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# Some Remarks on One-Dimensional Local Domains

By

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Let $R$ be a one-dimensional (noetherian) local domain with field of quotients $Q$. Then any ring extension $S$ of $R$ in $Q$ is obtained as a ring of quotients of some integral extension $C$ of $R$. Here, if $S$ is local and if $C$ can be chosen to be finite over $R$, then we call $S$ an $R$-locality.

If $R$ is analytically ramified, then $R$ does not satisfy the finiteness condition for integral extensions in $Q$ (cf. [3], p. 122, Exercise 1). In other words, $R$ possesses at least one latent singularity with respect to a certain analytic branch of $R$ which can not be resolved by any quadratic dilatations.

The purpose of this note is to give a necessary and sufficient condition for the finiteness of ring extensions $S$ as $R$-modules, and to prove a characterization of $R$-localities by making use of the concept of latent multiplicity found in [1] and [4] (more detailed accounts of this theory may be found in [3]).

## §0. Terminology and Preliminaries

(0.1) By a ring we mean a commutative noetherian ring with identity. The maximal ideal of a local ring $R$ is denoted by $M(R)$ and the multiplicity of $R$ is denoted by $e(R)$.

The integral closure of a ring $R$ in its total quotient ring is called the normalization of $R$ and denoted by $\bar{R}$. A local domain $R$ is said

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to be unibranched if the normalization $\bar{R}$ has exactly one maximal ideal.

The Henselization of a local ring $R$ is denoted by $R^h$; also the completion of $R$ is denoted by $R^*$. Let $R$ be a local domain. Then it is known that there exists a one to one correspondence between the maximal ideals $N$ of $\bar{R}$ and the minimal prime ideals $P$ of $R^h$ such that if $N$ corresponds to $P$, then the normalization of $R^h/P$ is canonically isomorphic to the Henselization of $\bar{R}_N$ and we have

$$P = \mathrm{Ker}(R^h \longrightarrow (\bar{R}_N)^h) \qquad \text{(cf. [2], Theorem 6)}.$$

If $R$ is a 1-dimensional local domain, then the similar assertion holds for the completion $R^*$ of $R$.

(0.2) Let $R$ be a 1-dimensional local domain with field of quotients $Q$ and let $S$ be an integral domain such that $R \subset S \subsetneqq Q$. Then $S$ is a (noetherian) semi-local ring by the theorem of Krull-Akizuki.

For a discrete valuation ring $V$ such that $R \subset V \subset Q$, we define the latent multiplicity $n_V(S)$ of $S$ with respect to $V$ in the following way.

If $S \not\subset V$, then we put $n_V(S) = 0$.

Suppose that $S \subset V$ and let $O$ be the ring of quotients of $S$ with respect to the maximal ideal $M(V) \cap S$. Then $P = \mathrm{Ker}(O^* \to V^*)$ is the rank 0 prime ideal of $O^*$ which canonically corresponds to $V$ (cf. (0.1)). Then we define $n_V(S) = e(O^*_P)$.

(0.3) We next explain certain terminology and results given in [1] which will be used later in this note.

Let $R$, $S$, $Q$ be as in (0.2). If $S$ is local and satisfies the following conditions:

$$M(R)S = M(S), \quad S/M(S) = R/M(R)$$

then $S$ is said to be strongly unramified over $R$. This is equivalent to saying that $S/R$ is a proper divisible $R$-submodule of $Q/R$.

Let $M$ be an Artinian $R$-module. Then $M$ has a composition series of divisible submodules:

$$O = D_0 \subsetneqq D_1 \subsetneqq \cdots\cdots \subsetneqq D_r = d(M)$$

where $d(M)$ is the largest divisible submodule of M. Every composition series of $M$ has the same number of terms and we denote the number by $L_R(M)$ which we call the divisible length of $M$. It is easy to see that $L(M)=0$ if and only if $M$ has a finite length. For a exact sequence of Artinian $R$-modules:

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0,$$

we have $L(M)=L(L)+L(N)$.

Since an Artinian $R$-module is a torsion $R$-module, we have a canonical isomorphism: $M\otimes_R R^*=M$. Thus $M$ has the structure of an $R^*$-module and this structure is unique (cf. [1], Theorem 2.7).

For a divisible submodule $D$ of $M$. if the annihilator $\mathrm{Ann}_{R^*}(D)$ is a primary ideal associated with some rank 0 prime ideal $P$ of $R^*$, then $D$ is said to be $P$-primary and in this case we denote the largest $P$-primary divisible submodule of $M$ by $M(P)$. If there exists no $P$-primary divisible submodule in $M$, we put $M(P)=0$. Then we have $L(M)=\sum L(M(P))$, where $P$ runs over all rank 0 prime ideals in $R^*$. Finally, we note the following two facts which may be used tacitly.

(A). $K=Q/R$ is an Artinian $R$-module and there exists a one to one, order-preserving, correspondence between the proper divisible submodules $D=A/R$ of $K$ and the rank 0 unmixed ideals $I$ of $R^*$ given by $D=K\otimes_R I$, $I=\mathrm{Hom}_R(K, D)$, $A^*=R^*/I$, where $A$ is a strongly unramified extension ring of $R$ (cf. [1], Theorem 6.6).

(B). $D=K\otimes I$ is $P$-primary if and only if $I$ and the zero ideal of $R^*$ have the same primary components except for the $P$-primary components. (cf. [1], pp. 126–128)

## §1.

Let $R$ be a 1-dimensional local domain and let $Q$ be its field of quotients. By $N(R)$ we denote the set of integral domains $S$ such that $R\subset S\subsetneqq Q$ and by $B(R)$ we denote the set of discrete valuation rings $V$ in $N(R)$.

**Lemma 1.1.** *Let $S\in N(R)$, let $V\in B(R)$ and let $P$ be the rank 0 prime ideal of $R^*$ which corresponds to $V$. Then*

$$n_V(R)-n_V(S)=L_R((S/R)(P)).$$

*Proof.* We first assume that $S\subset V$.

Let $(0)=I\cap N$ where $I$ is the $P$-primary component of the zero ideal of $R^*$ and $N:P=N$. Put $d(S/R)=A/R$. Then $L(S/A)=0$, hence $S$ is a finite $A$-module. Let $J$ be the rank 0 unmixed ideal of $R^*$ which corresponds to the divisible submodule $A/R$ and let $J=I'\cap N'$ where $I'$ is the $P$-primary component of $J$ and $N':P=N'$ (since $A\subset V$, $J:P\neq J$). Then $K\otimes(I'\cap N)$ is a $P$-primary divisible submodule of $K=Q/R$ (cf. (0.3), (B)). Let $(S/R)(P)=K\otimes(I_1\cap N)$ where $I_1$ is a $P$-primary ideal. Since $K\otimes(I'\cap N)\subset K\otimes(I_1\cap N)$, we have $I'\subset I_1$. On the other hand, since

$$K\otimes(I_1\cap N)\subset d(S/R)=K\otimes(I'\cap N'),$$

we have $I_1\subset I'$. Therefore, we have $I_1=I'$ and $(S/R)(P)=K\otimes(I'\cap N)$. Since $A^*=R^*/J$ (cf. (0.3), (A)) and

$$P/J=\mathrm{Ker}(A^*\longrightarrow V^*),$$

we see that $\bar{P}=P/J$ is the rank 0 prime ideal of $A^*$ which corresponds to $V$ and $I'/J$ is the $\bar{P}$-primary component of the zero ideal of $A^*$.

Let

$$I=I_1\subsetneqq I_2\subsetneqq\cdots\cdots\subsetneqq I_r=I'\subsetneqq I_{r+1}\subsetneqq\cdots\cdots\subsetneqq I_t=P$$

be a saturated chain of $P$-primary ideals. Then

$$I'/J\subsetneqq I_{r+1}/J\subsetneqq\cdots\cdots\subsetneqq I_t/J=\bar{P}$$

is a saturated chain of $\bar{P}$-primary ideals in $A^*$. Hence we have $n_V(R)=t$ and $n_V(A)=t-(r-1)$. Since $(S/R)(P)=K\otimes(I_r\cap N)$, we see that

$$L((S/R)(P))=r-1=t-(t-(r-1))=n_V(R)-n_V(A).$$

We next prove that $n_V(A)=n_V(S)$.

Let $O$ be the ring of quotients of $S$ with respect to the maximal ideal $M(V)\cap S$. Since $S$ is a finite $A$-module, the Henselization $O^h$ of $O$ is a finite $A^h$-module. Let

$$P_1 = \mathrm{Ker}(A^h \longrightarrow V^h), \quad P_2 = \mathrm{Ker}(O^h \longrightarrow V^h),$$

then we have $A^h/P_1 \subset O^h/P_2 \subset V^h$ and these three local domains are 1-dimensional, unibranched, local domains with the same field of quotients (cf. [2] p. 13). It is easy to see that $n_{V^h}(A^h/P_1) = n_V(A)$ and $n_{V^h}(O^h/P_2) = n_V(O) = n_V(S)$. Therefore we may assume that $A$ is unibranched. Let $P'$ (resp. $P''$) be the unique prime ideal of rank 0 of $A^*$ (resp. $O^* = S^*$) which corresponds to $V$. Since $S$ is a finite $A$-module and $R \subset A \subset S \subset Q$, we see that $zS \subset A$ for some $z \in A$, $z \neq 0$ and since $z$ is not a zero-divisor in $A^*$, $A^*$ and $S^*$ have the same total quotient ring which is equal to $A^*_{P'} = S^* \otimes A^*_{P'} = S^*_{P''}$.

Hence, we have $n_V(A) = n_V(S)$ and $L((S/R)(P)) = n_V(R) - n_V(A) = n_V(R) - n_V(S)$. Thus, the assertion is proved in the case $S \subset V$.

We next assume that $S \not\subset V$. Then $n_V(S) = 0$ from the definition and in the above notation, we have $J = N'$, $N' : P = N'$. Then, $(S/R)(P) = K \otimes N$ and $L((S/R)(P)) = t$ by the similar argument as above. Then $L((S/R)(P)) = n_V(R) = n_V(R) - n_V(S)$. Thus the proof is completed. Q. E. D.

**Theorem 1.2.** *Let $S \in N(R)$. Then*
(1) *$n_V(S) \leqq n_V(R)$ for any $V \in B(R)$,*
(2) *$S$ is a finite $R$-module if and only if*

$$n_V(R) = n_V(S) \qquad \textit{for each} \quad V \in B(R).$$

*Proof.* (1): The proof is immediate from Lemma 1.1.
(2): Since $L(S/R) = \sum L((S/R)(P))$, $L(S/R) = 0$ if and only if $L((S/R)(P)) = 0$ for each rank 0 prime ideal $P$ of $R^*$. By lemma 1.1, $L((S/R)(P)) = 0$ if and only if $n_V(R) = n_V(S)$ where $P$ corresponds to $V \in B(R)$. Since $L(S/R) = 0$ if and only if $S$ is a finite module over $R$, the assertion is proved. Q. E. D.

## §2.

Let $R$, $N(R)$, $B(R)$ be as in §1.

If $S \in N(R)$ is local and is equal to a ring of quotients of some finitely generated ring $R[a_1, \dots, a_n]$ where $a_i \in Q$, then $S$ is called an $R$-locality. Then, for an $R$-locality $S$, there exists an ideal $I$ of $R$ and

$V \in B(R)$ such that $S$ is the quadratic dilatation of $R$ by $I$ with respect to $V$ (cf. [3] p. 141). Also, we note that for a given $R$-locality $S$, we can choose the $a_i$ to be integral over $R$.

**Theorem 2.1.** *Let $S \in N(R)$ and assume that $S$ is local. Let*

$$P_1, \dots, P_n \qquad (\text{resp. } Q_1, \dots, Q_m)$$

*be the rank* 0 *prime ideals of $R^*$ (resp. $S^*$). Then*:

(1) $e(S) \leqq e(R), \qquad m \leqq n,$

(2) $e(R) = e(S)$ *if and only if the following conditions hold*:

(a) *$S$ is a finite $R$-module (hence, $m = n$).*

(b) *If $P_i = Q_i \cap R^*$ for $i = 1, \dots, n$ (by virtue of* (a)), *then we have $e(R^*/P_i) = e(S^*/Q_i)$ $(i = 1, \dots, n)$.*

*Proof.* (1): For $V \in B(R)$ (resp. $V \in B(R)$ such that $S \subset V$), let $P_V$ (resp. $Q_V$) be the rank 0 prime ideal of $R^*$ (resp. $S^*$) which corresponds to $V$. Then we have

$$e(R) = \sum e(R^*/P_V) n_V(R), \quad e(S) = \sum e(S^*/Q_V) n_V(S) \quad (\text{cf. [3] (23.5)}).$$

For each $V \in B(R)$ such that $S \subset V$, we have

$$R^*/P_V \subset S^*/Q_V \subset V^*,$$

and these three local domains have the same field of quotients.

Since $R^*/P_V$ is complete, $S^*/Q_V$ is a finite module over $R^*/P_V$.

Then it is easy to see that $e(S^*/Q_V) \leqq e(R^*/P_V)$ for each $V$ such that $S \subset V$. Since $n_V(S) \leqq n_V(R)$ by Theorem 1.2, we have

$$e(S^*/Q_V) n_V(S) \leqq e(R^*/P_V) n_V(R)$$

for each $V$ such that $S \subset V$. Consequently, we have $e(S) \leqq e(R)$.

Thus the proof of (1) is completed.

(2): Assume that $e(R) = e(S)$.

By the proof of (1), we see that

$$m = n, \quad \text{and} \quad e(R^*/P_V) = e(S^*/Q_V), \quad n_V(S) = n_V(R)$$

for any $V \in B(R)$. Hence, by Theorem 1.2, we see that $S$ is a finite

$R$-module.

The converse of (2) is immediate (by virtue of Theorem 1.2) and we omit the proof. Q. E. D.

**Theorem 2.2.** *The following two statments are equivalent:*

(a) $e(R)=e(S)$ *for any* $R$*-locality* $S$ *which is contained in a fixed* $V\in B(R)$.

(b) $R$ *is unibranched and* $R^*/P$ *is a discrete valuation ring, where* $P$ *is the unique rank* 0 *prime ideal of* $R^*$.

*Proof.* Assume first that (a) is true. Let $x$ be any element in $V$ and let $T$ be the ring of quotients of $R[x]$ with respect to the maximal ideal $M(V)\cap R[x]$. Then $e(R)=e(T)$ by our assumption and hence, by Theorem 2.1, $T$ is integral over $R$. Therefore, we see that $V$ is integral over $R$, i.e., $R$ is unibranched. Let $P$ be the rank 0 prime ideal of $R^*$ which corresponds to the normalization $V=\bar{R}$. Then $R^*/P$ and $V^*$ have the same field of quotients and identifying as $V\subset V^*$, $V^*=R^*/P[V]$. Choose $a_1,\dots, a_m\in V$ such that $V^*=R^*/P[a_1,\dots, a_m]$ and put $S=R[a_1,\dots, a_m]$. Then $S$ is unibranched. Let $Q$ be the unique rank 0 prime ideal of $S^*$. Since $S^*/Q=R^*/P[a_1,\dots, a_m]=R^*/P[V]=V^*$, we have $e(S^*/Q)=1$.

Then, by Theorem 2.1 (2), our assumption implies $e(R^*/P)=1$, i.e., $R^*/P$ is a discrete valuation ring. Conversely, assume that (b) is true and let $S$ be an $R$-locality. Then $S$ is unibranched and letting $Q$ be the rank 0 prime ideal of $S^*$, we have $e(S)=e(S^*/Q)n_V(S)$.

Since $n_V(R)=n_V(S)$ by Theorem 1.2 and $R^*/P=S^*/Q$ by our assumption, we have

$$e(S)=e(S^*/Q)n_V(S)=e(R^*/P)n_V(R)=e(R).$$

Thus the proof is completed. Q. E. D.

**Remark.** (1) By virtue of the above proof, if the statement (b) holds for $R$, we have $e(R)=e(S)$ for any $R$-locality $S$.

(2) In our case, $e(R)=1$ implies that $R$ is regular. In general, let $(R, M)$ be a Macaulay local ring of Krull-dimension $d$ and let $n=\dim_{R/M} M/M^2$. Let $r$ be the number of irreducible components of an ideal $I$ generated by a system of parameters of $R$.

Then it is known that the integer $r$ is an invariant of $R$ and is equal to $\dim_{R/M}(I: M)/I$. Then we have the following:

$e(R) \geqq n-d+1$ and the equality holds if and only if

$$(1) \quad R \text{ is regular}, \quad \text{or} \quad (2) \quad e(R)=r+1\,.$$

*Proof*: First assume that $R/M$ is an infinite field. Then there exists a system of parameters $a_1, \ldots, a_d$ of $R$ such that $e(R)=\text{length}_R R/I$ where $I=(a_1, \ldots, a_d)$ and the $a_i$ modulo $M^2$ are linearly independent over $R/M$ (cf. [3], (24.1), (24.3)). Then

$$e(R)=\text{length}_R R/I \geqq \text{length}_R R/I+M^2$$

$$=1+\text{length}_R M/M^2-\text{length}_R I+M^2/M^2=n-d+1\,.$$

The equality holds if and only if $I+M^2=I$, i.e., $M^2 \subset I$ which implies that $I=M$ or $I: M=M$. If $I=M$, then $R$ is regular and if $I: M=M$, then we have

$$e(R)=\text{length}_R M/I+1=\text{length}_R I: M/I+1=r+1\,.$$

Next, if $R/M$ is a finite field, then take a transcendental element $x$ and consider $R(x)=R[x]_{M[x]}$. Then, by the validity of the theorem of transition for $R$ and $R(x)$ (cf. [3], § 18, § 19), our assertion is also true in this case. Q.E.D.

**Lemma 2.3.** *Let $S \in N(R)$ and $V \in B(R)$. Assume that $S$ is local. Then, $n_V(R)=n_V(S)$ if and only if there an $R$-locality $T$ such that $S \subset T \subset V$.*

*Proof.* First, we assume that $n_V(R)=n_V(S)$. We can easily find an $R$-locality $R_0 \subset V$ such that $R_0$ is unibranched and the normalization of $R_0$ is equal to $V$. Put $T=R_0[S]$. Then $T$ is local and since $R_0$ is an $R$-locality, $T$ is an $S$-locality. By Theorem 1.2, we see that $n_V(R)=n_V(R_0)$ and $n_V(S)=n_V(T)$. Consequently, we have $n_V(R_0)=n_V(T)$ by our assumption. Since $R_0$ is unibranched, $T$ is a finite module over $R_0$ by Theorem 1.2. Hence we see that $T$ is an $R$-locality and $S \subset T \subset V$.

Next, we assume that $T$ is an $R$-locality such that $S \subset T \subset V$. Then, by Theorem 1.2, we see that $n_V(R)=n_V(T)$ and $n_V(S)=n_V(T)$. Hence we get that $n_V(R)=n_V(S)$. Thus the proof is completed. Q.E.D.

**Theorem 2.4.** *Let $V\in B(R)$ and $S_i\in N(R)$ $(i=1, 2)$ and assume that $S_i\subset V$ $(i=1, 2)$. Let $P$ be the rank 0 prime ideal of $R^*$ which corresqonds to $V$.*

*Then, $(S_1/R)(P)=(S_2/R)(P)$ if and only if there exists $T\in N(R)$ such that $T$ is an $S_i$-locality for $i=1, 2$ and $T\subset V$.*

*Proof.* Assume first that $(S_1/R)(P)=(S_2/R)(P)$.

Let $d(S_i/R)=A_i/R$ $(i=1, 2)$. Then $S_i$ is a finite $A_i$-module for $i=1, 2$ and $(A_1/R)(P)=(A_2/R)(P)$ by our assumption. Therefore, we may assume that the $S_i$ are strongly unramified extension rings of $R$. By our assumption, the unmixed ideals of rank 0 of $R^*$ which correspond to the $S_i$ have the same $P$-primary component $J$. Put $K\otimes(J\cap N)=S/R$ where $K=Q/R$ and $N$ is the intersection of the primary components of the zero ideal of $R^*$ except for the $P$-primary component. Then $S$ is a strongly unramified extension ring of $R$ and $S\subset S_i$ for $i=1, 2$. Since $(S_i/R)(P)=(S/R)(P)$ from our construction, we see that $n_V(S_i)=n_V(R)-L((S_i/R)(P))=n_V(R)-L((S/R)(P))=n_V(S)$ $(i=1, 2)$ by Lemma 1.1. Replacing $R$ by $S$ in Lemma 2.3, we see that there exist $S$-localities $T_1, T_2$ such that $S_i\subset T_i\subset V(i=1, 2)$. Let $T$ be the ring of quotients of $C=T_1[T_2]$ with respect to the maximal ideal $M(V)\cap C$. Since the $T_i$ are $S$-localities, $T$ is also an $S$-locality, hence $T$ is an $S_i$-locality for $i=1, 2$. Since $T\subset V$, our assertion is proved.

Conversely, assume the existence of a local domain $T$ as above. Then $(S_i/R)(P)\subset(T/R)(P)$ for $i=1, 2$ and by Lemma 1.1 and Lemma 2.3, we have

$$L((S_i/R)(P))=n_V(R)-n_V(S_i)=n_V(R)-n_V(T)=L((T/R)(P)).$$

Hence, $(S_i/R)(P)=(T/R)(P)$ for $i=1, 2$. Consequently, we have $(S_1/R)(P)=(S_2/R)(P)$ and the proof is completed. Q.E.D.

**Remark.** In the above theorem, suppose that $R$ is a homomorphic image of a two-dimensional regular local ring. Then it is easy to see that the $P$-primary ideals of $R^*$ are linearly ordered by inclusion. Hence, in this case, a local domain $T$ as in the above theorem exists i and only if $n_V(S_1)=n_V(S_2)$.

But this is not true for general case.

**Theorem 2.5.** *Let $S\in N(R)$ and suppose that $S$ is local. Then the following two statements are equivalent*:

(a) *$S$ is an $R$-locality.*

(b) *$n_V(R)=n_V(S)$ for any $V\in B(R)$ such that $S\subset V$.*

*Proof.* The implication (a)$\Rightarrow$(b) is an immediate consequence of Lemma 2.3. Next, assume that (b) holds. Let $C$ be the integral closure of $R$ in $S$. Then the ring of quotients of $C$ with respect to the maximal ideal $N=M(S)\cap C$ is integrally closed in $S$, and hence is equal to $S$ (cf. [3] (33.1)). Since $C$ is semi-local there exists an element $x\in N$ which is not contained in any maximal ideals of $C$ other than $N$. Let $T$ be the ring of quotients of $R[x]$ with respect to the maximal ideal $M(S)\cap R[x]$. Then, $T\subset S$ and $V(\in B(R))$ contains $S$ if and only if $V$ contains $T$.

By Lemma 2.3 and our assumption, we have

$$n_V(T)=n_V(R)=n_V(S)$$

for any $V\in B(R)$ such that $T\subset V$. Hence, by Theorem 1.2, $S$ is a finite module over $T$. Since $T$ is an $R$-locality, $S$ is also an $R$-locality. Thus the proof is completed. Q. E. D.

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