$ and we consider the product $S_{K_{j-1}} S_{k_j \Delta_{m-1+2j}}$.

As in the proofs of Corollaries 3.3, 3.5, we put

$$I(X) = k \Delta_2 + \sum_{c=0}^{j-1} k_c \Delta_{m-1+2c}, \quad J(X) = k_j \Delta_{m-1+2j} \quad \text{and} \quad h(K(X)) \leq m .$$

Then we obtain by Theorem 3.1 that

$$S_{K_{j-1}} S_{k_j \Delta_{m-1+2j}} = \sum \prod_{c=j}^{[m/2]+1} \binom{m+2j}{c-j}^{k_c} S_{K'} + \sum_{h(K) > 2m+1} a_K S_K ,$$

where $K' = \bar{k}\varDelta_2 + \sum_{c=0}^{j-1} k_c \varDelta_{m-1+2c} + \sum_{c=j}^{[m/2]+1} \bar{k}_c \varDelta_{m-1+2c}$, $a_K \in Z$, and the first sum on the right runs over all $\{\bar{k}, \bar{k}_c\}$ with $\sum_{c=0}^{j-1} k_c + \sum_{c=j}^{[m/2]+1} \bar{k}_c = 2^{a+1}$ and $\bar{k} + \sum_{c=1}^{j-1} ck_c + \sum_{c=j}^{[m/2]+1} c\bar{k}_c = 2^a$. Thus the inductive hypothesis and $Q(i-1)$ yield

$$0 \equiv S_{K_{j-1}} S_{k_j \varDelta_{m-1+2j}} \equiv S_{K_j}.$$

This completes the proof of $R(j)$, for all $[m/2] + 1 \geq j \geq 2$.

Proof of Proposition 4.13 *Case* (3) (*continued*).
By Corollary 3.4 and the assertions $R(j)$, $[m/2] + 1 \geq j \geq 2$,

(4.2) $$S_{2^a \varDelta_2} S_{2^{a+1} \varDelta_{m-1}} \equiv \sum \binom{k+k_0}{k}^{\delta_{m,3}} m^{k_1} S_{k \varDelta_2 + k_0 \varDelta_{m-1} + k_1 \varDelta_{m+1}}$$

with $k_0 + k_1 = 2^{a+1}$ and $k + k_1 = 2^a$. By Theorem 3.1 and $Q(i-1)$

(4.3) $$S_{k\varDelta_2 + k_0 \varDelta_{m-1}} S_{k_1 \varDelta_{m+1}} \equiv S_{k\varDelta_2 + k_0 \varDelta_{m-1} + k_1 \varDelta_{m+1}}$$

$$+ \delta_{k_1, 2^a} (m+2)^{2^a} S_{2^a \varDelta_{2m}}.$$

Therefore, by (4.1), (4.2), and (4.3),

$$0 \equiv [S_{2^{a+1} \varDelta_{m-1}}, S_{2^a \varDelta_2}] \equiv (3^{2^a} - (m^{2^a} - 3^{2^a})(m+2)^{2^a}) S_{2^a \varDelta_{2m}}.$$

The coefficient of $S_{2^a \varDelta_{2m}}$ in the equation above is odd and so it is relatively prime to 2^a. Thus by Lemma 4.12 we have $S_{p^a \varDelta_{2m}} \equiv 0$.

This completes the proof of the inductive step and the assertions $Q(i)$ are proved for all $i \geq 1$. Thus the proof of Proposition 4.13 is completed.

Proof of Theorem 4.1.

Proof of (1) *and* (3): *Case* (1): $K = p^a \varDelta_e$, for some prime p, $a \geq 1$, $e = 1$ or 2, $(p, a) \neq (2, 1)$. By Corollary 3.2, we have

$$S_{p^{a-1} \varDelta_e} S_{(p^a - p^{a-1}) \varDelta_e} = \sum \binom{k+k_0}{k} 2^{k_1} S_K,$$

where $k_0 + k_1 + k_2 = p^a - p^{a-1}$, $k + k_1 + 2k_2 = p^{a-1}$, $k, k_i \geq 0$, and $K = (k + k_0)\varDelta_e + k_1 \varDelta_{2e} + k_2 \varDelta_{3e}$. If either two of three integers $k + k_0$, k_1 and k_2 are positive, then by Corollaries 4.3 and 3.5 we have

$$S_K \equiv \sum c_{q,b,n} S_{q^b \Delta_n}, \quad c_{q,b,n} \in Z,$$

where $n > h(K) \geq 2$, q is a prime, $b \geq 1$ and $ep^a = q^b n$. By Proposition 4.13 each $S_{q^b \Delta_n} \equiv 0$, and hence $S_K \equiv 0$. If $k = k_0 = k_1 = 0$, then $p^{a-1} = 2k_2 = 2(p^a - p^{a-1}) = 2p^{a-1}$. This is a contradiction. If $k = k_0 = k_2 = 0$, then $p^{a-1} = k_1 = p^a - p^{a-1} > p^{a-1}$ except for $p = 2$. In case $p = 2$, since $2^{a-1} \geq a - 1$, we have $2^{2^{a-1}} S_{2^{a-1} \Delta_2} \equiv 0$ by Lemma 4.12. Therefore we have

$$S_{p^{a-1} \Delta_e} S_{(p^a - p^{a-1}) \Delta_e} \equiv \binom{p^a}{p^{a-1}} S_{p^a \Delta_e}.$$

By Lemma 4.12, $p^a S_{p^a \Delta_e} \equiv 0$. Since $\binom{p^a}{p^{a-1}} \equiv 0 (p)$, $\not\equiv 0(p^2)$, we have $p S_{p^a \Delta_e} \equiv 0$.

Case (2): $K = 2p^a \Delta_1$, for some odd prime p, $a \geq 1$. By Corollary 4.3 (1), (2) and Proposition 4.13, we may express $S_K \equiv c S_{p^a \Delta_2}$, for some integer c. Therefore in S we can express

$$S_K = c S_{p^a \Delta_2} + \sum_{I,J} c_{I,J} S_I S_J,$$

where the last sum runs over all pairs (I, J) such that $I \neq 0$, $J \neq 0$, $\|I\| + \|J\| = 2p^a$. Evaluate the both sides by $x^{p^{a+1} - p^a}$. Then by Proposition 4.11 we have

$$\binom{p^{a+1} - p^a}{2p^a} = c \binom{p^{a+1} - p^a}{p^a} + \Sigma c_{I,J} \binom{p^{a+1} - p^a + \|J\|}{I} \binom{p^{a+1} - p^a}{J}.$$

Therefore $c \equiv -1 \pmod{p}$. Thus $S_K \equiv (mp - 1) S_{p^a \Delta_2}$ for some integer m. The result of Case (1) yields $S_K \equiv -S_{p^a \Delta_2}$. Thus S_K is of order p in $\bar{S}/\bar{S}^2$.

Case (3): $K \neq p^a \Delta_e$, $2p^a \Delta_1$, for any prime p, $a \geq 1$, $e = 1$ or 2. By Corollary 4.3 (1), (2) and Proposition 4.13 we have $S_K \equiv 0$.

Proof of (2). Let V be either of the sets mentioned in Theorem 4.1 (2). Then the proof of (1) and (3) implies that we can express

$$(4.4) \qquad\qquad S = B + \bar{S}^2,$$

where B is a subalgebra of S generated by V and 1. If $S \neq B$, then

there is an element g of the smallest degree such that $g \in S$ and $g \notin B$. By (4.4) we can express

$$g = g' + \sum_i a_i a'_i, \qquad g' \in B, \qquad a_i, a'_i \in \bar{S}.$$

Since $\deg(a_i) > 0$ and $\deg(a'_i) > 0$, $\deg(a_i) < \deg(g)$ and $\deg(a'_i) < \deg(g)$. Since g is of the smallest degree, a_i and a'_i are in B for all i. Therefore g is in B. This is contrary to the assumption. Thus $S = B$.

Proof of (4) *and* (6): They are just the mod p reductions of (1) and (3).

Proof of (5): Similar to the proof of (2).
Thus the proof of Theorem 4.1 is completed.
We denote $\overline{A^G} = \ker(A^G \to \Lambda)$.

Corollary 4.14. Let V be a subset of A^G consisting only of monomials. Then V is a minimal subset such that V generates A^G as a left A^G-module if and only if V is either of the sets mentioned in Theorem 4.1 (2).

Proof. Since $\overline{A^G} = \Lambda \hat{\otimes} \bar{S}$ and the product in A^G is a semi-tensor product of Λ and S (that is, for $\lambda, \lambda' \in \Lambda$,

$$(\lambda \otimes S_I)(\lambda' \otimes S_J) = \sum_{I'+I''=I} (\lambda \cdot S_{I'}(\lambda')) \otimes S_{I''} S_J,$$

where $S_{I'}(\lambda')$ is the action of $S_{I'} \in S$ on $\lambda' \in \Lambda$), the following two statements on a subset $V \subset \bar{S} \subset \overline{A^G}$ are equivalent:

(1) V is a minimal subset such that V generates $\bar{S}$ as a left S-module.

(2) V is a minimal subset such that V generates $\overline{A^G}$ as a left A^G-module.

(1) is equivalent to (3) below:

(3) V is a minimal set of generators of S.

(We note that the proof of (1)$\Rightarrow$(3) is similar to that of Theorem 4.1 (2).)

Thus the proof is completed.

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