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On the Cohomology of the Classifying Spaces of $PSU(4n+2)$ and $PO(4n+2)$

By

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§0. Introduction

The quotients of $SU(m)$ and $SO(2m)$ by their centers $\Gamma_m = \left\{e^{\frac{2\pi j\sqrt{-1}}{m}}\begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix};\ 0\leqq j<m\right\}$ and $\Gamma_2=\left\{\pm\begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix}\right\}$ are denoted by $PU(m)$ and $PO(2m)$ respectively.

The purpose of this paper is to determine the module structure of the cohomology mod 2 of the classifying spaces $BPU(4n+2)$ and $BPO(4n+2)$.

The method is first to determine the E_2-term of the Eilenberg-Moore spectral sequence by constructing an injective resolution for $H^*(G; \boldsymbol{Z}_2)$, $(G=SU(4n+2)/\Gamma_2,\ PO(4n+2))$. Then by making use of naturality of the Eilenberg-Moore spectral sequence we show that the spectral sequence with $\boldsymbol{Z}_2$-coefficient collapses for these G.

Our results are

Theorem. *As a module*

$$H^*(BPU(4n+2); \boldsymbol{Z}_2)\cong \boldsymbol{Z}_2[a_2, a_3, x'_{8l+8}, y(\mathrm{I})]/R,$$

where $1\leqq l\leqq 2n$ *and* R *is an ideal generated by* $a_3y(I)$, $y(I)^2+\sum_{j=1}^{r} x_{8i_1+8}\cdots a^2_{4i_j+2}\cdots x_{8i_r+8}$ *and* $y(I)y(J)+\Sigma f_i y(I_i)$.

Theorem. *As a module*

$$H^*(BPO(4n+2); \boldsymbol{Z}_2)\cong \boldsymbol{Z}_2[a_2, x_{4l+4}, y'(I)]/R,$$

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where $1 \leqq l \leqq 2n$ *and* R *is an ideal generated by* $a_2 y'(I)$, $y'(I)^2 + \Sigma x_{4i_1+4} \cdots a^2_{i_j+1} \cdots x_{4i_r+4}$ *and* $y'(I)y'(J) + \Sigma f'_i y'(I_i)$.

In the above theorems I runs over all sequences of integers $(i_1, \dots, i_r)$ satisfying $1 \leqq r \leqq 2n$ and $1 \leqq i_1 < \cdots < i_r \leqq 2n$. (For details see §5.)

The paper is organized as follows:

In the first section we show that there exists a sort of "stability" in $H^*(BG; Z_2)$. §2 is used to calculate $H^*(U(n)/\Gamma_p; \boldsymbol{Z}_p)$. In §3 we determine the E_2-term of the Eilenberg-Moore spectral sequence, Cotor $^{H^*(G;Z_2)}(\boldsymbol{Z}_2, \boldsymbol{Z}_2)$, for $G = PO(4n+2), PU(4n+2)$. In the next section, §4, we show that the Eilenberg-Moore spectral sequence (with $\boldsymbol{Z}_2$-coefficient) collapses for these G. §5 is devoted to showing that the elements a_i's in the above theorems, namely Theorems 4.9 and 4.12, are in the trangression image. In the last section, the generators x_{8l+8} and x_{4l+4} in Theorems 4.9 and 4.12 are shown to be represented by certain exterior power representations.

Throughout the paper the map $BH \to BG$ induced from a homomorphism $H \to G$ of groups is denoted by the same symbol.

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§1. Quotients of $SU(n)$ and $SO(n)$

Notation. $I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in U(n)$ the identity matrix,

$$C(n) = \{\alpha I_n;\ |\alpha| = 1 \quad \text{and} \quad \alpha \in \mathbf{C}\},$$

$$\Gamma_m = \{w I_n;\ w^m = 1 \quad \text{and} \quad w \in \mathbf{C}\} \subset C(n).$$

Then $C(n)$ (resp. Γ_m) is the center of the unitary group $U(n)$ (resp. $SU(n)$). In particular we have the inclusions

$$\Gamma_2 = \left\{ \pm \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} \subset SO(2n) \subset SU(2n).$$

Hereafter we use the following

Notation.

$G_l(m)=SU(m)/\Gamma_l$ for a subgroup Γ_l of the center Γ_m,

$G_m(m)=PU(m)=PSU(m)\cong U(m)/C(m)$,

$G(2n)=G_2(2n)=SU(2n)/\Gamma_2$,

$PO(2n)=SO(2n)/\Gamma_2$.

Denote by π the natural projections $SU(m)\to G_l(m)$ and $SO(2n)\to PO(2n)$

Consider the k-fold diagonal map:

$$\Delta_k:\ SU(n)\longrightarrow (SU(n))^k\longrightarrow SU(nk),$$

$$\Delta_k:\ SO(n)\longrightarrow (SO(n))^k\longrightarrow SO(nk),$$

where Δ_k is the diagonal embedding:

$$\Delta_k(A)=\begin{pmatrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{pmatrix}.$$

For the identity matrix I_n then we have

$$\Delta_k(I_n)=I_{nk} \quad \text{and} \quad \Delta_k(-I_n)=-I_{nk}.$$

So for even n there exist maps $G(n)\to G(nk)$ and $PO(n)\to PO(nk)$ such that the following diagrams commute:

$$\begin{array}{ccc} SU(n) & \xrightarrow{\Delta_k} & SU(nk) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ G(n) & \longrightarrow & G(nk) \end{array} \qquad \begin{array}{ccc} SO(n) & \longrightarrow & SO(nk) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ PO(n) & \longrightarrow & PO(nk), \end{array}$$

We denote them by the same symbol:

$$\Delta_k:\ G(n)\longrightarrow G(nk), \qquad \Delta_k:\ PO(n)\longrightarrow PO(nk).$$

Notation.

$$C(n,\ k)=SU(nk)/\Delta_k SU(n),$$

$$R(n,\ k)=SO(nk)/\Delta_k SO(n).$$

So we have fiberings:

(1.1) $$SU(n) \xrightarrow{\Delta_k} SU(nk) \xrightarrow{\rho} C(n, k).$$

(1.2) $$SO(n) \xrightarrow{\Delta_k} SO(nk) \xrightarrow{\rho} R(n, k),$$

Remark 1.3.

(1) $C(n, k)$ is homeomorphic to $G_l(nk)/\Delta_k G_l(n)$ for $l|n$.

(2) $R(2n, k)$ is homeomorphic to $PO(2nk)/\Delta_k PO(2n)$.

Now recall from [4] and [5] the following

Proposition 1.4.

(1) $$H^*(SU(n); \boldsymbol{Z}) \cong \Lambda(u_3, \dots, u_{2n-1}),$$

$$H^*(U(n); \boldsymbol{Z}) \cong \Lambda(u_1, u_3, \dots, u_{2n-1}),$$

where $\deg u_{2i-1} = 2i-1$ *and* u_{2i-1} *is universally transgressive with* $\tau(u_{2i-1}) = c_i$ *the i-th universal Chern class.*

(2) $$H^*(SO(n); \boldsymbol{Z}_2) \cong \Delta(v_1, \dots, v_{n-1}),$$

where $\deg v_{i-1} = i-1$ *and* v_{i-1} *is universally transgressive with* $\tau(v_{i-1}) = w_i$ *the i-th universal Stiefel-Whitney class.*

Then

Proposition 1.5. (1) *For any integer* $k>0$ *and any prime* p *with* $(k, p)=1$, *we have*

$$H^*(C(n, k); \boldsymbol{Z}_p) \cong \Lambda(\bar{x}_{2n+1}, \dots, \bar{x}_{2nk-1})$$

where $\deg \bar{x}_{2i+1} = 2i+1$ *and* $\rho^* \bar{x}_{2i+1} = u_{2i+1}$.

(2) *For any odd integer* $k>0$ *we have*

$$H^*(R(n, k); \boldsymbol{Z}_2) \cong \Delta(\bar{z}_n, \dots, \bar{z}_{nk-1})$$

where $\deg \bar{z}_i = i$ *and* $\rho^* \bar{z}_i = v_i$.

Proof. (1) The map $\Delta_k\colon SU(n)\to SU(nk)$ induces a map $\Delta_k\colon BSU(n)\to BSU(nk)$ which gives the k-fold Whitney sum of complex vector bundles. Thus

$$\Delta_k^*(c_i)=\sum_{i_1+\cdots+i_k=i} c_{i_1}\ldots c_{i_k}=kc_i+(\text{decomposables}). \tag{1.6}$$

For the Serre cohomology spectral sequence with $\boldsymbol{Z}_p$-coefficient $\{E_r^{*,*}\}$ of the fibering

$$SU(nk)\longrightarrow C(n,\,k)\longrightarrow BSU(n),$$

we have

$$E_2^{*,*}=\boldsymbol{Z}_p[c_2,\ldots,\,c_n]\otimes\Lambda(u_3,\ldots,\,u_{2nk-1})$$

and
$$E_\infty^{*,*}\cong\mathscr{G}r(H^*(C(n,\,k);\,\boldsymbol{Z}_p)).$$

Then it follows from Proposition 1.4 and (1.6) that

$$d_{2i}(1\otimes u_{2i-1})=kc_i\otimes 1\qquad\text{for}\quad 2\leqq i\leqq n$$

and all other differentials are trivial. So we get

$$\mathscr{G}r(H^*(C(n,\,k);\,\boldsymbol{Z}_p))\cong E_\infty^{*,*}\cong E_{2n+1}^{*,*}\cong\Lambda(u_{2n+1},\ldots,\,u_{2nk-1}).$$

Since $(k,\,p)=1$, (1.6) implies that $\Delta_k^*\colon H^*(SU(nk);\,\boldsymbol{Z}_p)\to H^*(SU(n);\,\boldsymbol{Z}_p)$ is epimorphic, and hence $SU(n)$ is totally non-homologous to zero in the fibering (1.1). Thus $\rho^*\colon H^*(C(n,\,k);\,\boldsymbol{Z}_p)\to H^*(SU(nk);\,\boldsymbol{Z}_p)$ is monomorphic.

(2) is proved quite similarly. Q.E.D.

Theorem 1.7. (1) *Let p be a prime, k an integer with $(k,\,p)=1$ and $l|n$. Then $\Delta_k^*\colon H^i(BG_l(nk);\,\boldsymbol{Z}_p)\to H^i(BG_l(n);\,\boldsymbol{Z}_p)$ is isomorphic for $i\leqq 2n$ and monomorphic for $i\leqq 2n+1$.*

(2) *Let k be an odd integer. Then $\Delta_k^*\colon H^i(BPO(2kn);\,\boldsymbol{Z}_2)\to H^i(BPO(2n);\,\boldsymbol{Z}_2)$ is isomorphic for $i\leqq n-1$ and monomorphic for $i\leqq n$.*

Proof. Proposition 1.5 applied with the Serre exact sequence (Proposition 5 of [12]) for the fiberings

$$C(n,\,k)\longrightarrow BG_l(n)\longrightarrow BG_l(nk)$$

$$R(2n, k) \longrightarrow BPO(2n) \longrightarrow BPO(2nk)$$

gives the results. Q.E.D.

Notation. For each rational number k, define $v_p(k)$ to be the exponent of p when k is expressed as a product of powers of distinct primes.

Corollary 1.8. (1) *If* $v_p(n)=v_p(m)$, *then as algebras there hold*

$$H^*(BG_l(n); \boldsymbol{Z}_p)\cong H^*(BG_l(m); \boldsymbol{Z}_p) \quad \textit{for} \quad *\leqq 2\min(m, n).$$

(2) *If* $v_2(m)=v_2(n)$, *then as algebras there hold*

$$H^*(BPO(2n); \boldsymbol{Z}_2)\cong H^*(BPO(2m); \boldsymbol{Z}_2) \quad \textit{for} \quad *\leqq \min(m, n).$$

In the below we denote by ϕ the diagonal map in $H^*(G; \boldsymbol{Z}_p)$ induced from the multiplication on a group G. Put $\bar{\phi}=(\eta\otimes\eta)\circ\phi$, where $\eta\colon H^*(G; \boldsymbol{Z}_p)\to \sum_{i>0} H^i(G; \boldsymbol{Z}_p)$ is the natural projection.

Now we recall from [3] and [5] the following facts:

Proposition 1.9. *Let* $n=p^r n'$ *with* $(p, n')=1$ *and* $l|n$. *Then*

$$H^*(G_l(n); \boldsymbol{Z}_p)\cong \boldsymbol{Z}_p[y]/(y^{p^r})\otimes \Lambda(x_1,\dots, \hat{x}_{2p^r-1},\dots, x_{2n-1}),$$

where $\deg y=2$ *and* $\deg x_{2i-1}=2i-1$.

Proposition 1.9′. *There exist generators* $y\in H^1(G(4n+2); \boldsymbol{Z}_2)$ *and* $x_{2i+1}\in H^{2i+1}(G(4n+2); \boldsymbol{Z}_2)$, $2\leqq i\leqq 4n+1$, *such that*

(1) $$H^*(G(4n+2); \boldsymbol{Z}_2)\cong \Lambda(y, y^2, x_5,\dots, x_{8n+3}),$$

(2) $$\bar{\phi}(y)=0, \quad \bar{\phi}(x_{4j+1})=0 \quad \textit{for} \quad 1\leqq j\leqq 2n,$$

$$\bar{\phi}(x_{4j+3})=x_{4j+1}\otimes y^2 \quad \textit{for} \quad 1\leqq j\leqq 2n,$$

(3) $$Sq^{2k}x_{2i-1}=(k, i-k-1)x_{2i+2k-1}.$$

Remark 1.9″. $\bar{\phi}(x_{4j+3}+\text{decomp.})\neq 0$.

Proposition 1.10. *There exist generators* $y\in H^1(PO(4n+2); \boldsymbol{Z}_2)$ *and*

$z_i \in H^i(PO(4n+2); \boldsymbol{Z}_2)$, $2 \leqq i \leqq 4n+1$, *such that*

(1) $$H^*(PO(4n+2); \boldsymbol{Z}_2) \cong \Delta(y, z_2, \dots, z_{4n+1}),$$

(2) $$\bar{\phi}(y)=0, \quad \bar{\phi}(z_{2k})=0 \qquad \text{for} \quad 1 \leqq k \leqq 2n,$$

$$\bar{\phi}(z_{2k+1})=z_{2k} \otimes y \qquad \text{for} \quad 1 \leqq k \leqq 2n,$$

(3) $$Sq^i z_k=(k-j, j) z_{j+k}.$$

Notation. $PS(X; p)$ = the Poincaré series of X over $\boldsymbol{Z}_p$, i.e.,

$$PS(X: p)=\sum_{i=0}^{\infty} \{\operatorname{rank} H^i(X; \boldsymbol{Z}_p)\} t^i.$$

Using this expression we obtain from Propositions 1.5, 1.9 and 1.10:

$$PS(G_l(n): p) \cdot PS(C(n, k): p)=PS(G_l(nk): p) \qquad \text{for} \quad (k, p)=1,$$

$$PS(PO(2n): 2) \cdot PS(R(n, k): 2)=PS(PO(2nk): 2).$$

Thus we have

Proposition 1.11. (1) *The cohomology Serre spectral sequence with $\boldsymbol{Z}_p$-coefficient for the fibering $G_l(n) \rightarrow G_l(nk) \rightarrow C(n, k)$ collapses if $(k, p)=1$.*
(2) *The cohomology Serre spectral sequence with $\boldsymbol{Z}_2$-coefficient for the fibering $PO(2n) \rightarrow PO(2nk) \rightarrow R(2n, k)$ collapses.*

Now we choose generators in $H^*(G(4n+2); \boldsymbol{Z}_2)$ and $H^*(PO(4n+2); \boldsymbol{Z}_2)$ appropriately.

Lemma 1.12. *In Proposition* 1.9′ *we may choose generators y, x_{2i+1}, $2 \leqq i \leqq 4n+1$, of $H^*(G(4n+2); \boldsymbol{Z}_2)$ by using the correspondent generators in $H^*(G(4n-2); \boldsymbol{Z}_2)$ and in $H^*(C(4n-2, 2n+1); \boldsymbol{Z}_2)$ as follows*:

$$y=\Delta_{2n-1}^* \circ \Delta_{2n+1}^{*}{}^{-1}(y),$$

$$x_{2i+1}=\Delta_{2n-1}^* \circ \Delta_{2n+1}^{*}{}^{-1}(x_{2i+1}), \qquad 2 \leqq i \leqq 4n-3,$$

$$x_{2i+1}=\Delta_{2n-1}^* \cup p^*(\bar{x}_{2i+1}), \qquad 4n-2 \leqq i \leqq 4n+1.$$

Proof. This is clear from Proposition 1.11. Q.E.D.

Similarly

Lemma 1.13. *In Proposition* 1.10 *we may choose generators* y, z_i, $2\leqq i\leqq 4n+1$, *of* $H^*(PO(4n+2); \boldsymbol{Z}_2)$ *by using the correspondent generators in* $H^*(PO(4n-2); \boldsymbol{Z}_2)$ *and in* $H^*(R(4n-2, 2n+1); \boldsymbol{Z}_2)$ *as follows*:

$$y=\Delta_{2n-1}^{*}\circ\Delta_{2n+1}^{*}{}^{-1}(y),$$

$$z_i=\Delta_{2n-1}^{*}\circ\Delta_{2n+1}^{*}{}^{-1}(z_i), \qquad 2\leqq i\leqq 4n-3,$$

$$z_i=\Delta_{2n-1}^{*}\circ p^*(z_i), \qquad 4n-2\leqq i\leqq 4n+1.$$

Proposition 1.14. (1) *In* $H^*(C(4n-2, 2n+1); \boldsymbol{Z}_2)\cong \Lambda(\bar{x}_{8n-3}, \bar{x}_{8n-1}, \bar{x}_{8n+1}, \bar{x}_{8n+3},\dots)$ *there hold* $Sq^4\bar{x}_{8n-3}=\bar{x}_{8n+1}$ *and* $Sq^4\bar{x}_{8n-1}=\bar{x}_{8n+3}$.
(2) *In* $H^*(R(4n-2, 2n+1); \boldsymbol{Z}_2)\cong \Delta(\bar{z}_{4n-2}, \bar{z}_{4n-1}, \bar{z}_{4n}, \bar{z}_{4n+1},\dots)$ *there hold* $Sq^2\bar{z}_{4n-2}=\bar{z}_{4n}$ *and* $Sq^2\bar{z}_{4n-1}=\bar{z}_{4n+1}$.

Proof. (1) and (2) follow from (3) of Proposition 1.9′ and (3) of Proposition 1.10 respectively. Q.E.D.

Remark. See [9] for the results of the symplectic case.

§2. Quotients of $U(n)$

In this section let p be a prime and n an integer such that $(n, p)=1$. Then obviously

$$H^*(BPU(n); \boldsymbol{Z}_p)\cong H^*(BSU(n); \boldsymbol{Z}_p). \tag{2.1}$$

The following are easily obtained:

$$H^*(BC(n); \boldsymbol{Z}_p)\cong \boldsymbol{Z}_p[\alpha] \quad \textit{with} \quad \deg\alpha=2. \tag{2.2}$$

$$H^*(B\Gamma_p; \boldsymbol{Z}_p)\cong \begin{cases} \boldsymbol{Z}_2[t] & \textit{with} \quad \deg t=1 \quad \textit{for} \quad p=2 \\ \boldsymbol{Z}_p[\mu]\otimes\Lambda(\lambda) & \textit{with} \quad \deg\mu=2,\ \deg\lambda=1, \\ & \delta\lambda=\mu \quad \textit{for} \quad p\textit{: odd}, \end{cases} \tag{2.3}$$

where δ is the Bockstein operator.

Consider the cohomology Serre spectral sequence with $\boldsymbol{Z}_p$-coefficient associated with the fibering:

(2.4) $$BC(n) \xrightarrow{i'} BU(n) \longrightarrow BPU(n),$$

where i' is induced from the natural inclusion $C(n) \subset U(n)$. The map i'^* is epimorphic since the spectral sequence collapses by (2.2) and by the fact that $H^3(BPU(n); \boldsymbol{Z}_p) = H^3(BSU(n); \boldsymbol{Z}_p) = 0$. Let $j: \Gamma_p \subset C(n)$ be the inclusion. Then

(2.5) $$\operatorname{Im} j^* \cong \begin{cases} \boldsymbol{Z}_p[\mu] & \text{for } p: \text{odd} \\ \boldsymbol{Z}_2[t^2] & \text{for } p=2. \end{cases}$$

Putting $i = i' \circ j$ and choosing μ (or t) suitably we get

(2.6) $$i^*(c_1) = \begin{cases} \mu & \text{for } p: \text{odd} \\ t^2 & \text{for } p=2. \end{cases}$$

Let $\{E_r^{*,*}\}$ be the cohomology Serre spectral sequence with $\boldsymbol{Z}_p$-coefficient associated with the fibering $U(n) \xrightarrow{\pi} U(n)/\Gamma_p \longrightarrow B\Gamma_p$. Since the generators in $H^*(U(n); \boldsymbol{Z}_p) \cong \Lambda(u_1, u_3, \dots, u_{2n-1})$ are universally transgressive, they are transgressive with respect to this fibering. In particular we have

$$\tau(u_1) = i^*(c_1)$$

where τ is the transgression.

Therefore $E_3^{a,b} = 0$ if $a \geqq 2$, and hence

(2.7) $$E_3 \cong E_\infty \cong \begin{cases} \Lambda(\lambda) \otimes \Lambda(u_3, u_5, \dots, u_{2n-1}) & \text{for } p: \text{odd} \\ \Lambda(t) \otimes \Lambda(u_3, u_5, \dots, u_{2n-1}) & \text{for } p=2. \end{cases}$$

Proposition 2.8. *$H^*(U(n)/\Gamma_p; \boldsymbol{Z})$ is p-torsion free and hence it is torsion free.*

Proof is left to the reader.

It follows from this proposition

Theorem 2.9. *Let* $(n, p)=1$. *Then*

$$H^*(U(n)/\Gamma_p; \boldsymbol{Z}_p) \cong \Lambda(\bar{\lambda}, u'_3, \ldots, u'_{2n-1})$$

such that

(1) $\bar{\lambda}$ *and* u'_{2i-1} *are universally transgressive* (*and hence they are primitive*),

(2) $\deg \bar{\lambda}=1$ *and* $\deg u'_{2i-1}=2i-1$,

(3) $\pi^*(u'_{2i-1})=u_{2i-1}$ *for the projection* $\pi: U(n) \to U(n)/\Gamma_p$.

Proof. (1) and (2) follow from (2.7) and the Borel's theorem (Theorem 13.1 of [4]). (3) is clear, since $\pi^*(u'_{2i-1}) \neq 0$ by (2.7) and since $\pi^*(u'_{2i-1})$ are universally transgressive. Q.E.D.

§3. The E_2-term of the Eilenberg-Moore Spectral Sequence

Put $A=H^*(G(4n+2); \boldsymbol{Z}_2)$ for simplicity and regard A as a coalgebra over $\boldsymbol{Z}_2$, where the coalgebra structure $\bar{\phi}$ is given by Proposition 1.9′.

Let L be a $\boldsymbol{Z}_2$-submodule of $A^+=\sum_{i>0} H^i(G(4n+2); \boldsymbol{Z}_2)$ generated by $\{y, y^2, x_{4j+1}, x_{4i+3}\}$, $1 \leqq i, j \leqq 2n$. Let $s: L \to sL$ be the suspension. We express the corresponding elements as $sL=\{a_2, a_3, a_{4j+2}, b_{4i+4}\}$, $1 \leqq i, j \leqq 2n$. Let $\iota: L \to A$ be the inclusion and $\theta: A \to L$ the projection

$$\begin{array}{ccccc} L & \longrightarrow & A & \longrightarrow & L \\ & {\scriptstyle s^{-1}}\nwarrow & \bar{\iota}\uparrow\ \downarrow\bar{\theta} & \swarrow{\scriptstyle s} & \\ & & sL & & \end{array}$$

such that $\theta \circ \iota = 1_L$. Define $\bar{\theta}: A \to sL$ by $\bar{\theta}=s \circ \theta$ and $\bar{\iota}: sL \to A$ by $\bar{\iota}=\iota \circ s^{-1}$. Consider the tensor algebra $T(sL)$. Denote by I the ideal of $T(sL)$ generated by $\mathrm{Im}((\bar{\theta} \otimes \bar{\theta}) \circ \bar{\phi}) \circ \mathrm{Ker}\, \bar{\theta}$. Put $\bar{X}=T(sL)/I$. Then $\bar{X} \cong \boldsymbol{Z}_2[a_2, a_3, a_{4j+2}, b_{4i+4}]$, $1 \leqq i, j \leqq 2n$.

The map $\bar{d}=(\bar{\theta} \otimes \bar{\theta}) \circ \bar{\phi} \circ \bar{\iota}$ on sL can be extended over $\bar{X}$, since $\bar{d}(I) \subset I$. Further, $\bar{d}$ satisfies $\bar{d} \circ \bar{d}=0$ on $\bar{X}$. So $\bar{X}$ is a differential algebra. Now we construct the twisted tensor product $X=A \otimes \bar{X}$ with respect to $\bar{\theta}$ following Brown (cf. [7], [8] or [13]). Then $X=A \otimes \bar{X}$ is a differential A-comodule with the differential operator $d=1\otimes\bar{d}+(1\otimes\psi)\circ(1\otimes\bar{\theta}\otimes 1)\circ\phi\otimes 1$, where ϕ is the diagonal structure in A and ψ is the multiplication in $\bar{X}$. More concretely,

$$dy=a_2,\quad dy^2=a_3,$$

$$dx_{4j+1}=a_{4j+2},\quad 1\leqq j\leqq 2n,$$

$$dx_{4i+3}=b_{4i+4}+x_{4i+1}a_3,\quad 1\leqq i\leqq n.$$

Now we define weight in X as follows:

A:	y	y^2	x_{4j+1}	x_{4i+3}
$\bar{\theta}\downarrow$				
$\bar{X}$:	a_2	a_3	a_{4j+2}	b_{4i+4}
weight	0	0	0	1

The weight of a monomial is a sum of the weight of each element. Put $F_i=\{x \mid \text{weight } x\leqq i\}$. Then

$$E_0X=\sum_i F_i/F_{i-1}$$
$$\cong \Lambda(y, y^2, x_{4j+1}, x_{4i+3})\otimes \boldsymbol{Z}_2[a_2, a_3, a_{4j+2}, b_{4i+4}],$$

where the induced differential operator d_0 is given by

$$d_0y=a_2,\quad d_0y^2=a_3,\quad d_0x_{4j+1}=a_{4j+2},\quad d_0x_{4i+3}=b_{4i+4}.$$

Thus E_0X is acyclic and hence X is acyclic. Namely $X=A\otimes\bar{X}$ is an injective resolution for A over $\boldsymbol{Z}_2$. Therefore by definition

$$H^*(\bar{X}:\bar{d})=\mathrm{Cotor}^A(\boldsymbol{Z}_2, \boldsymbol{Z}_2).$$

As described above the differential operator $\bar{d}$ in $\bar{X}=\boldsymbol{Z}_2[a_2, a_3, a_{4j+2}, b_{4i+4}]$ is given by

$$\bar{d}a_i=0\qquad \text{for}\quad i=2, 3, 4j+2\quad (1\leqq j\leqq 2n),$$

$$\bar{d}b_{4i+4}=a_{4i+2}a_3\qquad (1\leqq i\leqq 2n).$$

For simplicity we put $P=\boldsymbol{Z}_2[a_{4j+2};\, 1\leqq j\leqq 2n]$ and $Q=\boldsymbol{Z}_2[b^2_{4i+4};$

$1 \leqq i \leqq 2n]$. Let C be a submodule of $\bar{X}$ generated by $\{b_8^{\varepsilon_1} \dots b_{4n+4}^{\varepsilon_n};\ \varepsilon_i = 0 \text{ or } 1\}$. Then as a module

$$\bar{X} \cong \boldsymbol{Z}_2[a_2] \otimes Q \otimes \boldsymbol{Z}_2[a_3] \otimes P \otimes C.$$

We remark that as a chain complex, $\bar{X}$ may be thought of as a tensor product of $(\boldsymbol{Z}_2[a_2] \otimes Q)$ with a trivial differential operator d_0 and $(\boldsymbol{Z}_2[a_3] \otimes P \otimes C)$ with a differential operator d_1 such that $d_1(a_3) = d_1(a_{4j+2}) = 0$ and $d_1(b_{4i+4}) = a_3 a_{4i+2}$. Therefore

$$H(\bar{X}: \bar{d}) \cong H(\boldsymbol{Z}_2[a_2] \otimes Q: d_0) \otimes H(\boldsymbol{Z}_2[a_3] \otimes P \otimes C: d_1)$$

$$\cong \boldsymbol{Z}_2[a_2] \otimes Q \otimes H(\boldsymbol{Z}_2[a_3] \otimes P \otimes C: d_1).$$

For $f \in P \otimes C$ there hold $d_1(f) = a_3 \bar{f}$ for some $\bar{f} \in P \otimes C$. Then we define $\bar{d}_1: P \otimes C \to P \otimes C$ by $\bar{d}_1(f) = \dfrac{d_1(f)}{a_3}$.

Lemma 3.1. *The chain complex $(P \otimes C: \bar{d}_1)$ is acyclic.*

Proof. Consider the Koszul resolution of the exterior algebra $\Lambda(b_{4i+4};\ 1 \leqq i \leqq 2n)$. Q.E.D.

Proposition 3.2. *Let $f \in \boldsymbol{Z}_2[a_3] \otimes P \otimes C$. Then $d_1 f = 0$ iff there exists an element $g \in \boldsymbol{Z}_2[a_3] \otimes P \otimes C$ such that $d_1(g) = a_3 f$, or else $f = 1 \otimes 1 \otimes 1$.*

Proof. Sufficiency is clear, since $\bar{X}$ is a polynomial algebra.

(Necessity) It suffices to prove necessity for an element $f \in \boldsymbol{Z}_2 \otimes P \otimes C \cong P \otimes C$. Suppose $d_1(f) = 0$. Then $a_3 \bar{f} = 0$, and hence $\bar{f} = 0$. So by definition $\bar{d}_1(f) = 0$, from which we deduce that $f = 1 \otimes 1 \otimes 1$ or else by Lemma 3.1 that there is an element $g \in P \otimes C$ such that $\bar{d}_1(g) = f$. Thus $a_3 f = d_1(g)$. Q.E.D.

Let $I = (i_1, \dots, i_r)$ be a sequence of integers satisfying

$$(3.3) \qquad 1 \leqq r \leqq 2n \quad \textit{and} \quad 1 \leqq i_1 < \cdots < i_r \leqq 2n.$$

We put $y(I) = \dfrac{1}{a_3} \bar{d}(b_{4i_1+4} \dots b_{4i_r+4})$.

It follows from Proposition 3.2 that a system of generators of

$\operatorname{Ker}\bar{d}$ over $\boldsymbol{Z}_2[a_2, a_3, a_{4j+2}, b_{4i+4}^2]$ is $\{1, y(I)\}$, where I runs over all sequences satisfying (3.3).

Theorem 3.4. *For* $A=H^*(G(4n+2); \boldsymbol{Z}_2)$

$$\operatorname{Cotor}^A(\boldsymbol{Z}_2, \boldsymbol{Z}_2)\cong \boldsymbol{Z}_2[a_2, a_3, x'_{8l+8}, y(I)]/R,$$

where $x'_{8l+8}=\{b_{4l+4}^2\}$ *for* $1\leqq l\leqq 2n$ *and* I *runs over all sequences satisfying* (3.3). *Further* R *is the ideal generated by* $a_3y(I)$, $y(I)^2+\sum_{j=1}^{r} x'_{8i_1+8}\cdots a_{4i_j+2}^2\cdots x'_{8i_r+8}$ *and* $y(I)y(J)+\sum_i f_iy(I_i)$, *where* f *is a polynomial of* a_2, a_3, x'_{8i_j+8}*'s.*

Remark 3.5. $y(\{i\})=a_{4i+2}$. For $I=(i_1,\dots, i_r)$, $(r\geqq 2)$, $y(I)$ can be defined inductively. Put $I'=(i_1,\dots, i_{r-1})$. Suppose that $y(I')=\{\frac{1}{a_3}\bar{d}(b_{4i_1+4}\dots b_{4i_{r-1}+4})\}$ is defined. Then $y(I)=y(I', i_r)=\{(b_{4i_1+4}\dots b_{4i_{r-1}+4})a_{4i_r+2}+y(I')b_{4i_r+4}\}=\langle y(I'), a_3, a_{4i_r+2}\rangle$, the Massey product.

Remark 3.6. The relation $y(I)y(J)+\Sigma f_iy(I_i)$ can be obtained by calculation on the cochains, since $\{1, y(I)\}$ is a system of generators over $\boldsymbol{Z}_2[a_2, a_3, x'_{8l+8}]$.

Now we consider the case $A=H^*(PO(4n+2); \boldsymbol{Z}_2)$. By a similar argument to the before we have $\bar{X}=\boldsymbol{Z}_2\{a_2, a_{2j+1}, b_{2i+2}\}/R$, $1\leqq i, j\leqq 2n$, where R is the ideal generated by $[a_{2k+1}, b_{2k+2}]+a_{4k+1}a_2$, $1\leqq k\leqq n$ and $[r, s]$ for other pairs of generators (r, s) ($[x, y]=xy+yx$). We define weight in $X=A\otimes\bar{X}$, the twisted tensor product with respect to $\bar{\theta}$:

A	y	z_{2j}	z_{2i+1}
$\downarrow \bar{\theta}$			
$\bar{X}$	a_2	a_{2j+1}	b_{2i+2}
weight	0	0	1

Put $F_i=\{x\mid \text{weight } x\leqq i\}$ as before. Then

$$\begin{aligned} E_0X&=\Sigma F_i/F_{i-1}\\ &\cong \Lambda(y)\otimes\Lambda(z_{2j}, z_{2i+1})\otimes \boldsymbol{Z}_2[a_2, a_{2j+1}, b_{2i+2}],\end{aligned}$$

where the induced differential operator is given by $d_0 y=a_2$, $d_0 z_{2j}=a_{2j+1}$ and $d_0 z_{2i+1}=b_{2i+2}$. It shows that $E_0 X$ and hence X is acyclic.

The differential operator $\bar{d}$ in $\bar{X}$ is given by $\bar{d}a_j=0$ for any j and $\bar{d}b_{2i+2}=a_{2i+1}a_2$. By a similar, although a little bit complicated, calculation to the before, we obtain the following.

For a sequence of integers $I=(i_1,\ldots, i_r)$ satisfying (3.3) we put $y'(I)=\frac{1}{a_2}\bar{d}(b_{2i_1+2}\cdots b_{2i_r+2})$.

Theorem 3.7. *For* $A=H^*(PO(4n+2); \boldsymbol{Z}_2)$

$$\mathrm{Cotor}^A(\boldsymbol{Z}_2, \boldsymbol{Z}_2)\cong \boldsymbol{Z}_2[a_2, x'_{4l+4}, y'(I)]/R,$$

where $x'_{4l+4}=b^2_{2l+2}+a_2 b_{4l+2}$ *for* $1\leqq l\leqq n$ *and* $=b^2_{2l+2}$ *for* $n+1\leqq l\leqq 2n$ *and* I *runs over all sequences satisfying* (3.3). *Further* R *is the ideal generated by* $a_2 y'(I)$, $y'(I)^2+\sum_{j=1}^{r} x'_{4i_1+4}\cdots a^2_{2i_j+1}\cdots x'_{4i_r+4}$ *and* $y'(I)y'(J)+\sum_i f'_i y'(I_i)$.

Remark 3.8. $y'(\{i\})=a_{2i+1}$, For $I=(i_1,\ldots, i_r)$, $y'(I)$ can also be defined inductively. i.e.,

$$y'(I)=\langle y'(I'), a_2, a_{2i_r+1}\rangle, \text{ where } I=(I', i_r).$$

The following results can easily be obtained.

Proposition 3.9.

(1) $$\mathrm{Cotor}^{H^*(U(2n+1);\boldsymbol{Z}_2)}(\boldsymbol{Z}_2, \boldsymbol{Z}_2)\cong \boldsymbol{Z}_2[\bar{c}_1,\ldots, \bar{c}_{2n+1}],$$

with $\deg \bar{c}_i=2i$.

(2) $$\mathrm{Cotor}^{H^*(U(2n+1)/\Gamma_2;\boldsymbol{Z}_2)}(\boldsymbol{Z}_2, \boldsymbol{Z}_2)\cong \boldsymbol{Z}_2[a'_2, \bar{c}'_2,\ldots, \bar{c}'_{2n+1}],$$

with $\deg a'_2=2$ *and* $\deg \bar{c}'_i=2i$.

(3) $$\mathrm{Cotor}^{H^*(SO(4n+2);\boldsymbol{Z}_2)}(\boldsymbol{Z}_2, \boldsymbol{Z}_2)\cong \boldsymbol{Z}_2[\bar{w}_2, \bar{w}_3,\ldots, \bar{w}_{4n+2}],$$

with $\deg \bar{w}_i=i$.

(4) $$\mathrm{Cotor}^{H^*(Sp(2n+1);\boldsymbol{Z}_2)}(\boldsymbol{Z}_2, \boldsymbol{Z}_2)\cong \boldsymbol{Z}_2[\bar{q}_1,\cdots, \bar{q}_{2n+1}],$$

with $\deg \bar{q}_i=4i$.

(5) $$\mathrm{Cotor}^{H^*(PSp(2n+1);Z_2)}(\boldsymbol{Z}_2, \boldsymbol{Z}_2) \cong \boldsymbol{Z}_2[a'_2, a'_3, \bar{q}'_2, \cdots, \bar{q}'_{2n+1}],$$

with $\deg a'_i = i$ *and* $\deg \bar{q}'_i = 4i$.

(6) $$\mathrm{Cotor}^{H^*(SU(4n+2);Z_2)}(\boldsymbol{Z}_2, \boldsymbol{Z}_2) \cong \boldsymbol{Z}_2[\bar{c}_2, \ldots, \bar{c}_{2n+1}],$$

with $\deg \bar{c}_i = 2i$.

§4. Collapsing of the Eilenberg-Moore Spectral Sequence

Let G be a topological group. In 1959 Eilenberg-Moore constructed a new type of spectral sequence $\{E_r(G), d_r\}$ such that

(1) $$E_2(G) \cong \mathrm{Cotor}^{H^*(G,Z_p)}(\boldsymbol{Z}_p, \boldsymbol{Z}_p),$$

(2) $$E_\infty(G) \cong \mathscr{G}r H^*(BG; \boldsymbol{Z}_p).$$

Furthermore, this spectral sequence satisfies naturality for a homomorphism $f: G \to G'$. We denote by $f^*: E_r(G') \to E_r(G)$ the induced homomorphism.

In this section we will show that the Eilenberg-Moore spectral sequence collapses for various (G, p). In particular, we will show that for $G = G(4n+2)$ and $PO(4n+2)$ the Eilenberg-Moore spectral sequence with $\boldsymbol{Z}_2$-coefficient collapses.

The following directly follows from Theorem 2.9:

Proposition 4.1. *Let* $(n, p) = 1$. *Then the Eilenberg-Moore spectral sequence collapses for* $(G, p) = (U(n)/\Gamma_p, p)$.

By Kono [9] $H^*(PSp(2n+1); \boldsymbol{Z}_2)$ is transgressively generated and hence we have

Proposition 4.2. *The Eilenberg-Moore spectral sequence collapses for* $(G, p) = (PSp(2n+1), 2)$.

The following result will be used below. The proof is easy and left to the reader.

Proposition 4.3. (1) *The Eilenberg-Moore spectral sequence col-*

lapses for $G=U(2n+1)/\Gamma_2$, $SO(4n+2)$, $U(2n+1)$, $PSp(2n+1)$, $SU(4n+2)$ *and* $Sp(2n+1)$.
(2) *The elements* $\bar{c}_i$ *and* $\bar{w}_i$ *in Proposition* 3.9 *represent* c_i *and* w_i *respectively. The elements* $\bar{q}'_i$ *and* $\bar{c}'_i$ *do* q'_i *and* c'_i *in* $H^*(BG; \boldsymbol{Z}_2)$ *such that* $\pi^*(c'_i)=c_i+(\text{decomp.})$ *and* $\pi^*(q'_1)=q_i+(\text{decomp.})$, *where* π *is the covering homomorphism.*

For simplicity we use the following

Notation.

$$A_1=H^*(U(2n+1); \boldsymbol{Z}_2),$$

$$A_2=H^*(U(2n+1)/\Gamma_2; \boldsymbol{Z}_2),$$

$$A_3=H^*(SO(4n+2); \boldsymbol{Z}_2),$$

$$A_4=H^*(PO(4n+2); \boldsymbol{Z}_2),$$

$$B_1=H^*(Sp(2n+1); \boldsymbol{Z}_2),$$

$$B_2=H^*(PSp(2n+1); \boldsymbol{Z}_2),$$

$$B_3=H^*(SU(4n+2); \boldsymbol{Z}_2),$$

$$B_4=H^*(G(4n+2); \boldsymbol{Z}_2).$$

Case I. $H^*(PO(4n+2); \boldsymbol{Z}_2)$.

Consider the commutative diagram

$$\begin{array}{ccc} U(2n+1) & \xrightarrow{i} & SO(4n+2) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ U(2n+1)/\Gamma_2 & \xrightarrow{i} & PO(4n+2) \end{array}$$

where π is the projection and i's are the standard maps (cf. §6).

Lemma 4.4. *The elements* $a'_2\in \mathrm{Cotor}^{A_2}(\boldsymbol{Z}_2, \boldsymbol{Z}_2)$ *and* $a_2\in \mathrm{Cotor}^{A_4}(\boldsymbol{Z}_2, \boldsymbol{Z}_2)$ *are permanent cycles and* $i^\#(a_2)=a'_2$.

Proof. Recall $H^*(B\boldsymbol{Z}_2; \boldsymbol{Z}_2)\cong \boldsymbol{Z}_2[t]$. In the commutative diagram

$$\begin{array}{ccc} & BZ_2 & \\ \swarrow & & \searrow \\ BU(2n+1) & \longrightarrow & BSO(4n+2) \\ \downarrow \pi & & \downarrow \pi \\ B(U(2n+1)/Z_2) & \longrightarrow & BPO(4n+2) \end{array}$$

the elements a_2 and a'_2 represent the trangression images of t, and hence they are permanent cycles. For dimensional reason we have $i^{\#}(a_2)=a'_2$.
Q.E.D.

The following relations among the elements in Theorem 3.7 and Proposition 3.9 are easily checked to be true:

(4.5.1) $\pi^{\#}(x'_{4i})=\bar{w}_{2i}^2+W_i$, *where* W_i *is a sum of monomials containing elements of lower degree,*

(4.5.2) $$i^{\#}(\bar{w}_{2i})=\bar{c}_i+(\text{decomp.}),\quad (\text{see } \S 6),$$

(4.5.3) $$\pi^{\#}(\bar{c}'_i)=\bar{c}_i+(\text{decomp.}),$$

(4.5.4) $$\pi^{\#}(a_2)=\pi^{\#}(a'_2)=0.$$

Therefore

(4.6) $i^{\#}(x'_{4i})=\bar{c}'^2_i+\gamma_i$, *where* γ_i *is a sum of monomials containing elements of lower degree.*

Let $E_r(1)$ be the Eilenberg-Moore spectral sequence with Z_2-coefficient for $PO(4n+2)$ and $\{E_r(2), d_r\}$ be the cartesian product of the Eilenberg-Moore spectral sequences of $U(2n+1)/\Gamma_2$ and $SO(4n+2)$, i.e.,

$$E_r(2)=\text{Cotor}^{A_2}(Z_2, Z_2)\times\text{Cotor}^{A_3}(Z_2, Z_2) \text{ and } d_r=0$$

for all $r\geqq 2$. Then the map $i^{\#}\times\pi^{\#}$ induces a homomorphism between the spectral sequences:

$$E_r(1)\longrightarrow E_r(2) \quad \text{for } r\geqq 2.$$

Lemma 4.7. $i^{\#}\times\pi^{\#}$: $E_2(1)\to E_2(2)$ *is injective.*

Proof. Let f_1 be a sum of monomials containing a_2 and f_2 a sum of those not containing a_2. Suppose $(i^{\#}\times\pi^{\#})(f_1+f_2)=0$ from which $\pi^{\#}(f_1+f_2)=\pi^{\#}(f_2)=0$ and hence $f_2=0$ by (4.5.1). Meanwhile $(i^{\#}\times\pi^{\#})(f_1+f_2)=0$ implies $i^{\#}(f_1+f_2)=0$, which implies $i^{\#}(f_1)=0$, and hence $f_1=0$ by (4.6). Thus $i^{\#}\times\pi^{\#}$ is injective. Q.E.D.

Thus we have shown

Theorem 4.8. *The Eilenberg-Moore spectral sequence with $\boldsymbol{Z}_2$-coefficient collapses for $G=PO(4n+2)$.*

In fact, Lemma 4.7 indicates that all differentials in $E_r(1)$ are trivial. An immediate corollary is

Theorem 4.9. *As a module*

$$H^*(BPO(4n+2);\boldsymbol{Z}_2)\cong\boldsymbol{Z}_2[a_2,x'_{4l+4},y'(I)]/R,$$

where $1\leqq l\leqq 2n$, *I runs over all sequences satisfying* (3.3) *and R is the ideal generated by* $a_2y'(I)$, $y'(I)^2+\sum_{j=1}^{r}x_{4i_1+4}\cdots a^2_{2i_j+1}\cdots x_{4i_r+4}$ *and* $y'(I)y'(J)+\sum_i f'_iy'(I_i)$.

Case II. $H^*(G(4n+2);\boldsymbol{Z}_2)$.

Consider the commutative diagram

$$\begin{array}{ccc} Sp(2n+1) & \xrightarrow{\ i\ } & SU(4n+2) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ PSp(2n+1) & \xrightarrow{\ i\ } & G(4n+2) \end{array}$$

where π is the projection and i's are the standard maps.

Lemma 4.4′. *The elements $a_i\in\mathrm{Cotor}^{B_4}(\boldsymbol{Z}_2,\boldsymbol{Z}_2)$ and $a'_i\in\mathrm{Cotor}^{B_2}(\boldsymbol{Z}_2,\boldsymbol{Z}_2)$ are permanent cycles and $i^{\#}(a_i)=a'_i$ for $i=2,3$.*

Proof is similar to that of Lemma 4.4.

The following relations among the elements in Theorem 3.4 and Proposition 3.9 are easily checked to be true:

(4.10.1) $\pi^{\#}(x'_{8l+8})=\bar{c}^2_{2l+2}+v_l$,

(4.10.2) $\pi^{\#}(\bar{q}'_i)=\bar{q}_i+(\text{decomp.})$,

where v_l *is a sum of monomials containing elements of lower degree,*

(4.10.3) $\pi^{\#}(a_i)=\pi^{\#}(a'_i)=0$ *for* $i=2, 3$,

(4.11) $i^{\#}(\bar{c}_{2i})=\bar{q}_i+(\text{decomp.})$.

Lemma 4.7′. *Let* $f\in \mathrm{Cotor}^{B_4}(\boldsymbol{Z}_2, \boldsymbol{Z}_2)$ *such that* $\deg f$ *is odd. Then* $i^{\#}(f)=0$ *iff* $f=0$.

Proof. $i^{\#}(x'_{8l+8})=\bar{q}'^2_{l+1}+Q_l$,

where Q_l is a sum of monomials containing elements of lower degree. So the elements $i^{\#}(x'_{8l+8})$, $1\leqq l\leqq 2n$, $i^{\#}a_3$ and $i^{\#}a_2$ are algebraically independent. Q.E.D.

Theorem 4.10. *The Eilenberg-Moore spectral sequence with* $\boldsymbol{Z}_2$-*coefficient collapses for* $G=G(4n+2)$.

Proof. Recall that a_2 and a_3 are permanent cycles. All generators of $\mathrm{Cotor}^{B_4}(\boldsymbol{Z}_2, \boldsymbol{Z}_2)$ except a_3 are of even degree. So $d_r(\alpha)$ is of odd degree for $\alpha\in\{y(I), x'_{8l+8}\}$. By naturality $i^{\#}d_r(\alpha)=d_r i^{\#}(\alpha)=0$. Hence by Lemma 4.7′ $d_r(\alpha)=0$. Thus all generators survive into E_∞. Q.E.D.

Immediate corollaries are

Theorem 4.11. *As a module*

$$H^*(BG(4n+2); \boldsymbol{Z}_2)\cong \boldsymbol{Z}_2[a_2, a_3, x'_{8l+8}, y(I)]/R,$$

where $x'_{8l+8}=\{b^2_{4l+4}\}$ *for* $1\leqq l\leqq 2n$ *and* I *runs over all sequences satisfying* (3.3) *and* R *is the ideal generated by* $a_3y(I)$, $y(I)^2+\sum_{j=1}^{r} x_{8i_1+8}\cdots a^2_{4i_j+2}\cdots x_{8i_r+8}$ *and* $y(I)y(J)+\sum_i f_i y(I_i)$.

Theorem 4.12. *As a module*

$$H^*(BPU(4n+2); \boldsymbol{Z}_2)\cong \boldsymbol{Z}_2[a_2, a_3, x'_{8l+8}, y(I)]/R$$

with the same l, I *and* R *as in Theorem* 4.11.

§5. Some Generators in $H^*(BG(4n+2); Z_2)$ and $H^*(BPO(4n+2); Z_2)$

Let G be a compact, connected Lie group and H its closed subgroup. Let EG and EH be the total spaces of the universal G- and H-bundles respectively. Then the following diagram is commutative:

$$\begin{array}{ccccc} H & \longrightarrow & EH & \longrightarrow & BH \\ \downarrow{\scriptstyle j} & & \downarrow & & \downarrow \\ G & \longrightarrow & EG & \longrightarrow & BG \\ \downarrow{\scriptstyle p} & & \downarrow & & \| \\ G/H & \longrightarrow & BH & \longrightarrow & BG \end{array}$$

Then naturality of the transgression implies

Lemma 5.1. *Let* k *be a commutative field.*

(1) *If* $x \in H^*(G/H; k)$ *is transgressive with respect to the bottom fibering, then* $p^*(x) \in H^*(G; k)$ *is universally transgressive.*

(2) *If* $x \in H^*(G; k)$ *is universally transgressive, so is* $j^*(x) \in H^*(H; k)$.

(3) *Suppose* $H^i(G/H; k)=0$ *for* $i<n$. *Let* $x \in H^i(G; k)$ *and* $i<n-1$. *If* $j^*(x)$ *is universally transgressive, so is* x.

Recall the following:

(5.2) $$G(2)=SO(3),$$

(5.3) $H^*(SO(3); Z_2)=Z_2[a]/(a^4)$ *where* a *is universally transgressive.*

Now we prove

Proposition 5.4. *The elements* a *and* x_{4j+1} *of* $H^*(G(4n+2); Z_2)$, $1 \leqq j \leqq 2n$, *are all universally transgressive.*

Proof. Proof is induction on n. The case $n=0$ is clear from (5.2) and (5.3). Suppose as the inductive hypothesis that the elements a and x_{4j+1}, $1 \leqq j \leqq 2n-1$, are universally transgressive in $H^*(G(4n-2);$

$\boldsymbol{Z}_2$). It follows from Proposition 1.5 and (2), (3) of Lemma 5.1 that the elements a and x_{4j+1}, $1 \leqq j \leqq 2n-1$, are universally transgressive. Clearly the element $\bar{x}_{8n-3}$ is transgressive with respect to the fibering $C(4n-2, 2n+1) \to BG(4n-2) \to BG((4n-2)(2n+1))$, and hence so is $\bar{x}_{8n+1}$, since $\bar{x}_{8n+1} = Sq^4 x_{8n-3}$ by Proposition 1.14. Thus by (1) of Lemma 5.1 the elements x_{8n-3} and x_{8n+1} are universally transgressive. Q.E.D.

It follows from (2.2) and (2.3) that $H^*(BG(2); \boldsymbol{Z}_2) \cong H^*(BSO(3); \boldsymbol{Z}_2) \cong \boldsymbol{Z}_2[a_2, a_3]$, where $a_2 = \tau(a)$ and $a_3 = \tau(a^2)$ with $\deg a_i = i$. As Δ_{2n+1}^*: $H^i(BG(4n+2); \boldsymbol{Z}_2) \to H^i(BG(2); \boldsymbol{Z}_2)$ is isomorphic for $i \leqq 4$ by (1) of Theorem 1.7, we denote by $a_2 = \tau(a)$ and $a_3 = \tau(a^2)$ the generators of $H^i(BG(4n+2); \boldsymbol{Z}_2) \cong \boldsymbol{Z}_2$ for $i = 2, 3$.

Lemma 5.5. $Sq^1 a_3 = 0$ and $Sq^2 a_3 = a_2 a_3$ in $H^*(BG(4n+2); \boldsymbol{Z}_2)$.

Proof. We obtain the above formula by virtue of the Wu formula, since a_i is the inverse image of Δ_{2n+1}^* of the i-th Stiefel-Whitney class. Q.E.D.

Proposition 5.6. *There exist elements* a_{4j+2}, $1 \leqq j \leqq 2n$, *in* $H^*(BG(4n+2); \boldsymbol{Z}_2)$ *such that*

(1) $$\deg a_{4j+2} = 4j+2,$$

(2) $$a_{4j+2} \equiv \tau(x_{4j+1}) \quad \text{mod (decomp.)},$$

(3) $$a_3 a_{4j+2} = 0.$$

Proof. Proof is induction on n. The case $n=0$ is clear from (2.2) and (2.3). Suppose that the assertion is true for $BG(4n-2)$. By Theorem 1.7 the homomorphism $\Delta_{2n+\varepsilon}^*$ is injective for $\deg \leqq 8n - 4\varepsilon + 1$ with $\varepsilon = \pm 1$. Put $a_i = \Delta_{2n-1}^* \circ \Delta_{2n+1}^{*-1}(a_i)$ for $i \leqq 8n-6$. Then a_i satisfies the properties (1), (2), (3) by the inductive hypothesis. For the transgression τ of the fibering

(5.7) $$C(4n-2, 2n+1) \longrightarrow BG(4n-2) \longrightarrow BG((4n-2)(2n+1))$$

we put $a_{8n-2} = \Delta_{2n-1}^* \tau(\bar{x}_{8n-3})$. The element $x_{8n-1} \in H^*(G(4n+2); \boldsymbol{Z}_2)$ is not universally transgressive, since it is not primitive by Proposition

1.9′. So the corresponding element $\bar{x}_{8n-1}$ of $H^*(C(4n-2, 2n+1); \boldsymbol{Z}_2)$ is not transgressive in the fibering (5.7). That is, in the cohomology Serre spectral sequence $\{E_r^{*,*}, d_r\}$ with $\boldsymbol{Z}_2$-coefficient of (5.7) we have $d_3(1\otimes\bar{x}_{8n-1})=a_3\otimes\bar{x}_{8n-3}$, from which we get $a_3\tau(\bar{x}_{8n-3})=0$. Applying Δ_{2n-1}^* we obtain $a_3a_{8n-2}=0$. Thus the element a_{8n-2} satisfies (1), (2), (3). Next, we put

$$a_{8n+2}=\Delta_{2n-1}^*\circ\tau(\bar{x}_{8n+1})+a_2Sq^2a_{8n-2}+a_3Sq^1a_{8n-2}.$$

Then
$$\begin{aligned}a_3a_{8n+2}&=a_3(Sq^4a_{8n-2}+a_2Sq^2a_{8n-2}+a_3Sq^1a_{8n-2})\\&=Sq^4(a_3a_{8n-2})\\&=0.\end{aligned}$$

So the element a_{8n+2} satisfies (1), (2), (3). Q.E.D.

Quite similarly one can prove

Proposition 5.8. *There exist elements* $a_2, a_{2j+1}, 1\leqq j\leqq 2n$, *in* $H^*(BPO(4n+2); \boldsymbol{Z}_2)$ *such that*

(1) $\deg a_2=2, \quad \deg a_{2j+1}=2j+1,$

(2) $a_2=\tau(y), \quad a_{2j+1}\equiv\tau(z_{2j}), \quad 1\leqq j\leqq 2n,$

(3) $a_2a_{2j+1}=0.$

Remark 5.9. The elements a_i in Theorems 4.9, 4.11 and 4.12 are thus the transgression images of some generators in $H^*(G(4n+2); \boldsymbol{Z}_2)$, $H^*(PU(4n+2); \boldsymbol{Z}_2)$ or $H^*(PO(4n+2); \boldsymbol{Z}_2)$. The relations among them are given in Propositions 5.6 and 5.8.

§6. Exterior Power Representations

To begin with we recall the definition of the exterior power representation (p. 90 of [14]).

Let G be a group and k a commutative field. Denote by $GL(n, k)$ the general linear group. Let $A=(a_{ij})\colon G\to GL(n, k)$ be a matrix representation. For a pair of sequences of r integers $I=(i_1,\dots, i_r)$ and $J=(j_1,\dots, j_r)$ such that

$$(*)\qquad 1\leqq i_1<\cdots<i_r\leqq n\,,\qquad 1\leqq j_1<\cdots<j_r\leqq n\,,$$

we define

$$a_{IJ}(x)=\det\begin{pmatrix} a_{i_1j_1}(x)\cdots\cdots\cdots\cdots & a_{i_rj_1}(x)\\ \vdots & \vdots\\ a_{i_1j_r}(x)\cdots\cdots\cdots\cdots & a_{i_rj_r}(x)\end{pmatrix}\qquad \text{for}\quad x\in G\,.$$

Definition 6.1. Let $1\leqq r\leqq n$. We define a representation $A^{(r)}(x)$: $G\to GL(\binom{n}{r}, k)$ by

$$A^{(r)}(x)=\begin{pmatrix} & \overset{J}{\vdots} & \\ I\ \cdots\cdots\cdots & a_{IJ}(x) & \\ & & \end{pmatrix}$$

where I and J run over all sequences satisfying $(*)$. We call $A^{(r)}$ *the exterior power representation of degree r of G.*

If G is a topological group and $k=\mathbf{R}$ or $\mathbf{C}$ and if $A\colon G\to GL(n, k)$ is continuous, so is $A^{(r)}$, namely, $A^{(r)}$ is a representation of G.

When G is a compact group and $k=\mathbf{C}$ (resp. $\mathbf{R}$), we may suppose

$$A^{(r)}\colon G\longrightarrow U(\tbinom{n}{r})\qquad (\text{resp. } A^{(r)}\colon G\longrightarrow O(\tbinom{n}{r}))$$

by making use of the G-invariant Hermitian (resp. Riemannian) metric (see [2]).

Proposition 6.2. *Let G be a subgroup of $GL(n, k)$. Let $A\colon G\to GL(n, k)$ be an inclusion. For $G\ni x=\begin{pmatrix}-1 & & 0\\ & \ddots & \\ 0 & & -1\end{pmatrix}$ we have $A^{(r)}(x)=\begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix}\in GL(\binom{n}{r}, k)$ if r is even.*

Proof. By definition

$$a_{IJ}(x)=\begin{cases}(-1)^r & \text{if}\quad I=J\\ 0 & \text{if}\quad I\neq J\,.\end{cases}$$

Q.E.D.

In the below we regard the identity map $\lambda: G=U(n)\to U(n)$ (or the inclusion $\lambda: SU(n)\to U(n)$) as an n-dimensional complex representation.

Corollary 6.3. *Let n be even. Then there exists a map $\bar{\lambda}^{(2)}$ such that the right diagram commutes*:

$$\begin{array}{ccc} SU(n) & \xrightarrow{\lambda^{(2)}} & U(\binom{n}{2}) \\ & {\scriptstyle\pi}\searrow \quad \nearrow{\scriptstyle\bar{\lambda}^{(2)}} & \\ & G(n) \; . & \end{array}$$

Let t_k be a generator of $H^2(BT^n; \boldsymbol{Z})$ corresponding to the torus

$$T^1 = \left\{ k \left(\begin{array}{ccccc} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 \; e^{i\theta} & & \\ & & & 1 & \\ & & & & \ddots \\ 0 & & & & 1 \end{array} \right) ; 0 \leqq \theta < 2\pi \right\} \subset T_n \subset U(n).$$

Then according to Borel-Hirzebruch (p. 492 of [6]) the total Chern class $c(\lambda^{(2)})$ of the second exterior power representation $\lambda^{(2)}$ is given by

(6.4) $$c(\lambda^{(2)}) = \prod_{1\leqq i<j\leqq n} (1+t_i+t_j) \in H^*(BU(n); \boldsymbol{Z}).$$

Remark 6.5. $t_1+\cdots+t_n=0$ if $G=SU(n)$.

Let α_i, $1\leqq i\leqq n$, be indeterminates with $\deg\alpha_i=1$. Express

$$\prod_{1\leqq i<j\leqq n} (1+\alpha_i+\alpha_j) = \beta_1+\cdots+\beta_n+(\text{higher terms}),$$

where β_k is a homogeneous term of degree k. Denoting by σ_k the k-th elementary symmetric function, we have $\beta_k=a_k\sigma_k(\alpha_1,\dots,\alpha_n)+(\text{decomp.})$ for some integer a_k.

Lemma 6.6. *If n is odd, a_i is odd for $2\leqq i\leqq n$.*

(A proof will be given at the end of the section.)

Let $i: Sp(n)\to SU(2n)$ be the usual inclusion map defined by

$$q_{ij}=\alpha_{ij}+j\beta_{ij} \mapsto c_{ij}=\begin{pmatrix} \alpha_{ij} & -\bar{\beta}_{ij} \\ \beta_{ij} & \bar{\alpha}_{ij} \end{pmatrix},$$

where $\alpha_{i,j}, \beta_{i,j} \in \mathbf{C}$.

Let s_i be a generator of $H^2(BT^n; \boldsymbol{Z})$ corresponding to the torus

$$T^1 = \left\{ \underset{i}{\smile} \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 \ e^{i\theta} & & \\ & & & 1 & \\ & & & & \ddots \\ 0 & & & & 1 \end{pmatrix} \in \underset{;0 \leqq \theta < 2\pi}{Sp(n)} \right\} \subset T^n \subset Sp(n).$$

Then

$$i^*(t_{2i-1}) = s_i \quad \textit{and} \quad i^*(t_{2i}) = -s_i. \tag{6.7}$$

Consider the composite of the maps

$$BSp(n) \xrightarrow{i} BSU(2n) \xrightarrow{\lambda^{(2)}} BU(\tbinom{2n}{2}).$$

Proposition 6.8. *The* mod 2 *reduction of* $i^*c(\lambda^{(2)})$ *is given by*

$$i^*c(\lambda^{(2)}) = \prod_{1 \leqq i < j \leqq n} (1 + s_i^4 + s_j^4) \in H^*(BSp(n); \boldsymbol{Z}_2).$$

Proof.

$$\begin{aligned} i^*c(\lambda^{(2)}) &= i^*(\prod_{1 \leqq i < j \leqq 2n} (1 + t_i + t_j)) && \text{by (6.4)} \\ &= \prod_{1 \leqq i < j \leqq n} (1 + s_i + s_j)^4 && \text{by (6.7)} \\ &= \prod_{1 \leqq i < j \leqq n} (1 + s_i^4 + s_j^4). \end{aligned}$$

Q.E.D.

Next we consider the commutative diagram:

$$\begin{array}{ccccc} BSp(2n+1) & \xrightarrow{i} & BSU(4n+2) & \xrightarrow{\lambda^{(2)}} & BU(\tbinom{4n+2}{2}) \\ \downarrow \pi & & \downarrow \pi & \nearrow \bar{\lambda}^{(2)} & \\ BPSp(2n+1) & \xrightarrow{i} & BG(4n+2) & & \end{array} \tag{6.9}$$

For the mod 2 reduction of the Chern class $c_{4i} \in H^{8i}(BU(\tbinom{4n+2}{2}); \boldsymbol{Z}_2)$ we put

$$x_{8i} = \bar{\lambda}^{(2)*}(c_{4i}) \in H^{8i}(BG(4n+2); \boldsymbol{Z}_2), \qquad 2 \leqq i \leqq 2n+1.$$

Then by the commutativity of the diagram (6.9)

$$i^*\pi^*\Sigma x_{8i} = i^*\pi^*\bar{\lambda}^{(2)*}(\Sigma c_{4i})$$

$$= i^*\lambda^{(2)*}(\Sigma c_{4i})$$

$$= i^* c(\lambda^{(2)})$$

$$= \prod_{1 \leqq i < j \leqq 2n+1} (1 + s_i^4 + s_j^4) \in H^*(BSp(2n+1); \boldsymbol{Z}_2).$$

Apply Lemma 6.6 and we obtain

$$i^*\pi^* x_{8i} = \sigma_i(s_1^4, \dots, s_{2n+1}^4) + (\text{decomp.}).$$

Denoting by q_i the mod 2 reduction of the i-th symplectic Pontrjagin class, we have

$$i^*\pi^* x_{8i} = q_i^2 + P,$$

where P is a sum of monomials containing q_j $(j<i)$.

On the other hand, since $i^*\colon H^m(BSU(4n+2); \boldsymbol{Z}_2) \to H^m(BSp(2n+1); \boldsymbol{Z}_2)$ is trivial for $m \not\equiv 0 \pmod 4$, we have

$$i^*\pi^*(a_2) = i^*\pi^*(a_3) = i^*\pi^*(a_{4j+2}) = 0, \quad \text{and hence}$$

$$i^*\pi^*(y(I)) = 0.$$

Thus we have shown

Theorem 6.10. *There exist non-decomposable elements* $x_{8i+8} \in H^{8i+8}(BG(4n+2); \boldsymbol{Z}_2)$, $1 \leqq i \leqq 2n$, *such that* $i^*\pi^*(x_{8i+8}) = q_{i+1}^2 + P$, *where* P *is a sum of monomials containing* q_j $(j<i+1)$.

Now we turn to the orthogonal case.

Let $\lambda\colon SO(n) \to O(n)$ be the natural inclusion and regard it as a real representation. As before we consider its exterior power representation $\lambda^{(2)}\colon SO(n) \to O(\binom{n}{2})$. The total Stiefel-Whitney class is then given as

$$w(\lambda^{(2)}) = \prod_{1 \leqq i < j \leqq 2n} (1 + t_i + t_j),$$

where t_i is a generator of $H^1(B(\boldsymbol{Z}_2)^n; \boldsymbol{Z}_2)$ corresponding to

$$\boldsymbol{Z}_2 = \left\{ \underset{\smile}{i} \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \cdots & \cdots & & \varepsilon & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} ; \varepsilon = \pm 1 \right\} \subset (\boldsymbol{Z}_2)^n \subset O(n).$$

Remark. $$t_1+\cdots+t_n=0.$$

Let $i\colon U(n)\to SO(2n)$ be the inclusion defined by the correspondence $b+c\sqrt{-1}\mapsto\begin{pmatrix} b & -c \\ c & b\end{pmatrix}$. Let s_i be a generator of $H^1(B(\boldsymbol{Z}_2^n);\boldsymbol{Z}_2)$ corresponding to

$$\boldsymbol{Z}_2=\left\{\underset{\smile}{i}\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \cdots & \cdots & & \varepsilon & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1\end{pmatrix};\ \varepsilon=\pm 1\right\}\subset(\boldsymbol{Z}_2)^n\subset U(n).$$

Then

(6.11) $$i^*(t_{2i-1})=i^*(t_{2i})=s_i.$$

Let w_i be the Stiefel-Whitney class. Then

$i^*(w_{2i-1})=0,$

$i^*(w_{2i})=c_i$, the mod 2 reduction of the i-th Chern class.

Consider the following commutative diagram

$$\begin{array}{ccccc} BU(2n+1) & \xrightarrow{\ i\ } & BSO(4n+2) & \xrightarrow{\ \lambda^{(2)}\ } & BO(\binom{4n+2}{2}) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle \bar{\lambda}^{(2)}} & \\ B(U(2n+1)/\Gamma_2) & \xrightarrow{\ i\ } & BPO(4n+2) & & \end{array}$$

where π is the natural projection and $\bar{\lambda}^{(2)}$ the one induced from $\lambda^{(2)}$. Then

$$i^*\pi^*\bar{\lambda}^{(2)*}\Big(\sum_{i=0}^{l} w_i\Big)=i^*(w(\lambda^{(2)}))\qquad\text{with } l=\binom{4n+2}{2},$$

where $w(\lambda^{(2)})=\prod_{1\leqq i<j\leqq 4n+2}(1+t_i+t_j)$.

So by Lemma 4.6 we have

$$i^*\pi^*\bar{\lambda}^{(2)*}\Big(\sum_{i=0}^{l} w_i\Big)=\prod_{1\leqq i<j\leqq 2n+1}(1+s_i^4+s_j^4).$$

Thus by a similar argument to the unitary case we have

Theorem 6.12. *There exist non-decomposable elements* $x_{4j+4} \in H^{4j+4}(BPO(4n+2); \boldsymbol{Z}_2)$, $1 \leqq j \leqq 2n$, *such that* $i^*\pi^* x_{4j+4} = c_{j+1}^2 + P$, *where* P *is a sum of monomials containing* c_k $(k < j+1)$.

First we consider the case $G = G(4n+2)$. The projection π: $SU(4n+2) \to G(4n+2)$ induces $\pi^\#$: $\mathrm{Cotor}^{B_4}(\boldsymbol{Z}_2, \boldsymbol{Z}_2) \to \mathrm{Cotor}^{B_3}(\boldsymbol{Z}_2, \boldsymbol{Z}_2)$ on the E_2-level of the Eilenberg-Moore spectral sequence. By naturality we have

$$\begin{aligned}\pi^\# x'_{8i+8} &= \pi^\# b_{4i+4}^2 \\ &= c_{2i+2}^2 \quad \text{for} \quad 1 \leqq i \leqq 2n,\end{aligned}$$

which survives in the $E_\infty(SU(4n+2))$-term, since $E_2(SU(4n+2) \cong E_\infty(SU(4n+2)) \cong \mathscr{G}r H^*(BSU(4n+2); \boldsymbol{Z}_2)$ by Proposition 4.3. On the other hand, since $q_{i+1} = i^* c_{2i+2}$, it follows from Theorem 6.10 that for π^*: $H^*(BG(4n+2); \boldsymbol{Z}_2) \to H^*(BSU(4n+2); \boldsymbol{Z}_2)$ we have

$$\pi^* x_{8i+8} = c_{2i+2}^2 + P'. \qquad 1 \leqq i \leqq 2n,$$

where P' is a sum of monomials containing c_j $(j < i+1)$.

Thus we obtain

Theorem 6.13. *The element* $x'_{8i+8} \in \mathrm{Cotor}^{B_4}(\boldsymbol{Z}_2, \boldsymbol{Z}_2)$ *survives in the* $E_\infty(G(4n+2)$*-term and represents* $x_{8i+8} \in H^*(BG(4n+2); \boldsymbol{Z}_2)$.

Similarly,

Theorem 6.13′. *The element* $x'_{4i+4} \in \mathrm{Cotor}^{A_4}(\boldsymbol{Z}_2, \boldsymbol{Z}_2)$ *survives in the* $E_\infty(PO(4n+2))$*-term and represents* $x_{4i+4} \in H^*(BPO(4n+2); \boldsymbol{Z}_2)$.

Proof of Lemma 6.6. Let m be an odd integer. We regard the identity map λ: $U(m) \to U(m)$ as an m-dimensional complex representation as before. Let t_k be a generator of $H^2(BT^m; \boldsymbol{Z})$ corresponding to the torus

$$T^1 = \left\{ \underset{\smile}{k} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & e^{i\theta} & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \; 0 \leqq \theta < 2\pi \right\} \subset T^m \subset U(m).$$

Then by (6.4) the total Chern class of the exterior representation of degree 2 of λ is given by

$$c(\lambda^{(2)})=\prod_{1\leqq i<j\leqq m}(1+t_i+t_j)\in H^*(BU(m);\boldsymbol{Z}).$$

We will show that the integer a_k is odd by taking $\alpha_i=t_i$ and $\beta_i=c_i(\lambda^{(2)})$, the i-th Chern class of $\lambda^{(2)}$.

Let Φ^k be the Adams operation on representations and ch_q the Chern character. Denote by λ^2 the tensor product $\lambda\otimes\lambda$.

Lemma 6.14. (1) $\quad ch_q\Phi^2(\lambda)=2^q ch_q(\lambda)$.

(2) $\Phi^2(\lambda)=\lambda^2-2\lambda^{(2)}$.

(3) $ch_q(\lambda^2)=2m\,ch_q(\lambda)+(\text{decomp.})$.

(4) *Let* $m\geq 3$. *For* $\eta=\lambda$ *or* $\lambda^{(2)}$

$$ch_q(\eta)=\frac{(-1)^q}{(q-1)!}c_q(\eta)+(\text{decomp.}).$$

Proof. (1), (2), (3) follow directly from the definition (also see [1]). (4) follows from the Newton formula. Q.E.D.

By this lemma we have

$$\begin{aligned}ch_i(\lambda^{(2)})&=\frac{1}{2}\{ch_i(\lambda^2)-ch_i(\Phi^2(\lambda))\}\\&=\frac{1}{2}\{2(n-2^{i-1})ch_i(\lambda)\}+(\text{decomp.})\\&=(n-2^{i-1})ch_i(\lambda)+(\text{decomp.}).\end{aligned}$$

Now by (4) we obtain

$$\begin{aligned}c_i(\lambda^{(2)})&=(n-2^{i-1})c_i(\lambda)+(\text{decomp.})\\&=(n-2^{i-1})\sigma_i(t_1,\dots,t_n)+(\text{decomp.}),\end{aligned}$$

where $(n-2^{i-1})$ is odd if $i\geqq 2$. Q.E.D.

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