Publ. RIMS, Kyoto Univ.
11 (1975), 31-50

Flows Associated with Ergodic Non-Singular Transformation Groups

By

Toshihiro HAMACHI,* Yukimasa OKA** and Motosige OSIKAWA*

§0. Introduction

In this paper we shall give a nice invariant for the weak equivalence of ergodic non-singular transformation groups. It is a one-parameter ergodic non-singular flow associated with an ergodic non-singular transformation group. Since in 1960 an example of an ergodic non-singular transformation without σ-finite invariant measures was given in Ergodic theory [16], the structure and the classification of ergodic non-singular transformations have been studied by many authors ([2], [4], [5], [6], [7], [8]$\sim$[11] and [13]). Among these works, Krieger's weak equivalence theory is fundamental in the classification problem of ergodic non-singular transformation groups without σ-finite invariant measures. This classification is closely connected with the classification of type III factors in the theory of von Neumann algebras ([15]).

The Tomita-Takesaki theory of generalized Hilbert algebras ([18]) plays important roles in the analysis of type III factors. Using this theory, A. Connes [3] introduced algebraic invariants — the S-set S(M) and the T-set T(M) — for a factor M and obtained a classification of type III factors. M. Takesaki [19] introduced the dual action of the modular automorphism group and obtained the structure theorem of type III factors. In the classification problem of ergodic non-singular transformation groups G, W. Krieger [10] and the present authors [6] introduced invariants r(G) and T(G) respectively, both of which are

Communicated by H. Araki, August 5, 1974.
* Department of Mathematics, College of General Education, Kyushu University, Fukuoka.
** Department of Mathematics, Faculty of Science, Kyushu University, Fukuoka.

closely related to the existence problem of σ-finite invariant measures. The invariant r(G) and T(G) are nothing but the S-set and T-set of the group measure space construction factor M_G of G. They are also corresponding to the Araki-Woods invariant $r_\infty(M_G)$ and $\rho(M_G)$ ([1]) of the factor M_G in the case of an infinite product type transformation group G.

By the metrical properties of the associated flows, we can obtain much more informations about non-singular transformation groups. The T-set is the point spectrum of the associated flow and the S-set is illustrated by only the periodic motion of the associated flow. It is shown by Krieger's skew-product method ([9]) that any ergodic measurable measure preserving flow is realized as the associated flow of an ergodic non-singular transformation group. We study the weakly equivalent classes of the product $G \times G'$ of ergodic non-singular transformation groups by using its associated flow and we introduce a new class (type III$^\Gamma$) of non-singular transformation groups of type III. Also we can obtain examples of ergodic non-singular transformation groups G of type III satisfying that $G \times G$ is not weakly equivalent with G. We show the Araki-Woods characterization theorem of the asymptotic ratio set and the ρ-set in the sense of weak equivalence of non-singular transformation groups. By a characterization of a dissipative non-singular flow, we give another proof of the existence of an invariant measure under the condition T(G)$=\mathbf{R}$ ([6]).

Professor H. Araki showed us that Takesaki's dual action ([19]) of the modular automorphism group of the group measure space construction factor is realized by the associated flow. Professor W. Krieger informed us that he introduced a non-singular flow for an ergodic non-singular transformation and proved the one-to-one correspondence between the weak equivalence of ergodic non-singular transformations and the strong equivalence of flows ([12]).

§ 1. Preliminaries

Let $(\Omega, \mathfrak{F}, P)$ be a Lebesgue measure space. Two measures μ and ν on the measurable space $(\Omega, \mathfrak{F})$ are equivalent $\mu \sim \nu$, when $\mu(A)=0$ if and only if $\nu(A)=0$, $A \in \mathfrak{F}$. A 1–1 mapping g from Ω onto itself is a non-singular transformation if it is bi-measurable (i.e. $g^{-1}\mathfrak{F}\subset\mathfrak{F}$ and $g\mathfrak{F}\subset\mathfrak{F}$) and $Pg\sim P$ where $Pg(A)=P(gA)$, $A\in\mathfrak{F}$. Let G be a countable group of non-singular transformations of $(\Omega, \mathfrak{F}, P)$. A measure μ defined on $(\Omega, \mathfrak{F})$ is G-invariant if $\mu g=\mu$, $g\in G$ and a measurable function $f(\omega)$ is G-invariant if $f(g\omega)=f(\omega)$, $g\in G$, a.e.ω. G is ergodic if every G-invariant function on $(\Omega, \mathfrak{F}, P)$ is a constant a.e. We denote by $[G]$ the group of all non-singular transformations g of $(\Omega, \mathfrak{F}, P)$ satisfying that there exist measurable sets $A_n\in\mathfrak{F}$, $n=1, 2,\dots$ and non-singular transformations $g_n\in G$, $n=1, 2,\dots$ such that $\Omega=\bigcup_{n=1}^{\infty} A_n$ (disjoint) and $g\omega=g_n\omega$, a.e.$\omega\in A_n$, $n=1, 2,\dots$. The group $[G]$ is said to be the full group of G. Two countable non-singular transformation groups G and G' of $(\Omega, \mathfrak{F}, P)$ and $(\Omega', \mathfrak{F}', P')$ respectively, are called weakly equivalent if there exists a bi-measurable 1–1 mapping φ from Ω onto Ω' such that $\varphi[G]\varphi^{-1}=[G']$ and $P\sim P'\varphi$.

Let us now define the ratio set $\mathrm{r}(G)$ and the T-set $\mathrm{T}(G)$ of a countable non-singular transformation group G of $(\Omega, \mathfrak{F}, P)$. The ratio set $\mathrm{r}(G)$ is the closure of the set of all positive numbers r satisfying that for any $\varepsilon>0$ and any measurable set A with $P(A)>0$ there exists a measurable subset B of A with $P(B)>0$ and $g\in G$ such that $gB\subset A$ and $re^{-\varepsilon}<\frac{dPg}{dP}(\omega)<re^{\varepsilon}$, $\omega\in B$ ([10]), and the T-set $\mathrm{T}(G)$ is the set of all real numbers t satisfying that there exists a measurable function $\exp i\xi(\omega)$ such that $\exp i\{\xi(g\omega)-\xi(\omega)\}=\exp it\log\frac{dPg}{dP}(\omega)$, $g\in G$, a.e.ω ([6]). The set $\mathrm{r}(G)\backslash\{0\}$ is a multiplicative subgroup of positive numbers and $\mathrm{T}(G)$ is an additive subgroup of $\mathbf{R}$. These two sets are invariants for the weak equivalence.

For a countable ergodic non-singular transformation group G of $(\Omega, \mathfrak{F}, P)$ a pair (μ, H) is said to be an admissible pair of G if μ is a σ-finite measure equivalent with P and if H is an ergodic subgroup consisting of μ-preserving transformations of $[G]$. The set $\Delta(\mu, H)$ $=\{r>0 \mid \text{for any } \varepsilon>0 \text{ there exists } g\in[G] \text{ such that } P(re^{-\varepsilon}<\frac{d\mu g}{d\mu}(\omega)<re^{\varepsilon})>0\}$ is a closed multiplicative subgroup of $\mathbf{R}_+$ if (μ, H) is an admissible pair of G. This set is independent of the choice of an admissible pair ([6]).

For a countable ergodic non-singular transformation group G of $(\Omega, \mathfrak{F}, P)$ we consider the following cases:

(a) There exists an equivalent σ-finite invariant measure μ, or equivalently (μ, G) is an admissible pair of G.

(b_λ) There exist an admissible pair (μ, H) and $0<\lambda<1$ such that $\Delta(\mu, H)=\{\lambda^n | -\infty<n<\infty\}$.

(c) There exists an admissible pair (μ, H) such that $\Delta(\mu, H)=(0, \infty)$.

(d) There is no admissible pair.

These cases are exclusive and exaustive.

Definition ([6]). Let G be a countable ergodic non-singular transformation group of $(\Omega, \mathfrak{F}, P)$. (1) We say that G is of type III if G has no equivalent σ-finite invariant measures. (2) We say that G is of type III_λ, $0<\lambda<1$, III_1 or III_0 accordingly as the case (b_λ), (c) or (d) happens.

Note that the type of G is an invariant under the weakly equivalent relation and that if G is of type III_1 then for any null set $N \in \mathfrak{F}$ the set $\{\log\frac{d\mu g}{d\mu}(\omega) | g \in G, \omega \notin N\}$ contains at least two rationally independent real numbers.

§2. The Associated Flow $\{\tilde{T}_s\}_{-\infty<s<+\infty}$

Definition 1. A one-parameter group $\{U_s\}_{-\infty<s<+\infty}$ of non-singular transformations of $(X, \mathfrak{B}_X, \mu_X)$, which we call simply a non-singular flow, is measurable if the mapping $\mathbf{R}\times X \ni (s, x) \to U_s x \in X$ is measurable.

Let G be a countable non-singular transformation group acting on a Lebesgue measure space $(\Omega, \mathfrak{F}, P)$. We denote by $\tilde{G}$ the group of following non-singular transformations $\tilde{g}$ on $(\Omega\times\mathbf{R}, \mathfrak{F}\times\mathfrak{B}(\mathbf{R}), dP\times du)$;

$$\tilde{g}(\omega, u)=\left(g\omega,\ u+\log\frac{dPg}{dP}(\omega)\right), \qquad g\in G.$$

Let $\zeta(\tilde{G})$ be the measurable partition ([17]) generated by all $\tilde{G}$-invariant measurable sets. For $-\infty<s<+\infty$, put $T_s(\omega, u)=(\omega, u+s)$, $(\omega, u)\in\Omega\times\mathbf{R}$. Since $\{T_s\}_{-\infty<s<+\infty}$ commutes with $\tilde{G}$, we can define the factor flow $\{\tilde{T}_s\}_{-\infty<s<+\infty}$ of $\{T_s\}_{-\infty<s<+\infty}$ on the quotient space $\Omega\times\mathbf{R}/\zeta(\tilde{G})$.

$\tilde{T}_s(-\infty < s < +\infty)$ is a non-singular transformation with respect to any σ-finite measure equivalent with the image measure of $\mathrm{d}P \times \mathrm{d}u$ and $\{\tilde{T}_s\}_{-\infty < s < +\infty}$ is a measurable flow.

Definition 2. We call the factor flow $\{\tilde{T}_s\}_{-\infty < s < +\infty}$ the non-singular flow associated with the non-singular transformation group G or simply the associated flow of G.

We note that the associated flow $\{\tilde{T}_s\}_{-\infty < s < +\infty}$ of G is ergodic if and only if G is ergodic.

Definition 3. Non-singular flows $(X, \mathfrak{B}_X, \mu_X; \{U_s\}_{-\infty < s < +\infty})$ and $(Y, \mathfrak{B}_Y, \mu_Y; \{V_s\}_{-\infty < s < +\infty})$ are mutually strongly equivalent if there exists a bi-measurable 1–1 mapping ψ from X onto Y such that $\mu_X \sim \mu_Y \psi$ and for all $-\infty < s < +\infty$, $\psi U_s x = V_s \psi x$, a.e.x.

We note that the strong equivalence among ergodic non-singular flows is the same as the metrically isomorphic equivalence if they admit finite equivalent invariant measures.

Theorem 1. *If ergodic non-singular transformation groups* $(\Omega, \mathfrak{F}, P; G)$ *and* $(\Omega', \mathfrak{F}', P'; G')$ *are mutually weakly equivalent, then their associated flows are mutually strongly equivalent.*

Proof. Let φ be a bi-measurable 1–1 mapping from Ω onto Ω' such that $\varphi[G]\varphi^{-1} = [G']$ and $P \sim P'\varphi$. Put $\psi(\omega, u) = (\varphi\omega, u + \log\frac{\mathrm{d}P'\varphi}{\mathrm{d}P}(\omega))$. Then ψ is a 1–1 mapping from $\Omega \times \mathbf{R}$ onto $\Omega' \times \mathbf{R}$ and satisfies that for $-\infty < s < +\infty$, $\psi T_s(\omega, u) = T'_s\psi(\omega, u)$, a.e.$(\omega, u) \in \Omega \times \mathbf{R}$. It is enough to show that $f(\psi(\omega, u))$ is a $\tilde{G}$-invariant measurable function for any $\tilde{G}'$-invariant measurable function $f(\omega', u)$. For $g \in [G]$ and $g' = \varphi g \varphi^{-1} \in [G']$,

$$f\left(\psi\left(g\omega, u + \log\frac{\mathrm{d}Pg}{\mathrm{d}P}(\omega)\right)\right) = f\left(\varphi g\omega, u + \log\frac{\mathrm{d}Pg}{\mathrm{d}P}(\omega) + \log\frac{\mathrm{d}P'\varphi}{\mathrm{d}P}(g\omega)\right)$$

$$= f\left(g'\varphi\omega, u + \log\frac{\mathrm{d}Pg}{\mathrm{d}P}(\omega) + \log\frac{\mathrm{d}P'g'\varphi}{\mathrm{d}Pg}(\omega)\right)$$

$$= f\left(g'\varphi\omega, u + \log\frac{\mathrm{d}P'g'}{\mathrm{d}P'}(\varphi\omega) + \log\frac{\mathrm{d}P'\varphi}{\mathrm{d}P}(\omega)\right)$$

$$=f\left(\tilde{g}'\left(\varphi\omega,\ u+\log\frac{dP'\varphi}{dP}(\omega)\right)\right)$$

$$=f(\psi(\omega,\ u)). \qquad \text{Q.E.D.}$$

Definition 4. Let $(X,\ \mathfrak{B}_X,\ \mu_X;\ \{U_s\}_{-\infty<s<+\infty})$ be a non-singular flow. A real number t belongs to the set $\sigma(\{U_s\})$, which is called the point spectrum of $\{U_s\}_{-\infty<s<+\infty}$, if there exists a measurable function $\exp i\xi(x)$ such that for all $-\infty<s<+\infty$

$$\exp i\xi(U_s x)=\exp its\cdot\exp i\xi(x), \qquad \text{a.e. } x.$$

Theorem 2. *Let G be a countable non-singular transformation group and $\{\tilde{T}_s\}_{-\infty<s<+\infty}$ be its associated flow. Then the* T-*set is* $\mathrm{T}(G)=\sigma(\{\tilde{T}_s\})$.

Proof. Let $t\in\mathrm{T}(G)$. Then

$$\exp i\{\xi(g\omega)-\xi(\omega)\}=\exp it\log\frac{dPg}{dP}(\omega), \qquad g\in G, \qquad \text{a.e. } \omega,$$

for some measurable function $\exp i\xi(\omega)$. If we put $f(\omega,\ u)=\exp i\{tu-\xi(\omega)\}$, then $f(\omega,\ u+s)=\exp its\cdot f(\omega,\ u)$ and

$$f\left(g\omega,\ u+\log\frac{dPg}{dP}(\omega)\right)=\exp\{-i\xi(g\omega)\}\exp it\left(u+\log\frac{dPg}{dP}(\omega)\right)$$

$$=\exp itu\cdot\exp\{-i\xi(\omega)\}$$

$$=f(\omega,\ u), \quad g\in G, \quad \text{a.e.}(\omega,\ u).$$

Conversely, let $t\in\sigma(\{\tilde{T}_s\})$. Then

$$\exp i\eta(\omega,\ u+s)=\exp its\cdot\exp i\eta(\omega,\ u), \qquad -\infty<s<+\infty,$$

for some $\tilde{G}$-invariant measurable function $\exp i\eta(\omega,\ u)$. If we put $\exp\{-i\xi(\omega)\}=\exp i\eta(\omega,\ 0)$, then

$$\exp i\{\xi(g\omega)-\xi(\omega)\}=\exp it\log\frac{dPg}{dP}(\omega), \qquad g\in G, \qquad \text{a.e.}\,\omega. \qquad \text{Q.E.D.}$$

Theorem 3. *Let $\{\tilde{T}_s\}_{-\infty<s<+\infty}$ be the non-singular flow associ-*

ated with an ergodic countable non-singular transformation group G of $(\Omega, \mathfrak{F}, P)$. Then
(1) *G admits an equivalent σ-finite invariant measure if and only if $\{\tilde{T}_s\}_{-\infty<s<+\infty}$ is strongly equivalent with the translation*

$$\mathbf{R} \ni u \to u+s, \qquad -\infty < s < +\infty,$$

(2) *G is of type III_λ $(0<\lambda<1)$ if and only if $\{\tilde{T}_s\}_{-\infty<s<+\infty}$ is strongly equivalent with the periodic flow* $[0, -\log\lambda) \ni u \to u+s (\text{mod. } -\log\lambda)$, $-\infty<s<+\infty$,
(3) *G is of type III_1 if $\{\tilde{T}_s\}_{-\infty<s<+\infty}$ is the trivial flow, and the converse is true under the assumption that G is a cyclic group, and*
(4) *G is of type III_0 if and only if $\{\tilde{T}_s\}_{-\infty<s<+\infty}$ is an ergodic, aperiodic and conservative flow, under the assumption that G is a cyclic group.*

Proof. (1) Let μ be an equivalent σ-finite G-invariant measure. Then the measurable partition $\zeta(\tilde{G})$ is equal to $\{\Omega \times \{u\}; u \in \mathbf{R}\}$ since $\tilde{G} = \left\{\tilde{g};\ \tilde{g}(\omega, u) = \left(g\omega, u + \log\frac{\mathrm{d}\mu g}{\mathrm{d}\mu}(\omega)\right) = (g\omega, u), g \in G\right\}$. So the factor flow of $\{T_s\}_{-\infty<s<+\infty}$ is the translation on $\mathbf{R}$. Conversely let ψ be a bi-measurable 1–1 mapping from $\Omega \times \mathbf{R}/\zeta(\tilde{G})$ onto $\mathbf{R}$ such that $\{\tilde{T}_s\}_{-\infty<s<+\infty}$ commutes with the translation on $\mathbf{R}$ under the mapping ψ, and π be the canonical mapping from $\Omega \times \mathbf{R}$ onto $\Omega \times \mathbf{R}/\zeta(\tilde{G})$. For the measurable partition $\{\pi^{-1} \circ \psi^{-1}(u); u \in \mathbf{R}\}$ of $\Omega \times \mathbf{R}$ it holds that for almost all $\psi^{-1}(u)$ and for almost all ω, $\pi^{-1} \circ \psi^{-1}(u)$ intersects $\{\omega\} \times \mathbf{R}$ in exactly one point $(\omega, u - \psi(\pi(\omega, 0)))$ from the assumption. Denote for a fixed cross section $\pi^{-1} \circ \psi^{-1}(u)$, $\mathrm{d}\mu(\omega) = \exp\{-u + \psi(\pi(\omega, 0))\}\mathrm{d}P(\omega)$. Notice that a ω-function $-u + \psi(\pi(\omega, 0))$ is measurable. Then the equivalent measure μ is G-invariant. Indeed, since $\tilde{g}(\omega, u - \psi(\pi(\omega, 0))) = \left(g\omega, u - \psi(\pi(\omega, 0)) + \log\frac{\mathrm{d}Pg}{\mathrm{d}P}(\omega)\right) \in \pi^{-1} \circ \psi^{-1}(u)$, we have $\psi(\pi(g\omega, 0)) = \psi(\pi(\omega, 0)) - \log\frac{\mathrm{d}Pg}{\mathrm{d}P}(\omega)$, a.e.$\omega$, $g \in G$. Therefore

$$\mathrm{d}\mu(g\omega) = \exp\{-u + \psi(\pi(g\omega, 0))\}\mathrm{d}P(g\omega)$$

$$= \exp\left\{-u + \psi(\pi(\omega, 0)) - \log\frac{\mathrm{d}Pg}{\mathrm{d}P}(\omega)\right\}\mathrm{d}P(g\omega)$$

$$=\exp\{-u+\psi(\pi(\omega, 0))\}\mathrm{d}P(\omega)$$

$$=\mathrm{d}\mu(\omega).$$

(2) Let (μ, H) be an admissible pair of type III_λ $(0<\lambda<1)$. Then the measurable partition $\zeta(\tilde{G})$ is $\{\bigcup_{-\infty<n<+\infty} \Omega\times\{u-n\log\lambda\};\ 0\leqq u<-\log\lambda\}$, since H is ergodic and $\tilde{h}(\omega, u)=(h\omega, u)$, $h\in H$ and since for almost all ω and for any integer n there exists $g\in G$ such that $\log\frac{\mathrm{d}\mu g}{\mathrm{d}\mu}(\omega)=n\log\lambda$ (Section 6 of [6]). Therefore $\{\tilde{T}_s\}_{-\infty<s<+\infty}$ is the periodic flow $[0, -\log\lambda)\ni u\to u+s$ (mod. $-\log\lambda$), $-\infty<s<+\infty$. Conversely let ψ be a bi-measurable 1–1 mapping from $\Omega\times\mathbf{R}/\zeta(\tilde{G})$ onto $[0, -\log\lambda)$ such that $\psi T_s(\omega, u)=\psi(\omega, u)+s$ (mod. $-\log\lambda$), a.e. (ω, u). Then for almost all $\psi^{-1}(u)$ and for almost all ω, $\pi^{-1}\circ\psi^{-1}(u)$ intersects each set $\{\omega\}\times[-n\log\lambda, -(n+1)\log\lambda)$, $-\infty<n<+\infty$, in exactly one point. For a fixed u, the function $f(\omega)=\inf\{v;\ v\geqq 0$ and $(\omega, v)\in\pi^{-1}\circ\psi^{-1}(u)\cap(\{\omega\}\times \mathrm{R})\}$ is measurable. We define an equivalent measure $\mathrm{d}\mu(\omega)=\exp\{-f\}(\omega)\mathrm{d}P(\omega)$. $\tilde{G}$ acts ergodically on each ergodic component $\pi^{-1}\circ\psi^{-1}(u)$. Let $\tilde{H}$ be the induced transformation group of this action on the set $\{(\omega, f(\omega));\ \omega\in\Omega\}$. Then H is ergodic and μ-preserving and $\frac{\mathrm{d}\mu g}{\mathrm{d}\mu}(\omega)\in\{\lambda^n;\ -\infty<n<+\infty\}$, $g\in G$, a.e. ω.

(3) Let (μ, H) be an admissible pair of a non-singular transformation group G of type III_1. If $f(\omega, u)$ is a $\tilde{G}$-invariant measurable function, then for almost all (ω, u), $f(h\omega, u)=f(\omega, u)$, $h\in H$. From the ergodicity of H, $f(\omega, u)=f(u)$, a.e. (ω, u) and so for almost all (ω, u), $f(u+\log\frac{\mathrm{d}\mu g}{\mathrm{d}\mu}(\omega))=f(u)$, $g\in G$. Let $\Gamma=\{t\in\mathbf{R};\ f(u+t)=f(u)$, a.e. $u\}$ then Γ is a subgroup of $\mathbf{R}$ and contains at least two rationally independent real numbers by the definition of type III_1. Thus Γ is a dense subgroup. Therefore $f(u)=$constant, a.e. u. Conversely let G be a cyclic group and let G be ergodic. Then for any set $A\in\mathfrak{F}$ with positive measure, any positive number r and any positive number ε, there exist a measurable subset $\tilde{B}\subset A\times[-\varepsilon, \varepsilon]$ with positive measure and $g\in G$ such that $\tilde{g}\tilde{B}\subset A\times[\log r-\varepsilon, \log r+\varepsilon]$. Put $B=\{\omega\in A;\ (\omega, u)\in\tilde{B}$ for some $u\}$. Then $P(B)>0$, $gB\subset A$ and $r\exp\{-2\varepsilon\}<\frac{\mathrm{d}Pg}{\mathrm{d}P}(\omega)<r\exp 2\varepsilon$, $\omega\in B$. Therefore the ratio set is $r(G)=[0, +\infty)$. By Theorem 2.8 of [10], G is of type III_1.

(4) Since an ergodic and dissipative non-singular flow on a Lebesgue space is metrically isomorphic with the translation on $\mathbf{R}$, (4) follows from (1), (2) and (3) in the case that G is a cyclic group. Q.E.D.

Problem. Is G of type III_1 if $\tilde{G}$ is ergodic, without the assumption that G is a cyclic group?

Theorem 4. *Let $(\Omega, \mathfrak{F}, P; G)$ be of type III_1 with an admissible pair (P, H) and $(X, \mathfrak{B}_X, \mu_X; \{U_s\}_{-\infty<s<+\infty})$ be a measurable ergodic measure preserving flow and let $G_{\{U\}}=\{g_{\{U\}}; g\in G\}$ where*

$$g_{\{U\}}(\omega, x)=(g\omega, U_{-\log\frac{dPg}{dP}(\omega)}x), \qquad g\in G.$$

Then the associated flow $\{\tilde{T}_s\}_{-\infty<s<+\infty}$ of the ergodic non-singular transformation group $G_{\{U\}}$ is metrically isomorphic with $\{U_s\}_{-\infty<s<+\infty}$.

Proof. Let $f(\omega, x, u)$ be a $\tilde{G}_{\{U\}}$-invariant bounded measurable function. Then $f(h\omega, x, u)=f(\omega, x, u)$, $h\in H$, a.e.(ω, x, u) and the ergodicity of H implies $f(\omega, x, u)=f(x, u)$, a.e.(ω, x, u). So we have for almost all (ω, x, u), $f\left(U_{-\log\frac{dPg}{dP}(\omega)}x, u+\log\frac{dPg}{dP}(\omega)\right)=f(x, u)$, $g\in G$. Let $\Gamma=\{t\in\mathbf{R}|$ $f(U_{-t}x, u+t)=f(x, u)$, a.e.$(x, u)\}$. Then Γ is a subgroup of $\mathbf{R}$ and contains at least two rationally independent real numbers by the definition of type III_1. Thus Γ is a dense subgroup and it follows from Lemma 2 (2) of Section 4 that for any $-\infty<s<+\infty$ $f(U_{-s}x, u+s)$ $=f(x, u)$, a.e.(x, u). Conversely a $\{U_{-s}\times\tau_s\}_{-\infty<s<+\infty}$-invariant measurable function is $\tilde{G}_{\{U\}}$-invariant, where $\tau_s u=u+s$, $u\in\mathbf{R}$, $-\infty<s<+\infty$, and then $\zeta(\tilde{G}_{\{U\}})=\Omega\times\zeta(\{U_{-s}\times\tau_s\})$. Since $\{U_s\}_{-\infty<s<+\infty}$ is a measurable flow, for $E\in\mathfrak{B}_X$ the set

$$\{(U_{-s}x, s); x\in E, -\infty<s<+\infty\}=\{(x, s); U_{-s}x\in E, -\infty<s<+\infty\}$$

is $\{U_{-s}\times\tau_s\}$-invariant and measurable in $X\times\mathbf{R}$. For a countably separating base $\{E_n\}_{n\geqq 1}$ of $(X, \mathfrak{B}_X, \mu_X)$, $\{\pi(\{(U_{-s}x, s); x\in E_n, -\infty<s<+\infty\})\}_{n\geqq 1}$ is a countably separating base of $X\times\mathbf{R}/\zeta(\{U_{-s}\times\tau_s\})$, where π is the canonical mapping from $X\times\mathbf{R}$ onto $X\times\mathbf{R}/\zeta(\{U_{-s}\times\tau_s\})$. Therefore the measurable partition $\zeta(\{U_{-s}\times\tau_s\})$ is given by $\{\{(U_{-r}x, r);$ $-\infty<r<+\infty\}; x\in X\}$. We define a mapping ψ from $\Omega\times X\times\mathbf{R}/\zeta(\tilde{G}_{\{U\}})$

onto X by $\psi(\Omega\times\{(U_{-r}x, r); -\infty<r<+\infty\})=x$. The mapping ψ is bi-measurable 1–1 and satisfies

$$\psi(\Omega\times\{(U_{-r}x, r+s); -\infty<r<+\infty\})$$

$$=U_s\psi(\Omega\times\{(U_{-r}x, r); -\infty<r<+\infty\}), \qquad -\infty<s<+\infty.$$

Therefore the associated flow $\{\tilde{T}_s\}_{-\infty<s<+\infty}$ is metrically isomorphic with $\{U_s\}_{-\infty<s<+\infty}$ under the mapping ψ. Q.E.D.

We note that, in the case that G of Theorem 4 is a cyclic group of type III_1, $G_{\{U\}}$ is of type III_0 if an ergodic measure preserving flow $\{U_s\}_{-\infty<s<+\infty}$ is aperiodic and conservative, and that furthermore if $\{U_s\}_{-\infty<s<+\infty}$ is a weakly mixing flow then $\mathrm{T}(G_{\{U\}})=\{0\}$ (cf. [3], [6]).

Let $(\Omega, \mathfrak{F}, P; G)$ be of type III_λ, $0<\lambda<1$, and $n_g(\omega)$ be an integer-valued function such that $\frac{\mathrm{d}Pg}{\mathrm{d}P}(\omega)=\lambda^{n_g(\omega)}$ and let $(X, \mathfrak{B}_X, \mu_X; U)$ be an ergodic measure preserving transformation. W. Krieger ([11]) constructed an ergodic non-singular transformation group $G_U=\{g_U; g\in G\}$ where $g_U(\omega, x)=(g\omega, U^{n_g(\omega)}x)$. It is easy to show that the associated flow of G_U is metrically isomorphic with the flow built under the constant function $-\log\lambda$ with the basic transformation U. From this it follows that for ergodic finite measure preserving transformations U and V, G_U is weakly equivalent with G_V if and only if U is metrically isomorphic with V ([9]).

§3. The Joint Flow and the Product of Non-Singular Transformation Groups

Let $\{U_s\}_{-\infty<s<+\infty}$ and $\{V_s\}_{-\infty<s<+\infty}$ be measurable non-singular flows acting on Lebesgue spaces $(X, \mathfrak{B}_X, \mu_X)$ and $(Y, \mathfrak{B}_Y, \mu_Y)$, respectively. Let $\{U_s\times \mathrm{I}\}_{-\infty<s<+\infty}$ be a flow defined by $(U_s\times\mathrm{I})(x, y)=(U_sx, y)$. Since $\{U_s\times\mathrm{I}\}_{-\infty<s<+\infty}$ commutes with $\{U_s\times V_{-s}\}_{-\infty<s<+\infty}$, we can define the factor flow of the flow $\{U_s\times\mathrm{I}\}_{-\infty<s<+\infty}$ on the quotient space $X\times Y/\zeta(\{U_s\times V_{-s}\})$ and denote it by $\{(U, V)_s\}_{-\infty<s<+\infty}$. $\{(U, V)_s\}_{-\infty<s<+\infty}$ is a measurable non-singular flow with respect to the image measure of $\mu_X\times\mu_Y$ on $X\times Y/\zeta(\{U_s\times V_{-s}\})$.

Definition 4. We call $\{(U, V)_s\}_{-\infty<s<+\infty}$ the joint flow of $\{U_s\}_{-\infty<s<+\infty}$ and $\{V_s\}_{-\infty<s<+\infty}$.

We note that $\{(U, V)_s\}_{-\infty<s<+\infty}$ is strongly equivalent with $\{(V, U)_s\}_{-\infty<s<+\infty}$ and that $\sigma(\{(U, V)_s\})=\sigma(\{U_s\})\cap\sigma(\{V_s\})$.

Theorem 5. *Let $G\times G'=\{g\times g';\ g\in G, g'\in G'\}$ be the product non-singular transformation group of countable non-singular transformation groups G and G', where $g\times g'(\omega, \omega')=(g\omega, g'\omega')$, $\omega\in\Omega$, $\omega'\in\Omega'$. Then the associated flow of $G\times G'$ is strongly equivalent with the joint flow of each associated flows.*

Proof. We define a mapping ψ from $\Omega\times\mathbf{R}\times\Omega'\times\mathbf{R}$ onto $\Omega\times\Omega'\times\mathbf{R}$ as follows

$$\psi(\omega, u, \omega', u')=(\omega, \omega', u+u').$$

Since

$$\tilde{g}\times\tilde{g}'(\omega, u, \omega', u')=\left(g\omega,\ u+\log\frac{\mathrm{d}Pg}{\mathrm{d}P}(\omega),\ g'\omega',\ u'+\log\frac{\mathrm{d}P'g'}{\mathrm{d}P'}(\omega')\right)$$

and

$$\widetilde{g\times g'}(\omega, \omega', u)=\left(g\omega,\ g'\omega',\ u+\log\frac{\mathrm{d}Pg}{\mathrm{d}P}(\omega)+\log\frac{\mathrm{d}P'g'}{\mathrm{d}P'}(\omega')\right),$$

we have

$$\psi\cdot(\tilde{g}\times\tilde{g}')=\widetilde{g\times g'}\cdot\psi,\qquad g\in G,\quad g'\in G'.$$

Hence ψ induces a mapping from the product space of the quotient spaces $(\Omega\times\mathbf{R}/\zeta(\tilde{G}))\times(\Omega'\times\mathbf{R}/\zeta(\tilde{G}'))$ onto the quotient space $\Omega\times\Omega'\times\mathbf{R}/\zeta(\widetilde{G\times G'})$. Since $\psi(\omega, u+s, \omega', u'-s)=(\omega, \omega', u+u')$ and $\psi(\omega, u+s, \omega', u')=(\omega, \omega', u+u'+s)$, ψ induces a strongly equivalent mapping between the joint flow of the associated flows of G and G' and the associated flow of $G\times G'$. Q.E.D.

Definition 5. For a countable additive subgroup Γ of $\mathbf{R}$, a countable ergodic non-singular transformation group G is of type III^{Γ} if the associated flow $\{\tilde{T}_s\}_{-\infty<s<+\infty}$ of G is ergodic finite measure preserving and if it has the pure point spectrum Γ.

We note $\mathrm{III}^{\frac{2\pi}{\log\lambda}Z}=\mathrm{III}_\lambda$, and $\mathrm{III}^{\{0\}}=\mathrm{III}_1$ (in case of cyclic group) by Theorem 3 and von Neumann's theorem that ergodic finite measure preserving flows with the same pure point spectrum are mutually metrically isomorphic ([14]).

Theorem 6. (1) *Let G be of type III^Γ and G' be any countable ergodic non-singular transformation group. Then $G\times G'$ is of type $III^{\Gamma\cap T(G')}$.*
(2) *Let G and G' be countable ergodic non-singular transformation groups whose associated flows have finite invariant measures. Then $G\times G'$ is of type $III^{T(G)\cap T(G')}$.*

Proof. The proof follows from the next lemma.

Lemma 1. (1) *Let $(X, \mathfrak{B}_X, \mu_X; \{U_s\}_{-\infty<s<+\infty})$ be an ergodic finite measure preserving flow which has the pure point spectrum and $(Y, \mathfrak{B}_Y, \mu_Y; \{V_s\}_{-\infty<s<+\infty})$ be an ergodic non-singular flow. Then the joint flow $\{(U, V)_s\}_{-\infty<s<+\infty}$ is ergodic finite measure preserving and has the pure point spectrum $\sigma(\{U_s\})\cap\sigma(\{V_s\})$.*
(2) *Let $(X, \mathfrak{B}_X, \mu_X; \{U_s\}_{-\infty<s<+\infty})$ and $(Y, \mathfrak{B}_Y, \mu_Y; \{V_s\}_{-\infty<s<+\infty})$ be ergodic finite measure preserving flows. Then the joint flow $\{(U, V)_s\}_{-\infty<s<+\infty}$ is ergodic finite measure preserving and has the pure point spectrum $\sigma(\{U_s\})\cap\sigma(\{V_s\})$.*

Proof. If μ_X is a $\{U_s\}_{-\infty<s<+\infty}$-invariant finite measure and if μ_Y is a finite measure, $\{(U, V)_s\}_{-\infty<s<+\infty}$ preserves the image measure of $\mu_X\times\mu_Y$ on the quotient space $X\times Y/\zeta(\{U_s\times V_{-s}\})$. There exist measurable functions $\exp i\xi_t(x)$ for $t\in\sigma(\{U_s\})$ such that $\exp i\xi_t(U_sx)=\exp its\cdot\exp i\xi_t(x)$ and measurable functions $\exp i\eta_t(y)$ for $t\in\sigma(\{V_s\})$ such that $\exp i\eta_t(V_sy)=\exp its\cdot\exp i\eta_t(y)$. Since

$$\begin{aligned}\exp i\xi_t(U_sx)\exp i\eta_t(V_{-s}y)&=\exp its\cdot\exp i\xi_t(x)\times\exp\{-its\}\exp i\eta_t(y)\\&=\exp i\xi_t(x)\exp i\eta_t(y)\end{aligned}$$

for $t \in \sigma(\{U_s\}) \cap \sigma(\{V_s\})$, $\exp i\xi_t(x) \exp i\eta_t(y)$ is a $\{U_s \times V_{-s}\}_{-\infty<s<+\infty}$-invariant function.

We will show that the set of all $\{U_s \times V_{-s}\}_{-\infty<s<+\infty}$-invariant square integrable functions are generated by $\{\exp i\xi_t(x) \exp i\eta_t(x);\ t \in \sigma(\{U_s\}) \cap \sigma(\{V_s\})\}$. Let $f(x, y)$ be a bounded $\{U_s \times V_{-s}\}_{-\infty<s<+\infty}$-invariant measurable function and assume

$$\langle f(\cdot, \cdot), \exp i\xi_t(\cdot) \exp i\eta_t(\cdot)\rangle_{L^2(\mu_X \times \mu_Y)} = 0$$

for $t \in \sigma(\{U_s\}) \cap \sigma(\{V_s\})$. Define

$$\hat{f}_t(y) = \langle f(\cdot, y), \exp i\xi_t(\cdot)\rangle_{L^2(\mu_X)}, \qquad y \in Y,$$

for $t \in \sigma(\{U_s\})$. Then we have

$$\begin{aligned}\hat{f}_t(V_s y) &= \langle f(\cdot, V_s y), \exp i\xi_t(\cdot)\rangle_{L^2(\mu_X)} \\ &= \langle f(U_s\cdot, y), \exp i\xi_t(\cdot)\rangle_{L^2(\mu_X)} \\ &= \langle f(\cdot, y), \exp i\xi_t(U_{-s}\cdot)\rangle_{L^2(\mu_X)} \\ &= \langle f(\cdot, y), \exp\{-its\}\cdot\exp i\xi_t(\cdot)\rangle_{L^2(\mu_X)} \\ &= \exp\{its\}\hat{f}_t(y), \qquad t \in \sigma(\{U_s\}).\end{aligned}$$

Since $\{V_s\}_{-\infty<s<+\infty}$ is ergodic, we have

$$\hat{f}_t(y) = \begin{cases} c_t\cdot\exp i\xi_t(y) & \text{if } t \in \sigma(\{U_s\}) \cap \sigma(\{V_s\}) \\ 0 & \text{if } t \in \sigma(\{U_s\}) \backslash \sigma(\{V_s\}), \end{cases}$$

where c_t is a constant. Hence we have

$$\hat{f}_t(y) = 0, \qquad \text{a.e. } y,$$

for any $t \in \sigma(\{U_s\})$ from the assumption on f.

Next consider the case (2) and take a measurable bounded function $\xi(x)$ which is orthogonal to every $\exp i\xi_t(x)$, $t \in \sigma(\{U_s\})$. Defining $\hat{f}_\xi(y) = \langle f(\cdot, y), \xi\rangle_{L^2(\mu_X)}$, we will see $\hat{f}_\xi(y) = 0$, a.e. y. Indeed

$$\begin{aligned}\hat{f}_\xi(V_s y) &= \langle f(\cdot, V_s y), \xi\rangle_{L^2(\mu_X)} \\ &= \langle f(U_s\cdot, y), \xi\rangle_{L^2(\mu_X)} \\ &= \langle f(\cdot, y), \xi(U_{-s}\cdot)\rangle_{L^2(\mu_X)}.\end{aligned}$$

From Stone's spectral decomposition theorem,

$$\xi(U_s\cdot)=\int_{-\infty}^{\infty}\exp is\lambda\cdot dE(\lambda)\xi .$$

Then

$$\hat{f}_\xi(V_s y)=\int_{-\infty}^{\infty}\exp is\lambda\cdot d<f(\cdot,\ y),\ E(\lambda)\xi(\cdot)>_{L^2(\mu_X)} .$$

We put $dF(\lambda)=d<f(\cdot,\ y),\ E(\lambda)\xi(\cdot)>_{L^2(\mu_X)}$. This measure is non-atomic since $\xi(\cdot)$ is orthogonal to all eigenfunctions of $\{U_s\}_{-\infty<s<+\infty}$. Therefore

$$\frac{1}{s}\int_0^s |\hat{f}_\xi(V_s y)|^2 ds=\iint\frac{\exp is(\lambda'-\lambda)}{s(\lambda'-\lambda)}dF(\lambda)dF(\lambda'), \qquad \text{a.e. } y.$$

The right term converges to 0 as $s\to\infty$ by Lebesgue's convergence theorem. Since $\{V_s\}_{-\infty<s<+\infty}$ is ergodic finite measure preserving, from Birkhoff's pointwise ergodic theorem we have $\hat{f}_\xi(y)=0$, a.e. y.

Thus for almost all $y, f(x, y)$ is orthogonal to any $\mathfrak{B}_X$-measurable bounded function and so $f(x, y)=0$, a.e. (x, y). This means that the all $\{U_s\times V_{-s}\}_{-\infty<s<+\infty}$-invariant square integrable functions are generated by $\{\exp i\xi_t(x)\exp i\eta_t(y);\ t\in\sigma(\{U_s\})\cap\sigma(\{V_s\})\}$ and that $\{(U,\ V)_s\}_{-\infty<s<+\infty}$ has the pure point spectrum $\sigma(\{U_s\})\cap\sigma(\{V_s\})$. Q.E.D.

Corollary 1 (cf. [1]). Let G_λ be of type III_λ $(0<\lambda<1)$ and G be a cyclic ergodic non-singular transformation group. Then

(1) G is of type $\mathrm{III}_{\lambda^{1/k}}$ for some integer k or is of type III_1 if $G_\lambda\times G$ is weakly equivalent with G, and

(2) $G_\lambda\times G$ is of type III_λ if and only if $\frac{2\pi}{\log\lambda}\in \mathrm{T}(G)$.

Proof. The proof is clear from Theorem 6.

Corollary 2 ([6]). Let U and V be ergodic finite measure preserving transformations of $(X, \mathfrak{B}_X, \mu_X)$ and $(Y, \mathfrak{B}_Y, \mu_Y)$ respectively and let G be an ergodic non-singular transformation group of type III_λ $(0<\lambda<1)$. Then $G_U\times G_V$ is of type III_λ if and only if $U\times V^{-1}$ is ergodic, where G_U is the ergodic non-singular transformation group whose definition is given in Section 2.

Proof. We have $T(G_U)=\frac{2\pi}{\log\lambda}\sigma_U$ and $T(G_V)=\frac{2\pi}{\log\lambda}\sigma_V$, where σ_U is the point spectrum set of U, that is the set of all $t\in\mathrm{R}$ such that there exists an $L^2(\mu_X)$-function ξ satisfying $\xi(Ux)=\exp 2\pi it\cdot\xi(x)$, a.e. x. By Theorem 6, $G_U\times G_V$ is of type III_λ if and only if $\sigma_U\cap\sigma_V=Z$. By Lemma 1, $\sigma_U\cap\sigma_V=Z$ if and only if $U\times V^{-1}$ is ergodic.

§4. A Remark on the Existence of an Invariant Measure

In [6] we showed the existence of an invariant measure under the condition that the T-set $T(G)=\mathbf{R}$. It is a measure-theoretical proof of the Connes' theorem for von Neumann algebras ([3]).

Here we give another proof of this theorem by using the associated flow.

Lemma 2. *Let* $\{U_s\}_{-\infty<s<+\infty}$ *be a measurable non-singular flow of a probability space* $(X, \mathfrak{B}, \mu)$. *Then*

$$(1)\qquad \int\left|f(U_sx)\sqrt{\frac{d\mu U_s}{d\mu}(x)}-f(U_{s_0}x)\sqrt{\frac{d\mu U_{s_0}}{d\mu}(x)}\right|^2 d\mu(x)\to 0$$

as $s\to s_0$ *for* $f\in L^2(X, \mathfrak{B}, \mu)$.

$$(2)\qquad \int|h(U_sx)-h(U_{s_0}x)|^2 d\mu(x)\to 0$$

as $s\to s_0$ *for* $h\in L^\infty(X, \mathfrak{B}, \mu)$.

Proof. (1) We put $\alpha(s, x)=\frac{d\mu U_s}{d\mu}(x)$ and for $N>0$, $\alpha_N(s, x)=\alpha(s, x)$ if $0<\alpha(s, x)\leq N$ and $\alpha_N(s, x)=N$ if $\alpha(s, x)>N$ which are (s, x)-measurable functions. We denote by $\mathbf{U}_s$ the unitary operator $(\mathbf{U}_sf)(x)=f(U_sx)\sqrt{\frac{d\mu U_s}{d\mu}(x)}$. For $|s|<1$ and $f\in L^\infty(X, \mathfrak{B}, \mu)$ with $|f(x)|<C$,

$$\|\mathbf{U}_sf(x)-f(x)\|_{L^2(\mu)}=\int_0^1\|\mathbf{U}_{s+u}f(x)-\mathbf{U}_uf(x)\|_{L^2(\mu)}du$$

$$\leq 2C\int_{-1}^{2}\|\sqrt{\alpha(u, x)}-\sqrt{\alpha_N(u, x)}\|_{L^2(\mu)}du$$

$$+\int_0^1\|f(U_{s+u}(x))\sqrt{\alpha_N(s+u, x)}-f(U_u(x))\sqrt{\alpha_N(u, x)}\|_{L^2(\mu)}du$$

For any $\varepsilon>0$ there exists N such that

$$\int_{-1}^{2}\|\sqrt{\alpha(u,x)}-\sqrt{\alpha_N(u,x)}\|_{L^2(\mu)}\mathrm{d}u<\varepsilon.$$

From Fubini's theorem

$$\int_0^1\|f(U_{s+u}(x))\sqrt{\alpha_N(s+u,x)}-f(U_u(x))\sqrt{\alpha_N(u,x)}\|_{L^2(\mu)}\mathrm{d}u$$

$$\leqq\left\{\int \mathrm{d}\mu(x)\int_0^1|f(U_{s+u}(x))\sqrt{\alpha_N(s+u,x)}-f(U_u(x))\sqrt{\alpha_N(u,x)}|^2\mathrm{d}u\right\}^{\frac{1}{2}}.$$

From the Riemann-Lebesgue theorem

$$\int_0^1|f(U_{s+u}(x))\sqrt{\alpha_N(s+u,x)}-f(U_u(x))\sqrt{\alpha_N(u,x)}|^2\mathrm{d}u\to 0$$

as $s\to 0$, a.e. x. Therefore from Lebesgue's convergence theorem

$$\int \mathrm{d}\mu(x)\int_0^1|f(U_{s+u}(x))\sqrt{\alpha_N(s+u,x)}-f(U_u(x))\sqrt{\alpha_N(u,x)}|^2\mathrm{d}u\to 0$$

as $s\to 0$. Since $\mathbf{U}_s$ is a unitary operator,

$$\|\mathbf{U}_s f-f\|_{L^2(\mu)}\to 0 \qquad \text{as} \quad s\to 0$$

for any $f\in L^2(X,\mathfrak{B},\mu)$.

(2) For $h\in L^\infty(X,\mathfrak{B},\mu)$ with $|h(x)|<C$,

$$\|h(U_{s+s_0}x)-h(U_{s_0}x)\|_{L^2(\mu)}$$

$$\leqq\|h(U_{s+s_0}x)-\mathbf{U}_s h(U_{s_0}x)\|_{L^2(\mu)}+\|\mathbf{U}_s h(U_{s_0}x)-h(U_{s_0}x)\|_{L^2(\mu)}$$

$$\leqq C\|1-\mathbf{U}_s\cdot 1\|_{L^2(\mu)}+\|\mathbf{U}_s h(U_{s_0}x)-h(U_{s_0}x)\|_{L^2(\mu)}.$$

From (1),

$$\|h(U_s x)-h(U_{s_0}x)\|_{L^2(\mu)}\to 0 \qquad \text{as} \quad s\to s_0.$$

Q.E.D.

Proposition 1. *Let* $\{U_s\}_{-\infty<s<+\infty}$ *be a measurable ergodic non-singular flow of a Lebesgue measure space* $(X,\mathfrak{B},\mu)$. *If* $\sigma(\{U_s\})=\mathbf{R}$, *then* $\{U_s\}_{-\infty<s<+\infty}$ *is metrically isomorphic with the translation on* $\mathbf{R}$.

Proof. The proof is similar to the one of Theorem 3 of [6] and we use the well-known lemmas quoted in [6]. First we show that there exists a (t, x)-measurable function $\exp i\xi(t, x)$ such that for $-\infty<s<+\infty$

$$(*)\qquad \exp i\xi(t, U_s x)=\exp ist\cdot\exp i\xi(t, x), \qquad \text{a.e.}\,(t, x).$$

We may assume that μ is a probability measure. Let Γ be the set of all complex valued measurable functions with absolute value 1 on $(X, \mathfrak{B}, \mu)$ and Γ_0 be the set of constant functions of Γ. Γ is a complete separable metric space under the relative $L^2(\mu)$-topology on Γ. Under the multiplication, Γ is a topological group with respect to this topology and Γ_0 is its closed subgroup. From Lemma 5 ([6]) there exists a Borel subset B of Γ that intersects each coset of the quotient space Γ/Γ_0 in exactly one point. We denote by π the canonical mapping from Γ onto Γ/Γ_0 and denote by $\pi|_B$ the restriction to B. For each $-\infty<s<+\infty$ and $E\in\mathfrak{B}$ we denote by $\tau_{s,E}$ a function

$$\exp i\xi(\cdot)\Gamma_0\to\int_E \exp i\{\xi(U_s x)-\xi(x)\}\sqrt{\frac{d\mu U_s}{d\mu}(x)}\,d\mu(x)$$

defined on Γ/Γ_0. Since

$$\left|\int_E\{\exp i\{\xi'(U_s x)-\xi'(x)\}-\exp i\{\xi(U_s x)-\xi(x)\}\}\sqrt{\frac{d\mu U_s}{d\mu}(x)}\,d\mu(x)\right|$$
$$\leqq 2\left(\int|\exp i\xi'(x)-\exp i\xi(x)|^2 d\mu(x)\right)^{\frac{1}{2}},$$

the function $\tau_{s,E}\circ\pi|_B$ defined on B is continuous under the relative $L^2(\mu)$-topology on B. Let $\mathfrak{C}$ be the smallest σ-algebra of Γ/Γ_0 such that every function $\tau_{s,E}$, $-\infty<s<+\infty$, $E\in\mathfrak{B}$, is measurable. We prove that $\mathfrak{C}$ has a countably separating base. It is enough to show that for a countably separating base $\{E_n\}_{n\geqq 1}$ of $\mathfrak{B}$ and a countable dense set K of R, $\mathfrak{C}$ is generated by τ_{s,E_n}, $s\in K$, $n\geqq 1$. From Lemma 2, for $s\in$ R and $E\in\mathfrak{B}$ there exist $s_n\in K$ and E_{m_n} such that

$$\tau_{s_n,E_{m_n}}(\exp i\xi(\cdot)\Gamma_0)\to\tau_{s,E}(\exp i\xi(\cdot)\Gamma_0), \qquad \text{as}\quad n\to\infty.$$

Since $\tau_{s,E}\circ\pi|_B$, $-\infty<s<+\infty$, $E\in\mathfrak{B}$ is continuous, $\pi|_B$ is measurable under the σ-algebra generated by the relative $L^2(\mu)$-topology of B and the σ-algebra $\mathfrak{C}$. From Lemma 7 ([6]) the inverse mapping $\pi|_B^{-1}$ is also measurable. For each $t \in \mathrm{R}$, let Γ_t be the set of all $\mathfrak{B}$-measurable solution $\exp i\xi(\cdot)$ of the equation, for $-\infty < s < +\infty$, $\exp i\xi(U_s x) = \exp its \cdot \exp i\xi(x)$, a.e. x. Then Γ_t is a coset in Γ/Γ_0. By α we denote a mapping $t \to \Gamma_t$ from $\mathbf{R}$ into Γ/Γ_0. Since the function

$$\tau_{s,E} \circ \alpha(t) = \int_E \exp its \sqrt{\frac{\mathrm{d}\mu U_s}{\mathrm{d}\mu}(x)}\,\mathrm{d}\mu(x)$$

is t-continuous for each $-\infty < s < +\infty$, $E \in \mathfrak{B}$, the mapping α is measurable. For each $E \in \mathfrak{B}$ we denote by γ_E the function $\exp i\xi(\cdot) \to \int_E \exp i\xi(x)\mathrm{d}\mu(x)$ defined on B. The function γ_E is continuous under the $L^2(\mu)$-topology. Since α, $\pi|_B^{-1}$ and γ_E are all measurable, the function $\gamma_E \circ \pi|_B^{-1} \circ \alpha(t) = \int_E \exp i\xi_t(x)\mathrm{d}\mu(x)$ is t-measurable for each $E \in \mathfrak{B}$, where $\exp i\xi_t(\cdot) = \pi|_B^{-1}\Gamma_t$. From Lemma 3.1 of [20], there exists a (t, x)-measurable function $\exp i\xi(t, x)$ such that for almost all t, $\exp i\xi(t, x) = \exp i\xi_t(x)$ holds except a x-null set. Then $\exp i\xi(t, x)$ satisfies the equation $(*)$. For a fixed $x_0 \in X$ the function

$$\exp i\bar{\xi}(t, x) = \exp i\{\xi(t, x) - \xi(t, x_0)\}$$

belongs to the coset Γ_t and satisfies for almost all x

$$\exp i\bar{\xi}(t+\tau, x) = \exp i\bar{\xi}(t, x) \cdot \exp i\bar{\xi}(\tau, x), \qquad \text{a.e.}\,(t, \tau).$$

From Lemma 9 ([6]) there exists a real measurable function $\xi(x)$ such that for almost all x

$$\exp i\bar{\xi}(t, x) = \exp it\xi(x), \qquad \text{a.e.}\, t.$$

Since $\exp i\bar{\xi}(t, \cdot) \in \Gamma_t$, for $-\infty < s < +\infty$ and for almost all x

$$\exp it\xi(U_s x) = \exp its \cdot \exp it\xi(x), \qquad \text{a.e.}\, t.$$

Therefore we have for $-\infty < s < +\infty$

$$\xi(U_s x) = \xi(x) + s, \qquad \text{a.e.}\, x.$$

Then the flow $\{U_s\}_{-\infty < s < +\infty}$ is metrically isomorphic with the translation on R under the mapping $x \to \xi(x)$ from X onto $\mathbf{R}$. Q.E.D.

Remark. If $\{U_s\}_{-\infty<s<+\infty}$ is not necessarily ergodic, then $\{U_s\}_{-\infty<s<+\infty}$ is metrically isomorphic with a flow $(x_0, u) \to (x_0, u+s)$, $-\infty<s<+\infty$, defined on a measure space $X_0 \times \mathbf{R}$ under the same condition of Proposition 1.

Corollary ([6], Theorem 3). Let G be a countable non-singular transformation group. If the T-set $\mathrm{T}(G)=\mathbf{R}$, then G admits an equivalent σ-finite invariant measure.

Proof. The proof follows from Theorem 2, Proposition 1 and Theorem 3.

References

[1] Araki, H. and Woods, E. J., A classification of factors, *Publ. RIMS, Kyoto Univ.*, **4** (1968), 51–130.

[2] Arnold, L. K., On σ-finite invariant measures, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete,* **9** (1968), 85–97.

[3] Connes, A., Une classification de facteurs de type III, *Ann. Sci. École. Norm. Sup.*, **6** (1973), 133–252.

[4] Hajian, A. B., Ito, Y. and Kakutani, S., Invariant measures and orbits of dissipative transformations, *Advances in Math.*, **9** (1972), 52–65.

[5] Hamachi, T., Equivalent measures on product spaces, *Mem. Fac. Sci. Kyushu Univ.,* **27** (1973), 335–341.

[6] Hamachi, T., Oka, Y. and Osikawa, M., A classification of ergodic non-singular transformation groups, *Mem. Fac. Sci. Kyushu Univ.*, **28** (1974), 113–133.

[7] Hill, D., σ-finite invariant measures on infinite product spaces, *Trans. Amer. Math. Soc.*, **153** (1971), 347–370.

[8] Krieger, W., On non-singular transformations of a measure space I, II, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete,* **11** (1969), 83–91, 98–119.

[9] ——— On hyperfinite factors and non-singular transformations of a measure space, *preprint.*

[10] ——— On the Araki-Woods' asymptotic ratio set and non-singular transformations of a measure space, *Lecture notes in Math.*, **160** (1970), 158–177.

[11] ——— On a class of hyperfinite factors that arise from null-recurrent Markov chains, *J. Functional Analysis,* **7** (1971), 27–42.

[12] ——— On ergodic flows and isomorphism of factors, *preprint.*

[13] Moore, C. C., Invariant measures on product spaces, *Fifth Berkley Symposium on Math. Stat. and Prob.* (1967), 447–459.

[14] von Neumann, J., Zur operatoren Methode in der klassischen Mechanik, *Ann. of Math.,* **33** (1932), 587–642.

[15] von Neumann, J., On rings of operators, III, *Ann. of Math.,* **41** (1940), 94–161.

[16] Ornstein, D. S., On invariant measures, *Bull. Amer. Math. Soc.*, **66** (1960), 297–300.

[17] Rohlin, V. A., On the fundamental ideas of a measure theory, *Mat. Sb.*, **25** (67) (1949), 107–150=*Amer. Math. Soc. Transl.*, **71** (1952).

[18] Takesaki, M., Tomita's theory of modular Hilbert algebras, *Lecture notes in Math.*, **128** (1970).

[19] ———— Duality for crossed products and the structure of von Neumann algebras of type III, *Acta Math.*, **131** (1973), 249–310.

[20] Mackey., G. W. A theorem of Stone and von-Neumann, *Duke Math. J.*, **16** (1949), 313–326.