Publ. RIMS, Kyoto Univ.
10 (1975), 619-668

Boundary Value Problems with Oblique Derivative

By

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§0. Introduction

Let Ω be a bounded open set in $\mathbf{R}^n$ with a smooth boundary Γ. Let us consider the boundary value problem

$$
\text{(0.1)} \qquad \begin{cases} A(x, D_x)u=f & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu}=g & \text{on} \quad \Gamma, \end{cases}
$$

where $A(x, D_x)$ is a second-order elliptic differential operator in $\bar{\Omega}$, and ν is a smooth non-vanishing real vector field on Γ. When ν is nowhere tangent to Γ, the problem (0.1) is, so-called, of coercive type and satisfactory results are obtained (e.g., see [7]).

Egorov and Kondrat'ev in [2] have considered (0.1) when ν is tangent to Γ on its submanifold Γ_0, and have classified the problem into three cases in the following way.

First class: ν leaves Ω through Γ_0;

Second class: ν enters Ω through Γ_0;

Third class: ν neither leaves nor enters Ω through Γ_0,

(for details, see [2] or §1 of our paper). In the first class the problem (0.1) has an infinite-dimensional kernel. Therefore, adding the Dirichlet condition $u|_{\Gamma_0}$ to (0.1), they have shown that the problem

$$
\begin{cases} A(x, D_x)u=f & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu}=g & \text{on} \quad \Gamma, \\ u=h_0 & \text{on} \quad \Gamma_0 \end{cases}
$$

Communicated by S. Matsuura, June 10, 1974.
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becomes Noetherian (cf. §4 of our paper). But they have not mentioned the solvability for the second class, and their method shown there concerning the solvability for the first class does not work for the second and third classes.

In the present paper we assume that $A(x, D_x)$ is strongly elliptic and that the vector field ν is tangent to Γ of finite order, and we shall study mainly, for each class, the unique solvability of the problem with a parameter $\mu(\geqq 0)$:

$$(0.2)\qquad \begin{cases} A(x, D_x)u+\mu^2 u=f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu}=g & \text{on } \Gamma. \end{cases}$$

However, we add the Dirichlet condition $u|_{\Gamma_0}$ to (0.2) in the first class as well as Egorov and Kondrat'ev have done and the coboundary condition $B_\mu(\rho\otimes\delta_{\Gamma_0})$ in the second class, because the problem (0.2) has an infinite-dimensional kernel in the first class and an infinite-dimensional cokernel in the second class (see §4 and §5). In order to solve uniquely (0.2), we construct the similar regularizer to that of Agranovich-Višik [1] by modifying the method in Višik-Grušin [10]. In short, their method can be stated as follows. Let $\mathscr{P}$ denote the Poisson operator of the Dirichlet problem

$$\begin{cases} A(x, D_x)u=0 & \text{in } \Omega, \\ u=h & \text{on } \Gamma. \end{cases}$$

Then, $T\colon h\mapsto\dfrac{\partial}{\partial \nu}(\mathscr{P}h)|_\Gamma$ is an operator acting on Γ, and the solvability of the problem (0.1) can be reduced to that of T.

Èskin [3], Višik-Grušin [10], etc. have considered more general boundary value problems than ours, and have stated that the problems are Noetherian. However, they have not studied the unique solvability. Maz'ja [8] has studied the unique solvability of the similar problem to ours by the method different from ours. His results imply that there exists a unique solution u of (0.2) for any (f, g) in some spaces, but the mapping $u\mapsto(f, g)$ is not continuous. In our paper we show that the mapping $u\mapsto(f, g)$ is a topological isomorphism between two spaces with some appropriate norms when μ is sufficiently large.

Finally, I would like to thank Professor H. Kumano-go, who has given me many useful pieces of advice.

§1. Preliminaries

Let $\mathbf{R}^n$ be the n-dimensional Euclidean space and $\mathbf{R}^n_+$ be the half space $\{x=(x_1,\dots,x_n)\in\mathbf{R}^n;\ x_n>0\}$. $C^m(G)$ denotes the set of functions continuously differentiable in G of order m $(m=0,1,\dots)$. $C^m_0(G)$ is the set of functions of $C^m(G)$ with compact supports. $\mathscr{B}^m(\mathbf{R}^n)$ is the of set $\{u\in C^m(\mathbf{R}^n);\ \sup_{x\in\mathbf{R}^n}|D^\alpha u(x)|<+\infty$ for $|\alpha|\leqq m\}$. $\mathscr{S}=\mathscr{S}(\mathbf{R}^n)$ is the space rapidly decreasing functions, and $\mathscr{S}'$ is its dual space. $\hat{f}(\xi)=\mathscr{F}[f]$ denotes the Fourier transform of $f(x)$, which is defined by

$$\hat{f}(\xi)=\int e^{-ix\cdot\xi}f(x)dx,\qquad f\in\mathscr{S}.$$

The inverse Fourier transform $\mathscr{F}^{-1}[f]$ is expressed by

$$\mathscr{F}^{-1}[f](x)=\int e^{i\xi\cdot x}f(\xi)đ\xi,\qquad f\in\mathscr{S}$$

where $đ\xi=\left(\frac{1}{2\pi}\right)^n d\xi$. For a multi-index $\alpha=(\alpha_1,\dots,\alpha_n)$ (α_j is a non negative integer), we set

$$|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n,\qquad \xi^\alpha=\xi_1^{\alpha_1}\xi_2^{\alpha_2}\dots\xi_n^{\alpha_n},$$

$$D_x^\alpha=(-i)^{|\alpha|}\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}.$$

We define the operator $p(x,D_x)$ by

$$p(x,D_x)f(x)=\mathscr{F}^{-1}_{\xi\to x}[p(x,\xi)\hat{f}(\xi)]$$

for a function $p(x,\xi)$ on $\mathbf{R}^n_x\times\mathbf{R}^n_\xi$, and call $p(x,\xi)$ the symbol of $p(x,D_x)$. We denote the Sobolev space by $H_s(G)$ where $s\in\mathbf{R}$ and G is an open set in $\mathbf{R}^n$. That is, $H_s(\mathbf{R}^n)=\{f\in\mathscr{S}';\ (1+|D_x|^2)^{\frac{s}{2}}f\in L^2(\mathbf{R}^n)\}$ and $H_s(G)$ is the restriction of $H_s(\mathbf{R}^n)$ to G. $\|f\|_{s,G}$ denotes the norm of $H_s(G)$.

Now, let Ω be a bounded open set in $\mathbf{R}^n$ $(n\geqq3)$ with a connected C^∞ smooth boundary Γ. We assume that Γ is separated into two connected components Γ_-, Γ_+ by an $(n-2)$-dimensional C^∞ submanifold Γ_0. Let v be a C^∞ smooth non-vanishing real vector field. We assume that *v is tangent to Γ just on Γ_0 and not tangent to Γ_0 there.* We denote by Γ_+ the part of the two components which is on the positive side for the direction v on Γ_0, and the other by Γ_- (see Figure (1)). We decompose v into the two components:

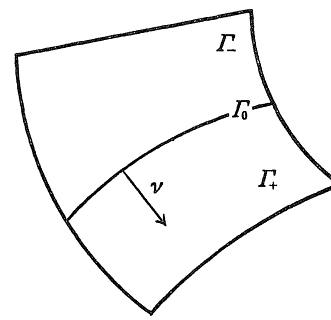

Figure (1)

$$v = v_t + v_n \tag{1.1}$$

where v_t is tangent to Γ and v_n is perpendicular to Γ (the interior direction is positive). Then we see that three cases are possible. Namely,

First class: v_n is positive in Γ_- and negative in Γ_+;
Second class: v_n is negative in Γ_- and positive in Γ_+;
Third class: v_n is positive or negative in both Γ_- and Γ_+, (cf. Egorov-Kondrat'ev [2]).

Proposition 1.1. *For any point $x_0 \in \Gamma_0$, there exists a C^∞ diffeomorphism Φ (local coordinates) defined in a neighborhood $U(x_0)$ of x_0 satisfying the following four conditions. Set $\Phi(x) = (t, y, z) = (t, y_1, \dots, y_{n-2}, z)$.*

(1) *$U(x_0)$ is transformed to an open ball of $\mathbf{R}^n$, and x_0 to the origin.*

(2) *$U(x_0) \cap \Gamma$ is transformed to the surface given by the equation* $z=0$.

(3) *$U(x_0) \cap \Gamma_0$ is transformed to the surface given by the equation* $t=0$, $z=0$.

(4) *$v_t(x)$ $(x \in U(x_0))$ is transformed to $\left(\overrightarrow{\frac{\partial}{\partial t}}\right)$, and the positive normal vector of Γ (near x_0) is transformed to $\left(\overrightarrow{\frac{\partial}{\partial z}}\right)$.*

The above diffeomorphism Φ transforms $\frac{\partial}{\partial v}$ to

$$\frac{\partial}{\partial v'} \equiv \frac{\partial}{\partial t} + v'_n(t, y)\frac{\partial}{\partial z},$$

where $v'_n(t, y)$ is a real valued C^∞-function defined near the origin. In this paper we assume that *the vector field* v *is tangent to* Γ *of finite order* k *at every point of* Γ_0. That is, for any point $(0, y, 0) \in \Phi(U(x_0) \cap \Gamma_0)$ we have

$$v'_n(0, y) = \frac{\partial v'_n}{\partial t}(0, y) = \cdots = \frac{\partial^{k-1} v'_n}{\partial t^{k-1}}(0, y) = 0 \quad \text{and} \quad \frac{\partial^k v'_n}{\partial t^k}(0, y) \neq 0$$

where k is a constant positive integer independent of the choice of x_0 and Φ. Then we obtain

Proposition 1.2. *The following* (1), (2) *and* (3) *are equivalent to the fact that* v *is of first, second and third class respectively.*

(1) k *is odd and* $\frac{\partial^k v'_n}{\partial t^k}(0, y) < 0$;

(2) k *is odd and* $\frac{\partial^k v'_n}{\partial t^k}(0, y) > 0$;

(3) k *is even.*

From now on, we shall study the boundary value problem

$$(1.2) \qquad \begin{cases} A(x, D_x)u + \mu^2 u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial v} = g & \text{on } \Gamma. \end{cases}$$

Here μ is a parameter ($\geqq 0$), and $A(x, D_x)$ is a second order differential operator in Ω with coefficients belonging to $C^\infty(\bar{\Omega})$ ($\bar{\Omega}$ is the closure of Ω) and independent of μ. We assume that *there exists a positive constant* δ *such that*

$$\mathrm{Re}\,[A_0(x, \xi)] \geqq \delta |\xi|^2$$

holds for every $\xi \in \mathbf{R}^n$ *and every* $x \in \bar{\Omega}$ where $A_0(x, \xi)$ is the principal symbol of $A(x, D_x)$.

Let $A'(t, y, z; D_t, D_y, D_z)$ be the transformed operator of $A(x, D_x)$ by the diffeomorphism Φ stated in Proposition 1.1, and we denote its principal symbol by $A_0'(t, y, z; \tau, \eta, \omega)$. We denote by $r_{(t,y,z)}(\tau, \eta, \mu)$ the root of the quadratic equation in ω

$$A_0'(t, y, z; \tau, \eta, \omega)+\mu^2=0$$

with a positive imagenary part. We set

$$N_\mu=\{u\in H_0(\mathbf{R}^n_+);\ A_0'(0; D_t, D_y, D_z)u+\mu^2u=0\}\quad (\mu>0).$$

Proposition 1.3. *Suppose* $s\geqq 0, \mu>0$ *and* $u\in N_\mu$, *and put* $h(t, y)=u|_{z=0}$. *Then* u *belongs to* $H_s(\mathbf{R}^n_+)$ *if and only if* $h\in H_{s-\frac{1}{2}}(\mathbf{R}^{n-1})$, *and we can express it uniquely by the form*

$$u(t, y, z)=\mathscr{F}^{-1}_{(\tau,\eta)\to(t,y)}[e^{ir_0(\tau,\eta,\mu)z}\hat{h}(\tau, \eta)]\,. \tag{1.3}$$

Conversely, if we define $u(t, y, z)$ *by* (1.3) *for* $h\in H_{s-\frac{1}{2}}(\mathbf{R}^{n-1})$, *then* $u(t, y, z)$ *belongs to* $H_s(\mathbf{R}^n_+)\cap N_\mu$.

Let us set

$$\frac{\partial u}{\partial \nu_0}=\left(\frac{\partial u}{\partial t}+\frac{1}{k!}\ \frac{\partial^k \nu_n'}{\partial t^k}(0)t^k\frac{\partial u}{\partial z}\right)\Big|_{z=0}.$$

From Proposition 1.3 we get

$$\frac{\partial u}{\partial \nu_0}=\frac{\partial}{\partial t}(u|_{z=0})+n_0t^k\mathscr{F}^{-1}_{(\tau,\eta)\to(t,y)}[ir_0(\tau, \eta, \mu)\widehat{u|_{z=0}}(\tau, \eta)]\,,$$

$$\left(n_0=\frac{1}{k!}\ \frac{\partial^k \nu_n'}{\partial t^k}(0)\right)$$

for any $u\in N_\mu$ $(\mu>0)$. The operator

$$T:\ u|_{z=0}\longmapsto\frac{\partial u}{\partial \nu_0} \tag{1.4}$$

(where $u\in N_\mu$ and $\mu>0$) is defined on $H_{-\frac{1}{2}}(\mathbf{R}^{n-1})$, and is a pseudo-differential operator with the symbol

$$T_\mu(t, \tau, \eta)=i\tau+n_0t^kir_0(\tau, \eta, \mu)\,.$$

Setting

$$P_\mu(t, \tau, \eta)=i\tau+n_0t^kir_0(0, \eta, \mu)\,,$$

we reduce the solvability of the problem (1.2) to the solvability and the estimates for the operator $P_\mu(t, D_t, D_y)$ on $\mathbf{R}^{n-1}$.

In §2 we fix μ and η, and investigate the ordinary differential operator $P_\mu(t, D_t, \eta)$. In §3, §4 and §5 we consider the problem (1.2) in the case of third, first and second class respectively. Finally in Appendix we summarize several elementary theorems for the pseudo-differential operator with a parameter which we often use in this paper.

Remark 1.1. Hereafter, whenever we consider $P_\mu(t, \tau, \eta)$, we assume that there exist positive constants M_1, M_2 independent of the choice of x_0 and Φ in Proposition 1.1 such that

$$M_1 \leqq |\mathrm{Re}\, a|, \qquad |a| \leqq M_2 \tag{1.5}$$

where $a = \dfrac{n_0 i r_0(0, \eta, \mu)}{(|\mu|^2 + \mu^2)^{\frac{1}{2}}}$ (, which is possible since Γ_0 is compact). Then we can take the constant in the a priori estimate for $P_\mu(t, D_t, D_y)$ which is independent of the choice of Φ (e.g., see Theorem 3.1). Furthermore, we can assume that the constants in estimates for the operators transformed by Φ are all independent of the choice of Φ (e.g., see Lemma 3.3).

§2. Basic Theorems for an Ordinary Differential Operator

In this section we consider an ordinary differential operator $\dfrac{d}{dt} + at^k$. The theorems below play a basic role in the following sections.

Let us set

$$p(t, D_t) = \frac{d}{dt} + at^k\,,$$

where k is a positive integer and the coefficient a is a constant satisfying $\mathrm{Re}\, a \neq 0$. Noting Proposition 1.2 and $\mathrm{Re}[ir_0(0, \eta, \mu)] < 0$, we see easily that the following case (2.1), (2.2) and (2.3) correspond to the first, the second and the third class respectively.

(2.1) $\quad k$ is odd and $\mathrm{Re}\, a > 0$;

(2.2) $\quad k$ is odd and $\mathrm{Re}\, a < 0$;

(2.3) k is even.

For $s\in\mathbf{R}$ we define

$$W_s^{(k)}(\mathbf{R}^1)=\{v(t)\in H_s(\mathbf{R}^1)\,;\ t^k v\in H_{s-1}(\mathbf{R}^1)\}\,,$$

$$\|v\|_{W_s^{(k)}}=(\|v\|^2_{s,\mathbf{R}^1}+\|t^k v\|^2_{s-1,\mathbf{R}^1})^{\frac{1}{2}}. \tag{2.4}$$

Lemma 2.1. i) $p(t, D_t)$: $W_s^{(k)}(\mathbf{R}^1)\to H_{s-1}(\mathbf{R}^1)$ *is continuous and Noetherian (Fredholm type).*

ii) *Suppose that* $v(t)\in\mathscr{S}'$ *and* $p(t, D_t)v\in\mathscr{S}$, *then* $v\in\mathscr{S}$. *The same statement is obtained for the formally adjoint* $p^{(*)}(t, D_t)=-\frac{d}{dt}+\bar{a}t^k$.

iii) *Let* K^*_{s-1} *denote the cokernel of* $p(t, D_t)$: $W_s^{(k)}(\mathbf{R}^1)\to H_{s-1}(\mathbf{R}^1)$. *Then we have*

$$K^*_{s-1}=(1+D_t^2)^{-s+1}\Big(\{v\in\mathscr{S}'(\mathbf{R}^1);\ p^{(*)}(t, D_t)v=0\}\Big).$$

Proof. We refer the proofs of i) and ii), for example, to Grušin [4], [5]. So we shall prove only iii). From i) we have the orthogonal decomposition of $H_{s-1}(\mathbf{R}^1)$:

$$H_{s-1}(\mathbf{R}^1)=p(t, D_t)W_s^{(k)}(\mathbf{R}^1)\oplus K^*_{s-1}\,.$$

Since $(1+D_t^2)^{\frac{s-1}{2}}: H_{s-1}(\mathbf{R}^1)\to L^2(\mathbf{R}^1)$ is isometric, we get the orthogonal decomposition of $L^2(\mathbf{R}^1)$:

$$L^2(\mathbf{R}^1)=(1+D_t^2)^{\frac{s-1}{2}}\circ p(t, D_t)W_s^{(k)}\oplus(1+D_t^2)^{\frac{s-1}{2}}K^*_{s-1}\,.$$

Hence it follows that

$$(1+D_t^2)^{\frac{s-1}{2}}K^*_{s-1}=\{v\in L^2(\mathbf{R}^1)\,;\,p^{(*)}(t, D_t)\circ(1+D_t^2)^{\frac{s-1}{2}}v=0\}\,.$$

Therefore, from ii) we have

$$K^*_{s-1}=(1+D_t^2)^{-s+1}\Big(\{v\in\mathscr{S}';\ p^{(*)}(t, D_t)v=0\}\Big).$$

Theorem 2.1. *Let* $s\in\mathbf{R}$, *and suppose that* k *is even and* $\mathrm{Re}\,a\neq 0$. *Then, for the operator*

$$p(t, D_t)\colon W_s^{(k)}(\mathbf{R}^1)\longrightarrow H_{s-1}(\mathbf{R}^1)\,,$$

the kernel and the cokernel are both $\{0\}$, *and we have the estimate*

$$C^{-1}\|v\|_{W_s^{(k)}} \leqq \|p(t, D_t)v\|_{s-1,\mathbf{R}^1} \leqq C\|v\|_{W_s^{(k)}}, \qquad v \in W_s^{(k)}(\mathbf{R}^1).$$

Here the constant C *depends only on positive numbers* M_1, M_2 *when* $M_1 \leqq |\mathrm{Re}\, a|$, $|a| \leqq M_2$.

Proof. If $v \in W_s^{(k)}(\mathbf{R}^1)$ and $p(t, D_t)v=0$, then $v \in C^\infty(\mathbf{R}^1)$ and

$$v(t) = v(0)\exp\left(-\frac{a}{k+1}\, t^{k+1}\right).$$

From the hypotheses, $v(t)$ belongs to $W_s^{(k)}(\mathbf{R}^1)$ if and only if $v(0)=0$. Therefore the kernel is $\{0\}$. By iii) of Lemma 2.1, in the same way we see that the cokernel is $\{0\}$.

It is easy to see that $\|p(t, D_t)v\|_{s-1,\mathbf{R}^1} \leqq C\|v\|_{W_s^{(k)}}$. So we indicate only that

$$\|v\|_{W_s^{(k)}} \leqq C\|p(t, D_t)v\|_{s-1,\mathbf{R}^1}. \tag{2.5}$$

Since the kernel and the cokernel are $\{0\}$, (2.5) holds at any fixed a by the Banach theorem. Fix a_0 such that $M_1 \leqq |\mathrm{Re}\, a_0|$, $|a_0| \leqq M_2$, and assume that

$$\|v\|_{W_s^{(k)}} \leqq C_0\left\|\frac{dv}{dt} + a_0 t^k v\right\|_{s-1,\mathbf{R}^1}.$$

Then we get

$$\|v\|_{W_s^{(k)}} \leqq 2C_0\left\|\frac{dv}{dt} + a t^k v\right\|_{s-1,\mathbf{R}^1}$$

if $|a-a_0|$ is sufficiently small. Noting that the set $\{a \in \mathbf{C};\, M_1 \leqq |\mathrm{Re}\, a|, |a| \leqq M_2\}$ is compact, we can take the constant C in (2.5) independently of a provided that $M_1 \leqq |\mathrm{Re}\, a|$, $|a| \leqq M_2$. The theorem is proved.

Theorem 2.2. *Let* $s \in \mathbf{R}$, *and suppose that* k *is odd and* $\mathrm{Re}\, a > 0$. *Then, for the operator*

$$p(t, D_t)\colon W_s^{(k)}(\mathbf{R}^1) \longrightarrow H_{s-1}(\mathbf{R}^1), \tag{2.6}$$

the kernel K *is one-dimensional and the cokernel is* $\{0\}$.

Corollary. *For* $s\geqq 1$ *we define the operator*

$$\mathfrak{A}_0:\ W_s^{(k)}(\mathbf{R}^1)\longrightarrow H_{s-1}(\mathbf{R}^1)\times\mathbf{C} \tag{2.7}$$

by $\mathfrak{A}_0(v)=(p(t, D_t)v, v(0))$. *Then the kernel and the cokernel are both* $\{0\}$, *and we have the estimate*

$$C^{-1}\|v\|_{W_s^{(k)}}\leqq\|p(t, D_t)v\|_{s-1,\mathbf{R}^1}+|v(0)|\leqq C\|v\|_{W_s^{(k)}},\qquad v\in W_s^{(k)}(\mathbf{R}^1).$$

Here the constant C *depends only on positive numbers* M_1, M_2 *when* $M_1\leqq\operatorname{Re}a,\ |a|\leqq M_2$.

We can show the theorem in the same way as in Theorem 2.1, and have

$$K=\left\{v(t);\ v(t)=\rho_0\exp\left(-\frac{a}{k+1}t^{k+1}\right),\ \rho_0\in\mathbf{C}\right\}. \tag{2.8}$$

Proof of the corollary. From (2.8) the kernel of (2.7) is $\{0\}$. We take $(f, \rho_0)\in H_{s-1}(\mathbf{R}^1)\times\mathbf{C}$ arbitrarily. Since the cokernel of (2.6) is $\{0\}$, we can find an element $w(t)\in W_s^{(k)}(\mathbf{R}^1)$ such that $p(t, D_t)w=f$. Set

$$v(t)=w(t)+(\rho_0-w(0))\exp\left(-\frac{a}{k+1}t^{k+1}\right),$$

then it follows that $v\in W_s^{(k)}(\mathbf{R}^1)$ and $\mathfrak{A}_0(v)=(f, \rho_0)$. Hence the cokernel of (2.7) is $\{0\}$. The estimate can be obtained in the same way as in Theorem 2.1. The corollary is proved.

Theorem 2.3. *Let* $s\in\mathbf{R}$ *and suppose that* k *is odd and* $\operatorname{Re}a<0$. *Then, for the operator*

$$p(t, D_t):\ W_s^{(k)}(\mathbf{R}^1)\longrightarrow H_{s-1}(\mathbf{R}^1),$$

the kernel is $\{0\}$ *and the cokernel* K_{s-1}^* *is one-dimensional.*

Corollary. *Let* $g_0(t)$ *be a given element of* $H_{s-1}(\mathbf{R}^1)$ *not orthogonal to* K_{s-1}^* *in* $H_{s-1}(\mathbf{R}^1)$, *and we define the operator*

$$\pi:\ W_s^{(k)}(\mathbf{R}^1)\times\mathbf{C}\longrightarrow H_{s-1}(\mathbf{R}^1) \tag{2.9}$$

by $\pi(v, \rho_0)=p(t, D_t)v+\rho_0 g_0(t)$. *Then the kernel and the cokernel are both* $\{0\}$, *and we have the estimate*

$$C^{-1}(\|v\|_{W_s^{(k)}}+|\rho_0|)\leqq\|\pi(v,\ \rho_0)\|_{s-1,\mathbf{R}^1}\leqq C(\|v\|_{W_s^{(k)}}+|\rho_0|),$$
$$(v,\ \rho_0)\in W_s^{(k)}(\mathbf{R}^1)\times\mathbf{C}.$$

We can show the theorem in the same way as in Theorem 2.1.

Proof of the corollary. Let $(f, g)_{s-1}$ denote the inner product $\int(1+\tau^2)^{s-1}\hat{f}(\tau)\overline{\hat{g}(\tau)}d\tau$ and $e_{s-1}(t)$ be a base of K_{s-1}^* satisfying $(e_{s-1}, e_{s-1})_{s-1}=1$. Suppose that $\pi(v, \rho_0)=0$ where $(v, \rho_0)\in W_s^{(k)}(\mathbf{R}^1)\times\mathbf{C}$, then

$$\rho_0=0,\qquad v=0,$$

because $g_0(t)$ is not orthogonal to $e_{s-1}(t)$. That is, the kernel of (2.9) is $\{0\}$.

We decompose $H_{s-1}(\mathbf{R}^1)$ into K_{s-1}^* and its orthogonal complement $(K_{s-1}^*)^{\perp}$. Noting that

$$f-(f,\ e_{s-1})_{s-1}e_{s-1},\quad g_0-(g_0,\ e_{s-1})_{s-1}e_{s-1}\in(K_{s-1}^*)^{\perp},$$

we can find $v\in W_s^{(k)}(\mathbf{R}^1)$ such that

$$p(t,\ D_t)v=\{f-(f,\ e_{s-1})_{s-1}e_{s-1}\}-\frac{(f,\ e_{s-1})_{s-1}}{(g_0,\ e_{s-1})_{s-1}}\{g_0-(g_0,\ e_{s-1})_{s-1}e_{s-1}\}.$$

Here, put $\rho_0=\frac{(f,\ e_{s-1})_{s-1}}{(g_0,\ e_{s-1})_{s-1}}$, then

$$\pi(v,\ \rho_0)=f.$$

Hence the cokernel of (2.9) is $\{0\}$. The estimate can be obtained in the same way as in Theorem 2.1. The corollary is proved.

Remark 2.1. We have proved Theorem 2.1, 2.2 and 2.3 (also Lemma 3.1, 4.1 and 5.1) in the same way as in Grušin [4], [5], Višik-Grušin [10]. Otherwise, representing the solution of $p(t, D_t)v=f$:

$$v(t)=\{v(0)+\int_0^t\exp\Big(\frac{a}{k+1}\sigma^{k+1}\Big)f(\sigma)d\sigma\}\exp\Big(-\frac{a}{k+1}t^{k+1}\Big),$$

we can verify them as well as Èskin [3] has done.

At the end of this section we shall state an interpolational inequality which is often used later.

Proposition 2.1. *Let λ be a parameter (>0) and s be a real number satisfying $|s|\leqq s_0$. Suppose that k is a positive integer. Then, for any positive number ε we have*

$$(2.10)\qquad \|(\tau^2+\lambda^2)^{\frac{1}{2}(s+j)}D_\tau^j w(\tau)\|_{L_\tau^2}^2 \leqq \varepsilon\|(\tau^2+\lambda^2)^{\frac{1}{2}(s+k)}D_\tau^k w\,(\tau)\|_{L_\tau^2}^2 + C\|(\tau^2+\lambda^2)^{\frac{s}{2}}w(\tau)\|_{L_\tau^2}^2,\quad w\in\mathscr{S}(\mathbf{R}^1)$$

where $j=0, 1,\dots, k-1$. Here, the constant C depends only on s_0, k and ε.

Proof. We shall prove the proposition by the induction with respect to k. Set $\chi(\tau, \lambda)=(\tau^2+\lambda^2)^{\frac{1}{2}}$. When $k=1$, (2.10) is trivial. Suppose that (2.10) is proved when $k=k_0$, then for any ε_1 (>0) we get

$$(2.11)\quad \|\chi(\tau, \lambda)^{s+j}D_\tau^j w\|_{L^2}^2 \leqq \varepsilon_1\|\chi^{s+k_0+1}D_\tau^{k_0+1}w\|_{L^2}^2 + C(\varepsilon_1, k_0, s_0)\|\chi^{s+1}D_\tau w\|_{L^2}^2$$

where $j=1, 2,\dots, k_0$. By partial integration,

$$\left|\int\chi^{2s+2}D_\tau w\overline{D_\tau w}d\tau\right| \leqq \left|\int\chi^{2s+2}D_\tau^2 w\bar{w}d\tau\right| + 2(s_0+1)\left|\int\chi^{2s+1}D_\tau w\,\bar{w}d\tau\right|.$$

Using this inequality and (2.11) where $\varepsilon_1=1$, we have

$$\|\chi^{s+1}D_\tau w\|_{L^2}^2 \leqq \|\chi^{s+k_0+1}D_\tau^{k_0+1}w\|_{L^2}\|\chi^s w\|_{L^2} + C_1(k_0, s_0)\|\chi^{s+1}D_\tau w\|_{L^2}\|\chi^s w\|_{L^2}.$$

Hence, for any ε_2 (>0) independent of ε_1, it follows that

$$\|\chi^{s+1}D_\tau w\|_{L^2}^2 \leqq \varepsilon_2\|\chi^{s+k_0+1}D_\tau^{k_0+1}w\|_{L^2}^2 + C_2(\varepsilon_2, k_0, s_0)\|\chi^s w\|_{L^2}^2.$$

Combining this inequality and (2.11), we see that (2.10) is valid when $k=k_0+1$. Therefore, the proposition is proved.

§3. The Third Class

In this section we shall consider the problem (1.2) where the vector field v is of third class. In this case we obtain the same results as in the coercive case by introducing the weighted Sobolev space (see Theorem 3.2).

We set (for $g \in H_s(G)$)

$$|||g|||_{s,G} = \|g\|_{s,G} + \mu^s \|g\|_{0,G}, \tag{3.1}$$

where $s \geqq 0$ and μ is a parameter ($\geqq 0$). Agranovich and Višik in [1] have used the norms of this type. The interpolational inequality

$$\mu^{s-j}\|g\|_{j,G} \leqq C(s)(\|g\|_{s,G} + \mu^s\|g\|_{0,G}) = C(s)|||g|||_{s,G} \tag{3.2}$$

is obtained where $0 \leqq j \leqq s$ and the constant $C(s)$ does not depend on μ. The similar inequality

$$\mu^{s-j}\|g\|_{j,G} \leqq C'(s)(\|g\|_{s,G} + \mu^{s+1}\|g\|_{-1,G}) \tag{3.3}$$

is also valid where $-1 \leqq j \leqq s$ and the constant $C'(s)$ does not depend on μ. We define

$$H_s^{(k)}(\mathbf{R}^{n-1}) = \{h(t, y) \in H_{s-1}(\mathbf{R}^{n-1});\ D_t h \in H_{s-1}(\mathbf{R}^{n-1}),$$
$$(1+|D_y|)t^k h \in H_{s-1}(\mathbf{R}^{n-1})\},$$

$$\|h\|_{s,\mathbf{R}^{n-1}}^{(k)} = \|D_t h\|_{s-1,\mathbf{R}^{n-1}} + \|\,|D_y|t^k h\|_{s-1,\mathbf{R}^{n-1}} + \mu\|t^k h\|_{s-1,\mathbf{R}^{n-1}}, \tag{3.4}$$

$$|||h|||_{s,\mathbf{R}^{n-1}}^{(k)} = \|h\|_{s,\mathbf{R}^{n-1}}^{(k)} + \mu^{s-1}\|h\|_{1,\mathbf{R}^{n-1}}^{(k)}$$

(where $s \geqq 1$ and $\mu > 0$). When μ is fixed, obviously the norms $\|h\|_{s,\mathbf{R}^{n-1}}^{(k)}$ and $|||h|||_{s,\mathbf{R}^{n-1}}^{(k)}$ are equivalent to the norm $\|D_t h\|_{s-1,\mathbf{R}^{n-1}} + \|(1+|D_y|)\cdot t^k h\|_{s-1,\mathbf{R}^{n-1}} + \|h\|_{s-1,\mathbf{R}^{n-1}}$, which gives the topology to $H_s^{(k)}(\mathbf{R}^{n-1})$.

First, we investigate the operator $P_\mu(t, D_t, D_y)$:

Theorem 3.1. *Let* $s \geqq 1$ *and* $\mu \geqq \mu_0$ (μ_0 *is an arbitrary positive constant*). *We have*

i) $C^{-1}|||h|||_{s,\mathbf{R}^{n-1}}^{(k)} \leqq |||P_\mu(t, D_t, D_y)h|||_{s-1,\mathbf{R}^{n-1}} \leqq C|||h|||_{s,\mathbf{R}^{n-1}}^{(k)}$, $h \in H_s^{(k)}(\mathbf{R}^{n-1})$,

where the constant C *does not depend on* μ.

ii) *For the operator*

$$P_\mu(t, D_t, D_y)\colon H_s^{(k)}(\mathbf{R}^{n-1}) \longrightarrow H_{s-1}(\mathbf{R}^{n-1}),$$

the kernel and the cokernel are both $\{0\}$.

Corollary. $P_\mu(t, D_t, D_y)$ ($\mu > 0$) *has the inverse* $G^1\colon H_0(\mathbf{R}^{n-1}) \longrightarrow H_1^{(k)}(\mathbf{R}^{n-1})$. *Furthermore, if* $g \in H_{s-1}(\mathbf{R}^{n-1})$ ($s \geqq 1$), *then* $G^1 g \in H_s^{(k)}(\mathbf{R}^{n-1})$, *and we have the estimate*

$$|||G^1 g|||^{(k)}_{s,\mathbf{R}^{n-1}} \leqq C|||g|||_{s-1,\mathbf{R}^{n-1}}.$$

The corollary is clear from the theorem.

Before proving the theorem, we verify

Lemma 3.1. *Set*

$$p_\lambda(t, D_t) = \frac{d}{dt} + a\lambda t^k,$$

where λ is a parameter (>0), k is even and a satisfies $(0<)M_1 \leqq |\mathrm{Re}\, a|$ and $|a| \leqq M_2$. Then, for a real number s there is a constant C independent of a and λ such that

$$\begin{aligned} &C^{-1}\{\|(D_t^2+\lambda^{\frac{2}{k+1}})^{\frac{s}{2}} D_t v\|^2_{0,\mathbf{R}^1} + \|(D_t^2+\lambda^{\frac{2}{k+1}})^{\frac{s}{2}} \lambda t^k v\|^2_{0,\mathbf{R}^1}\} \\ (3.5)\quad &\leqq \|(D_t^2+\lambda^{\frac{2}{k+1}})^{\frac{s}{2}} p_\lambda(t, D_t) v\|^2_{0,\mathbf{R}^1} \leqq C(\|(D_t^2+\lambda^{\frac{2}{k+1}})^{\frac{s}{2}} D_t v\|^2_{0,\mathbf{R}^1} \\ &+ \|(D_t^2+\lambda^{\frac{2}{k+1}})^{\frac{s}{2}} \lambda t^k v\|^2_{0,\mathbf{R}^1}), \quad v \in \mathscr{S}(\mathbf{R}^1). \end{aligned}$$

Proof. By the change of variable: $t = \lambda^{-\frac{1}{k+1}} t'$, we have

$$\|(D_t^2+\lambda^{\frac{2}{k+1}})^{\frac{s}{2}} p_\lambda(t, D_t) v\|^2_{0,\mathbf{R}^1} = \lambda^{\frac{2s+1}{k+1}} \|p_1(t, D_t) v(\lambda^{-\frac{1}{k+1}} t)\|^2_{s,\mathbf{R}^1},$$

$$\|(D_t^2+\lambda^{\frac{2}{k+1}})^{\frac{s}{2}} D_t v\|^2_{0,\mathbf{R}^1} = \lambda^{\frac{2s+1}{k+1}} \|D_t(v(\lambda^{-\frac{1}{k+1}} t))\|^2_{s,\mathbf{R}^1},$$

$$\|(D_t^2+\lambda^{\frac{2}{k+1}})^{\frac{s}{2}} \lambda t^k v\|^2_{0,\mathbf{R}^1} = \lambda^{\frac{2s+1}{k+1}} \|t^k v(\lambda^{-\frac{1}{k+1}} t)\|^2_{s,\mathbf{R}^1}.$$

Therefore, noting that the norm $\|v\|_{W^{(k)}_{s+1}}$ (see (2.4)) is equivalent to $(\|D_t v\|^2_{s,\mathbf{R}^1} + \|t^k v\|^2_{s,\mathbf{R}^1})^{\frac{1}{2}}$, we obtain (3.5) by Theorem 2.1.

Proof of Theorem 3.1. Let us prove i). From the definitions (3.1) and (3.4), it suffices to indicate that the inequality

$$\begin{aligned} &C^{-1}\iint(\tau^2+|\eta|^2+\mu^2)^{s-1}\{|\widehat{D_t h}(\tau, \eta)|^2 + (|\eta|^2+\mu^2)|\widehat{t^k h}(\tau, \eta)|^2\} d\tau d\eta \\ (3.6)\quad &\leqq \iint(\tau^2+|\eta|^2+\mu^2)^{s-1}|\mathscr{F}[P_\mu h](\tau, \eta)|^2 d\tau d\eta \leqq C\iint(\tau^2+|\eta|^2+\mu^2)^{s-1} \\ &\times\{|\widehat{D_t h}(\tau, \eta)|^2 + (|\eta|^2+\mu^2)|\widehat{t^k h}(\tau, \eta)|^2\} d\tau d\eta \end{aligned}$$

holds for $h(t, y) \in \mathscr{S}$ where the constant C does not depend on $\mu(\geqq \mu_0)$. We put

$$\lambda = (|\eta|^2 + \mu^2)^{\frac{1}{2}} \quad \text{and} \quad p_\lambda(t, D_t) = P_\mu(t, D_t, \eta)$$

(note that (1.5) is satisfied), and apply Lemma 3.1. Then we have the inequalities:

$$C_1^{-1} \int (\tau^2 + (|\eta|^2 + \mu^2)^{\frac{1}{k+1}})^{s-1} \{|\widehat{D_t h}(\tau, \eta)|^2 + (|\eta|^2 + \mu^2)|\widehat{t^k h}(\tau, \eta)|^2\} d\tau$$

$$\leqq \int (\tau^2 + (|\eta|^2 + \mu^2)^{\frac{1}{k+1}})^{s-1} |\mathscr{F}[P_\mu h](\tau, \eta)|^2 d\tau$$

$$\leqq C_1 \int (\tau^2 + (|\eta|^2 + \mu^2)^{\frac{1}{k+1}})^{s-1} \{|\widehat{D_t h}(\tau, \eta)|^2 + (|\eta|^2 + \mu^2)|\widehat{t^k h}(\tau, \eta)|^2\} d\tau,$$

$$C_2^{-1} (|\eta|^2 + \mu^2)^{s-1} \int \{|\widehat{D_t h}(\tau, \eta)|^2 + (|\eta|^2 + \mu^2)|\widehat{t^k h}(\tau, \eta)|^2\} d\tau$$

$$\leqq (|\eta|^2 + \mu^2)^{s-1} \int |\mathscr{F}[P_\mu h](\tau, \eta)|^2 d\tau$$

$$\leqq C_2 (|\eta|^2 + \mu^2)^{s-1} \int \{|\widehat{D_t h}(\tau, \eta)|^2 + (|\eta|^2 + \mu^2)|\widehat{t^k h}(\tau, \eta)|^2\} d\tau,$$

where the constants C_1 and C_2 are independent of η and $\mu\ (\geqq \mu_0)$. Therefore, since $C_3^{-1}(\tau^2 + |\eta|^2 + \mu^2)^{s-1} \leqq (\tau^2 + (|\eta|^2 + \mu^2)^{\frac{1}{k+1}})^{s-1} + (|\eta|^2 + \mu^2)^{s-1} \leqq C_3(\tau^2 + |\eta|^2 + \mu^2)^{s-1}$ $(\mu \geqq \mu_0)$, there is a constant C_4 independent of η and $\mu(\geqq \mu_0)$ such that

$$C_4^{-1} \int (\tau^2 + |\eta|^2 + \mu^2)^{s-1} \{|\widehat{D_t h}(\tau, \eta)|^2 + (|\eta|^2 + \mu^2)|\widehat{t^k h}(\tau, \eta)|^2\} d\tau$$

$$\leqq \int (\tau^2 + |\eta|^2 + \mu^2)^{s-1} |\mathscr{F}[P_\mu h](\tau, \eta)|^2 d\tau$$

$$\leqq C_4 \int (\tau^2 + |\eta|^2 + \mu^2)^{s-1} \{|\widehat{D_t h}(\tau, \eta)|^2 + (|\eta|^2 + \mu^2)|\widehat{t^k h}(\tau, \eta)|^2\} d\tau.$$

Thus we get (3.6).

Next we show ii). It is clear from i) that the kernel is $\{0\}$. Let us take $g(t, y) \in H_{s-1}(\mathbf{R}^{n-1})$ arbitrarily. Then, by Theorem 2.1 we can find $v_\eta(t)$ for almost every η such that

$$P_\mu(t, D_t, \eta)v_\eta(t) = \mathscr{F}_{y\to\eta}[g](t, \eta).$$

Set

$$h(t, y) = \mathscr{F}^{-1}_{\eta\to y}[v_\eta(t)],$$

then $h(t, y)$ belongs to $H_s^{(k)}(\mathbf{R}^{n-1})$ and satisfies

$$P_\mu(t, D_t, D_y)h = g.$$

Hence the cokernel is $\{0\}$. The theorem is proved.

Let $d(x)$ be the distance between x and Γ_0. Let $\alpha(x)$ $(\in C^\infty(\bar{\Omega}))$ satisfy $\alpha(x)=1$ near Γ_0 and $\text{supp}(\alpha)$ be sufficiently small. We introduce the space

$$H_l^{(k)}(\Omega) = \Big\{u \in H_{l-1}(\Omega)\,;\ \frac{\partial}{\partial v_t}(\alpha u) \in H_{l-1}(\Omega),\ d(x)^k D_x^\gamma(\alpha u) \in L^2(\Omega) \quad \text{for } |\gamma| \leqq l, \quad (1-\alpha)u \in H_l(\Omega)\Big\}$$

($l = 0, 1, 2, \dots$; v_t is defined in (1.1)). Clearly this space depends on the vector field v. Fixing $\alpha(x)$, we employ the following norms of $H_l^{(k)}(\Omega)$:

$$\|u\|_{l,\Omega}^{(k)} = \Big\|\frac{\partial}{\partial v_t}(\alpha u)\Big\|_{l-1,\Omega} + \sum_{|\gamma|\leqq l}\|d(x)^k D_x^\gamma(\alpha u)\|_{0,\Omega} + \|(1-\alpha)u\|_{l,\Omega} + \|u\|_{l-1,\Omega},$$

$$|||u|||_{l,\Omega}^{(k)} = \|u\|_{l,\Omega}^{(k)} + \mu^l \|u\|_{0,\Omega}^{(k)}.$$

Remark 3.1. We have

$$H_l(\Omega) \subset H_l^{(k)}(\Omega) \subset H_{l-\frac{k}{k+1}}(\Omega),$$

and if $u \in H_l^{(k)}(\Omega)$ and $\text{supp}(u) \cap \Gamma_0 = \phi$, then $u \in H_l(\Omega)$.

We denote the Sobolev space on the manifold M by $H_s(M)$ $(s \in \mathbf{R})$ and its norm by $\|g\|_{s,M}$. We set

$$|||g|||_{s,M} = \|g\|_{s,M} + \mu^s \|g\|_{0,M} \qquad (s \geqq 0).$$

The following theorem implies that the problem (1.2) is uniquely solvable for large μ.

Theorem 3.2. *Let l be an integer $\geqq 2$. We obtain*

i) *Estimate*: a) *We have*

$$|||A(x, D_x)u+\mu^2 u|||^{(k)}_{l-2,\Omega}+\left|\left|\left|\frac{\partial u}{\partial \nu}\right|\right|\right|_{l-\frac{3}{2},\Gamma}\leqq C_1|||u|||^{(k)}_{l,\Omega}, \quad u\in H^{(k)}_l(\Omega)$$

where the constant C_1 does not depend on μ.

b) *There is a constant μ_1 such that provided $\mu\geqq\mu_1$,*

$$|||u|||^{(k)}_{l,\Omega}\leqq C_2\left(|||A(x, D_x)u+\mu^2 u|||^{(k)}_{l-2,\Omega}+\left|\left|\left|\frac{\partial u}{\partial \nu}\right|\right|\right|_{l-\frac{3}{2},\Gamma}\right), \; u\in H^{(k)}_l(\Omega)$$

where the constant C_2 does not depend on μ.

ii) *Solvability*: *There is a constant μ_2 independent of l such that if $\mu\geqq\mu_2$, a solution u of* (1.2) *is found in $H^{(k)}_l(\Omega)$ for any $(f, g)\in H^{(k)}_{l-2}(\Omega)\times H_{l-\frac{3}{2}}(\Gamma)$.*

iii) *Regularity*: *We fix $\mu(\geqq 0)$ in* (1.2) *arbitrarily. Suppose that u is a solution of* (1.2) *in $H^{(k)}_2(\Omega)$ for $(f, g)\in H^{(k)}_{l-2}(\Omega)\times H_{l-\frac{3}{2}}(\Gamma)$, then u belongs to $H^{(k)}_l(\Omega)$.*

Remark 3.2. The regularity follows so long as $A(x, D_x)$ is elliptic. Furthermore, the problem is Noetherian in the above spaces (cf. Višik-Grušin [10]).

To begin with, we shall present several lemmas and propositions.

Let $x_j (j=1,\dots, N)$ be points in Γ_0 and fix a diffeomorphism Φ stated in Proposition 1.1 for each x_j under (1.5). Let $\{\varphi_j\}_{j=1,\dots,N}$ denote a partition of unity near Γ_0, and assume that each $\operatorname{supp}(\varphi_j)$ is sufficiently small and contains x_j. For the function $f(x)$ we set

$$f'(t, y, z)=f(\Phi^{-1}(t, y, z)).$$

For a non-negative integer l we define

$$H^{(k)}_l(\mathbf{R}^n_+)=\{u(t, y, z)\in H_{l-1}(\mathbf{R}^n_+);\ D_t u\in H_{l-1}(\mathbf{R}^n_+),\ (t^k+iz^k)u\in H_l(\mathbf{R}^n_+)\},$$

$$\|u\|^{(k)}_{l,\mathbf{R}^n_+}=\|D_t u\|_{l-1,\mathbf{R}^n_+}+\|(t^k+iz^k)u\|_{l,\mathbf{R}^n_+},$$

$$|||u|||^{(k)}_{l,\mathbf{R}^n_+}=\|u\|^{(k)}_{l,\mathbf{R}^n_+}+\mu^l\|u\|^{(k)}_{0,\mathbf{R}^n_+}.$$

(The norm $|||u|||^{(k)}_{l,\mathbf{R}^n_+}$ differs from (3.4)). Obviously $\|u\|^{(k)}_{l,\mathbf{R}^n_+}$ is equivalent to the norm $\|D_t u\|_{l-1,\mathbf{R}^n_+}+\|(t^k+iz^k)u\|_{l,\mathbf{R}^n_+}+\|u\|_{l-1,\mathbf{R}^n_+}$, which gives the topology to $H_l^{(k)}(\mathbf{R}^n_+)$. From Lemma 3.2 below it is easily seen that $H_l^{(k)}(\Omega)$ is equal to the set

$$\{u\in H_{l-1}(\Omega);\ (1-\sum_j\varphi_j)u\in H_l(\Omega),\ (\varphi_j u)'(t,\ y,\ z)\in H_l^{(k)}(\mathbf{R}^n_+)\ \text{for any } j\}.$$

Therefore, if the partition of unity $\{\varphi_j\}_{j=1,\dots,N}$ is fixed, the norm $\|u\|^{(k)}_{l,\Omega}$ is equivalent to

$$\|(1-\sum_j\varphi_j)u\|_{l,\Omega}+\sum_j\|(\varphi_j u)'(t,\ y,\ z)\|^{(k)}_{l,\mathbf{R}^n_+}.$$

Hereafter, we make $\{\varphi_j\}_{j=1,\dots,N}$ fine enough to have the later statements, and define the norms $\|u\|^{(k)}_{l,\Omega}$, $|||u|||^{(k)}_{l,\Omega}$ by

$$\|(1-\sum_j\varphi_j)u\|_{l,\Omega}+\sum_j\|(\varphi_j u)'(t,\ y,\ z)\|^{(k)}_{l,\mathbf{R}^n_+},$$

$$|||(1-\sum_j\varphi_j)u|||_{l,\Omega}+\sum_j|||(\varphi_j u)'(t,\ y,\ z)|||^{(k)}_{l,\mathbf{R}^n_+} \tag{3.7}$$

respectively. Similarly, assume that the norm $|||g|||_{s,\Gamma}$ is defined with a sufficiently fine partition of unity.

Remark 3.3. Using Lemma 3.2 below, we have easily for any $\alpha\,(|\alpha|\leqq l)$

$$|||D^\alpha_{(t,y,z)}u|||^{(k)}_{l-|\alpha|,\mathbf{R}^n_+}\leqq C_1|||u|||^{(k)}_{l,\mathbf{R}^n_+}.$$

So it follows that

$$|||D^\alpha_x u|||^{(k)}_{l-|\alpha|,\Omega}\leqq C_2|||u|||^{(k)}_{l,\Omega}.$$

This implies that the operator D^α_x: $H_l^{(k)}(\Omega)\to H^{(k)}_{l-|\alpha|}(\Omega)$ is continuous.

Proposition 3.1. *Let l be an integer ($\geqq 2$). The trace operator γ: $u(t,\ y,\ z)\mapsto u|_{z=0}$ is a continuous one from $H_l^{(k)}(\mathbf{R}^n_+)$ to $H^{(k)}_{l-\frac{1}{2}}(\mathbf{R}^{n-1})$ and has the estimate*

$$|||\gamma(u)|||^{(k)}_{l-\frac{1}{2},\mathbf{R}^{n-1}}\leqq C|||u|||^{(k)}_{l,\mathbf{R}^n_+}$$

where the constant C does not depend on μ.

Proof. It suffices only to show the estimate for $u \in C_0^\infty(\overline{\mathbf{R}_+^n})$. From (3.4) and (3.2), it follows that

$$|||\gamma(u)|||^{(k)}_{l-\frac{1}{2},\mathbf{R}^{n-1}} \leqq C_1(\|\gamma(D_t u)\|_{l-\frac{3}{2},\mathbf{R}^{n-1}} + \mu^{l-\frac{3}{2}}\|\gamma(D_t u)\|_{0,\mathbf{R}^{n-1}}$$

$$+ \|\gamma(t^k u)\|_{l-\frac{1}{2},\mathbf{R}^{n-1}} + \mu^{l-\frac{1}{2}}\|\gamma(t^k u)\|_{0,\mathbf{R}^{n-1}}).$$

For a positive integer l' we have

$$\|\gamma(u)\|_{l'-\frac{1}{2},\mathbf{R}^{n-1}} + \mu^{l'-\frac{1}{2}}\|\gamma(u)\|_{0,\mathbf{R}^{n-1}} \leqq C_2(\|u\|_{l',\mathbf{R}_+^n} + \mu^{l'}\|u\|_{0,\mathbf{R}_+^n}), \quad u \in C_0^\infty(\overline{\mathbf{R}_+^n})$$

where the constant C_2 does not depend on μ, which is stated in Agranovich-Višik [1]. Therefore,

$$|||\gamma(u)|||^{(k)}_{l-\frac{1}{2},\mathbf{R}^{n-1}} \leqq C_3(\|D_t u\|_{l-1,\mathbf{R}_+^n} + \mu^{l-1}\|D_t u\|_{0,\mathbf{R}_+^n} + \|t^k u\|_{l,\mathbf{R}_+^n} + \mu^l\|t^k u\|_{0,\mathbf{R}_+^n}).$$

Furthermore, using (3.3) and the inequality

$$\|t^k u\|_{l',\mathbf{R}_+^n} \leqq C_4(\|D_t u\|_{l'-1,\mathbf{R}_+^n} + \|(t^k + iz^k)u\|_{l',\mathbf{R}_+^n}) \quad (l'=0, 1,\dots), \tag{3.8}$$

we get $|||\gamma(u)|||^{(k)}_{l-\frac{1}{2},\mathbf{R}^{n-1}} \leqq C|||u|||^{(k)}_{l,\mathbf{R}_+^n}$. The inequality (3.8) is easily obtained by means of the following lemma.

Lemma 3.2. *Let l be a positive integer. The commutator $[D^\alpha_{(t,y,z)}, t^k]$ $(1 \leqq |\alpha| \leqq l)$ has the estimates*:

i) $\|[D^\alpha, t^k]u\|_{0,\mathbf{R}_+^n} \leqq C_1(\|u\|_{l-1,\mathbf{R}_+^n} + \sum_{|\alpha'| \leqq l-1}\|(t^k + iz^k)D^{\alpha'}u\|_{0,\mathbf{R}_+^n})$,

ii) $\|[D^\alpha, t^k]u\|_{0,\mathbf{R}_+^n} \leqq C_2(\|u\|_{l-1,\mathbf{R}_+^n} + \|(t^k + iz^k)u\|_{l-1,\mathbf{R}_+^n})$.

(These estimates are valid also on $\mathbf{R}^n$, and $[D^\alpha, z^k]$ has also the same estimates.)

Noting that

$$\|t^j u\|_{0,\mathbf{R}_+^n} \leqq \|t^k u\|_{0,\mathbf{R}_+^n} + \|u\|_{0,\mathbf{R}_+^n} \qquad (j=1, 2,\dots, k),$$

we can easily prove the lemma.

In order to construct the regularizer of the problem (1.2), we try to solve the problem

$$\begin{cases} A_0'(0; D_t, D_y, D_z)u + \mu^2 u = f & \text{in} \quad \mathbf{R}^n_+, \\ \dfrac{\partial u}{\partial \nu_0} = g & \text{on} \quad \mathbf{R}^{n-1} \end{cases}$$

by the well-known procedure. First, we extend $f(t, y, z)$ to $\mathbf{R}^n$ and, ignoring the boundary value, define a function $v(t, y, z)$ on $\mathbf{R}^n$ satisfying $A_0'(0; D_t, D_y, D_z)v + \mu^2 v = f$ in $\mathbf{R}^n_+$ by means of the Fourier transformation (, which are described in Proposition 3.2 and Lemma 3.3). Consequently we have only to consider the equation

$$\begin{cases} A_0'(0; D_t, D_y, D_z)v' + \mu^2 v' = 0 & \text{in} \quad \mathbf{R}^n_+, \\ \dfrac{\partial v'}{\partial \nu_0} = g' & \text{on} \quad \mathbf{R}^{n-1} \end{cases}$$

$\left(\text{where } g' = g - \dfrac{\partial v}{\partial \nu_0}\right)$. We reduce this equation to the problem on the boundary $\mathbf{R}^{n-1}$ by employing intermediately the Dirichlet problem

$$\begin{cases} A_0'(0; D_t, D_y, D_z)w + \mu^2 w = 0 & \text{in} \quad \mathbf{R}^n_+, \\ w = h & \text{on} \quad \mathbf{R}^{n-1}, \end{cases}$$

(cf. Proposition 1.3).

For a non-negative integer l we define

$$\mathscr{H}_l^{(k)}(\mathbf{R}^n) = \{u(t, y, z) \in H_{l-1}(\mathbf{R}^n);\ D_t u \in H_{l-1}(\mathbf{R}^n), \quad (t^k + iz^k)u \in H_l(\mathbf{R}^n)\},$$

$$\|u\|_{\mathscr{H}_l^{(k)}} = \|D_t u\|_{l-1,\mathbf{R}^n} + \|(t^k + iz^k)u\|_{l,\mathbf{R}^n},$$

$$|||u|||_{\mathscr{H}_l^{(k)}} = \|u\|_{\mathscr{H}_l^{(k)}} + \mu^l \|u\|_{\mathscr{H}_0^{(k)}}.$$

Restrict $u(t, y, z) \in \mathscr{H}_l^{(k)}(\mathbf{R}^n)$ to $\mathbf{R}^n_+$, then $u \in H_l^{(k)}(\mathbf{R}^n_+)$ and $\|u\|^{(k)}_{l,\mathbf{R}^n_+} \leqq \|u\|_{\mathscr{H}_l^{(k)}}$, $|||u|||^{(k)}_{l,\mathbf{R}^n_+} \leqq |||u|||_{\mathscr{H}_l^{(k)}}$.

Proposition 3.2. *Let us define Ef (for $f \in C_0^\infty(\overline{\mathbf{R}^n_+})$) by*

$$Ef(t, y, z) = \begin{cases} f(t, y, z) & (z \geqq 0) \\ \sum_{j=1}^{l_0+1} a_j f(t, y, -jz) & (z < 0) \end{cases}$$

where $\{a_j\}_{j=1,\dots,l_0+1}$ *satisfy* $\sum_{j=1}^{l_0+1} a_j(-j)^l = 1$ *for* $l = -1, 0, 1, \dots, l_0 - 1$.

Then we have

i) *There is a constant* C_1 *depending only on* l_0 *such that*

$$\|Ef\|_{l,\mathbf{R}^n} \leqq C_1 \|f\|_{l,\mathbf{R}^n_+}, \qquad f \in C_0^\infty(\overline{\mathbf{R}^n_+})$$

for $l=-1, 0, 1,\dots, l_0$. *(This estimate holds for* $l=l_1, l_1+1,\dots, l_1+l_0+1$ *if* $\sum_{j=1}^{l_0+1} a_j(-j)^l=1$ *is satisfied for* $l=l_1, l_1+1,\dots, l_1+l_0$. *Here* l_0 *is a negative integer.)*

ii) *There is a constant* C_2 *depending only on* l_0 *such that*

$$|||Ef|||_{\mathscr{H}_l^{(k)}} \leqq C_2 |||f|||^{(k)}_{l,\mathbf{R}^n_+}, \qquad f \in C_0^\infty(\overline{\mathbf{R}^n_+})$$

for $l=0, 1,\dots, l_0$.

Proof. Let us prove only ii). From the definition of $|||\cdot|||_{\mathscr{H}_l^{(k)}}$,

$$|||Ef|||_{\mathscr{H}_l^{(k)}} = \{\|D_tEf\|_{l-1,\mathbf{R}^n} + \mu^l \|D_tEf\|_{-1,\mathbf{R}^n}\}$$

$$+\{\|(t^k+iz^k)Ef\|_{l,\mathbf{R}^n} + \mu^l\|(t^k+iz^k)Ef\|_{0,\mathbf{R}^n}\} \equiv I_1 + I_2 .$$

The estimate in i) yields

$$I_1 \leqq C_3(\|D_tf\|_{l-1,\mathbf{R}^n_+} + \mu^l\|D_tf\|_{-1,\mathbf{R}^n_+}) \leqq C_3 |||f|||^{(k)}_{l,\mathbf{R}^n_+} .$$

It is easily seen that for any α ($|\alpha| \leqq l$)

$$\|(t^k+iz^k)D^\alpha_{(t,y,z)}Ef\|_{0,\mathbf{R}^n} \leqq C_4 \|(t^k+iz^k)D^\alpha_{(t,y,z)}f\|_{0,\mathbf{R}^n_+} .$$

Using the inequality in i), this inequality and Lemma 3.2, we get for any α ($|\alpha| \leqq l$, $1 \leqq l$)

$$\|D^\alpha_{(t,y,z)}((t^k+iz^k)Ef)\|_{0,\mathbf{R}^n} \leqq C_5(\|t^k+iz^k)f\|_{l,\mathbf{R}^n_+} + \|f\|_{l-1,\mathbf{R}^n_+}) .$$

Hence,

$$I_2 \leqq C_6(\|(t^k+iz^k)f\|_{l,\mathbf{R}^n_+} + \mu^l\|(t^k+iz^k)f\|_{0,\mathbf{R}^n_+} + \|D_tf\|_{l-1,\mathbf{R}^n_+})$$

$$\leqq C_6 |||f|||^{(k)}_{l,\mathbf{R}^n_+} .$$

The proof is complete.

From this proposition we can extend E to a continuous operator from $H_l^{(k)}(\mathbf{R}^n_+)$ to $\mathscr{H}_l^{(k)}(\mathbf{R}^n)$ ($l=0, 1,\dots, l_0$).

Lemma 3.3. *Let l be an integer $(\geqq 2)$ and $\mu \geqq \mu_0$ (μ_0 is an arbitrary positive constant). Define Qf (for $f \in H_{l-2}^{(k)}(\mathbf{R}_+^n)$) by*

$$Qf = \mathscr{F}^{-1}\left[\frac{1}{A_0'(0;\tau,\eta,\omega)+\mu^2}\widehat{Ef}(\tau,\eta,\omega)\right].$$

Then we obtain

i) $Qf \in H_l^{(k)}(\mathbf{R}_+^n)$, $\{A_0'(0; D_t, D_y, D_z)+\mu^2\}Qf=f$ *in* $\mathbf{R}_+^n$, *and there is a constant* C_1 *independent of* μ *such that*

$$|||Qf|||_{l,\mathbf{R}_+^n}^{(k)} \leqq C_1 |||f|||_{l-2,\mathbf{R}_+^n}^{(k)}, \qquad f \in H_{l-2}^{(k)}(\mathbf{R}_+^n).$$

ii) *Suppose* $\varphi, \psi \in \mathscr{B}^\infty(\mathbf{R}^n)$ *and* $\operatorname{supp}(\varphi) \cap \operatorname{supp}(\psi) = \phi$, *then we have* $\varphi Q\psi f \in H_{l+1}^{(k)}(\mathbf{R}_+^n)$ *and*

$$|||\varphi Q\psi f|||_{l+1,\mathbf{R}_+^n}^{(k)} \leqq C_2 |||f|||_{l-2,\mathbf{R}_+^n}^{(k)}, \qquad f \in H_{l-2}^{(k)}(\mathbf{R}_+^n)$$

where the constant C_2 *is independent of* μ.

Proof. Let us prove i). In virtue of ii) of Proposition 3.2, we have only to show that the estimate

$$\left|\left|\left|\mathscr{F}^{-1}\left[\frac{\hat{g}(\tau,\eta,\omega)}{A_0'(0;\tau,\eta,\omega)+\mu^2}\right]\right|\right|\right|_{\mathscr{H}_l^{(k)}} \leqq C_3 |||g|||_{\mathscr{H}_{l-2}^{(k)}}, \quad g \in \mathscr{S}(\mathbf{R}^n)$$

holds for a constant C_3 independent of μ $(\geqq \mu_0)$. From the definition of $|||\cdot|||_{\mathscr{H}_l^{(k)}}$,

$$\left|\left|\left|\mathscr{F}^{-1}\left[\frac{\hat{g}(\tau,\eta,\omega)}{A_0'(0;\tau,\eta,\omega)+\mu^2}\right]\right|\right|\right|_{\mathscr{H}_l^{(k)}}$$

$$\leqq C_4(\mu_0)\Big[\Big\{\iiint\Big((\tau^2+|\eta|^2+\omega^2+1)^{l-1}$$

$$+\mu^{2l}(\tau^2+|\eta|^2+\omega^2+1)^{-1}\Big)\frac{|\widehat{D_t g}|^2}{|A_0'+\mu^2|^2}d\tau d\eta d\omega\Big\}^{\frac{1}{2}}$$

$$+\Big\{\iiint(\tau^2+|\eta|^2+\omega^2+\mu^2)^l\Big|D_\tau^k\frac{\hat{g}}{A_0'+\mu^2}\Big|^2 d\tau d\eta d\omega\Big\}^{\frac{1}{2}}$$

$$+\Big\{\iiint(\tau^2+|\eta|^2+\omega^2+\mu^2)^l\Big|D_\omega^k\frac{\hat{g}}{A_0'+\mu^2}\Big|^2 d\tau d\eta d\omega\Big\}^{\frac{1}{2}}\Big]$$

$$\equiv C_4(\mu_0)(I_1+I_2+I_3).$$

Obviously there is a constant C_5 independent of μ such that

$$\left| D_\tau^j \frac{1}{A_0'(0;\tau,\eta,\omega)+\mu^2} \right| \leqq C_5(\tau^2+|\eta|^2+\omega^2+\mu^2)^{\frac{1}{2}(-2-j)},$$
$$(j=0, 1,\dots, k).$$

By this inequality we have

$$I_1 \leqq C_6(\|D_t g\|_{l-3,\mathbf{R}^n} + \mu^{l-2}\|D_t g\|_{-1,\mathbf{R}^n}) \leqq C_6 |||g|||_{\mathscr{H}_{l-2}^{(k)}},$$

$$I_2 \leqq C_7\Big(\sum_{j=0}^{k}\iint d\eta d\omega \int (\tau^2+|\eta|^2+\omega^2+\mu^2)^{l-2-k+j}|D_\tau^j \hat{g}|^2 d\tau\Big)^{\frac{1}{2}}.$$

Furthermore, using Proposition 2.1 and (3.8), we get

$$I_2 \leqq C_8\Big\{\iiint(\tau^2+|\eta|^2+\omega^2+\mu^2)^{l-2}|D_\tau^k \hat{g}|^2 d\tau d\eta d\omega$$
$$+\iiint(\tau^2+|\eta|^2+\omega^2+\mu^2)^{l-2-k}|\hat{g}|^2 d\tau d\eta d\omega\Big\}^{\frac{1}{2}}$$
$$\leqq C_9 |||g|||_{\mathscr{H}_{l-2}^{(k)}}.$$

I_3 is also estimated in the same way. Thus i) is obtained.

Similarly, noting the following fact, we can prove ii). Set

$$\lambda_\mu(\tau,\eta,\omega)=(\tau^2+|\eta|^2+\omega^2+\mu^2)^{\frac{1}{2}},$$

then $\psi'(t,y,z)$ and $\varphi'(t,y,z)\dfrac{1}{A_0'(0;\tau,\eta,\omega)+\mu^2}$ (where $\psi',\varphi'\in\mathscr{B}^\infty(\mathbf{R}^n)$) belong to $S_{\lambda_\mu}^0$ and $S_{\lambda_\mu}^{-2}$ respectively ($S_{\lambda_\mu}^m$ is defined in Appendix). Therefore, if $\operatorname{supp}(\psi')\cap\operatorname{supp}(\varphi')=\phi$, the estimate

$$\|\lambda_\mu(D_t,D_y,D_z)^s\circ\varphi'(A_0'(0,D_t,D_y,D_z)+\mu^2)^{-1}\circ\psi' u\|_{0,\mathbf{R}^n}$$
$$\leqq C_{10}\|\lambda_\mu(D_t,D_y,D_z)^{s'}u\|_{0,\mathbf{R}^n}$$

is derived for any $s, s'\in\mathbf{R}$ from Theorem A.2 and A.3 in Appendix. The lemma is proved.

In view of Proposition 1.3, $u\in N_\mu$ $(\mu>0)$ corresponds to the trace γu one to one. The mapping $\mathscr{P}:\gamma u\mapsto u$ is a continuous operator from $H_{s-\frac{1}{2}}(\mathbf{R}^{n-1})$ $(s\geqq 0)$ to $H_s(\mathbf{R}_+^n)$ ($\mathscr{P}$ is called the Poisson operator). Moreover we have

Lemma 3.4. *Let* $\psi(t)\in C_0^\infty(\mathbf{R}^1)$ *and* $\mu\geqq\mu_0$ (μ_0 *is an arbitrary positive constant*). *Then, for an integer* $l(\geqq 2)$ *there is a constant* C *independent of* μ *such that*

$$|||\psi\mathscr{P}h|||_{l,\mathbf{R}_+^n}^{(k)}\leqq C|||h|||_{l-\frac{1}{2},\mathbf{R}^{n-1}}^{(k)},\qquad h\in H_{l-\frac{1}{2}}^{(k)}(\mathbf{R}^{n-1}).$$

Proof. From the definition of $|||\cdot|||_{l,\mathbf{R}_+^n}^{(k)}$,

$$\begin{aligned}|||\psi\mathscr{P}h|||_{l,\mathbf{R}_+^n}^{(k)}\leqq&\{\|D_t(\psi\mathscr{P}h)\|_{l-1,\mathbf{R}_+^n}+\mu^l\|D_t(\psi\mathscr{P}h)\|_{-1,\mathbf{R}_+^n}\}\\&+\{\|z^k\psi\mathscr{P}h\|_{l,\mathbf{R}_+^n}+\mu^l\|z^k\psi\mathscr{P}h\|_{0,\mathbf{R}_+^n}\}\\&+\{\|t^k\psi\mathscr{P}h\|_{l,\mathbf{R}_+^n}+\mu^l\|t^k\psi\mathscr{P}h\|_{0,\mathbf{R}_+^n}\}\equiv I_1+I_2+I_3.\end{aligned}$$

For any integer s we have later the estimates:

$$\text{(i)}\quad \|\mathscr{P}h\|_{s,\mathbf{R}_+^n}\leqq C_1\|(D_t^2+|D_y|^2+\mu^2)^{\frac{1}{2}(s-\frac{1}{2})}h\|_{0,\mathbf{R}^{n-1}};$$

$$(3.9)\quad\text{(ii)}\quad \|z^k\mathscr{P}h\|_{s,\mathbf{R}_+^n}\leqq C_2\|(D_t^2+|D_y|^2+\mu^2)^{\frac{1}{2}(s-k-\frac{1}{2})}h\|_{0,\mathbf{R}^{n-1}};$$

$$\begin{aligned}\text{(iii)}\quad \|t^k\mathscr{P}h\|_{s,\mathbf{R}_+^n}\leqq C_3\Big(&\|D_t^2+|D_y|^2+\mu^2)^{\frac{1}{2}(s-\frac{1}{2})}(t^kh)\|_{0,\mathbf{R}^{n-1}}\\&+\|(D_t^2+|D_y|^2+\mu^2)^{\frac{1}{2}(s-k-\frac{1}{2})}h\|_{0,\mathbf{R}^{n-1}}\Big),\end{aligned}$$

where the constants $C_1\sim C_3$ do not depend on $\mu\,(\geqq\mu_0)$. Then, the estimate (i) and (ii) yield

$$\begin{aligned}I_1\leqq C_4(&\|D_th\|_{l-\frac{3}{2},\mathbf{R}^{n-1}}+\mu^{l-\frac{3}{2}}\|D_th\|_{0,\mathbf{R}^{n-1}}+\|h\|_{l-\frac{3}{2},\mathbf{R}^{n-1}}\\&+\mu^{l-\frac{3}{2}}\|h\|_{0,\mathbf{R}^{n-1}})\leqq C_5|||h|||_{l-\frac{1}{2},\mathbf{R}^{n-1}}^{(k)},\end{aligned}$$

$$I_2\leqq C_6(\|h\|_{l-k-\frac{1}{2},\mathbf{R}^{n-1}}+\mu^{l-k-\frac{1}{2}}\|h\|_{0,\mathbf{R}^{n-1}})\leqq C_7|||h|||_{l-\frac{1}{2},\mathbf{R}^{n-1}}^{(k)}.$$

Let $\varphi(t)\,(\in C_0^\infty(\mathbf{R}^1))$ satisfy $\varphi(t)=1$ in a neighborhood of $\mathrm{supp}(\psi)$. We have

$$\begin{aligned}I_3\leqq& C_8\{(\|t^k\mathscr{P}(\varphi h)\|_{l,\mathbf{R}_+^n}+\mu^l\|t^k\mathscr{P}h\|_{0,\mathbf{R}_+^n})+\|\psi\mathscr{P}((1-\varphi)h)\|_{l,\mathbf{R}_+^n}\}\\\equiv& C_8(I_4+I_5).\end{aligned}$$

From (iii) of (3.9), it follows that

$$I_4 \leqq C_9 |||h|||^{(k)}_{l-\frac{1}{2},\mathbf{R}^{n-1}}.$$

Furthermore, combining (i) of (3.9) and Lemma 3.5 below, we obtain

$$I_5 \leqq C_{10} \|(D_t^2 + |D_y|^2 + \mu^2)^{\frac{1}{2}(l-\frac{3}{2})} h\|_{0,\mathbf{R}^{n-1}} \leqq C_{11} |||h|||^{(k)}_{l-\frac{1}{2},\mathbf{R}^{n-1}}.$$

Finally, we shall prove the estimates (3.9). There is a constant δ such that

$$\delta(\tau^2 + |\eta|^2 + \mu^2)^{\frac{1}{2}} \leqq \mathrm{Im}\,[r_0(\tau, \eta, \mu)] \leqq |r_0(\tau, \eta, \mu)| \leqq \delta^{-1}(\tau^2 + |\eta|^2 + \mu^2)^{\frac{1}{2}}.$$

Hence, for any non-negative integer s we have

$$\begin{aligned} \|\mathscr{P}h\|_{s,\mathbf{R}^n_+} &\leqq C_{12}\left\{\iiint_{z>0}\Big(|(\tau^2 + |\eta|^2 + \mu^2)^{\frac{s}{2}} e^{ir_0(\tau,\eta,\mu)z} \hat{h}(\tau, \eta)|^2 \right. \\ &\qquad \left. + |D_z^s e^{ir_0(\tau,\eta,\mu)z} \hat{h}|^2\Big) d\tau d\eta dz\right\}^{\frac{1}{2}} \\ &\leqq C_{12} C_{13}\left\{\iint (\tau^2 + |\eta|^2 + \mu^2)^s |\hat{h}|^2 d\tau d\eta \right. \\ &\qquad \left. \times \int_0^{+\infty} \exp[-2\delta(\tau^2 + |\eta|^2 + \mu^2)^{\frac{1}{2}} z] dz\right\}^{\frac{1}{2}} \\ &\leqq C_{14} \|(D_t^2 + |D_y|^2 + \mu^2)^{\frac{1}{2}(s-\frac{1}{2})} h\|_{0,\mathbf{R}^{n-1}}. \end{aligned}$$

For any negative integer s, we obtain

$$\begin{aligned} \|\mathscr{P}h\|_{s,\mathbf{R}^n_+} &= \sup_{\substack{\|\varphi\|_{-s,\mathbf{R}^n_+} < 1 \\ \varphi \in C_0^\infty(\mathbf{R}^n_+)}} |<\mathscr{P}h, \varphi>| = \sup |<\{ir_0(\tau, \eta, \mu)\}^s e^{ir_0 z} \hat{h}, \\ &\qquad \frac{\partial^{-s}}{\partial z^{-s}} \mathscr{F}^{-1}_{(t,y)\to(\tau,\eta)}[\varphi](\tau, \eta, z)>| \\ &\leqq \sup \|\mathscr{P}(r_0(D_t, D_y, \mu)^s h)\|_{0,\mathbf{R}^n_+} \|\varphi\|_{-s,\mathbf{R}^n_+} \\ &\leqq C_{15} \|(D_t^2 + |D_y|^2 + \mu^2)^{\frac{1}{2}(s-\frac{1}{2})} h\|_{0,\mathbf{R}^{n-1}}. \end{aligned}$$

The proof of (ii) is similar. Let us show (iii). We can express

$$\mathscr{F}_{(t,y)\to(\tau,\eta)}[t^k \mathscr{P}h] = \sum_{k_1=0}^{k}\left(\sum_{k_2=0}^{k_1} z^{k_1-k_2} a^{k_1}_{k_2}(\tau, \eta, \mu)\right) e^{ir_0(\tau,\eta,\mu)z} D_\tau^{k-k_1} \hat{h},$$

where $a^{k_1}_{k_2}(\tau, \eta, \mu)$ is homogeneous of order $-k_2$ in (τ, η, μ). Therefore for any non-negative integer s,

$$\|t^k \mathscr{P} h\|_{s,\mathbf{R}^n_+} \leqq C_{16}\left\{\sum_{k_1=0}^{k}\iint(\tau^2+|\eta|^2+\mu^2)^{s-k_1-\frac{1}{2}}|D_\tau^{k-k_1}\hat{h}(\tau, \eta)|^2 d\tau d\eta\right\}^{\frac{1}{2}}.$$

Furthermore, by Proposition 2.1 we have

$$\|t^k \mathscr{P} h\|_{s,\mathbf{R}^n_+} \leqq C_{17}\{\|(D_t^2+|D_y|^2+\mu^2)^{\frac{1}{2}(s-\frac{1}{2})} t^k h\|_{0,\mathbf{R}^{n-1}}$$
$$+\|(D_t^2+|D_y|^2+\mu^2)^{\frac{1}{2}(s-\frac{1}{2}-k)} h\|_{0,\mathbf{R}^{n-1}}\}.$$

For any negative integer, we can obtain the same estimate in the similar way to the proof of (i). The proof is thus complete.

Lemma 3.5. *Let* $\psi(t)$, $\varphi(t)$ $(\in \mathscr{B}^\infty(\mathbf{R}^1))$ *satisfy* $\operatorname{supp}(\psi) \cap \operatorname{supp}(\varphi) = \phi$ *and* $\mu \geqq \mu_0$ (μ_0 *is an arbitrary positive constant*). *Then, for any non-negative integer* s *we have*

$$\|\varphi \mathscr{P}(\psi h)\|_{s,\mathbf{R}^n_+} \leqq C\|(D_t^2+|D_y|^2+\mu^2)^{\frac{1}{2}(s-\frac{3}{2})} h\|_{0,\mathbf{R}^{n-1}}, \quad h \in \mathscr{S}(\mathbf{R}^{n-1}),$$

where the constant C *does not depend on* μ

The lemma is proved in Appendix.

Let us set

$$V^n_{\varepsilon,+} = \{(t, y, z) \in \mathbf{R}^n;\ |t| < \varepsilon,\ |y_i| < \varepsilon,\ 0 < z < \varepsilon\},$$

$$V^{n-1}_\varepsilon = \{(t, y) \in \mathbf{R}^{n-1};\ |t| < \varepsilon,\ |y_i| < \varepsilon\}.$$

Lemma 3.6. *Suppose* $0 < \varepsilon \leqq 1$, $l = 2, 3, \ldots$ *and* $\mu \geqq \mu_0$ (μ_0 *is an arbitrary positive constant*), *then we have*

$$|||u|||^{(k)}_{l,\mathbf{R}^n_+} \leqq C_1\Big(|||A_0'(0; D_t, D_y, D_z)u + \mu^2 u|||^{(k)}_{l-2,\mathbf{R}^n_+}$$
$$+|||\frac{\partial u}{\partial v_0}|||_{l-\frac{3}{2},\mathbf{R}^{n-1}}\Big) + C_2\Big(\varepsilon + \frac{1}{\mu}\Big)|||u|||^{(k)}_{l,\mathbf{R}^n_+}$$

for $u \in H^{(k)}_l(\mathbf{R}^n_+)$ *whose support lies in* $V^n_{\varepsilon,+}$. *Here the constants* C_1 *and* C_2 *are independent of* ε *and* μ.

Proof. We can assume $u \in C_0^\infty(\overline{\mathbf{R}_+^n})$ without loss of generality. Let $\psi(t)$ $(\in C_0^\infty(\mathbf{R}^1))$ satisfy $\psi(t)=1$ in a neighborhood of $\{t; |t| \leqq 1\}$, and set

$$f(t, y, z) = A_0'(0; D_t, D_y, D_z)u + \mu^2 u .$$

We have

$$|||u|||_{l,\mathbf{R}_+^n}^{(k)} \leqq |||\psi Q f|||_{l,\mathbf{R}_+^n}^{(k)} + |||\psi(u - Qf)|||_{l,\mathbf{R}_+^n}^{(k)} \equiv I_1 + I_2 ,$$

(Q is defined in Lemma 3.3). i) of Lemma 3.3 yields

$$I_1 \leqq C_3 |||f|||_{l-2,\mathbf{R}_+^n}^{(k)} .$$

Let us note $u - Qf \in N_\mu$. From Lemma 3.4 and Theorem 3.1 it follows that

$$I_2 \leqq C_4 |||\gamma(u - Qf)|||_{l-\frac{1}{2},\mathbf{R}^{n-1}}^{(k)} \leqq C_5 |||P_\mu(t, D_t, D_y)(\gamma(u - Qf))|||_{l-\frac{3}{2},\mathbf{R}^{n-1}} .$$

Furthermore, since $T_\mu(\gamma(u - Qf)) = \dfrac{\partial}{\partial v_0}(u - Qf)$ (see (1.4)), we have

$$\begin{aligned} I_2 &\leqq C_5\left(\left|\left|\left| \frac{\partial}{\partial v_0}(u - Qf) \right|\right|\right|_{l-\frac{3}{2},\mathbf{R}^{n-1}} + |||(P_\mu - T_\mu)(\gamma(u - Qf))|||_{l-\frac{3}{2},\mathbf{R}^{n-1}} \right) \\ &\equiv C_5(I_3 + I_4). \end{aligned}$$

By i) of Lemma 3.3, it is easily seen that

$$I_3 \leqq C_6\left(\left|\left|\left| \frac{\partial u}{\partial v_0} \right|\right|\right|_{l-\frac{3}{2},\mathbf{R}^{n-1}} + |||f|||_{l-2,\mathbf{R}_+^n}^{(k)} \right).$$

We can write

$$\begin{aligned} &\{P_\mu(t, D_t, D_y) - T_\mu(t, D_t, D_y)\} h(t, y) \\ &\qquad = n_0 t^k \mathscr{F}^{-1}[i\beta(\tau, \eta, \mu)\tau \hat{h}(\tau, \eta)](t, y) \end{aligned} \tag{3.10}$$

where $\beta(\tau, \eta, \mu) = \displaystyle\int_0^1 \frac{\partial r_0}{\partial \tau}(\theta\tau, \eta, \mu) d\theta$, which is homogeneous of order 0 in (τ, η, μ). By this expression and Proposition 2.1, we obtain

$$\begin{aligned} |||(P_\mu - T_\mu)h|||_{l-\frac{3}{2},\mathbf{R}^{n-1}} &\leqq C_7(|||t^k D_t h|||_{l-\frac{3}{2},\mathbf{R}^{n-1}} \\ &\quad + \|(D_t^2 + |D_y|^2 + \mu^2)^{\frac{1}{2}(l-\frac{3}{2}-k)} D_t h\|_{0,\mathbf{R}^{n-1}}). \end{aligned} \tag{3.11}$$

Therefore, from the fact that $\operatorname{supp}(u) \subset V^n_{\varepsilon,+}$, we see that

$$|||(P_\mu - T_\mu)\gamma(u)|||_{l-\frac{3}{2},\mathbf{R}^{n-1}} \leqq C_8\Big(\varepsilon + \frac{1}{\mu}\Big)|||u|||^{(k)}_{l,\mathbf{R}^n_+}.$$

Let $\varphi(t, y, z)\ (\in C_0^\infty(\overline{\mathbf{R}^n_+}))$ satisfy $\varphi(t, y, z)=1$ in a neighborhood of $\overline{V^n_{1,+}}$. Then, using (3.11), Proposition 3.1 and Lemma 3.3, we have

$$\begin{aligned}|||(P_\mu - T_\mu)\gamma(Qf)|||_{l-\frac{3}{2},\mathbf{R}^{n-1}} &\leqq |||(P_\mu - T_\mu)\gamma(\varphi Qf)|||_{l-\frac{3}{2},\mathbf{R}^{n-1}} \\ &\quad + |||(P_\mu - T_\mu)\gamma((1-\varphi)Qf)|||_{l-\frac{3}{2},\mathbf{R}^{n-1}} \\ &\leqq C_9 |||f|||^{(k)}_{l-2,\mathbf{R}^n_+}.\end{aligned}$$

Thus,

$$I_4 \leqq C_{10}|||f|||^{(k)}_{l-2,\mathbf{R}^n_+} + C_8\Big(\varepsilon + \frac{1}{\mu}\Big)|||u|||^{(k)}_{l,\mathbf{R}^n_+}.$$

Therefore the lemma is proved.

Lemma 3.7. *Suppose* $0<\varepsilon\leqq\frac{1}{2}$, $l=2, 3,\dots$ *and* $\mu\geqq\mu_0$ (μ_0 *is an arbitrary positive constant*). *Let* $\psi(t)\ (\in C^\infty(\mathbf{R}^1))$ *satisfy* $\psi(t)=0$ *in* $\{t;\ |t|>2\}$, $\psi(t)=1$ *in a neighborhood of* $\{t;\ |t|\leqq 1\}$ *and* $0\leqq\psi(t)\leqq 1$. *Put*

$$\psi_\varepsilon(t, y, z) = \psi\Big(\frac{t}{\varepsilon}\Big)\psi\Big(\frac{y_1}{\varepsilon}\Big)\dots\psi\Big(\frac{y_{n-2}}{\varepsilon}\Big)\psi\Big(\frac{z}{\varepsilon}\Big). \tag{3.12}$$

We define $R^1(f, g)$ *by*

$$R^1(f, g) = Qf + \psi_1 \mathscr{P} G^1\Big(g - \frac{\partial}{\partial\nu_0}Qf\Big),$$

where $(f, g)\in H^{(k)}_{l-2}(\mathbf{R}^n_+)\times H_{l-\frac{3}{2}}(\mathbf{R}^{n-1})$ *and* $\operatorname{supp}(f)$, $\operatorname{supp}(g)$ *lie in* $V^n_{\varepsilon,+}$, V^{n-1}_ε *respectively*. (G^1 *is defined in Corollary of Theorem* 3.1). *Then we obtain*

i) *There is a constant* C_1 *independent of* ε *and* μ *such that*

$$|||R^1(f, g)|||^{(k)}_{l,\mathbf{R}^n_+} \leqq C_1(|||f|||^{(k)}_{l-2,\mathbf{R}^n_+} + |||g|||_{l-\frac{3}{2},\mathbf{R}^{n-1}}).$$

ii) *Set*

$$S_1^1(f, g)=\Big(A_0'(t, y, z; D_t, D_y, D_z)+\mu^2\Big)\psi_\varepsilon R^1(f, g)-f,$$

$$S_2^1(f, g)=\frac{\partial}{\partial v'}\psi_\varepsilon R^1(f, g)|_{z=0}-g$$

(v' denotes the transformed vector field of v by the diffeomorphism Φ stated in Proposition 1.1). *Then we have*

$$|||S_1^1(f, g)|||_{l-2,\mathbf{R}_+^n}^{(k)}\leqq\Big(\delta_1(\varepsilon)+\frac{C_2(\varepsilon)}{\mu}\Big)(|||f|||_{l-2,\mathbf{R}_+^n}^{(k)}+|||g|||_{l-\frac{3}{2},\mathbf{R}^{n-1}}),$$

$$|||S_2^1(f, g)|||_{l-\frac{3}{2},\mathbf{R}^{n-1}}\leqq\Big(\delta_2(\varepsilon)+\frac{C_3(\varepsilon)}{\mu^{\frac{1}{2}}}\Big)(|||f|||_{l-2,\mathbf{R}_+^n}^{(k)}+|||g|||_{l-\frac{3}{2},\mathbf{R}^{n-1}}),$$

where $\delta_1(\varepsilon), \delta_2(\varepsilon)\to 0$ uniformly in μ as $\varepsilon\to 0$, and the constants $C_2(\varepsilon)$, $C_3(\varepsilon)$ do not depend on μ.

Proof. The proof of i) is similar to that of Lemma 3.6. Let us prove ii). We write

$$\begin{aligned} S_1^1(f, g)=&[A'(t, y, z; D_t, D_y, D_z),\ \psi_\varepsilon]\circ R^1(f, g)\\ &+\psi_\varepsilon\{A'(t, y, z; D_t, D_y, D_z)-A_0'(t, y, z; D_t, D_y, D_z)\}R^1(f, g)\\ &+\psi_\varepsilon\{A_0'(t, y, z; D_t, D_y, D_z)-A_0'(0; D_t, D_y, D_z)\}R^1(f, g)\\ &+\psi_\varepsilon\{(A_0'(0; D_t, D_y, D_z)+\mu^2)R^1(f, g)-f\}\\ \equiv& I_1+I_2+I_3+I_4. \end{aligned}$$

Obviously $I_4=0$. Noting that $[A_0'(t, y, z; D_t, D_y, D_z),\ \psi_\varepsilon]$ and $\{A'(t, y, z; D_t, D_y, D_z)-A_0'(t, y, z; D_t, D_y, D_z)\}$ are first order operators, we have

$$\begin{aligned} |||I_1|||_{l-2,\mathbf{R}_+^n}^{(k)}&\leqq C_4(\varepsilon)|||R^1(f, g)|||_{l-1,\mathbf{R}_+^n}^{(k)}\\ &\leqq\frac{C_5(\varepsilon)}{\mu}(|||f|||_{l-2,\mathbf{R}_+^n}^{(k)}+|||g|||_{l-\frac{3}{2},\mathbf{R}^{n-1}}), \end{aligned}$$

$$\begin{aligned} |||I_2|||_{l-2,\mathbf{R}_+^n}^{(k)}&\leqq C_6(\varepsilon)|||R^1(f, g)|||_{l-1,\mathbf{R}_+^n}^{(k)}\\ &\leqq\frac{C_7(\varepsilon)}{\mu}(|||f|||_{l-2,\mathbf{R}_+^n}^{(k)}+|||g|||_{l-\frac{3}{2},\mathbf{R}^{n-1}}), \end{aligned}$$

by means of (3.8), (3.2) and (3.3). Let A_0' be written in the form

$$A_0'(t, y, z; D_t, D_y, D_z)=\sum_{|\alpha|=2} a_\alpha(t, y, z)D^\alpha_{(t,y,z)}$$

and put

$$\delta_3(\varepsilon)=\sup_{\substack{|\alpha|=2\\(t,y,z)\in V^n_{2\varepsilon,+}}} |a_\alpha(t, y, z)-a_\alpha(0)|,$$

then $\delta_3(\varepsilon)\to 0$ as $\varepsilon\to 0$. From (3.2) and (3.3) it follows that

$$\begin{aligned}|||I_3|||^{(k)}_{l-2,\mathbf{R}^n_+} &\leqq C_8\delta_3(\varepsilon)|||R^1(f, g)|||^{(k)}_{l,\mathbf{R}^n_+}+C_9(\varepsilon)|||R^1(f, g)|||^{(k)}_{l-1,\mathbf{R}^n_+}\\ &\leqq\left(C_{10}\delta_3(\varepsilon)+\frac{C_{11}(\varepsilon)}{\mu}\right)(|||f|||^{(k)}_{l-2,\mathbf{R}^n_+}+|||g|||_{l-\frac{3}{2},\mathbf{R}^{n-1}}).\end{aligned}$$

Therefore, we obtain the estimate for $S_1^1(f, g)$.

Next let us examine $S_2^1(f, g)$.

$$\begin{aligned}|||S_2^1(f, g)|||_{l-\frac{3}{2},\mathbf{R}^{n-1}} &\leqq\left|\left|\left|\left(\frac{\partial}{\partial v'}-\frac{\partial}{\partial v_0}\right)\psi_\varepsilon R^1(f, g)\right|\right|\right|_{l-\frac{3}{2},\mathbf{R}^{n-1}}\\ &\quad+\left|\left|\left|\frac{\partial}{\partial v_0}\psi_\varepsilon R^1(f, g)-g\right|\right|\right|_{l-\frac{3}{2},\mathbf{R}^{n-1}}\equiv I_5+I_6.\end{aligned}$$

We can write $v'(t, y)=\overrightarrow{\left(\frac{\partial}{\partial t}\right)}+v_n'(t, y)\overrightarrow{\left(\frac{\partial}{\partial z}\right)}$. Set

$$\delta_4(\varepsilon)=\sup_{(t,y)\in V^{n-1}_{2\varepsilon}}\left|k\int_0^1\frac{\partial^k v_n'}{\partial t^k}(\theta t, y)(1-\theta)^{k-1}d\theta-\frac{\partial^k v_n'}{\partial t^k}(0)\right|,$$

then $\delta_4(\varepsilon)\to 0$ as $\varepsilon\to 0$. By (3.2),

$$\begin{aligned}I_5 &\leqq\left(\delta_4(\varepsilon)+\frac{C_{12}(\varepsilon)}{\mu^{1/2}}\right)|||t^k D_z(\psi_\varepsilon R^1(f, g))|||_{l-\frac{3}{2},\mathbf{R}^{n-1}}\\ &\leqq C_{13}\left(\delta_4(\varepsilon)+\frac{C_{12}(\varepsilon)}{\mu^{1/2}}\right)(|||f|||^{(k)}_{l-2,\mathbf{R}^n_+}+|||g|||_{l-\frac{3}{2},\mathbf{R}^{n-1}}).\end{aligned}$$

We have

$$I_6\leqq\left|\left|\left|\psi_\varepsilon\frac{\partial}{\partial v_0}Qf-g+\psi_\varepsilon P_\mu(t, D_t, D_y)G^1\left(g-\frac{\partial}{\partial v_0}Qf\right)\right|\right|\right|_{l-\frac{3}{2},\mathbf{R}^{n-1}}$$

$$+\left\|\!\left|\frac{\partial\psi_\varepsilon}{\partial\nu_0}R^1(f,g)\right|\!\right\|_{l-\frac{3}{2},\mathbf{R}^{n-1}}+\left\|\!\left|\psi_\varepsilon(T_\mu-P_\mu)G^1\Big(g-\frac{\partial}{\partial\nu_0}Qf\Big)\right|\!\right\|_{l-\frac{3}{2},\mathbf{R}^{n-1}}$$

$$\equiv I_7+I_8+I_9\,.$$

From the definition of G^1, $I_7=0$ follows. It is easily seen that

$$I_8\leqq C_{14}(\varepsilon)|||(t^k+iz^k)R^1(f,g)|||_{l-1,\mathbf{R}^n_+}$$

$$\leqq\frac{C_{15}(\varepsilon)}{\mu}(|||f|||^{(k)}_{l-2,\mathbf{R}^n_+}+|||g|||_{l-\frac{3}{2},\mathbf{R}^{n-1}})\,.$$

By (3.10) we have

$$I_9\leqq\left\|\!\left|\psi_\varepsilon t^k\beta(D_t,D_y,\mu)D_tG^1\Big(g-\frac{\partial}{\partial\nu_0}Qf\Big)\right|\!\right\|_{l-\frac{3}{2},\mathbf{R}^{n-1}}$$

$$\leqq\Big(\varepsilon C_{16}+\frac{C_{17}(\varepsilon)}{\mu^{1/2}}\Big)\left\|\!\left|G^1\Big(g-\frac{\partial}{\partial\nu_0}Qf\Big)\right|\!\right\|^{(k)}_{l-\frac{1}{2},\mathbf{R}^{n-1}}$$

$$\leqq C_{18}\Big(\varepsilon C_{16}+\frac{C_{17}(\varepsilon)}{\mu^{1/2}}\Big)(|||f|||^{(k)}_{l-2,\mathbf{R}^n_+}+|||g|||_{l-\frac{3}{2},\mathbf{R}^{n-1}})\,.$$

Therefore, we get the estimate for $S^1_2(f,g)$. The proof is thus complete.

Proof of Theorem 3.2. In view of Remark 3.3, it is easy to show a) of i). Let us prove b) of i). Let $\varphi(x)=\sum_{j=1}^N\varphi_j(x)$ be the partition of unity in (3.7) and satisfy $0\leqq\varphi_j(x)\leqq 1$ for any j. Set $\varepsilon=\max_j\{\text{diameter}$ $(\operatorname{supp}\varphi_j)\}$. From the definition of $|||\cdot|||^{(k)}_{l,\Omega}$ (see (3.7)),

$$|||u|||^{(k)}_{l,\Omega}=\sum_{j=1}^N|||(\varphi_ju)'(t,y,z)|||^{(k)}_{l,\mathbf{R}^n_+}+|||(1-\varphi)u|||_{l,\Omega}\,.$$

Since ν is not tangent to Γ in $\Gamma-\Gamma_0$, we can use there the method in Agranovich-Višik [1] (see Theorem 4.1 of [1]). Hence,

$$(3.13)\qquad \begin{aligned}|||(1-\varphi)u|||_{l,\Omega}\leqq C_3\Big(&|||(A(x,D_x)+\mu^2)(1-\varphi)u|||_{l-2,\Omega}\\&+\left\|\!\left|\frac{\partial}{\partial\nu}(1-\varphi)u\right|\!\right\|_{l-\frac{3}{2},\Gamma}\Big)+\frac{C_4(\varphi)}{\mu}|||(1-\varphi)u|||_{l,\Omega}\end{aligned}$$

where the constants C_3 and $C_4(\varphi)$ does not depend on μ. By Lemma 3.6 we have

$$
\begin{aligned}
&|||(\varphi_j u)'(t, y, z)|||^{(k)}_{l,\mathbf{R}^n_+} \\
&\leqq C_5\Big(|||(A'_0(0; D_t, D_y, D_z)+\mu^2)(\varphi_j u)'|||^{(k)}_{l-2,\mathbf{R}^n_+} \\
&\quad + \Big|\Big|\Big| \frac{\partial}{\partial \nu_0}(\varphi_j u)' \Big|\Big|\Big|_{l-\frac{3}{2},\mathbf{R}^{n-1}}\Big) + C_6\Big(\varepsilon + \frac{1}{\mu}\Big)|||(\varphi_j u)'|||^{(k)}_{l,\mathbf{R}^n_+} \qquad (3.14)\\
&\leqq C_7 |||\Big(A'(t, y, z; D_t, D_y, D_z)+\mu^2)(\varphi_j u)'|||^{(k)}_{l-2,\mathbf{R}^n_+} \\
&\quad + \Big|\Big|\Big| \frac{\partial}{\partial \nu'}(\varphi_j u)' \Big|\Big|\Big|_{l-\frac{3}{2},\mathbf{R}^{n-1}}\Big) + \Big(\delta(\varepsilon) + \frac{C_8(\varphi)}{\mu^{1/2}}\Big)|||(\varphi_j u)'|||^{(k)}_{l,\mathbf{R}^n_+}
\end{aligned}
$$

where $\delta(\varepsilon)\to 0$ uniformly in μ and the choice of the partition of unity $\sum_{j=1}^{N} \varphi_j$ as $\varepsilon\to 0$. Let the partition of unity $\sum_{j=1}^{N} \varphi_j$ be fine enough, then from (3.13) and (3.14) it follows that

$$
\begin{aligned}
|||u|||^{(k)}_{l,\Omega} &\leqq C_9\Big(|||A(x, D_x)u+\mu^2 u|||^{(k)}_{l-2,\Omega} + \Big|\Big|\Big| \frac{\partial u}{\partial \nu}\Big|\Big|\Big|_{l-\frac{3}{2},\Gamma}\Big) \\
&\quad + \frac{C_{10}}{\mu^{1/2}}|||u|||^{(k)}_{l,\Omega},
\end{aligned}
$$

which proves b) of i).

Next let us show ii). In $\Gamma-\Gamma_0$ we can apply the method in Agranovich-Višik [1]. At any point of Γ_0 we have obtained Lemma 3.7. Therefore, we can construct an operator

$$
\mathscr{R}^1 : H^{(k)}_{l-2}(\Omega)\times H_{l-\frac{3}{2}}(\Gamma) \longrightarrow H^{(k)}_l(\Omega)
$$

by the same procedure as in Agranovich-Višik [1] (see Theorem 5.1 of [1]) such that

$$
(3.15)\quad \begin{cases}
|||\mathscr{R}^1(f, g)|||^{(k)}_{l,\Omega} \leqq C_{11}(|||f|||^{(k)}_{l-2,\Omega} + |||g|||_{l-\frac{3}{2},\Gamma}); \\
|||(A(x, D_x)+\mu^2)\mathscr{R}^1(f, g) - f|||^{(k)}_{l-2,\Omega} \\
\qquad\qquad \leqq \Big(\frac{1}{2} + \frac{C_{12}}{\mu}\Big)(|||f|||^{(k)}_{l-2,\Omega} + |||g|||_{l-\frac{3}{2},\Gamma}); \\
\Big|\Big|\Big| \frac{\partial}{\partial \nu}\mathscr{R}^1(f, g) - g\Big|\Big|\Big|_{l-\frac{3}{2},\Gamma} \leqq \Big(\frac{1}{2} + \frac{C_{13}}{\mu^{1/2}}\Big)(|||f|||^{(k)}_{l-2,\Omega} + |||g|||_{l-\frac{3}{2},\Gamma}),
\end{cases}
$$

where the constants C_{12}, C_{13} do not depend on μ. We define the opeator $\mathfrak{S}^1(f, g)$ in $H_{l-2}^{(k)}(\Omega) \times H_{l-\frac{3}{2}}(\Gamma)$ by

$$\mathfrak{S}^1(f, g) = \left((A+\mu^2)\mathscr{R}^1(f, g) - f, \frac{\partial}{\partial \nu}\mathscr{R}^1(f, g)|_\Gamma - g\right).$$

From (3.15) there is a constant $\mu_2'(l)$ such that for any $\mu \geqq \mu_2'(l)$ the operator

$$I + \mathfrak{S}^1 : H_{l-2}^{(k)}(\Omega) \times H_{l-\frac{3}{2}}(\Gamma) \longrightarrow H_{l-2}^{(k)}(\Omega) \times H_{l-\frac{3}{2}}(\Gamma)$$

has the continuous inverse. Therefore, noting iii), we have ii) when $\mu \geqq \mu_2 \equiv \max(\mu_1, \mu_2'(2))$.

Finally let us prove iii). We fix μ $(\geqq 0)$ arbitrarily, and assume that u is a solution of (1.2) in $H_2^{(k)}(\Omega)$ for $(f, g) \in H_{l-2}^{(k)}(\Omega) \times H_{l-\frac{3}{2}}(\Gamma)$. Then, obviously u is a solution of the equation

$$\begin{cases} A(x, D_x)v + (\mu^2 + \mu_3^2)v = f + \mu_3^2 u & \text{in} \quad \Omega, \\ \dfrac{\partial v}{\partial \nu} = g & \text{on} \quad \Gamma, \end{cases}$$

where μ_3 is sufficiently large. On the other hand, by the proof of ii) we can find a solution v of this equation in $H_{\min(l,4)}^{(k)}(\Omega)$. From b) of i) we have $v = u$. Hence $u \in H_{\min(l,4)}^{(k)}(\Omega)$. By induction, we see that $u \in H_l^{(k)}(\Omega)$. The theorem is proved.

§4. The First Class

In this section we shall consider the problem (1.2) where the vector field ν is of first class. In this case the problem (1.2) has an infinite-dimensional kernel (see Remark 4.2). But, adding the Dirichlet condition $u|_{\Gamma_0}$ to (1.2), we obtain the same results as in the third class (see Theorem 4.2).

To begin with, we shall investigate $P_\mu(t, D_t, D_y)$ as we have done in §3. Theorem 2.2 implies that the operator

$$P_\mu(t, D_t, D_y) : H_s^{(k)}(\mathbf{R}^{n-1}) \longrightarrow H_{s-1}(\mathbf{R}^{n-1})$$

has an infinite-dimensional kernel. But, adding the Dirichlet condition

$h|_{t=0} \equiv \gamma_0(h)$, the kernel and the cokernel are both $\{0\}$ (, which is stated in Theorem 4.1). Therefore, we consider the problem

$$(4.1) \qquad \begin{cases} A(x, D_x)u + \mu^2 u = f & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on} \quad \Gamma, \\ u = h_0 & \text{on} \quad \Gamma_0 \end{cases}$$

instead of (1.2). Then we can repeat the same procedure as in § 3.

Theorem 4.1. *Let* $s \geqq 1$ *and* $\mu \geqq \mu_0$ (μ_0 *is an arbitrary positive constant). We have*

i) $$C^{-1} |||h|||^{(k)}_{s,\mathbf{R}^{n-1}} \leqq |||P_\mu(t, D_t, D_y)h|||_{s-1,\mathbf{R}^{n-1}} + |||\gamma_0(h)|||_{s-1+\frac{1}{2(k+1)},\mathbf{R}^{n-2}_y}$$
$$\leqq C|||h|||^{(k)}_{s,\mathbf{R}^{n-1}}, \qquad h \in H^{(k)}_s(\mathbf{R}^{n-1})$$

where the constant C *does not depend on* μ.

ii) *We define the operator*

$$\mathfrak{A}_1 : H^{(k)}_s(\mathbf{R}^{n-1}) \longrightarrow H_{s-1}(\mathbf{R}^{n-1}) \times H_{s-1+\frac{1}{2(k+1)}}(\mathbf{R}^{n-2}_y)$$

by $\mathfrak{A}_1 h = (P_\mu(t, D_t, D_y)h, \gamma_0(h))$. *Then the kernel and the cokernel are both* $\{0\}$.

Corollary 1. $\mathfrak{A}_1$ *has the inverse* $G^2 : H_0(\mathbf{R}^{n-1}) \times H_{\frac{1}{2(k+1)}}(\mathbf{R}^{n-2}) \longrightarrow H^{(k)}_1(\mathbf{R}^{n-1})$. *Furthermore if* $(g, h_0) \in H_{s-1}(\mathbf{R}^{n-1}) \times H_{s-1+\frac{1}{2(k+1)}}(\mathbf{R}^{n-1})$, *then* $G^2(g, h_0) \in H^{(k)}_s(\mathbf{R}^{n-1})$ *and we have the estimate*

$$|||G^2(g, h_0)|||^{(k)}_{s,\mathbf{R}^{n-1}} \leqq C(|||g|||_{s-1,\mathbf{R}^{n-1}} + |||h_0|||_{s-1+\frac{1}{2(k+1)},\mathbf{R}^{n-2}}).$$

Corollary 2. *The trace operator* $\gamma_0 : H^{(k)}_s(\mathbf{R}^{n-1}) \to H_{s-1+\frac{1}{2(k+1)}}(\mathbf{R}^{n-2}_y)$ *is continuous and has the estimate*

$$|||\gamma_0(h)|||_{s-1+\frac{1}{2(k+1)},\mathbf{R}^{n-2}} \leqq C|||h|||^{(k)}_{s,\mathbf{R}^{n-1}}.$$

The corollaries are clear from the theorem.

Proof of the theorem. From the following Lemma 4.1, we derive

$$C_1^{-1}\int(\tau^2+|\eta|^2+\mu^2)^{s-1}\{|\widehat{D_t h}(\tau,\eta)|^2+(|\eta|^2+\mu^2)|\widehat{t^k h}(\tau,\eta)|^2\}d\tau$$

$$\leqq\int(\tau^2+|\eta|^2+\mu^2)^{s-1}|\mathscr{F}[P_\mu h](\tau,\eta)|^2 d\tau$$

$$+(|\eta|^2+\mu^2)^{s-1+\frac{1}{2(k+1)}}|\mathscr{F}_{y\to\eta}[h](0,\eta)|^2$$

$$\leqq C_1\int(\tau^2+|\eta|^2+\mu^2)^{s-1}\{|\widehat{D_t h}(\tau,\eta)|^2+(|\eta|^2+\mu^2)|\widehat{t^k h}(\tau,\eta)|^2\}d\tau$$

where the constant C_1 is independent of η and $\mu\,(\geqq\mu_0)$. Therefore, i) is proved in the same way as in Theorem 3.1. By Corollary of Theorem 2.2 we can show ii) in the same fashion as in Theorem 3.1.

Lemma 4.1. *Set*

$$p_\lambda(t,D_t)=\frac{d}{dt}+a\lambda t^k$$

where λ is a parameter (>0), k is odd and a satisfies $(0<)M_1\leqq\mathrm{Re}\,a$, $|a|\leqq M_2$. Then, for a real number $s\,(\geqq 0)$ there is a constant C independent of a and λ such that

$$C^{-1}\{\|(D_t^2+\lambda^{\frac{2}{k+1}})^{\frac{s}{2}}D_t v\|_{0,\mathbf{R}^1}^2+\|(D_t^2+\lambda^{\frac{2}{k+1}})^{\frac{s}{2}}\lambda t^k v\|_{0,\mathbf{R}^1}^2\}$$

$$\leqq\|(D_t^2+\lambda^{\frac{2}{k+1}})^{\frac{s}{2}}p_\lambda(t,D_t)v\|_{0,\mathbf{R}^1}^2+\lambda^{\frac{2s+1}{k+1}}|v(0)|^2$$

$$\leqq C\{\|(D_t^2+\lambda^{\frac{2}{k+1}})^{\frac{s}{2}}D_t v\|_{0,\mathbf{R}^1}^2+\|(D_t^2+\lambda^{\frac{2}{k+1}})^{\frac{s}{2}}\lambda t^k v\|_{0,\mathbf{R}^1}^2\},\qquad v\in\mathscr{S}(\mathbf{R}^1).$$

By means of Corollary of Theorem 2.2, the lemma is proved in the same way as in Lemma 3.1.

The following theorem is the main result in this section, which corresponds to Theorem 3.2.

Theorem 4.2. *Let l be an integer $\geqq 2$. We obtain*

i) *Estimate*: a) *We have*

$$|||A(x,D_x)u+\mu^2 u|||_{l-2,\Omega}^{(k)}+\left|\left|\left|\frac{\partial u}{\partial\nu}\right|\right|\right|_{l-\frac{3}{2},\Gamma}+|||u|||_{l-\frac{3}{2}+\frac{1}{2(k+1)},\Gamma_0}$$

$$\leqq C_1|||u|||_{l,\Omega}^{(k)}\quad,\quad u\in H_l^{(k)}(\Omega),$$

where the constant C_1 does not depend on μ.

b) *There is a constant μ_1 such that provided $\mu \geqq \mu_1$,*

$$|||u|||_{l,\Omega}^{(k)} \leqq C_2\Big(|||A(x, D_x)u+\mu^2 u|||_{l-2,\Omega}^{(k)} + \Big|\Big|\Big|\frac{\partial u}{\partial \nu}\Big|\Big|\Big|_{l-\frac{3}{2},\Gamma}$$

$$+ |||u|||_{l-\frac{3}{2}+\frac{1}{2(k+1)},\Gamma_0}\Big), \qquad u \in H_l^{(k)}(\Omega)$$

where the constant C_2 does not depend on μ.

ii) *Solvability: There is a constant μ_2 independent of l such that if $\mu \geqq \mu_2$, a solution u of (4.1) is found in $H_l^{(k)}(\Omega)$ for any $(f, g, h_0) \in H_{l-2}^{(k)}(\Omega) \times H_{l-\frac{3}{2}}(\Gamma) \times H_{l-\frac{3}{2}+\frac{1}{2(k+1)}}(\Gamma_0)$.*

iii) *Regularity: We fix $\mu\,(\geqq 0)$ in (4.1) arbitrarily. Suppose that u is a solution of (4.1) in $H_2^{(k)}(\Omega)$ for $(f, g, h_0) \in H_{l-2}^{(k)}(\Omega) \times H_{l-\frac{3}{2}}(\Gamma) \times H_{l-\frac{3}{2}+\frac{1}{2(k+1)}}(\Gamma_0)$, then u belongs to $H_l^{(k)}(\Omega)$.*

Remark 4.1 The regularity follows so long as $A(x, D_x)$ is elliptic. Furthermore, the problem is Noetherian in the above spaces (cf. Egorov-Kondrat'ev [2], Višik-Grušin [10]).

Remark 4.2. Let us show briefly that the problem (0.1) (and (1.2)) has an infinite-dimensional kernel (where ν is of first class). Set

$$\mathfrak{A}(u) = \Big(A(x, D_x)u, \frac{\partial u}{\partial \nu}\Big|_{\Gamma}, u|_{\Gamma_0}\Big),$$

and assume that $A(x, D_x)$ is elliptic. Then, we know that the operator

$$\mathfrak{A}: H_l^{(k)}(\Omega) \longrightarrow H_{l-2}^{(k)}(\Omega) \times H_{l-\frac{3}{2}}(\Gamma) \times H_{l-\frac{3}{2}+\frac{1}{2(k+1)}}(\Gamma_0)$$

is Noetherian for $l=2, 3,\dots$ (e.g., see [10]). Hence, we have infinite elements $\{h_n\}_{n=1,2,\dots}$ of $H_{l-\frac{3}{2}+\frac{1}{2(k+1)}}(\Gamma_0)$ linearly independent such that there exists a solution $u_n \in H_l^k(\Omega)$ of $\mathfrak{A}(u_n)=(0, 0, h_n)$ for any n. Then $\{u_n\}_{n=1,2,\dots}$ are linearly independent and satisfy

$$\begin{cases} A(x, D_x)u_n = 0 & \text{in } \Omega, \\ \dfrac{\partial u_n}{\partial \nu} = 0 & \text{on } \Gamma, \qquad (n=1, 2,\dots). \end{cases}$$

That is, the problem (0.1) has an infinite-dimensional kernel.

Proof of Theorem 4.2. Noting that from Corollary 2 of Theorem 4.1 the estimate $|||u|||_{l-\frac{3}{2}+\frac{1}{2(k+1)},\Gamma_0} \leqq C|||u|||^{(k)}_{l,\Omega}$ is obtained, we can easily show a) of i). By means of following Lemma 4.2 and Lemma 4.3 (, which correspond to Lemma 3.6 and Lemma 3.7 respectively), we can prove b) of i) and ii) respectively in the same way as in Theorem 3.2. The proof of iii) is also similar to that of iii) of Theorem 3.2.

Lemma 4.2. *Suppose* $0<\varepsilon\leqq 1$, $l=2,3,\ldots$ *and* $\mu\geqq\mu_0$ (μ_0 *is an arbitrary positive constant*), *then we have*

$$|||u|||^{(k)}_{l,\mathbf{R}^n_+} \leqq C_1\left(|||A'_0(0;\, D_t,\, D_y,\, D_z)u+\mu^2 u|||^{(k)}_{l-2,\mathbf{R}^n_+} + \left|\left|\left|\frac{\partial u}{\partial v_0}\right|\right|\right|_{l-\frac{3}{2},\mathbf{R}^{n-1}} \right.$$

$$\left. + |||u|||_{l-\frac{3}{2}+\frac{1}{2(k+1)},\mathbf{R}^{n-2}_y}\right) + C_2\left(\varepsilon+\frac{1}{\mu}\right)|||u|||^{(k)}_{l,\mathbf{R}^n_+}.$$

for $u\in H^{(k)}_l(\mathbf{R}^n_+)$ *whose support lies in* $V^n_{\varepsilon,+}$. *Here the constants* C_1 *and* C_2 *are independent of* ε, μ.

The proof is similar to that of Lemma 3.6.

Lemma 4.3. *Suppose* $0<\varepsilon\leqq\frac{1}{2}$, $l=2, 3,\ldots$ *and* $\mu\geqq\mu_0$ (μ_0 *is an arbitrary positive constant*). *Let* $\psi_\varepsilon(t, y, z)$ *be the function* (3.12). *We define* $R^2(f, g, h_0)$ *by*

$$R^2(f,\, g,\, h_0)=Qf+\psi_1\mathscr{P}G^2\left(g-\frac{\partial}{\partial v_0}Qf,\; h_0-Qf\,|_{\substack{z=0\\t=0}}\right),$$

where $(f, g, h_0)\in H^{(k)}_{l-2}(\mathbf{R}^n_+)\times H_{l-\frac{3}{2}}(\mathbf{R}^{n-1})\times H_{l-\frac{3}{2}+\frac{1}{2(k+1)}}(\mathbf{R}^{n-2}_y)$ *and* $\operatorname{supp}(f)$, $\operatorname{supp}(g)$, $\operatorname{supp}(h_0)$ *lie in* $V^n_{\varepsilon,+}$, V^{n-1}_ε, V^{n-2}_ε *respectively* (G^2 *is defined in Corollary* 1 *of Theorem* 4.1). *Then we have*

i) *There is a constant* C_1 *independent of* ε *and* μ *such that*

$$|||R^2(f,\, g,\, h_0)|||^{(k)}_{l,\mathbf{R}^n_+}\leqq C_1(|||f|||^{(k)}_{l-2,\mathbf{R}^n_+}+|||g|||_{l-\frac{3}{2},\mathbf{R}^{n-1}}$$

$$+|||h_0|||_{l-\frac{3}{2}+\frac{1}{2(k+1)},\mathbf{R}^{n-2}}).$$

ii) *Set*

$$S_1^2(f, g, h_0)=\{A'(t, y, z; D_t, D_y, D_z)+\mu^2\}\psi_\varepsilon R^2(f, g, h_0)-f,$$

$$S_2^2(f, g, h_0)=\frac{\partial}{\partial v'}\psi_\varepsilon R^2(f, g, h_0)|_{z=0}-g,$$

$$S_3^2(f, g, h_0)=\psi_\varepsilon R^2(f, g, h_0)|_{\substack{z=0\\t=0}}-h_0,$$

then we have

$$|||S_1^2(f, g, h_0)|||_{l-2,\mathbf{R}_+^n}^{(k)}\leqq\left(\delta_1(\varepsilon)+\frac{C_2(\varepsilon)}{\mu}\right)(|||f|||_{l-2,\mathbf{R}_+^n}^{(k)}$$

$$+|||g|||_{l-\frac{3}{2},\mathbf{R}^{n-1}}+|||h_0|||_{l-\frac{3}{2}+\frac{1}{2(k+1)},\mathbf{R}^{n-2}}),$$

$$|||S_2^2(f, g, h_0)|||_{l-\frac{3}{2},\mathbf{R}^{n-1}}\leqq\left(\delta_2(\varepsilon)+\frac{C_3(\varepsilon)}{\mu^{1/2}}\right)(|||f|||_{l-2,\mathbf{R}_+^n}^{(k)}$$

$$+|||g|||_{l-\frac{3}{2},\mathbf{R}^{n-1}}+|||h_0|||_{l-\frac{3}{2}+\frac{1}{2(k+1)},\mathbf{R}^{n-2}}),$$

$$S_3^2(f, g, h_0)=0,$$

where $\delta_1(\varepsilon)$, $\delta_2(\varepsilon)\to 0$ *uniformly in* μ *as* $\varepsilon\to 0$, *and the constants* $C_2(\varepsilon)$, $C_3(\varepsilon)$ *do not depend on* μ.

The proof is similar to that of Lemma 3.7.

§5. The Second Class

In this section we shall consider the problem (1.2) where the vector field v is of second class. In this case the problem (1.2) has an infinite-dimensional cokernel (see Remark 5.3). But, adding a coboundary condition (cf. [3], [5], [10]) to (1.2), we obtain the results in Theorem 5.2.

To begin with, we shall investigate $P_\mu(t, D_t, D_y)$ as we have done in §3 and §4. Theorem 2.3 implies that the operator

$$P_\mu(t, D_t, D_y)\colon H_s^{(k)}(\mathbf{R}^{n-1})\longrightarrow H_{s-1}(\mathbf{R}^{n-1})$$

has an infinite-dimensional cokernel. This fact suggests that the problem (1.2) has also an infinite-dimensional cokernel (, which is proved in Remark 5.3). Therefore, adding the coboundary condition $B_\mu(\rho\otimes\delta_{\Gamma_0})$ to (1.2), we consider the problem

$$(5.1)\qquad \begin{cases} A(x, D_x)u+\mu^2 u=f & \text{in } \Omega, \\ \left.\dfrac{\partial u}{\partial \nu}\right|_{\Gamma}+B_\mu(\rho\otimes\delta_{\Gamma_0})=g & \text{on } \Gamma, \end{cases}$$

instead of (1.2). Here, δ_{Γ_0} is the Dirac measure on Γ_0, ρ is a function on Γ_0 and B_μ is a pseudo-differential operator on Γ satisfying Assumption 5.1 below.

For any $x_0\in\Gamma_0$ we choose a diffeomorphism $\Phi_{x_0}(x)$ $(x\in U(x_0))$ stated in Proposition 1.1 such that, in addition to (1.5), for every pair of $\Phi_{x_0}(x)=(t, y, z)$ and $\Phi_{x_1}(x)=(t', y', z')$ $(x_0, x_1\in\Gamma_0$ and $x\in U(x_0)\cap U(x_1))$ the transition from (t', y', z') to (t, y, z) is given by the transformation in the form

$$t=t',\quad y_j=\varphi_j(y')\quad (j=1,\dots, n-2),\quad z=z'.$$

Throughout this section we fix

$$(5.2)\qquad \{\Phi_{x_0}(x)\}_{x_0\in\Gamma_0}.$$

Definition 5.1. Let μ be a parameter (>0). We say that a function $B(t, y; \tau, \eta, \mu)\in C^\infty(\mathbf{R}^{n-1}\times\mathbf{R}^{n-1}\times\mathbf{R}^1_+)$ belongs to $\mathbf{S}^m (m\in\mathbf{R})$ when for any multi-index α and β we have

$$|D^\alpha_{(t,y)}D^\beta_{(\tau,\eta,\mu)}B(t, y; \tau, \eta, \mu)|\leqq C_{\alpha\beta}\{\tau^2+(|\eta|^2+\mu^2)^{\frac{1}{k+1}}\}^{\frac{1}{2}(m-|\beta|)}$$

where the constant $C_{\alpha\beta}$ is independent of t, y, τ, η and μ $(\geqq\mu_0$, μ_0 is an arbitrary positive constant).

Proposition 5.1. *Let* $B(t, y; \tau, \eta, \mu)\in\mathbf{S}^m$ *and* $\mu\geqq\mu_0$ $(m<-\frac{1}{2}$ *and* μ_0 *is an arbitrary positive constant). Then we have*

$$|||B(t, y; D_t, D_y, \mu)(\delta(t)\otimes\rho(y))|||_{s,\mathbf{R}^{n-1}}$$

$$\leqq C\Big(|B|_{m,0}\|(|D_y|^2+\mu^2)^{\frac{1}{2}\left(s+\frac{m+1/2}{k+1}\right)}\rho\|_{0,\mathbf{R}^{n-2}}$$

$$+|B|_{m,l}\|(|D_y|^2+\mu^2)^{\frac{1}{2}\left(s+\frac{m}{k+1}\right)}\rho\|_{0,\mathbf{R}^{n-2}}\Big),\qquad \rho\in\mathscr{S}(\mathbf{R}^{n-2}),$$

where $0\leqq s<-m-\frac{1}{2}$ *and the constants* C, l *are independent of* μ *and the choice of* $B(t, y; \tau, \eta, \mu)$. $\Big(|B|_{m,l'}$ *denotes* $\displaystyle\sup_{\substack{t,y,\tau,\eta,\mu\geqq\mu_0\\|\alpha|+|\beta|\leqq l'}}$

$$\left|\frac{D^{\alpha}_{(t,y)}D^{\beta}_{(\tau,\eta)}B(t, y; \tau, \eta, \mu)}{\{\tau^2+(|\eta|^2+\mu^2)^{\frac{1}{k+1}}\}^{\frac{1}{2}(m-|\beta|)}}\right|\Bigg).$$

The proposition is proved in Appendix.

We assume that B_μ in the problem (5.1) satisfies

Assumption 5.1. B_μ is a pseudo-differential operator on Γ (e.g., see §IV of [9]) with the parameter μ, and for any Φ_{x_0} of (5.2) there exists a symbol $B(t, y; \tau, \eta, \mu) \in \mathbf{S}^m$ which is the local representation of B_μ in Φ_{x_0}. Here m is a constant (<-1) independent of the choice of Φ_{x_0}. Moreover $B(t, y; \tau, \eta, \mu)$ has an asymptotic expansion $\sum_{j=0}^{\infty} B_j(t, y; \tau, \eta, \mu)$ $\left(B(t, y; \tau, \eta, \mu)-\sum_{j=0}^{N-1} B_j(t, y; \tau, \eta, \mu) \in \mathbf{S}^{m-N}\right)$ satisfying

(i) Each $B_j(t, y; \tau, \eta, \mu) (\in \mathbf{S}^{m-j})$ possesses the property of quasi-homogeneity

$$B_j(t, y; \lambda\tau, \lambda^{k+1}\eta, \lambda^{k+1}\mu)=\lambda^{m-j}B_j(t, y; \tau, \eta, \mu), \quad \lambda>0;$$

(ii) For any (η, μ) such that $|\eta|^2+\mu^2=1$, the function

$$\mathscr{F}^{-1}_{\tau\to t}[B_0(0; \tau, \eta, \mu)](t)$$

is not orthogonal in $L^2(\mathbf{R}^1_t)$ to the solution of $P^{(*)}_\mu(t, D_t, \eta)v(t)=0$.

Actually there exists such B_μ:

Example 5.1. Let $\sum_{j=1}^{n_0} \alpha_j(x) (x \in \Gamma)$ be a partition of unity near Γ_0 and $\beta_j(x)$ $(\in C^\infty(\Gamma))$ satisfy $\beta_j(x)=1$ in a neighborhood of $\mathrm{supp}(\alpha_j)$. Let $\mathrm{supp}(\alpha_j)$ and $\mathrm{supp}(\beta_j)$ be small enough. We take x_j $(\in \mathrm{supp}(\alpha_j) \cap \Gamma_0)$ and denote by $\Phi'_j(x)$ the restriction to Γ of $\Phi_{x_j}(x)$ in (5.2). We set

$$b(\tau, \eta, \mu)=\int e^{-it\tau}|t|^{-m-1}\exp\{-N(|\eta|^2+\mu^2)^{\frac{1}{2}}t^{k+1}\}dt$$

where m is a constant (<-1) and N is a sufficiently large constant. Define $B_\mu(h)$ (for $h \in C^\infty(\Gamma)$) by

$$B_\mu(h)=\sum_{j=1}^{n_0} \beta_j(x)\times[b(D_t, D_y, \mu)\{\alpha_j h(\Phi'^{-1}_j(t, y))\}](\Phi'_j(x)),$$

then B_μ satisfies Assumption 5.1 when N is large enough.

Let us set

$$\Pi(h, \rho)=P_\mu(t, D_t, D_y)h(t, y)+B_0(0; D_t, D_y, \mu)(\delta(t)\otimes\rho(y))$$

where h and ρ are functions on $\mathbf{R}^{n-1}$ and $\mathbf{R}^{n-2}$ respectively and $\delta(t)$ is the Dirac function.

Theorem 5.1. *Suppose* $m<-\frac{1}{2}$, $1\leqq s<-m+\frac{1}{2}$ *and* $\mu\geqq\mu_0$ (μ_0 *is an arbitrary positive constant). We have*

i)
$$C^{-1}\Big(|||h|||^{(k)}_{s,\mathbf{R}^{n-1}}+\|(|D_y|^2+\mu^2)^{\frac{1}{2}(s-1+\frac{m+1/2}{k+1})}\rho\|_{0,\mathbf{R}^{n-2}}\Big)$$
$$\leqq |||\Pi(h, \rho)|||_{s-1,\mathbf{R}^{n-1}}$$
$$\leqq C\Big(|||h|||^{(k)}_{s,\mathbf{R}^{n-1}}+\|(|D_y|^2+\mu^2)^{\frac{1}{2}(s-1+\frac{m+1/2}{k+1})}\rho\|_{0,\mathbf{R}^{n-2}}\Big),$$
$$(h, \rho)\in H^{(k)}_s(\mathbf{R}^{n-1})\times H_{s-1+\frac{m+1/2}{k+1}}(\mathbf{R}^{n-2}_y)$$

where the constant C *does not depend on* μ.

ii) *For the operator*

$$\Pi\colon H^{(k)}_s(\mathbf{R}^{n-1})\times H_{s-1+\frac{m+1/2}{k+1}}(\mathbf{R}^{n-2})\longrightarrow H_{s-1}(\mathbf{R}^{n-1}),$$

the kernel and the cokernel are both $\{0\}$.

Corollary. *Π has the inverse* $G^3\colon H_0(\mathbf{R}^{n-1})\longrightarrow H^{(k)}_1(\mathbf{R}^{n-1})\times H_{\frac{m+1/2}{k+1}}(\mathbf{R}^{n-2})$. *Furthermore if* $g\in H_{s-1}(\mathbf{R}^{n-1})$, *then* $G^3g\equiv(G^3_1g, G^3_2g)\in H^{(k)}_s(\mathbf{R}^{n-1})\times H_{s-1+\frac{m+1/2}{k+1}}(\mathbf{R}^{n-2})$ *and we have the estimate*

$$|||G^3_1g|||^{(k)}_{s,\mathbf{R}^{n-1}}+\|(|D_y|^2+\mu^2)^{\frac{1}{2}(s-1+\frac{m+1/2}{k+1})}G^3_2g\|_{0,\mathbf{R}^{n-2}}\leqq C|||g|||_{s-1,\mathbf{R}^{n-1}}.$$

The corollary is clear from the theorem. Using Lemma 5.1 below, we can prove i) of the theorem in the same way as in Theorem 3.1. Noting ii) of Assumption 5.1, we can show ii) of the theorem in the same fashion by means of Corollary of Theorem 2.3.

Lemma 5.1. *Let us set*

$$\lambda=(|\eta|^2+\mu^2)^{\frac{1}{2}},\qquad a=\frac{n_0 i r_0(0, \eta, \mu)}{(|\eta|^2+\mu^2)^{\frac{1}{2}}},$$

and define $\pi_\lambda(v, \rho_0)$ (for $(v, \rho_0)\in\mathscr{S}(\mathbf{R}^1)\times\mathbf{C}$) *by*

$$\pi_\lambda(v, \rho_0)=\frac{dv}{dt}+a\lambda t^k v+\rho_0\mathscr{F}^{-1}_{\tau\to t}[B_0(0;\tau,\eta,\mu)](t)$$

where k *is odd and* $\lambda>0$. *Then, for a real number* $s\left(<-m-\frac{1}{2}\right)$ *there is a constant* C *independent of* a *and* λ *such that*

$$C^{-1}\left(\|(D_t^2+\lambda^{\frac{2}{k+1}})^{\frac{s}{2}}D_t v\|^2_{0,\mathbf{R}^1}+\|(D_t^2+\lambda^{\frac{2}{k+1}})^{\frac{s}{2}}\lambda t^k v\|^2_{0,\mathbf{R}^1}+\lambda^{\frac{2s+2m+1}{k+1}}|\rho_0|^2\right)$$

$$\leqq\|(D_t^2+\lambda^{\frac{2}{k+1}})^{\frac{s}{2}}\pi_\lambda(v, \rho_0)\|^2_{0,\mathbf{R}^1}$$

$$\leqq C\left(\|(D_t^2+\lambda^{\frac{2}{k+1}})^{\frac{s}{2}}D_t v\|^2_{0,\mathbf{R}^1}+\|(D_2^t+\lambda^{\frac{2}{k+1}})^{\frac{s}{2}}\lambda t^k v\|^2_{0,\mathbf{R}^1}+\lambda^{\frac{2s+2m+1}{k+1}}|\rho_0|^2\right),$$

$$(v, \rho_0)\in\mathscr{S}(\mathbf{R}^1)\times\mathbf{C}.$$

Proof. In view of i) of Assumption 5.1, we get

$$\|(D_t^2+\lambda^{\frac{2}{k+1}})^{\frac{s}{2}}\pi_\lambda(v, \rho_0)\|^2_{0,\mathbf{R}^1}=\lambda^{\frac{2s+1}{k+1}}\left\|\left(\frac{d}{dt}+at^k\right)v(\lambda^{-\frac{1}{k+1}}t)\right.$$

$$\left.+\mathscr{F}^{-1}_{\tau\to t}\left[B_0\left(0;\tau,\frac{\eta}{\lambda},\frac{\mu}{\lambda}\right)\right](t)\lambda^{\frac{m}{k+1}}\rho_0\right\|^2_{s,\mathbf{R}^1}$$

by a change of variable. Similarly

$$\|(D_t^2+\lambda^{\frac{2}{k+1}})^{\frac{s}{2}}D_t v\|^2_{0,\mathbf{R}^1}+\|(D_t^2+\lambda^{\frac{2}{k+1}})^{\frac{s}{2}}\lambda t^k v\|^2_{0,\mathbf{R}^1}+\lambda^{\frac{2s+2m+1}{k+1}}|\rho_0|^2$$

$$=\lambda^{\frac{2s+1}{k+1}}\{\|D_t(v(\lambda^{-\frac{1}{k+1}}t))\|^2_{s,\mathbf{R}^1}+\|t^k v(\lambda^{-\frac{1}{k+1}}t)\|^2_{s,\mathbf{R}^1}+\lambda^{\frac{2m}{k+1}}|\rho_0|^2\}.$$

Noting ii) of Assumption 5.1, by Corollary of Theorem 2.3 we have a constant $C(a, \zeta)$ at any fixed a and $\zeta\equiv\left(\frac{\eta}{\lambda},\frac{\mu}{\lambda}\right)$ such that

$$C(a,\zeta)^{-1}\{\|D_t(v(\lambda^{-\frac{1}{k+1}}t))\|^2_{s,\mathbf{R}^1}+\|t^k v(\lambda^{-\frac{1}{k+1}}t)\|^2_{s,\mathbf{R}^1}+\lambda^{\frac{2m}{k+1}}|\rho_0|^2\}$$

$$\leqq\left\|\left(\frac{d}{dt}+at^k\right)v(\lambda^{-\frac{1}{k+1}}t)+\mathscr{F}^{-1}_{\tau\to t}[B_0(0;\tau,\zeta)](t)\lambda^{\frac{m}{k+1}}\rho_0\right\|^2_{s,\mathbf{R}^1}\tag{5.3}$$

$$\leqq C(a,\zeta)\{\|D_t(v(\lambda^{-\frac{1}{k+1}}t))\|^2_{s,\mathbf{R}^1}+\|t^k v(\lambda^{-\frac{1}{k+1}}t)\|^2_{s,\mathbf{R}^1}+\lambda^{\frac{2m}{k+1}}|\rho_0|^2\}.$$

Let us fix a_0 and ζ_0. If $|a-a_0|$ and $|\zeta-\zeta_0|$ are small enough, the constant $C(a, \zeta)$ in (5.3) can be taken independently of a and ζ. Since (a, ζ) moves on a compact set, we can choose $C(a, \zeta)$ in (5.3) independent of a and ζ, which proves the lemma.

Let $\sum_{i=1}^{N} \varphi_i$ be the partition of unity in (3.7) and define the norm $|||\rho|||_{s,\Gamma_0}$ of $H_s(\Gamma_0)$ $(s \in \mathbf{R})$ by

$$|||\rho|||_{s,\Gamma_0} = \sum_{i=1}^{N} \|(|D_y|^2+\mu^2)^{\frac{s}{2}}(\varphi_i\rho)'(y)\|_{0,\mathbf{R}^{n-2}},$$

(where $\mu>0$ and $(\varphi_i\rho)'(y)=(\varphi_i\rho)[\Phi_{x_i}^{-1}(0, y, 0)]$).

Theorem 5.2. *Let l be an integer satisfying $2 \leqq l < -m+1$ (m is the constant in Assumption 5.1) and μ_0 be an arbitrary positive constant. Then we obtain*

i) *Estimate*: a) *We have*

$$|||A(x, D_x)u+\mu^2 u|||^{(k)}_{l-2,\Omega} + \left|\left|\left|\frac{\partial u}{\partial \nu} + B_\mu(\rho\otimes\delta_{\Gamma_0})\right|\right|\right|_{l-\frac{3}{2},\Gamma}$$

$$\leqq C_1(|||u|||^{(k)}_{l,\Omega} + |||\rho|||_{l-\frac{3}{2}+\frac{m+1/2}{k+1},\Gamma_0}),\ (u, \rho) \in H^{(k)}_l(\Omega)\times H_{l-\frac{3}{2}+\frac{m+1/2}{k+1}}(\Gamma_0)$$

where $\mu \geqq \mu_0$ and the constant C_1 does not depend on μ.

b) *There is a constant μ_1 such that provided $\mu \geqq \mu_1$,*

$$|||u|||^{(k)}_{l,\Omega} + |||\rho|||_{l-\frac{3}{2}+\frac{m+1/2}{k+1},\Gamma_0} \leqq C_2\Big(|||A(x, D_x)u+\mu^2 u|||^{(k)}_{l-2,\Omega}$$

$$+\left|\left|\left|\frac{\partial u}{\partial \nu} + B_\mu(\rho\otimes\delta_{\Gamma_0})\right|\right|\right|_{l-\frac{3}{2},\Gamma}\Big),\ (u, \rho) \in H^{(k)}_l(\Omega)\times H_{l-\frac{3}{2}+\frac{m+1/2}{k+1}}(\Gamma_0)$$

where the constant C_2 does not depend on μ.

ii) *Solvability: There is a constant μ_2 such that if $\mu \geqq \mu_2$, we have a solution (u, ρ) of* (5.1) *in $H^{(k)}_l(\Omega)\times H_{l-\frac{3}{2}+\frac{m+1/2}{k+1}}(\Gamma_0)$ for any $(f, g) \in H^{(k)}_{l-2}(\Omega)\times H_{l-\frac{3}{2}}(\Gamma)$.*

Noting that the estimate $|||B_\mu(\rho\otimes\delta_{\Gamma_0})|||_{l-\frac{3}{2},\Gamma} \leqq C|||\rho|||_{l-\frac{3}{2}+\frac{m+1/2}{k+1},\Gamma_0}$ is obtained from Proposition 5.1, we can easily show a) of i). Furthermore, we can prove b) of i) and ii) in the similar way to the proof of Theorem 3.2 by means of Lemma 5.2 and 5.3 below, which correspond to Lemma 3.6 and 3.7 respectively.

Remark 5.1. The same estimates and regularity of the problem (1.2) as in Theorem 3.2 are also valid without the coboundary condition.

Remark 5.2. Let us consider the problem

$$(5.4)\qquad \begin{cases} A(x, D_x)u=f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu}\Big|_\Gamma + B(\rho\otimes\delta_{\Gamma_0})=g & \text{on } \Gamma. \end{cases}$$

Here, $A(x, D_x)$ is elliptic, ν is of second class and B is a pseudo-differential operator satisfying Assumption 5.1 where $\mu=0$. We set

$$\mathfrak{A}_1(u, \rho)=\Big(A(x, D_x)u, \frac{\partial u}{\partial \nu}\Big|_\Gamma + B(\rho\otimes\delta_{\Gamma_0})\Big).$$

Then, by the similar method to that of this section, we see that the operator

$$\mathfrak{A}_1: H_l^{(k)}(\Omega)\times H_{l-\frac{3}{2}+\frac{m+1/2}{k+1}}(\Gamma_0) \longrightarrow H_{l-2}^{(k)}(\Omega)\times H_{l-\frac{3}{2}}(\Gamma)$$

is continuous and Noetherian (cf. Višik-Grušin [10]).

Remark 5.3. We shall prove briefly that the problem

$$(5.5)\qquad \begin{cases} A(x, D_x)u=f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu}=g & \text{on } \Gamma \end{cases}$$

has an infinite-dimensional cokernel where $A(x, D_x)$ is elliptic and ν is of second class. Set

$$\mathfrak{A}_2(u)=\Big(A(x, D_x)u, \frac{\partial u}{\partial \nu}\Big|_\Gamma\Big).$$

$\mathscr{H}_1\equiv\mathfrak{A}_2(H_l^{(k)}(\Omega))$ is closed in $H_{l-2}^{(k)}(\Omega)\times H_{l-\frac{3}{2}}(\Gamma)$. Decompose $H_{l-2}^{(k)}(\Omega)\times H_{l-\frac{3}{2}}(\Gamma)$ into $\mathscr{H}_1$ and its orthogonal complement $\mathscr{H}_1^\perp$. We have only to show that $\mathscr{H}_1^\perp$ is infinite-dimensional. From Remark 5.2, $\mathscr{H}_2\equiv \mathfrak{A}_1(0, H_{l-\frac{3}{2}+\frac{m+1/2}{k+1}}(\Gamma_0))$ is infinite-dimensional. Therefore, $\mathscr{H}_2\cap\mathscr{H}_1$ or $\mathscr{H}_2\cap\mathscr{H}_1^\perp$ is infinite-dimensional. Remark 5.2 yields that the dimension of $\mathscr{H}_2\cap\mathscr{H}_1$ is finite. Hence, $\mathscr{H}_2\cap\mathscr{H}_1^\perp$ is infinite-dimensional, and so is $\mathscr{H}_1^{\perp}$. The proof is complete.

Lemma 5.2. *Suppose* $0<\varepsilon\leqq 1$, $l=2, 3,\ldots$ *and* $\mu\geqq\mu_0$ (μ_0 *is an arbitrary positive constant), then we have*

$$|||u|||_{l,\mathbf{R}_+^n}^{(k)}+\|(|D_y|^2+\mu^2)^{\frac{1}{2}(l-\frac{3}{2}+\frac{m+1/2}{k+1})}\rho\|_{0,\mathbf{R}_y^{n-2}}$$

$$\leqq C_1\Big(|||A_0'(0;D_t,D_y,D_z)u+\mu^2 u|||_{l-2,\mathbf{R}_+^n}^{(k)}$$

$$+\Big|\Big|\Big|\frac{\partial u}{\partial v_0}+B_0(0;D_t,D_y,\mu)(\delta\otimes\rho)\Big|\Big|\Big|_{l-\frac{3}{2},\mathbf{R}^{n-1}}\Big)$$

$$+C_2\Big(\varepsilon+\frac{1}{\mu}\Big)|||u|||_{l,\mathbf{R}_+^n}^{(k)}$$

for $(u,\rho)\in H_l^{(k)}(\mathbf{R}_+^n)\times H_{l-\frac{3}{2}+\frac{m+1/2}{k+1}}(\mathbf{R}^{n-2})$ *satisfying* $\mathrm{supp}(u)\subset V_{\varepsilon,+}^n$. *Here the constants* C_1 *and* C_2 *are independent of* ε *and* μ.

By Theorem 5.1, we can prove the lemma in the same way as in Lemma 3.6.

Lemma 5.3. *Suppose* $0<\varepsilon\leqq\frac{1}{2}$, $l=2, 3,\ldots$ *and* $\mu\geqq\mu_0$ (μ_0 *is an arbitrary positive constant). Let* $\psi_\varepsilon(t, y, z)$ *be the function* (3.12). *We define* $R^3(f, g)\equiv(R_1^3(f, g), R_2^3(f, g))$ $(\in H_l^{(k)}(\mathbf{R}_+^n)\times H_{l-\frac{3}{2}+\frac{m+1/2}{k+1}}(\mathbf{R}^{n-2}))$ *by*

$$R_1^3(f, g)=Qf+\psi_1\mathscr{P}G_1^3\Big(g-\frac{\partial}{\partial v_0}Qf\Big),$$

$$R_2^3(f, g)=G_2^3\Big(g-\frac{\partial}{\partial v_0}Qf\Big),$$

where $(f, g)\in H_{l-2}^{(k)}(\mathbf{R}_+^n)\times H_{l-\frac{3}{2}}(\mathbf{R}^{n-1})$ *and* $\mathrm{supp}(f)$, $\mathrm{supp}(g)$ *lie in* $V_{\varepsilon,+}^n$, V_ε^{n-1} *respectively* (G_1^3, G_2^3 *are defined in Corollary of Theorem* 5.1). *Then we obtain*

i) *There is a constant* C_1 *independent of* ε *and* μ *such that*

$$|||R_1^3(f, g)|||_{l,\mathbf{R}_+^n}^{(k)}+\|(|D_y|^2+\mu^2)^{\frac{1}{2}(l-\frac{3}{2}+\frac{m+1/2}{k+1})}R_2^3(f, g)\|_{0,\mathbf{R}^{n-2}}$$

$$\leqq C_1(|||f|||_{l-2,\mathbf{R}_+^n}^{(k)}+|||g|||_{l-\frac{3}{2},\mathbf{R}^{n-1}}).$$

ii) *Set*

$$S_1^3(f, g)=\{A'(t, y, z; D_t, D_y, D_z)+\mu^2\}\psi_\varepsilon R_1^3(f, g)-f,$$

$$S_2^3(f, g)=\frac{\partial}{\partial\nu'}\psi_\varepsilon R_1^3(f, g)|_{z=0}$$

$$+\psi_{2\varepsilon}B(t, y; D_t, D_y, \mu)\psi_\varepsilon(\delta(t)\otimes R_2^3(f, g))-g,$$

then we have

$$|||S_1^3(f, g)|||_{l-2,\mathbf{R}_+^n}^{(k)} \leqq \left(\sigma_1(\varepsilon)+\frac{C_2(\varepsilon)}{\mu}\right)(|||f|||_{l-2,\mathbf{R}_+^n}^{(k)} + |||g|||_{l-\frac{3}{2},\mathbf{R}^{n-1}}),$$

$$|||S_2^3(f, g)|||_{l-\frac{3}{2},\mathbf{R}^{n-1}} \leqq \left(\sigma_2(\varepsilon)+\frac{C_3(\varepsilon)}{\mu^{1/2(k+1)}}\right)(|||f|||_{l-2,\mathbf{R}_+^n}^{(k)} + |||g|||_{l-\frac{3}{2},\mathbf{R}^{n-1}}),$$

where $\sigma_1(\varepsilon), \sigma_2(\varepsilon)\to 0$ *uniformly in* μ *as* $\varepsilon\to 0$, *and the constants* $C_2(\varepsilon)$, $C_3(\varepsilon)$ *do not depend on* μ.

By means of Proposition 5.1 and Theorem 5.1, we can prove the lemma in the similar way to the proof of Lemma 3.7.

Appendix

We shall define a class of pseudo-differential operators with a parameter and state several theorems. We can prove these theorems in the same way as in Kumano-go [6]. So we omit the proofs.

Let μ be a parameter and move on M.

Definition A.1. We say that a C^∞-function $\lambda_\mu(\xi)$ on $\mathbf{R}_\xi^n$ with the parameter μ is a basic weight function when $\lambda_\mu(\xi)$ satisfies

i) $\lambda_0(|\xi|^2+1)^{\frac{\sigma}{2}} \leqq \lambda_\mu(\xi)$

where the constants σ and λ_0 $(0<\sigma\leqq 1, 0<\lambda_0)$ do not depend on μ;

ii) For any multi-index α there is a constant A_α independent of μ such that

$$|D_\xi^\alpha\lambda_\mu(\xi)| \leqq A_\alpha\lambda_\mu(\xi)^{1-|\alpha|}.$$

Example A.1. Set $M=]\mu_0, +\infty[$ where μ_0 is an arbitrary positive constant.

i) $\lambda_\mu(\xi)=(|\xi|^2+\mu^2)^{\frac{1}{2}}$ (cf. Lemma 3.3).

ii) $\lambda_\mu(\tau, \eta)=\{\tau^2+(|\eta|^2+\mu^2)^{\frac{1}{k+1}}\}^{\frac{1}{2}}$ (k is a positive integer; cf. Definition 5.1).

Definition A.2. We say that a C^∞-function $p_\mu(x, \xi)$ on $\mathbf{R}_x^n \times \mathbf{R}_\xi^n$ with the parameter μ belongs to $S_{\lambda_\mu}^m$ ($m \in \mathbf{R}$) when for any multi-index α and β we have

$$|D_x^\alpha D_\xi^\beta p_\mu(x, \xi)| \leqq C_{\alpha\beta}\lambda_\mu(\xi)^{m-|\beta|}$$

where the constant $C_{\alpha\beta}$ does not depend on μ.

We set

$$|p_\mu|_{m,l}= \sup_{\substack{\mu\in M\\ (x,\xi)\in\mathbf{R}^{2n}\\ |\alpha|+|\beta|\leqq l}} |D_x^\alpha D_\xi^\beta p_\mu(x, \xi)/\lambda_\mu(\xi)^{m-|\beta|}|.$$

Definition A.3. We say that a C^∞-function $p_\mu(x, \xi, x', \xi')$ on $\mathbf{R}_x^n \times \mathbf{R}_\xi^n \times \mathbf{R}_{x'}^n \times \mathbf{R}_{\xi'}^n$ belongs to $S_{\lambda_\mu}^{m,m'}$ ($m, m' \in \mathbf{R}$) when for any multi-index α, β, α' and β' we have

$$|D_x^\alpha D_\xi^\beta D_{x'}^{\alpha'} D_{\xi'}^{\beta'} p_\mu(x, \xi, x', \xi')| \leqq C_{\alpha\beta\alpha'\beta'}\lambda_\mu(\xi)^{m-|\beta|}\lambda_\mu(\xi')^{m'-|\beta'|}$$

where the constant $C_{\alpha\beta\alpha'\beta'}$ does not depend on μ.

For $p_\mu(x, \xi, x', \xi') \in S_{\lambda_\mu}^{m,m'}$ we define

$$[p_\mu(x, D_x, x', D_{x'})u](x)$$
$$=\int d\!\!{}^-\xi\int\left\{\int e^{i(x-x')\cdot\xi+ix'\cdot\xi'}p_\mu(x, \xi, x', \xi')\hat{u}(\xi')d\!\!{}^-\xi'\right\}dx', \quad u \in \mathscr{S}$$

(cf. §1 of [6]).

Theorem A.1. *For $p_\mu(x, \xi, x', \xi') \in S_{\lambda_\mu}^{m,m'}$ we define $L(p_\mu)(x, \xi)$ by*

$$L(p_\mu)(x, \xi)=\int d\!\!{}^-\zeta\int e^{-iz\zeta}(1+|z|^2)^{-n_0}(1-\Delta_\zeta)^{n_0}\, p_\mu(x, \xi+\zeta, x+z, \xi)dz$$

(where n_0 is an integer and $2n_0 \geqq n+1$). Then, $L(p_\mu)(x, \xi) \in S_{\lambda_\mu}^{m+m'}$ and we have

$$p_\mu(x, D_x, x', D_x')u=L(p_\mu)(x, D_x)u, \qquad u \in \mathscr{S}.$$

Furthermore, L is a continuous operator from $S_{\lambda_\mu}^{m,m'}$ to $S_{\lambda_\mu}^{m+m'}$ (see Theorem 1.1 of [6]).

Theorem A.2. (*The asymptotic expansion formula*). *Let $p_\mu(x, \xi, x', \xi') \in S_{\lambda_\mu}^{m,m'}$, then we have $R_\mu^N(x, \xi)\ S_{\lambda_\mu}^{m+m'-N}$ for any positive integer N such that*

$$R_\mu^N(x, \xi) = L(p_\mu)(x, \xi) - \sum_{|\alpha|<N} \frac{1}{\alpha!} D_{x'}^\alpha \left(\frac{\partial}{\partial \xi}\right)^\alpha p_\mu(x, \xi, x', \xi') \Big|_{\substack{\xi'=\xi \\ x'=x}}.$$

Furthermore, the operator $p_\mu(x, \xi, x', \xi') \mapsto R_\mu^N(x, \xi)$ is a continuous one from $S_{\lambda_\mu}^{m,m'}$ to $S_{\lambda_\mu}^{m+m'-N}$ (see Theorem 4.1 of [6]).

Theorem A.3. *Let $\|u\|_s$ denote $\|\lambda_\mu(D_x)^s u\|_{0,\mathbf{R}^n}$ ($s \in \mathbf{R}$). Suppose $p_\mu(x, \xi) \in S_{\lambda_\mu}^m$, then we have*

$$\|p_\mu(x, D_x)u\|_s^2 \leqq |p_\mu|_{m,0}^2 \|u\|_{s+m}^2 + C|p_\mu|_{m,l}^2 \|u\|_{s+m-\frac{1}{2}}^2, \qquad u \in \mathscr{S}$$

where the constants C and l do not depend on the choice of $p_\mu(x, \xi)$ (see Theorem 5.2 of [6]).

Proof of Lemma 3.5. We set $\lambda_\mu(\tau, \eta) = (\tau^2 + |\eta|^2 + \mu^2)^{\frac{1}{2}}$. It is easily seen that

$$\begin{aligned} &\|\varphi \mathscr{P}(\psi h)\|_{s,\mathbf{R}_+^n}^2 \\ &\leqq C_1(\mu_0)\left\{\int_0^{+\infty} \|\lambda_\mu^s(D_t, D_y) \circ \varphi e^{ir_0(D_t, D_y, \mu)z} \circ \psi h\|_{0,\mathbf{R}_{(t,y)}^{n-1}}^2 dz \right. \\ &\left. + \int_0^{+\infty} \|\varphi \{ir_0(D_t, D_y, \mu)\}^s e^{ir_0(D_t, D_y, \mu)z} \circ \varphi h\|_{0,\mathbf{R}_{(t,y)}^{n-1}}^2 dz \right\} \\ &\equiv C_1(\mu_0)(I_1 + I_2). \end{aligned}$$

Let us fix z. We see that

$$\varphi(t) e^{ir_0(\tau,\eta,\mu)z} \in S_{\lambda_\mu}^0 \quad \text{and} \quad |\varphi(t) e^{ir_0(\tau,\eta,\mu)z}|_{0,l} \leqq C_2(l) e^{-\delta(\mu_0)z}$$

where the constants $C_2(l)$ and $\delta(\mu_0)(>0)$ do not depend on z. Theorem A.1 implies that there is a symbol $q_\mu(t, y; \tau, \eta) \in S_{\lambda_\mu}^0$ such that

$$q_\mu(t, y; D_t, D_y)h = \varphi e^{ir_0(D_t, D_y, \mu)z} \circ \psi h, \qquad h \in \mathscr{S}.$$

By Theorem A.2, we have for any integer $N>0$

$$S_{\lambda_\mu}^{-N}\ni q_\mu(t, y; \tau, \eta)-\sum_{j=0}^{N-1}\varphi(t)\Big(\frac{\partial^j}{\partial\tau^j}e^{iro(\tau,\eta,\mu)z}\Big)D_t^j\varphi(t)$$

$$=q_\mu(t, y; \tau, \eta)$$

and

$$|q_\mu|_{-N,l}\leqq C_3(N, l)e^{-\delta(\mu_0)z}$$

where the constant $C_3(N, l)$ does not depend on z. Therefore, from Theorem A.3 it follows that

$$I_1\leqq C_4\int_0^{+\infty}e^{-2\delta(\mu_0)z}dz\,\|\lambda_\mu^{s-N}(D_t, D_y)h\|_{0,\mathbf{R}^{n-1}}^2 .$$

I_2 is also estimated in the same way. Hence, the lemma is proved.

Proof of Proposition 5.1. Let us set $\lambda_\mu(\tau, \eta)=\{\tau^2+(|\eta|^2+\mu^2)^{\frac{1}{k+1}}\}^{\frac{1}{2}}$. We see easily that

$$|||B(t, y; D_t, D_y, \mu)h|||_{s,\mathbf{R}^{n-1}}\leqq C_1(\mu_0)\{\|\lambda_\mu^s(D_t, D_y)\circ B(t, y; D_t, D_y, \mu)h\|_{0,\mathbf{R}^{n-1}}$$

$$+\|(|D_y|^2+\mu^2)^{\frac{s}{2}}\circ B(t, y; D_t, D_y, \mu)h\|_{0,\mathbf{R}^{n-1}}\}\equiv C_1(\mu_0)(I_1+I_2)$$

where $h(t, y)=\delta(t)\otimes\rho(y)$. Theorem A.3 yields

$$(I_1)^2\leqq|B|_{m,0}^2\|\lambda_\mu^{s+m}(D_t, D_y)h\|_{0,\mathbf{R}^{n-1}}^2+C_2|B|_{m,l}^2\|\lambda_\mu^{s+m-\frac{1}{2}}(D_t, D_y)h\|_{0,\mathbf{R}^{n-1}}^2 .$$

For any $j\Big(<-m-\dfrac{1}{2}\Big)$ we get

$$\text{(A.1)}\qquad \|\lambda_\mu^{j+m}(D_t, D_y)(\delta(t)\otimes\rho(y))\|_{0,\mathbf{R}^{n-1}}$$

$$=C_3(j, m)\|(|D_y|^2+\mu^2)^{\frac{j+m+1/2}{2(k+1)}}\rho\|_{0,\mathbf{R}^{n-2}} .$$

Therefore,

$$I_1\leqq C_4\{|B|_{m,0}\|(|D_y|^2+\mu^2)^{\frac{s+m+1/2}{2(k+1)}}\rho\|_{0,\mathbf{R}^{n-2}}$$

$$+|B|_{m,l}\|(|D_y|^2+\mu^2)^{\frac{s+m}{2(k+1)}}\rho\|_{0,\mathbf{R}^{n-2}}\} .$$

Noting that $(|\eta|^2+\mu^2)^{\frac{s}{2}}\in S_{\lambda_\mu}^{(k+1)s}$, by Theorem A.2 we have $R_N(t, y; \tau, \eta,$

$\mu) \in S^{m+k(s+1)-N}_{\lambda_\mu}$ $(N>m+s(k+1)+1)$ such that

$$(|D_y|^2+\mu^2)^{\frac{s}{2}} \circ B(t, y; D_t, D_y, \mu)h$$

$$= \sum_{|\alpha|<N} \frac{1}{\alpha!} \mathscr{F}^{-1}\left[\left(\frac{\partial}{\partial \eta}\right)^\alpha (|\eta|^2+\mu^2)^{\frac{s}{2}} \times D_y^\alpha B(t, y; \tau, \eta, \mu)\hat{h}\right]$$

$$+R_N(t, y; D_t, D_y, \mu)h.$$

Using Theorem A.3 and (A.1), we obtain

$$I_2 \leqq C_5\{|B|_{m,0}\|(|D_y|^2+\mu^2)^{\frac{1}{2}\left(s+\frac{m+1/2}{k+1}\right)}\rho\|_{0,\mathbf{R}^{n-2}}$$

$$+|B|_{m,l'}\|(|D_y|^2+\mu^2)^{\frac{1}{2}\left(s+\frac{m}{k+1}\right)}\rho\|_{0,\mathbf{R}^{n-2}}\}.$$

Therefore the proposition is proved.

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