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On the Maximally Overdetermined System of Linear Differential Equations, I*

By

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Introduction

The purpose of this paper is to present finiteness theorems and several properties of cohomologies of holomorphic solution sheaves of maximally overdetermined systems of linear differential equations. The proof relies on the finiteness theorem for elliptic systems due to T. Kawai [4], as an analytic tool, and on the theory of stratifications of analytic sets introduced by H. Whitney [8] and [9] ,as a geometric tool.

Our goal is the following theorem.

Theorem (3.1) *Let $\mathfrak{M}$ be a maximally overdetermined system on a complex manifold X and $X = \cup X_\alpha$ be a stratification of X satisfying the regularity conditions of H. Whitney such that the singular support of $\mathfrak{M}$ is contained in the union of conormal projective bundles of the strata. Then the restriction of $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{O}_X)$ to each stratum is a locally constant sheaf of finite rank.*

We mention that an introduction of differential operators of infinite order becomes indispensable when we analyze more precisely the structure of maximally overdetermined systems. We, however, leave this theme to our next paper and restrict ourselves to the category of finite order differential operators in the present paper.

As for notions on systems of differential equations, we refer to

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Kashiwara [2] and Sato-Kawai-Kashiwara [6].

We will list up the notations used in this paper.

Notations

X : complex manifold

$\mathcal{D}=\mathcal{D}_X$: the sheaf of differential operators of *finite order* on X

$\mathfrak{M}$: a coherent $\mathcal{D}_X$-Module

$\mathcal{O}_X$: the sheaf of holomorphic functions on X, which is a left coherent $\mathcal{D}_X$-Module

$\mathfrak{B}_{Y|X}$: a left $\mathcal{D}_X$ Module $\mathcal{H}^d_Y(\mathcal{O}_X)$ where $d=\operatorname{codim} Y$

Ω^n_X : the sheaf of holomorphic n-forms, which is a right $\mathcal{D}_X$-Module if $n=\dim X$

M : real analytic manifold

$\mathbb{R}$: the right derived functor in the derived category

$f_!$: the direct image by the map f with proper support.

§ 1. Finiteness Theorem for Elliptic Systems

In this section, we recall the finiteness theorem for elliptic systems due to T. Kawai [4].

Let M be a real analytic manifold, and let X be its complexification. We will denote by $\mathcal{D}_X$ the sheaf of differential operators of *finite order*. $\mathcal{D}_M$ is the restriction of $\mathcal{D}_X$ to M. Let $\mathfrak{M}$ be an elliptic system of differential equations on M, that is, the coherent $\mathcal{D}_M$-Module whose singular support does not intersect $\sqrt{-1}\,S^*M=S^*_MX$. Let Ω be a relatively compact open subset in M whose boundary is a real analytic hypersurface in M. The following theorem is due to T. Kawai [4].

Theorem (1.1) *If $\mathfrak{M}$ is an elliptic system on M and if the system $\mathfrak{M}_{\partial\Omega}$ induced from $\mathfrak{M}$ to the boundary $\partial\Omega$ of Ω is an elliptic system on $\partial\Omega$, then*

$$\operatorname{Ext}^i_{\mathcal{D}_M}(\Omega;\mathfrak{M},\mathfrak{A}_M)=\operatorname{Ext}^i_{\mathcal{D}_M}(\Omega;\mathfrak{M},\mathfrak{B}_M)$$

are finite dimensional vector spaces, where $\mathfrak{A}_M$ (resp. $\mathfrak{B}_M$) is a sheaf of real analytic functions (resp. hyperfunctions) on M.

This theorem is not sufficient for our study of maximally overdetermined system. We must discuss the invariance of the cohomology group by the perturbation of domains.

Let $\{\Omega_c\}_{c\in \boldsymbol{R}}$ be a family of relatively compact open subsets in M which have real analytic hypersurfaces as their boundaries satisfying the following properties

(1.1) $\Omega_{c_1}\supset\bar{\Omega}_{c_2}$ for $c_1>c_2$

(1.2) For any c_0, Ω_{c_0} is a union of Ω_c where c runs over the set of all numbers strictly less than c_0.

(1.3) For any c_0, $\{\Omega_c;\ c>c_0\}$ is a neighborhood system of the closure $\bar{\Omega}_{c_0}$ of Ω_{c_0}.

Theorem (1.2) *Let $\{\Omega_c\}$ be a family satisfying the above conditions. If $\mathfrak{M}$ and $\mathfrak{M}_{\partial\Omega_c}$ are elliptic, then the restriction homomorphisms*

$$\mathrm{Ext}^i_{\mathcal{D}_M}(\Omega_c;\mathfrak{M},\mathfrak{A}_M)\longrightarrow \mathrm{Ext}^i_{\mathcal{D}_M}(\Omega_{c'};\mathfrak{M},\mathfrak{A}_M)$$

are isomorphisms for $c\geqq c'$.

Proof. Set $E^i_c=\mathrm{Ext}^i_{\mathcal{D}}(\Omega_c;\mathfrak{M},\mathfrak{A}_M)$.

Then there is a canonical map $E^i_c\to E^i_{c'}$, if $c\geqq c'$. Note the following lemma (we omit its proof).

Lemma (1.3) *Let $\{V_c\}_{c\in\boldsymbol{R}}$ be a family of finite dimensional vector spaces parametrized by $c\in\boldsymbol{R}$. For $c\geqq c'$, there are given the homomorphisms $\rho_{c'c}\colon V_c\to V_{c'}$ satisfying the following chain condition:*

(1.4.1) $\rho_{c_1c_2}\rho_{c_2c_3}=\rho_{c_1c_3}$ *for* $c_1\leqq c_2\leqq c_3$,

(1.4.2) $\rho_{cc}=$*identity*

Moreover, they satisfy the following property of continuity.

(1.5.1) $\varprojlim_{c<c_0} V_c \xleftarrow{\sim} V_{c_0}$ *for any* c_0

(1.5.2) $\varinjlim_{c>c_0} V_c \xrightarrow{\sim} V_{c_0}$ *for any* c_0

They all $\rho_{c'c}$ are isomorphisms.

By virture of this lemma, it suffices to show

$$\varinjlim_{c>0} E_c^i = E_0^i, \qquad \varprojlim_{c<0} E_c^i = E_0^i.$$

Lemma (1.4) $$\varinjlim_{c>0} E_c^i = E_0^i$$

Proof. Since $\mathfrak{M}$ is elliptic and $\mathfrak{M}_{\partial\Omega_0}$ is elliptic, we have

$$\boldsymbol{R}\Gamma(\partial\Omega_0;\ \boldsymbol{R}\Gamma_Z\boldsymbol{R}\ \mathcal{H}om(\mathfrak{M},\ \mathfrak{A}_M)|_{\partial\Omega_0})=0$$

by virture of Kashiwara-Kawai [3], where $Z=M-\Omega_0$. It follows that

$$\begin{aligned}&\varinjlim_{V} H^i_{Z\cap V}(V;\boldsymbol{R}\ \mathcal{H}om(\mathfrak{M},\ \mathfrak{A}_M))\\ =&\varinjlim_{V} \mathrm{Ext}^i_{Z\cap V}(V;\ \mathfrak{M},\ \mathfrak{A}_M)\\ =&0\end{aligned}$$

where V runs over the neighborhood system of $\partial\Omega_0$.

Therefore

$$\varinjlim_{c\downarrow 0} \mathrm{Ext}^i(\Omega_c \bmod \Omega_0;\ \mathfrak{M},\ \mathfrak{A}_M)=0.$$

The long exact sequence

$$\begin{aligned}\cdots \longrightarrow\ &\mathrm{Ext}^i(\Omega_c \bmod \Omega_0;\ \mathfrak{M},\ \mathfrak{A}_M) \longrightarrow \mathrm{Ext}^i(\Omega_c;\ \mathfrak{M},\ \mathfrak{A}_M)\\ \longrightarrow\ &\mathrm{Ext}^i(\Omega_0;\ \mathfrak{M},\ \mathfrak{A}_M) \longrightarrow \cdots\end{aligned}$$

implies the desired result. Lastly, we will prove the following

Lemma (1.5) $$\varprojlim_{c<0} E_c^i = E_0^i$$

Proof. Since E_c^{i-1} is finite dimensional, $\{E_c^{i-1}\}_{c<0}$ satisfies the condition (ML). Therefore $\varprojlim_{c<0} E_c^i = E_0^i$ (see [1]). Q.E.D.

This completes the proof of Theorem (1.2).

We will apply Theorem (1.2) to the special case.

Let X be a complex manifold, and let $\mathfrak{M}$ be a coherent $\mathcal{D}_X$-Module. A real analytic submanifold N of X is said to be non characteristic with respect to $\mathfrak{M}$ when the singular support of $\mathfrak{M}$ does not intersect the conormal bundle of N.

Theorem (1.6) *Let $\{\Omega_c\}_{c\in \boldsymbol{R}}$ be a family of relatively compact open subsets of X with real analytic hypersurfaces as boundary satisfying* (1.1), (1.2) *and* (1.3), *and let $\mathfrak{M}$ be a coherent $\mathcal{D}_X$ Module. Assume that every boundary $\partial\Omega_c$ is non characteristic with respect to $\mathfrak{M}$. Then $Ext^i_{\mathcal{D}_X}(\Omega_c;\mathfrak{M},\mathcal{O}_X)$ are finite dimensional and all the restriction homomorphisms*

$$\mathrm{Ext}^i_{\mathcal{D}_X}(\Omega_c;\mathfrak{M},\mathcal{O}_X)\longrightarrow \mathrm{Ext}^i_{\mathcal{D}_X}(\Omega_{c'};\mathfrak{M},\mathcal{O}_X)$$

are isomorphisms for $c\geqq c'$.

Proof. We will regarad X as a real analytic manifold and $\mathcal{O}_X$ be a solution of Cauchy-Riemann equation. We can take $X\times\bar{X}$ as a complex neighborhood of X where $\bar{X}$ is the complex conjugate of X.

Then

$$\mathcal{O}_X=\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_{X\times\bar{X}}}(\mathcal{D}_X\hat{\otimes}\mathcal{O}_{\bar{X}};\mathfrak{B}_X).$$

Therefore

$$\mathrm{Ext}^i_{\mathcal{D}_X}(\Omega_c;\mathfrak{M},\mathcal{O}_X)=\mathrm{Ext}^i_{\mathcal{D}_{X\times\bar{X}}}(\Omega_c;\mathfrak{M}\hat{\otimes}\mathcal{O}_{\bar{X}},\mathfrak{B}_X).$$

Since $\mathfrak{M}\hat{\otimes}\mathcal{O}_{\bar{X}}$ is an elliptic system on X and $\mathfrak{M}\hat{\otimes}\mathcal{O}_{\bar{X}}|_{\partial\Omega_c}$ is also elliptic, the theorem follows from Theorem (1.1) and (1.2). Q.E.D.

§ 2. Finitistic Sheaves

In order to clarify the main theorem of this paper, we will introduce the following notion. In this section, A always denotes a commutative noetherian ring and sheaves are A-Modules. In the following sections we take $\boldsymbol{C}$ as A.

Definition (2.1) Let S be an analytic space, and let F be a sheaf on S. We say that F is finitistic, if there is a stratification of S such that F

is locally constant on each stratum and every stalk is an A-module of finite type.

It is evident that the category of finitistic sheaves is an abelian category. If there is an exact sequence

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$$

of sheaves and two of them are finitistic, then so is the other clearly. If F and G are finitistic, then $\mathcal{T}or_k^A(F, G)$ is also finitistic.

Firstly, we will prove the following

Lemma (2.2) *Let F be a sheaf on S, $S = \cup X_\alpha$ be a stratification which satisfies the regularity conditions of Whitney, and F is locally constant on each stratum. Then, for every $x_0 \in X_\alpha$, there is a sufficiently small open neighborhood U of x_0 such that*

$$F(U) \longrightarrow F_x$$

is an isomorphism for every point x in X_α sufficiently near x_0, and $H^k(U, F) = 0$ for every $k > 0$.

Proof. We choose a local coordinate. It is sufficient to show that any ε-neighborhood of x_0 satisfies the condition of the lemma for any sufficiently small ε. Let X_β be an open stratum in the support of F and j be the inclusion map. Then,

$$0 \longrightarrow j_!(F|_{X_\beta}) \longrightarrow F \longrightarrow F|_{S-X_\beta} \longrightarrow 0$$

is an exact sequence in a neighborhood of X_α. Since the support of $j_!(F|_{X_\beta})$ is contained in X_β and that of $F|_{S-X_\beta}$ is contained in $S - X_\beta$, we can assume that the support of F is concentrated onto only one stratum X_β dominating X_α. Then, if U is an ε-neighborhood and x is a point in X_α sufficiently near x_0, then $V \cap X_\beta$ is a deformation retract of $U \cap X_\beta$ for sufficiently small open ball V with center x. It follows that $H^k(U, F) \xrightarrow{\sim} H^k(V, F)$ is an isomorphism, which shows this lemma. Q.E.D.

Lemma (2.3) *Let S be an analytic set and let F be a finitistic sheaf on S. Let W be the largest open set in S such that $F|_W$ is locally constant* *on W. Then $T=S-W$ is an analytic subset of S with codimension* $\geqq 1$.

Proof. It is evident that T is nowhere dense. So, it suffices to show that T is an analytic set. Let $S=\cup X_\alpha$ be a stratisfication of S satisfying the regularity conditions of Whitney so that F is locally constant on each stratum. It suffices to show that $W\cap X_\alpha \neq \phi$, then W contains X_α, because this implies that T is a union of strata. Since X_α is connected, it suffices to show that $W\cap X_\alpha$ is a closed subset. Let x_0 be a cluster point of $W\cap X_\alpha$ in X_α. By the induction of codimension of strata, we may assume that W contains all strata dominating X_α, which implies that F is locally constant on $U-X_\alpha$, where U is a sufficient small neighborhood of x_0. By the preceding lemma, we can assume that

$$F(U) \longrightarrow F_x$$

is an isomorphism for any $x\in X_\alpha$ sufficiently near x. Since F is locally constant on W, $F(U)\rightarrow F_x$ is an isomorphism for some point x in $W\cap X_\alpha \cap U$. It implies that F is locally constant in a neighborhood of x_0 because we can assume that $U\cap X_\beta$ is connected for any stratum X_β. It implies that x_0 is contained in W. Q.E.D.

By using above lemmas, we can show that finitisticity is a local property.

Proposition (**2.3**) *Let F be a sheaf on S. If there is an open covering of S such that F is finitistic on each cover, then F is finitistic.*

Proof. Let W be the largest open subset of S on which F is locally constant. Set $T=S-W$. T is a closed analytic set of S. Then, by the induction, we can assume that $F|_T$ is finitistic. Therefore F is finitistic. Q.E.D.

We will introduce the following notion.

Definition (**2.4**) Let F be a sheaf on S. The stratification $S=\cup X_\alpha$ of S is said to be regular with respect to F if it satisfies the regularity conditions of Whitney and F is locally constant on each stratum.

We will prove that "finitisticity" is invariant under "cohomology".

Propositiou (**2.5**) *Let $S=\cup X_\alpha$ be a stratification which satisfies the regularity conditions of Whitney, T be a closed analytic subset of S which is a union of strata and let j be the inclusion map $S-T\hookrightarrow S$.*

Let F be a finitistic sheaf on $S-T$ locally constant on any stratum. Then $R^kj_*(F)$ and $R^kj_!(F)$ are finitistic sheaves on S and locally constant on each stratum. Moreover, if S is a manifold and Z is a submanifold of S transversal to each stratum; *then*

$$R^kj_*(F)|_Z=R^kj_*(F|_Z)$$
$$R^kj_!(F)|_Z=R^kj_!(F|_Z)$$

on Z.

Proof. By the long exact sequence

$$\cdots \longrightarrow R^kj_!(F) \longrightarrow R^kj_*(F) \longrightarrow R^kj_*(F)|_T \longrightarrow \cdots$$

it is sufficient to show the statement on the direct image. We can suppose without loss of generality that S is a manifold and the support of F consists of only one stratum X_β. Let X_α be the stratum contained in T and dominated by X_β. Let d be the codimension of X_α and let H be the submanifold of dimension d transversal to X_α. Since (X_β, X_α) is locally isomorphic to $((X_\beta\cap H)\times X_\alpha, X_\alpha)$, $R^kj_*(F)|_{X_\alpha}$ is locally constant. Since $X_\beta\cap H$ is triangulated by finite cells, the stalk of $R^kj_*(F)$ is an A-module of finite type.

$(X_\beta\cap Z, Z\cap X_\alpha)$ being locally isomorphic to $((X_\beta\cap H)\times(Z\cap X_\alpha), Z\cap X_\alpha)$, we obtain $R^kj_*(F)|_Z=R^kj_*(F|_Z)$. Q.E.D.

As a corollary of this proposition we obtain the following

Proposition (**2.6**) *If F is a finitistic sheaf on S and T is a closed analytic set in S, then $\mathcal{H}^k_T(F)$ is also a finitistic sheaf on T. More precisely, if $S=\bigcup X_\alpha$ is a stratification of S regular with respect to F and if T is a union of strata, then it is a stratification regular with respect to $\mathcal{H}^k_T(F)$.*

Proof. Let j be the inclusion map $S-T\hookrightarrow S$. Then the exact sequences

$$0 \longrightarrow \mathcal{H}_T^0(F) \longrightarrow F \longrightarrow j_* j^{-1} F \longrightarrow \mathcal{H}_T^1(F) \longrightarrow 0$$

$$\mathcal{H}_T^k(F) = R^{k-1} j_* j^{-1} F \qquad (k>1)$$

and Proposition (2.5) imply the proposition.

By the same reasoning, we obtain the following

Proposition (2.7) *Let X be a complex manifold, Y be a submanifold, T be an analytic subset of X and let F be a finitistic sheaf on X. Assume that there exists a stratification of X regular with respect to F such that Y is transversal to each stratum and T is a union of strata, then*

$$\mathcal{H}_T^k(F)|_Y \xrightarrow{\sim} \mathcal{H}_{T\cap Y}^k(F|_Y).$$

Corollary (2.8) *Under the assumption as above, the support of $\mathcal{H}_T^k(F)$ is a locally closed analytic set of codimension $\geqq k/2$ in S.*

In fact, if X_α is a stratum of codimension $<k/2$ we can take Y so that its dimension $<k/2$. Therefore $\mathcal{H}_{T\cap Y}^k(F|_Y)=0$ because the real dimension of Y is less than k.

§ 3. Finiteness Theorem for Maximally Overdetermined System

Let X be a complex manifold of dimension n. We will show, in this section, the following main theorem.

Theorem (3.1) *If $\mathfrak{M}$ is a maximally overdetermined system on X, then $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{O}_X)$ are finitistic.*

Remember that a maximally overdetermined system on X is, by definition, a coherent $\mathcal{D}_X$ Module whose singular support is of codimension n in P^*X at each point in it.

From now on, $\mathfrak{M}$ always denotes a maximally overdetermined system.

Firstly, we will discuss the Lagrangean analytic set. Lagrangean analytic set is by the definition an involutory analytic subset in P^*X of dimension $(n-1)$ (a fortiori, purely $(n-1)$ dimensional). An analytic set of pure codimension n is Lagrangean if and only if the fundamental 1-

form on P^*X vanishes on the tangent cone of it at any point (equivalently on the non singular locus). The singular support of a maximally over-determined system is always Lagrangean.

Lemma (**3.2**) *Let Λ be a Lagrangean closed analytic set in P^*X. Then there is a stratification $X=\cup X_\alpha$ of X satisfying the regularity conditions a) and b) of Whitney* [8] *such that*

$$\Lambda \subset \cup P^*_{X_\alpha}X \tag{3.1}$$

where $P^*_{X_\alpha}X$ is the conormal projective bundle of X_α in X.

Proof. Let π be the projection from P^*X onto X. Set $X_0=X-\pi(\Lambda)$. X_0 is a dense open set in X because dim $\pi(\Lambda)$ is less than $(n-1)$. Let X'_1 be a set of non singular point of $\pi(\Lambda)$. Note the following fact.

Sublemma (**3.3**) *Let Λ be a Lagrangean closed analytic set in P^*X. Assume that $Y=\pi(\Lambda)$ is a non singular set. Then Λ contains P^*_YX and $\pi(\overline{\Lambda-P^*_YX})$ is an analytic subset of Y with codimension $\geqq 1$.*

Proof. The question being local in Y, we can assume that Y is defined by $x_1=\ldots=x_r=0$. Since Λ is involutory, Λ is invariant by the infinitesimal transformation $\partial/\partial\xi_1, \ldots, \partial/\partial\xi_r$ where ξ is cotangent vector. Therefore Λ containes $\{(x, \xi); x_1=\ldots=x_r=0, \xi_{r+1}=\ldots=\xi_n=0\}$, which equals to P^*_YX. Since $\overline{\Lambda-P^*_YX}$ is also Lagrangean, and $\overline{\Lambda-P^*_YX}$ does not contain P^*_YX, $\pi(\overline{\Lambda-P^*_YX})$ is not equal to Y. Q.E.D.

Set $\Lambda_1=\overline{\Lambda-P^*_{X'_1}X}$, $X_1=X'_1-\pi(\Lambda_1)$. Then $X=X_0\cup X_1\cup\pi(\Lambda_1)$. By the induction we define X_j, X'_j and Λ_{j+1} as follows. X'_j is a non singular locus of $\pi(\Lambda_j)$.

$$\Lambda_{j+1}=\overline{\Lambda_j-P^*_{X'_j}X}$$
$$X_j=X'_j-\pi(\Lambda_{j+1}).$$

Then, since dim X_j is strictly decreasing, $X_{n+1}=\phi$, and therefore $\Lambda_n=\phi$. It is clear that $\{X_j\}$ is a stratification of X and satisfies (3.1). By Whitney [8], there exists a refinement of $\{X_j\}$ which is a stratification satisfying the regularity conditions. It is evident that this stratification satisfies (3.1).

We remark that the regularity condition (a) of Whitney is equivalent to say that $\cup P^*_{X_\alpha}X$ is a closed analytic set of P^*X. Since the stratification which satisfies (3.1) appears frequently, we introduce the following notion.

Definition (3.4) Let $\mathfrak{M}$ be a maximally overdetermined system on X. The stratification of X is said to be regular with respect to $\mathfrak{M}$ if it satisfies the regularity conditions a) b) of Whitney and the singular support of $\mathfrak{M}$ is contained in the union of the conormal projective bundle of strata.

Now we will prove the following refined form of Theorem (3.1).

Theorem (3.5) *Let $\mathfrak{M}$ be a maximally overdetermined system on X, and let $X=\cup X_\alpha$ be a stratification of X which is regular with respect to $\mathfrak{M}$. Then $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{O}_X)|_{X_\alpha}$ is a locally constant $\boldsymbol{C}$-Module of finite rank.*

We will prove firstly the following preparatory lemma.

Lemma (3.6) *Let x_0 be a point in X_α, and choose a local coordinate near x_0. Then $(x, (\bar{x}-\bar{y})\infty)$ does not belong to $SS(\mathfrak{M})$ for $x\in X$, $y\in X_\alpha$ such that $|x-x_0|\ll 1$, $|y-x_0|\ll 1$ and $x\neq y$. (where $\bar{x}$ is the complex conjugate of x)*

Proof. We may assume $x\in X_\beta$, where X_β is a stratum whose closure contains X_α. If the lemma is false there are sequences $X_\beta\ni x_n$ and $X_\alpha\ni y_n$ which converge to x_0 satisfying $(x_n, (\bar{x}_n-\bar{y}_n)\infty)\in SS(\mathfrak{M})$. We may assume that $(TX_\beta)_{x_n}$ tends to T and $a_n(x_n-y_n)$ tends to a non zero vector v in T, where a_n is a sequence in $\boldsymbol{C}^*$. By the assumption $\bar{a}_n(\bar{x}_n-\bar{y}_n)\in(T^*_{X_\beta}X)_{x_n}$, which is an orthogonal vector space of $(TX_\beta)_{x_n}$. It follows that $\bar{v}$ belongs in the orthogonal of T, which implies $\langle v, \bar{v}\rangle=0$. This is a contradiction. Q.E.D.

Now we can prove Theorem (3.5). Let x_0 be a point in X_α. We chose a local coordinate of X such that x_0 is the origin and X_α is a vector space. Suppose that $(x, (\bar{x}-\bar{y})\infty)\notin SS(\mathfrak{M})$ for $y\in X_\alpha$, $x\in X$ such that $|x|\leqq c$, $|y|\leqq c$, $x\neq y$. Set $\varphi(t, x, y)=|x-(1-t)y|^2-t^2c^2/2$. Then $\partial_x\varphi(t, x, y)\neq 0$ and

$$(x, \partial_x\varphi(t, x, y))\notin SS(\mathfrak{M}) \quad \text{for} \quad 0<t\leqq 1,\ |y|<c/2,\ \varphi(t, x, y)=0.$$

It implies that the boundary of $\Omega_{t,y}=\{x; \varphi(t, x, y)<0\}$ is non characteristic with respect to $\mathfrak{M}$.

$$\Omega_{t_1,y}\subset\Omega_{t_2,y} \quad \text{if} \quad t_1\leqq t_2$$

and $\Omega_{1,y}$ is independent of y. By Theorem (1.6), the restriction homomorphism

$$\mathrm{Ext}^i(\Omega_{1,y}; \mathfrak{M}, \mathcal{O}_X) \longrightarrow \mathrm{Ext}^i(\Omega_{t,y}; \mathfrak{M}, \mathcal{O}_X)$$

are isomorphisms. Since $\{\Omega_{t,y}\}_{t>0}$ is a neighborhood system of y, the homomorphism

$$\mathrm{Ext}^i(\Omega_{1,0}; \mathfrak{M}, \mathcal{O}_X) \xrightarrow{\sim} \mathcal{E}xt^i(\mathfrak{M}, \mathcal{O}_X)_y.$$

is an isomorphism for every y. It follows that $\mathcal{E}xt^i(\mathfrak{M}, \mathcal{O}_X)|_{X_\alpha}$ is locally constant. Since $\mathrm{Ext}^i(\Omega_{1,0}; \mathfrak{M}, \mathcal{O}_X)$ is finite dimensional, $\mathcal{E}xt^i(\mathfrak{M}, \mathcal{O}_X)_y$ is finite dimensional. It completes the proof of Theorem (3.5).

Let Y be a complex submanifold of X of codimension d. $\mathfrak{B}_{Y|X}$ is, by the definition, a left $\mathcal{D}_X$-Module $\mathcal{H}_Y^d(\mathcal{O}_X)$. Remark that $\mathcal{H}_Y^k(\mathcal{O}_X)$ vanishes for any k except d. $\mathfrak{B}_{X|X}$ is nothing but $\mathcal{O}_X$.

Theorem (3.7) *Let $X=\cup X_\alpha$ be a regular stratification with respect to a maximally overdetermined system $\mathfrak{M}$, and let Y be a complex submanifold of X which is a union of strata. Then $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathfrak{M}, \mathfrak{B}_{Y|X})$ is finitistic on Y and locally constant on each stratum.*

This is a corollary of Theorem (3.5) and Proposition (2.6) because $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathfrak{M}, \mathfrak{B}_{Y|X})$ is equal to $\mathcal{H}_Y^{i+d}(\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}}(\mathfrak{M}, \mathcal{O}_X))$.

Theorem (3.8) *Under the same assumption as above,*

$$\mathcal{E}xt^i_{\mathcal{D}_X}(\mathfrak{M}, \mathfrak{B}_{Y|X})|_Z \xrightarrow{\sim} \mathcal{E}xt^i_{\mathcal{D}_Z}(\mathfrak{M}_Z, \mathfrak{B}_{Y\cap Z|Z})$$

for any i and any submanifold Z transversal to every stratum, where $\mathfrak{M}_Z$ is the induced system of $\mathfrak{M}$ onto Z.

This is a corollary of Theorem (3.7) and Proposition (2.7). (See Kashiwara [2] Th. 2.3.1.)

Lastly, we will remark the following propositions. $\mathfrak{B}^f_{Y|X}$ is, by the definition, a maximally overdetermined $\mathcal{D}_X$ Module $\varinjlim_k \mathcal{E}xt^r_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}^k; \mathcal{O}_X)$ where r is the codimension of Y and $\mathcal{J}$ is the defining ideal of Y. If Y is defined by $x_1=\dots=x_r=0$ for a local coordinate system $(x_1, \dots, x_n)$, $\mathfrak{B}^f_{Y|X}=\mathcal{D}_X/\mathcal{D}_X x_1+\dots+\mathcal{D}_X x_r+\mathcal{D}_X D_{r+1}+\dots+\mathcal{D}_X D_n$.

Proposition (3.9) *Let $\mathfrak{M}$ be a maximally overdetermined system whose support is contained in a submanifold Y and whose singular support is contained in $P^*_Y X$, then $\mathfrak{M}$ is locally isomorphic to the direct sum of finite copies of $\mathfrak{B}^f_{Y|X}$.*

Proof. We choose a local coordinate $(x_1, \dots, x_n)$ such that $Y=\{(x_1, \dots, x_n);\ x_1=\dots=x_r=0\}$.

Let u be a non zero section of $\mathfrak{M}$. Since $\mathcal{O}u$ is a coherent sheaf whose support is contained in Y, there is N such that $x_1^N u=\dots=x_r^N u=0$. Therefore, there is a non zero section u of $\mathfrak{M}$ such that $x_1 u=\dots=x_r u=0$. Since $\mathcal{E}xt^1(\mathfrak{B}^f_{Y|X}, \mathfrak{B}^f_{Y|X})=0$, if we prove the proposition for $\mathcal{D}u$ and $\mathfrak{M}/\mathcal{D}u$, then the proposition is true for $\mathfrak{M}$. Since $\mathfrak{M}$ is noetherian, it suffices to show it for $\mathcal{D}u$. Let $\mathcal{D}u=\mathcal{D}/\mathcal{J}$. $\mathcal{J}$ contains $x_1, \dots, x_r$. If $\mathcal{J}$ contains $P(x'', D)=\sum P_\alpha(x'', D'')D'^\alpha$ (where $x''=(x_{r+1}, \dots, x_n)$, $D'=(D_1, \dots, D_r)$, $D''=(D_{r+1}, \dots, D_n)$) then $\mathcal{J}$ contains all $P_\alpha(x'', D'')$ since $[D_i, x_i]=1$. It follows that $\mathcal{D}u=(\mathcal{D}'/\mathcal{D}'x_1+\dots+\mathcal{D}'x_r)\hat{\otimes}(\mathcal{D}''/\mathcal{J}'')$, where $\mathcal{D}'=\mathcal{D}_{C^r}$, $\mathcal{D}''=\mathcal{D}_{C^{n-r}}$. Since the singular support of $\mathcal{D}''/\mathcal{J}''$ is a void set, $\mathcal{D}''/\mathcal{J}''$ is a finite sum of $\mathcal{O}_{C^{n-r}}$ (See Kashiwara [2]). It follows the proposition.

Q.E.D.

Corollary (3.10) *If $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{O}_X)=0$ for every i, then $\mathfrak{M}=0$.*

Because $\mathcal{E}xt^r_{\mathcal{D}_X}(\mathfrak{B}^f_{Y|X}, \mathcal{O}_X)=\boldsymbol{C}_Y$ where r is the codimension of Y.

This corollary means that the functor $\mathfrak{M}\longmapsto \boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{O}_X)$ is a faithful functor. This will be investigated more precisely in the subsequent paper.

§ 4. Several Properties of Cohomologies of Holomorphic Solutions of Maximally Overdetermined Systems

Let $\mathfrak{M}$ be a maximally overdetermined system on a complex manifold

X of dimension n. We fix a stratification $X=\cup X_\alpha$ of X which is regular with respect to $\mathfrak{M}$. The support Supp $\mathcal{E}xt^i(\mathfrak{M}, \mathcal{O}_X)$ is a set of x where $\mathcal{E}xt^i(\mathfrak{M}, \mathcal{O}_X)_x\neq 0$. Since $\mathcal{E}xt^i(\mathfrak{M}, \mathcal{O}_X)$ is locally constant on the strata, its support is a union of strata, which implies that the support is a locally finite union of locally closed analytic sets (and its closure is an analytic set).

Theorem (4.1) *The support of $\mathcal{E}xt^i(\mathfrak{M}, \mathcal{O}_X)$ is a locally finite union of locally closed analytic sets of codimension $\geqq i$.*

Corollary (4.2) *If s is a section of $\mathcal{E}xt^i(\mathfrak{M}, \mathcal{O}_X)$, then the support of s is an analytic set of codimension $\geqq i$.*

Firstly we introduce the notation of the modified singular support $\widehat{SS}(\mathfrak{M})$. $\widehat{SS}(\mathfrak{M})=T^*X\cap SS(\mathfrak{M}\hat{\otimes}\mathcal{D}_C/\mathcal{D}_C t)$, where t is a coordinate of $\boldsymbol{C}$ and we embed T^*X into $P^*(X\times \boldsymbol{C})$ in the following way: $(x, \xi)\longmapsto (x, 0; (\xi, 1)\infty)$. $\widehat{SS}(\mathfrak{M})$ is a closed analytic set in the cotangent bundle T^*X. It is clear that $\widehat{SS}(\mathfrak{M})$ is a cone, that is, invariant by the multiplication of complex number. Note that the image of the map $\widehat{SS}(\mathfrak{M})-X\to P^*X$ coincides with $SS(\mathfrak{M})$. $\widehat{SS}(\mathfrak{M})=\phi$ if and only if $\mathfrak{M}=0$ and $\widehat{SS}(\mathfrak{M})$ is contained in the zero section if and only if $\mathfrak{M}$ is locally isomorphic to the finite number of direct sum of $\mathcal{O}_X$. If $\mathfrak{M}=\cup\mathfrak{M}_k$ is a good filtration of $\mathfrak{M}$, then $\widehat{SS}(\mathfrak{M})$ is the support of the coherent sheaf on T^*X corresponding to the graded Module $\oplus(\mathfrak{M}_k/\mathfrak{M}_{k-1})$. (See Kashiwara [2]).

Proposition (4.3) *In a neighborhood of x in X, we have $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{O}_X)=0$ for $i>dim\,(\widehat{SS}(\mathfrak{M})\cap T^*_xX)$ for any coherent $\mathcal{D}_X$-Module $\mathfrak{M}$ (not necessarily maximally overdetermined). T^*_xX is a fiber of x.*

Proof. Set $d=\dim(\widehat{SS}(\mathfrak{M})\cap T^*_xX)$. If $d=-\infty$, then this is clear. Suppose that $d\geqq 0$. Then there is a d-dimensional submanifold Y through x such that $P^*_YX\cap SS(\mathfrak{M})=\phi$ in a neighborhood of x which means that Y is non characteristic with respect to $\mathfrak{M}$. By Kashiwara [2], we have

$$\mathcal{E}xt^i_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{O}_X)_x=\mathcal{E}xt^i_{\mathcal{D}_Y}(\mathfrak{M}_Y, \mathcal{O}_Y)_x$$

where $\mathfrak{M}_Y$ is the induced system of $\mathfrak{M}$. Since the global dimension of

$\mathcal{D}_{Y,x}$ is $d=\dim Y$ (See Kashiwara [2]), $\mathcal{E}xt^i_{\mathcal{D}_Y}(\mathfrak{M}_Y, \mathcal{O}_Y)_x=0$ for $i>d$. Since $\dim(\widehat{SS}(\mathfrak{M})\cap T^*_xX)$ is upper semi-continuous, the proposition follows. Q.E.D.

Corollary (4.4) $\mathcal{E}xt^i(\mathfrak{M}, \mathcal{O}_X)|_{X_\alpha}=0$ *for* $i>$ *codim* X_α.

This corollary immediately implies Theorem (4.1). Let Y be a submanifold of X of codimension c. By taking a refinement of the regular stratification of Y, we may assume that Y is a union of strata.

By replacing $\mathcal{O}_X$ with $\mathfrak{B}_{Y|X}$, we obtain the same type of the preceding theorems.

Theorem (4.5) *The support of* $\mathcal{E}xt^i(\mathfrak{M}, \mathfrak{B}_{Y|X})$ *is a locally finite union of (locally closed) analytic subsets of* Y *of codimension* $\geqq(i-c)$ *in* Y.

Corollary (4.6) *The support of a global section of* $\mathcal{E}xt^i(\mathfrak{M}, \mathfrak{B}_{Y|X})$ *is an analytic subset of* Y *of codimension* $\geqq(i-c)$.

Since this theorem can be proved in the same way as before, we do not repeat it.

§ 5. Duality

Let $\mathfrak{M}$ be a maximally overdetermined system on a complex manifold X of dimension n and x_0 be a point in X. Then there exists a free resolution

$$(5.1)\qquad 0 \longleftarrow \mathfrak{M} \longleftarrow \mathcal{D}_X^{r_0} \xleftarrow{P_0(x,D)} \mathcal{D}_X^{r_1} \xleftarrow{P_1(x,D)} \cdots \xleftarrow{P_{N-1}(x,D)} \mathcal{D}_X^{r_N} \longleftarrow 0$$

in a neighborhood of x_0 where $P_i(x, D)$ is an $(r_{i+1}\times r_i)$ matrix of differential operators. Then $\mathcal{E}xt^i_{\mathcal{D}}(\mathfrak{M}, \mathcal{O})_{x_0}$ is an i-th cohomology of

$$(5.2)\qquad \mathcal{O}^{r_0}_{X,x_0} \xrightarrow{P_0(x,D)} \mathcal{O}^{r_1}_{X,x_0} \longrightarrow \cdots \xrightarrow{P_{N-1}(x,D)} \mathcal{O}^{r_N}_{X,x_0}$$

$\mathrm{Tor}_i^{\mathcal{D}}(\mathfrak{B}^{(n)}_{\{x_0\}|X}, \mathfrak{M})$ is an i-th homology of

$$(5.3)\qquad \mathfrak{B}^{(n)r_0}_{\{x_0\}|X} \longleftarrow \mathfrak{B}^{(n)r_1}_{\{x_0\}|X} \longleftarrow \cdots \longleftarrow \mathfrak{B}^{r_N}_{\{x_0\}|X}$$

where $\mathfrak{B}_{\{x_0\}|X}$ is a left $\mathcal{D}$-Module $\mathcal{H}^n_{\{x_0\}}(\mathcal{O}_X)$ and $\mathfrak{B}^{(n)}_{\{x_0\}|X}$ a right $\mathcal{D}_X$ Module defined by

$$\Omega_X^n \otimes_{\mathcal{O}_X} \mathfrak{B}_{\{x\}|X} = \mathcal{H}_{\{x_0\}}^n(\mathcal{O}_X).$$

Since $\mathcal{O}_{X,x_0}$ is an (DFS) topological vector space and $\mathfrak{B}_{\{x_0\}|X}^{(n)}$ is a (FS) topological vector space and they are dual to each other. Since the cohomology of (5.2) is finite dimensional, $\mathcal{E}xt_{\mathcal{D}}^i(\mathfrak{M}, \mathcal{O}_X)_{x_0}$ is a dual vector space of $\mathrm{Tor}_i^{\mathcal{D}}(\mathfrak{B}_{\{x_0\}|X}^{(n)}, \mathfrak{M})$. This duality is obtained by the canonical cup product

$$(5.4) \qquad \begin{aligned} &\mathcal{E}xt_{\mathcal{D}}^i(\mathfrak{M}, \mathcal{O}_X)_{x_0} \times \mathrm{Tor}_i^{\mathcal{D}}(\mathfrak{B}_{\{x_0\}|X}^{(n)}, \mathfrak{M}) \\ &\longrightarrow \mathfrak{B}_{\{x_0\}|X}^{(n)} \otimes_{\mathcal{D}} \mathcal{O}_{X,x_0} = \boldsymbol{C}. \end{aligned}$$

Setting $\mathfrak{M}^* = \mathcal{E}xt_{\mathcal{D}}^n(\mathfrak{M}, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \Omega_X^{n \otimes (-1)}$, we call it the adjoint system of $\mathfrak{M}$. $\mathfrak{M}^*$ is also a maximally overdetermined system and $(\mathfrak{M}^*)^* = \mathfrak{M}$, and $\mathfrak{M} \longmapsto \mathfrak{M}^*$ is a contravariant exact functor from the category of maximally overdetermined systems into itself. (See Kashiwara [2], Sato-Kawai-Kashiwara [6]). Remark that the existence of such a functor immediately implies that a stalk of maximally overdetermined system is a $\mathcal{D}$-module of finite length. Note that

$$\mathcal{O}_X^* = \mathcal{O}_X \quad \text{and} \quad \mathfrak{B}_{\{x_0\}|X}^* = \mathfrak{B}_{\{x_0\}|X}.$$

Since $\mathrm{Tor}_i^{\mathcal{D}}(\mathfrak{B}_{\{x_0\}|X}^{(n)}, \mathfrak{M}) = \mathrm{Ext}_{\mathcal{D}}^{n-i}(\mathfrak{M}^*, \mathfrak{B}_{\{x_0\}|X})$, we obtain the following

Proposition (5.1) *$\mathcal{E}xt_{\mathcal{D}}^{n-i}(\mathfrak{M}^*, \mathfrak{B}_{\{x_0\}|X})$ and $\mathcal{E}xt_{\mathcal{D}}^i(\mathfrak{M}, \mathcal{O}_X)_{x_0}$ are dual vector space of each other.*

We will generalize this proposition. Let Y be a submanifold of X of codimension d, x be a point in Y. $\mathfrak{B}_{Y|X}$ is by the definition the left $\mathcal{D}_X$-Module $\mathcal{H}_Y^d(\mathcal{O}_X)$ and $\mathfrak{B}_{Y|X}^{(n)}$ is the right $\mathcal{D}_X$-Module $\mathcal{H}_Y^d(\Omega_X^n)$. Since $\mathcal{T}or_i^{\mathcal{D}}(\mathfrak{B}_{Y|X}^{(n)}, \mathcal{O}_X) = \boldsymbol{C}_Y$ for $i = n-d$ and 0 for $i \neq n-d$, there is a cup product

$$(5.5) \qquad \begin{aligned} &\mathcal{E}xt_{\mathcal{D}}^i(\mathfrak{M}, \mathcal{O}_X)_x \times \mathcal{T}or_{n+i-d}(\mathfrak{B}_{Y|X}^{(n)}, \mathfrak{M})_x \\ &\longrightarrow \mathcal{T}or_{n-d}(\mathfrak{B}_{Y|X}^{(n)}, \mathcal{O}_X)_x = \boldsymbol{C}. \end{aligned}$$

Theorem (5.2) *Let $X = \bigcup X_\alpha$ be a regular stratification with respect to $\mathfrak{M}$. Let X_α be a stratum of codimension d and x be a point in X_α. Then, there is a canonical perfect pairing*

$$(5.6) \qquad \mathcal{E}xt^i_{\mathcal{D}}(\mathfrak{M}, \mathcal{O})_x \times \mathcal{E}xt^{d-i}_{\mathcal{D}}(\mathfrak{M}^*, \mathfrak{B}_{X_\alpha|X})_x \longrightarrow C$$

that is, the two cohomologies are dual to each other.

Proof. Since $\mathcal{E}xt^{d-i}(\mathfrak{M}^*; \mathfrak{B}_{X_\alpha|X}) = \mathcal{T}or_{n-d+i}(\mathfrak{B}^{(n)}_{X_\alpha|X}; \mathfrak{M})$, the pairing (5.6) is induced from (5.5). Let Z be a d-dimensional submanifold transversal to X_α and through x. Then by Theorem (3.8)

$$\mathcal{E}xt^i_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{O}_X)_x = \mathcal{E}xt^i_{\mathcal{D}_Z}(\mathfrak{M}_Z, \mathcal{O}_Z)_x$$

$$\mathcal{E}xt^{d-i}_{\mathcal{D}_X}(\mathfrak{M}^*, \mathfrak{B}_{X_\alpha|X})_x = \mathcal{E}xt^{d-i}_{\mathcal{D}_Z}(\mathfrak{M}^*_Z; \mathfrak{B}_{X_\alpha \cap Z|Z})_x.$$

Therefore the theorem is a corollary of Proposition (5.1). Q.E.D.

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